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Dichotomy spectrum and reducibility for mean hyperbolic systems

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The topological structure of 'mean dichotomy spectrum' is shown in this article, as an extension of Sacker–Sell spectrum and non-uniform dichotomy spectrum. With regard to mean hyperbolic systems, the coexistence of expansion and contraction behaviours can lead to non-hyperbolic phenomena during evolution process. To be precise, distinct from uniform and non-uniform hyperbolic cases, error hyperbolic degree $\varepsilon(t, \tau)$ is vital to depict the spectral manifolds. As application, the reducibility theorem for mean hyperbolic systems is provided to deduce block diagonalization.

Keywords: dichotomy spectrum; Sacker–Sell spectrum; mean hyperbolicity; non-uniform hyperbolicity; reducibility

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1. Introduction

The notion of dichotomy spectrum can be traced to Sacker and Sell [14] for analysing uniform exponential dichotomies about skew-product flows. For qualitative theory of dynamical systems, the dichotomy or Sacker–Sell spectrum plays a significant role in reducibility and bifurcation theory.

It is well-known that the autonomous system $\dot{x} = Ax$ can be transformed into block diagonal form $\dot{y} = T^{-1}ATy$ with Jordan normal form $T^{-1}AT$, where the blocks correspond to different eigenvalues. In order to extend the Jordan normal form to non-autonomous system, Siegmund [16] established a spectral theory for which the notions of eigenvalues and eigenspaces are generalized to spectral intervals and spectral manifolds. Furthermore, with the aid of dichotomy spectrum, the reducibility for non-autonomous system $\dot{x} = A(t)x$ with locally integrable function $A(t)$ has been shown in Siegmund [17].

Except for uniform dichotomy spectrum, with a suitably small error constant ε, the non-uniform dichotomy spectrum also has attracted considerable attention of researchers and experts. For example, Chu et al. offered the non-uniform dichotomy spectrum and proved reducibility for non-autonomous differential equations [2] and non-autonomous difference equations [3], respectively; Zhang depicted the normal form theory for non-autonomous differential systems by using non-uniform dichotomy spectrum in [19]. Noticing that the topological structure of non-uniform dichotomy spectrum can be applied to deduce block diagonalization and the reducibility result for non-autonomous differential equations. The aim of reduction is transforming a system of ordinary differential equation into another system, which is simpler to analyse and has the same qualitative behaviours. For more details about dichotomy spectrum and reducibility, see [1, 4, 10, 15] and references therein.

Inspired by quasi-hyperbolic orbit segments in Liao [9] and mean hyperbolic sets for autonomous systems in Sun et al. [18], mean hyperbolicity only requires fixed average contraction and expansion rates measured at sufficiently long evolution time length. Here, we refine the definition of mean hyperbolicity, such that non-hyperbolic behaviours are allowed along the trajectories. Owing to mean hyperbolicity, the coexistence of expansion and contraction behaviours in generalized stable or unstable spaces can cause very complicated dynamic phenomena, such as coexistence of multiple complicated attractors and chaos, even in low dimensions. Noticing that uniform and non-uniform hyperbolic systems can be viewed as special cases of mean hyperbolic systems. With regard to non-autonomous systems, we have constructed mean hyperbolic Smale horseshoe with infinite branches in [7]. Recently, the equivalence between mean hyperbolicity and admissibility is obtained for evolution equations in [8]. Therefore, the admissibility is a practical skill to verify mean hyperbolicity for non-autonomous systems. In addition, the Hartman–Grobman linearized theorem, the Hölder regularity, and continuous dependence on perturbation have been established for mean hyperbolic systems [6].

In this article, we investigate the mean dichotomy spectral theory, which contains the topological structure of spectrum and decomposition of spectral manifolds. To be precise, the mean hyperbolic system is kinematically similar to a block diagonal system with suitable change of variables. Based on mean dichotomy spectrum, the reducibility for mean hyperbolic system emphasizes that every block in diagonal system has corresponding spectral interval and spectral manifold.

The content of this article is as follows. In §2, we present some basic concepts and lemmas about mean hyperbolic systems. In §3, we describe the dynamical skeleton of mean hyperbolic systems by spectral intervals and spectral manifolds in Theorem 1. An explicit example of quasi-periodic system is provided, for which the non-trivial mean dichotomy spectrum can be computed. In §4, the reducibility theorem for mean hyperbolic systems is shown as application. More sophisticated models and predictive frameworks can be developed for analysing complex systems by mean hyperbolicity.

2. Preliminaries

In this section, the necessary notations, definitions, and lemmas are presented for mean hyperbolic systems. In detail, the difference and connection among uniform, non-uniform, and mean hyperbolic systems are the preparations for mean dichotomy spectral theory.

Let us consider the non-autonomous differential equation on R^N

$$\dot{x} = A(t)x, \ t \in R, \tag{2.1}$$

with locally integrable matrix function $A \in L^1_{loc}(R, R^{N\times N}), N \in \mathbb{N}$.

DEFINITION 1. *Denote $T(t,s)$ as the evolution operator of (Equation 2.1), which satisfies $T(t,s)x(s) = x(t)$ and the following:*

(a) $T(s,s) = \mathrm{I}$, *where* I *denotes the identity operator on* X;
(b) $T(t,\tau)T(\tau,s) = T(t,s), \quad \forall\ t,\tau,s \in R$;
(c) *for every fixed $x \in X$, $(t,s) \mapsto T(t,s)x$ is continuous mapping.*

DEFINITION 2. *The linear system (Equation 2.1) is called mean hyperbolic or mean exponential dichotomic of type $(K, L, \zeta, \varepsilon)$, if there exist dichotomic projections $P(t) : R \to R^{N\times N}$, constants $K > 0, L > 0, \ \zeta > 0$, and bounded function $0 \leqslant |\varepsilon(t,s)| \leqslant \varepsilon^*$ such that:*

(a) $T(t,s) \circ P(s) = P(t) \circ T(t,s), \ \forall\ t,s \in R.$
(b) *The restriction $T(t,s) \mid \mathcal{N}(P(s)), \ t \geqslant s$, is an isomorphism from $\mathcal{N}(P(s))$ to $\mathcal{N}(P(t))$, where $\mathcal{N}(P(s))$ denotes the null space of $P(s)$. We define $T(s,t)$ as the inverse*

$$T(s,t) := [T(t,s) \mid \mathcal{N}(P(s))]^{-1} : \mathcal{N}(P(t)) \to \mathcal{N}(P(s)), \ s \leqslant t.$$

(c) *There are error hyperbolic degree $\varepsilon(t,s)$ and average hyperbolic degree $\zeta > 0$ such that:*

$$\|T(t,s) \circ P(s)\| \leqslant Ke^{-\zeta(t-s)}e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)}, \quad t \geqslant s, \tag{2.2}$$

$$\|T(t,s) \circ [I - P(s)]\| \leqslant Ke^{\zeta(t-s)}e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)}, \quad t \leqslant s. \tag{2.3}$$

Here, we provide several remarks to describe the characteristics of mean hyperbolicity, for the convenience of discussing the differences and relations between mean hyperbolic, uniform, and non-uniform hyperbolic systems.

REMARK 1. As $t - s > L$, for the generalized stable space $\mathcal{R}(P(s))$, there exists a sequence of $\{\tau_k\}$ with $t = \tau_k \geqslant \tau_{k-1} \geqslant \cdots \geqslant \tau_0 = s$ such that $|\tau_j - \tau_{j-1}| \leqslant L$, $j = 1, 2, \ldots, k$, and

$$\begin{aligned}
&\|T(t,s) \circ P(s)\| \\
&= \|T(\tau_k, \tau_{k-1}) \circ \cdots \circ T(\tau_1, \tau_0) \circ P(\tau_0)\| \\
&\leqslant \|T(\tau_k, \tau_{k-1}) \circ P(\tau_{k-1})\| \cdot \|T(\tau_{k-1}, \tau_{k-2}) \circ P(\tau_{k-2})\| \cdot \cdots \cdot \|T(\tau_1, \tau_0) \circ P(s)\| \\
&\leqslant Ke^{-\zeta(t-s)}e^{\varepsilon(\tau_k,\tau_{k-1})|\tau_k-\tau_{k-1}|+\cdots+\varepsilon(\tau_1,\tau_0)|\tau_1-\tau_0|}.
\end{aligned}$$

Due to mean hyperbolicity, the error hyperbolic degree $\varepsilon(\tau_j, \tau_{j-1})$ may be positive or negative such that norm inequality (Equation 2.2) is well defined. In a similar manner, as $t-s<-L$, the norm inequality (Equation 2.3) for generalized unstable space $\mathcal{N}(P(s))$ is also reasonable.

Remark 2. Mean hyperbolic system emphasizes that the average contraction and expansion rates measured at sufficiently long time will be controlled. More specifically, for any given $L>0$, mean hyperbolic systems exhibit the average contraction and expansion rates as follows:

$$\lim_{|t-s|\to\infty}\frac{1}{|t-s|}\log\|T(t,s)\circ P(s)\|\leqslant -\zeta,\quad t\geqslant s,$$
$$\lim_{|t-s|\to\infty}\frac{1}{|t-s|}\log m(T(s,t)\circ[I-P(t)])\geqslant \zeta,\quad t\leqslant s,$$

where $m(\cdot)$ denotes the minimum of operator norm. The second inequality describes the average expansion behaviour on generalized unstable space $\mathcal{N}(P(t))$ along the positive direction of time.

Let us recall the concepts of uniform and non-uniform hyperbolicities [5, 11–13]. The system (Equation 2.1) is called non-uniform hyperbolic, if there exist constants $K>0,\zeta>0,\varepsilon>0$ satisfying the norm inequalities

$$\|T(t,s)\circ P(s)\|\leqslant Ke^{-\zeta(t-s)+\varepsilon|s|},\quad t\geqslant s,$$
$$\|T(t,s)\circ[I-P(s)]\|\leqslant Ke^{\zeta(t-s)+\varepsilon|s|},\quad t\leqslant s,$$

where constant ε indicates the non-uniform degree. Naturally, the uniform hyperbolicity is available as $\varepsilon=0$. Apparently, the non-uniform term $e^{\varepsilon|s|}$ depends on fixed non-uniform hyperbolic degree $\varepsilon>0$ and initial time $s\in R$, laying stress on inconsistency for different initial moments. It is noteworthy that non-uniform hyperbolic degree ε is less than dichotomic exponent ζ to guarantee the contraction trend as time $t\to\pm\infty$.

Drawing inspiration from averaging principle, one introduce mean hyperbolic systems with coexisting compression and expansion behaviours for generalized stable and unstable spaces. More precisely, compared to uniform and non-uniform hyperbolic systems, mean hyperbolic systems manifest distinct features: (a) Error term $e^{\varepsilon(t,s)|t-s|}$ relies on both initial time and end time of evolution operator $T(t,s)$; (b) The value of error hyperbolic degree $\varepsilon(t,s)$ may be larger than average hyperbolic degree ζ at some moments, which leads to non-hyperbolic behaviours within certain evolution intervals; (c) It admits fixed average contraction and expansion rates.

For the sake of clarity, with distinct initial moments $s\in R$, we provide the images of contraction tendency as $t\to+\infty$ for uniform, non-uniform, and mean hyperbolicity in Figure 2.1. Evidently, if $t\in[s,s+L]$, the upper bound of norm inequality (Equation 2.2) for mean hyperbolic system is not necessarily strictly decreasing as time t increases. The coexistence of expansion and contraction behaviours is allowable in generalized stable space $\mathcal{R}(P(s))$ and generalized unstable space $\mathcal{N}(P(s))$.

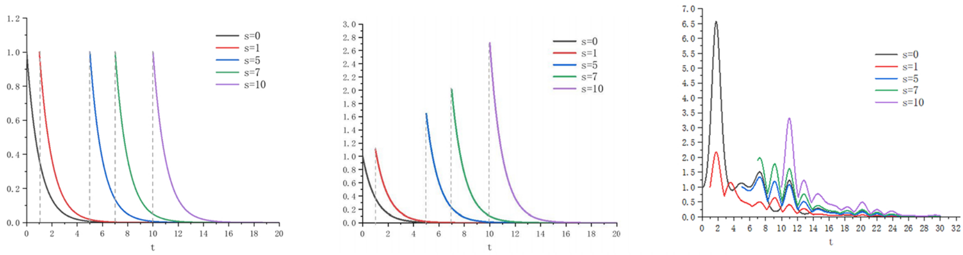

Figure 2.1. Uniform, non-uniform, and mean exponential dichotomy.

LEMMA 1. *Assume that system (Equation 2.1) is mean hyperbolic of type $(K, L, \zeta, \varepsilon)$, one can choose approximate fundamental matrix $X(t)$ of (Equation 2.1) and invariant projector*

$$\hat{P} = \begin{pmatrix} I_{N_1 \times N_1} & 0_{N_1 \times N_2} \\ 0_{N_2 \times N_1} & 0_{N_2 \times N_2} \end{pmatrix}$$

with $N_1 = \dim Im\hat{P}$ and $N_2 = \dim Ker\hat{P}$, such that

$$\|X(t)\hat{P}X^{-1}(s)\| \leqslant Ke^{-\zeta(t-s)}e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)}, \quad t \geqslant s, \tag{2.4}$$

$$\|X(t)[I-\hat{P}]X^{-1}(s)\| \leqslant Ke^{\zeta(t-s)}e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)}, \quad t \leqslant s. \tag{2.5}$$

Proof. For the sake of completeness, we will provide a brief proof. For any given $\tau \in R$, there exists a non-singular matrix $\mathcal{T} \in R^{N \times N}$ such that

$$\hat{P} := \mathcal{T}P(\tau)\mathcal{T}^{-1} = \begin{pmatrix} I_{N_1 \times N_1} & 0_{N_1 \times N_2} \\ 0_{N_2 \times N_1} & 0_{N_2 \times N_2} \end{pmatrix}.$$

Let us define $X(t) := T(t,\tau)\mathcal{T}^{-1}$ and

$$\begin{aligned} &\|X(t)\hat{P}X^{-1}(s)\| \\ &= \|T(t,\tau)\mathcal{T}^{-1}\hat{P}\mathcal{T}T^{-1}(s,\tau)\| \\ &= \|T(t,\tau)T(\tau,s)P(s)\| \\ &= \|T(t,s)\circ P(s)\|. \end{aligned}$$

Owing to (Equation 2.2)–(Equation 2.3), the mean exponential dichotomy or mean hyperbolicity of linear system (Equation 2.1) can be established with invariant projector $\hat{P}$. □

3. Dichotomy spectrum for mean hyperbolic systems

With respect to mean hyperbolic systems, the generalized stable manifold $S_{\gamma,\varepsilon,L}$ and unstable manifold $U_{\gamma,\varepsilon,L}$ relying on error function $\varepsilon(t,\tau)$ and $L > 0$, are the main content that constitutes spectral manifolds. For non-trivial mean dichotomy spectrum, we provide an explicit example with quasi-periodic coefficient matrix.

The spectrum of linear system (Equation 2.1) is closely related to the shifted system

$$\dot{x} = [A(t) - \gamma I]x, \quad \gamma \in R. \tag{3.1}$$

Definition 3. *The mean dichotomy spectrum of (Equation 2.1) is the set*

$$\Sigma_{MED}(A) := \{\gamma \in R \mid (3.1) \text{ admits no mean exponential dichotomy}\}$$

and the resolvent set $\rho_{MED}(A) := R \setminus \Sigma_{MED}(A)$.

Similar to Sacker–Sell and non-uniform dichotomy spectrums, spectral intervals and spectral manifolds are key components of spectral theory. It is widely known that there are compression and expansion behaviours for classical stable and unstable manifolds respectively. For mean hyperbolic systems, the error hyperbolic degree $\varepsilon(t,\tau)$ is essential to depict generalized stable and unstable manifolds.

Lemma 2. *For* $\gamma \in \rho_{MED}(A)$, *the shifted system (Equation 3.1) has mean hyperbolicity of type* $(K, L, \zeta, \varepsilon)$. *We conclude that*

$$S_{\gamma,\varepsilon,L} := \{(\tau,\xi) \in R \times R^N \mid \sup_{t \geqslant \tau}\{\|T(t,\tau)\xi\| e^{-\gamma t} e^{-\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)}\} < \infty\}$$

and

$$U_{\gamma,\varepsilon,L} := \{(\tau,\xi) \in R \times R^N \mid \sup_{t \leqslant \tau}\{\|T(t,\tau)\xi\| e^{-\gamma t} e^{-\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)}\} < \infty\}$$

are linear integral manifolds of system (Equation 2.1). As $\gamma_1 < \gamma_2$, *the integral manifolds satisfy* $S_{\gamma_1,\varepsilon,L} \subset S_{\gamma_2,\varepsilon,L}$ *and* $U_{\gamma_2,\varepsilon,L} \subset U_{\gamma_1,\varepsilon,L}$.

Proof. To keep this article as self-contained as possible, a simple proof is exhibited. If we take $(\tau,\xi) \in S_{\gamma,\varepsilon,L}$, then there is a constant $C_{\gamma,\varepsilon,L} \geqslant 0$ such that

$$\|T(t,\tau)\xi\| e^{-\gamma t} e^{-\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)} \leqslant C_{\gamma,\varepsilon,L}, \quad t \geqslant \tau.$$

Therefore, for any $s \in R$, due to $0 \leqslant |\varepsilon(t,s)| \leqslant \varepsilon^*$, one has

$$\begin{aligned}
&\sup_{s\in R}\sup_{t\geqslant\tau} \|T(t,s)T(s,\tau)\xi\| e^{-\gamma t} e^{-\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)} \\
&= \sup_{s\in R}\sup_{t\geqslant\tau} \|T(t,\tau)\xi\| e^{-\gamma t} e^{-\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)} \\
&\leqslant \sup_{s\in R}\sup_{t\geqslant\tau} C_{\gamma,\varepsilon,L} e^{\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)} e^{-\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)} \\
&\leqslant C_{\gamma,\varepsilon,L} e^{2\varepsilon^* L} \\
&< +\infty
\end{aligned}$$

and $(s, T(s,\tau)\xi) \in S_{\gamma,\varepsilon,L}$. Notice that the fibre $S_{\gamma,\varepsilon,L}(\tau) := \{\xi \in R^N \mid (\tau,\xi) \in S_{\gamma,\varepsilon,L}\}$ is a linear subspace. The same method can be applied to $U_{\gamma,\varepsilon,L}$. Above all, it can be asserted that invariant sets $S_{\gamma,\varepsilon,L}$ and $U_{\gamma,\varepsilon,L}$ are linear integral manifolds.

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As $\gamma_1 < \gamma_2$, the claims $S_{\gamma_1,\varepsilon,L} \subset S_{\gamma_2,\varepsilon,L}$ and $U_{\gamma_2,\varepsilon,L} \subset U_{\gamma_1,\varepsilon,L}$ follow easily from $e^{-\gamma_2 t} \leqslant e^{-\gamma_1 t}$ $(t \geqslant 0)$ and $e^{-\gamma_1 t} \leqslant e^{-\gamma_2 t}$ $(t \leqslant 0)$. The proof is completed. □

From discussion above, the invariant integral manifolds are well defined. Go a step further, the following result shows the direct sum decomposition about generalized stable and unstable manifolds.

LEMMA 3. *If shifted system (Equation 3.1) admits mean exponential dichotomy of type $(K, L, \zeta, \varepsilon)$ for $\gamma \in \rho_{MED}(A)$, then there exist invariant projectors $P : R \to R^{N\times N}$ satisfying*

$$S_{\gamma,\varepsilon,L} = ImP, \quad U_{\gamma,\varepsilon,L} = KerP, \quad S_{\gamma,\varepsilon,L} \oplus U_{\gamma,\varepsilon,L} = R \times R^N.$$

Proof. For any fixed $(\tau, \xi) \in S_{\gamma,\varepsilon,L}$, one has

$$\|T(t,\tau)\xi\| \leqslant C_{\gamma,\varepsilon,L} e^{\gamma t} e^{\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)}, \quad t \geqslant \tau.$$

If $\xi = \xi_1 + \xi_2$ with $\xi_1 \in ImP(\tau)$ and $\xi_2 \in KerP(\tau)$, then we claim that $\xi_2 = 0$.

Denote $T_\gamma(t,s) := e^{-\gamma(t-s)}T(t,s)$ as the evolution operator of shifted system (Equation 3.1). Owing to mean exponential dichotomy of system (Equation 3.1), there are positive constants $K, L, \zeta > 0$ and bounded function $0 \leqslant |\varepsilon(t,s)| \leqslant \varepsilon^*$ such that

$$\begin{aligned}
\|\xi_2\| &= \|T_\gamma(\tau,t)T_\gamma(t,\tau)[I - P(\tau)]\xi\| \\
&= \|T_\gamma(\tau,t)[I - P(t)]T_\gamma(t,\tau)\xi\| \\
&\leqslant K e^{\zeta(\tau-t)} e^{\varepsilon(\tau,t)|t-\tau|\chi_{[-L,L]}(t-\tau)} \|T_\gamma(t,\tau)\xi\| \\
&\leqslant C_{\gamma,\varepsilon,L} K e^{\zeta(\tau-t)} e^{\gamma\tau} e^{2\varepsilon^* L}.
\end{aligned}$$

Therefore, $\xi_2 = 0$ by letting $t \to +\infty$, which means that $S_{\gamma,\varepsilon,L} \subset ImP$.

On the other hand, for any fixed $\tau \in R$, if $\xi \in ImP(\tau)$, linking with mean exponential dichotomy of shifted system (Equation 3.1), then we gain

$$\|T_\gamma(t,\tau)P(\tau)\xi\| \leqslant K e^{-\zeta(t-\tau)} e^{\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)} \|\xi\|, \quad t \geqslant \tau,$$

and

$$\|T(t,\tau)\xi\| \leqslant K e^{(\gamma-\zeta)(t-\tau)} e^{\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)} \|\xi\|.$$

Naturally,

$$\sup_{t \geqslant \tau} \{\|T(t,\tau)\xi\| e^{-\gamma t} e^{-\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)}\} < \infty.$$

Hence $\xi \in S_{\gamma,\varepsilon,L}(\tau)$. It is apparent that $S_{\gamma,\varepsilon,L} = ImP$. Using a similar method, $U_{\gamma,\varepsilon,L} = KerP$ and $S_{\gamma,\varepsilon,L} \oplus U_{\gamma,\varepsilon,L} = R \times R^N$ can be easily verified. □

LEMMA 4. *The resolvent set $\rho_{MED}(A)$ is open and mean dichotomy spectrum $\Sigma_{MED}(A)$ is a closed set. More precisely, for every $\gamma \in \rho_{MED}(A)$, there exists*

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constant $\beta > 0$ such that $(\gamma - \beta, \gamma + \beta) \subset \rho_{MED}(A)$ and

$$S_{\kappa,\varepsilon,L} = S_{\gamma,\varepsilon,L}, \quad U_{\kappa,\varepsilon,L} = U_{\gamma,\varepsilon,L}, \quad \forall \kappa \in (\gamma - \beta, \gamma + \beta).$$

Proof. Let us consider the fundamental matrix $X_\gamma(t) := e^{-\gamma t} X(t)$ with $\gamma \in \rho_{MED}(A)$ for shifted system (Equation 3.1). According to the definition of resolvent set, there exist projection $\hat{P}$, positive constants $K, L, \zeta > 0$, and bounded function $\varepsilon(t, s)$ satisfying Lemma 1.

Without loss of generality, denote $X_\kappa(t) := e^{(-\kappa+\gamma)t} X_\gamma(t)$ and $\beta := \zeta/2 > 0$. For any $\kappa \in (\gamma - \beta, \gamma + \beta)$, it is apparent that as $t \geqslant s$,

$$\begin{aligned} &\|X_\kappa(t)\hat{P}X_\kappa^{-1}(s)\| \\ &= \|e^{(-\kappa+\gamma)t} X_\gamma(t)\hat{P}e^{(\kappa-\gamma)s} X_\gamma^{-1}(s)\| \\ &\leqslant K e^{(\gamma-\kappa-\zeta)(t-s)} e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)} \\ &\leqslant K e^{-\beta(t-s)} e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)}. \end{aligned} \tag{3.2}$$

As $t \leqslant s$, with projector $I - \hat{P}$, we conclude that

$$\begin{aligned} &\|X_\kappa(t)[I - \hat{P}]X_\kappa^{-1}(s)\| \\ &\leqslant K e^{(\gamma-\kappa+\zeta)(t-s)} e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)} \\ &\leqslant K e^{\beta(t-s)} e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)}. \end{aligned} \tag{3.3}$$

Thus, there are invariant projector $\hat{P}$, bounded function $\varepsilon(t, s)$, and $K, L, \beta > 0$ such that $\kappa \in \rho_{MED}(A)$. Owing to Lemma 3, $S_{\kappa,\varepsilon,L} = Im\hat{P} = S_{\gamma,\varepsilon,L}$ and $U_{\kappa,\varepsilon,L} = Ker\hat{P} = U_{\gamma,\varepsilon,L}$. To summary, the conclusions that $\rho_{MED}(A)$ is open set and $\Sigma_{MED}(A)$ is closed set are presented. □

LEMMA 5. *For $\gamma_1, \gamma_2 \in \rho_{MED}(A)$ with $\gamma_1 < \gamma_2$, denote integral manifold*

$$W := S_{\gamma_2,\varepsilon,L} \cap U_{\gamma_1,\varepsilon,L}.$$

We have the following equivalent statements:

(a) $W \neq R \times \{0\}$;
(b) $[\gamma_1, \gamma_2] \cap \Sigma_{MED}(A) \neq \emptyset$;
(c) $\dim S_{\gamma_1,\varepsilon,L} < \dim S_{\gamma_2,\varepsilon,L}$;
(d) $\dim U_{\gamma_1,\varepsilon,L} > \dim U_{\gamma_2,\varepsilon,L}$.

Proof. The proof of the equivalent conclusions above for mean hyperbolic systems is similar to the case of non-uniform exponential dichotomy in [2, 19].

$(a) \Rightarrow (b)$ By contradiction and Lemmas 3–4, if $[\gamma_1, \gamma_2] \subset \rho_{MED}(A)$ then

$$W := S_{\gamma_2,\varepsilon,L} \cap U_{\gamma_1,\varepsilon,L} = S_{\gamma_1,\varepsilon,L} \cap U_{\gamma_1,\varepsilon,L} = R \times \{0\}.$$

$(b) \Rightarrow (c)$ It follows from Lemma 2 that $S_{\gamma_1,\varepsilon,L} \subset S_{\gamma_2,\varepsilon,L}$ as $\gamma_1 < \gamma_2$. Assume that $\dim S_{\gamma_1,\varepsilon,L} \geqslant \dim S_{\gamma_2,\varepsilon,L}$, we conclude $S_{\gamma_1,\varepsilon,L} = S_{\gamma_2,\varepsilon,L}$, which equals to

$[\gamma_1,\gamma_2]\cap\Sigma_{MED}(A)=\emptyset$. The equivalence between (c) and (d) follows easily from Lemma 3.

$(d)\Rightarrow(a)$ Considering the set $(S_{\gamma_2,\varepsilon,L}\cup U_{\gamma_1,\varepsilon,L})\setminus(S_{\gamma_2,\varepsilon,L}\cap U_{\gamma_1,\varepsilon,L})\subset R\times R^N$ and

$$\dim(S_{\gamma_2,\varepsilon,L}+U_{\gamma_1,\varepsilon,L})=\dim S_{\gamma_2,\varepsilon,L}+\dim U_{\gamma_1,\varepsilon,L}-\dim(S_{\gamma_2,\varepsilon,L}\cap U_{\gamma_1,\varepsilon,L})\leqslant N,$$

therefore

$$\begin{aligned}\dim W&=\dim(S_{\gamma_2,\varepsilon,L}\cap U_{\gamma_1,\varepsilon,L})\\&\geqslant\dim S_{\gamma_2,\varepsilon,L}+\dim U_{\gamma_1,\varepsilon,L}-N\\&>\dim S_{\gamma_2,\varepsilon,L}+\dim U_{\gamma_2,\varepsilon,L}-N\\&=0.\end{aligned}$$

Hence $W\neq R\times\{0\}$. The proof is completed. □

The spectral intervals and decomposition of spectral manifolds are constructed for mean dichotomy spectrum, which are the foundation of reducibility and normal form theory for mean hyperbolic systems.

THEOREM 1. *The mean dichotomy spectrum $\Sigma_{MED}(A)$ is a disjoint union of n closed intervals with $0\leqslant n\leqslant N$. To be precise, $\Sigma_{MED}(A)=\emptyset$, or $\Sigma_{MED}(A)=R$, or $\Sigma_{MED}(A)$ is in one of the four cases*

$$\Sigma_{MED}(A)=\mathbf{I}_1\cup\mathbf{I}_2\cup\cdots\cup\mathbf{I}_{n-1}\cup\mathbf{I}_n,$$

with

$$\mathbf{I}_1:=[a_1,b_1]\ \textit{or}\ (-\infty,b_1],\ \mathbf{I}_k:=[a_k,b_k]\ (k=2,\ldots,n-1),\ \mathbf{I}_n:=[a_n,b_n]\ \textit{or}\ [a_n,+\infty),$$

where $a_1\leqslant b_1<a_2\leqslant b_2<\cdots<a_n\leqslant b_n$.

- *Set $b_0=-\infty$ and $a_{n+1}=+\infty$, if $\mathbf{I}_1$ and $\mathbf{I}_n$ are closed intervals, we can choose $\gamma_i\in(b_i,a_{i+1})$, $i=0,\ldots,n$.*
- *If $\mathbf{I}_1=(-\infty,b_1]$, we take $\gamma_0<b_1$ and set $S_{\gamma_0,\varepsilon,L}:=R\times\{0\}$, $U_{\gamma_0,\varepsilon,L}:=R\times R^N$.*
- *If $\mathbf{I}_n=[a_n,+\infty)$, we take $\gamma_n>a_n$ and set $S_{\gamma_n,\varepsilon,L}:=R\times R^N$, $U_{\gamma_n,\varepsilon,L}:=R\times\{0\}$.*

Then for every $i=1,\ldots,n$, the intersection

$$W_i:=S_{\gamma_i,\varepsilon,L}\cap U_{\gamma_{i-1},\varepsilon,L}$$

is a linear integral manifold with $\dim W_i\geqslant 1$. Denote $W_0:=S_{\gamma_0,\varepsilon,L}$ and $W_{n+1}:=U_{\gamma_n,\varepsilon,L}$. The spectral manifolds W_i should satisfy the following decomposition

$$W_0\oplus W_1\oplus\cdots\oplus W_{n+1}=R\times R^N.$$

Proof. According to Lemma 4, $\Sigma_{MED}(A)\subset R$ is a closed set. Except for empty set or whole axis R, here we discuss the non-trivial situation. Without loss of generality,

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we assume that it consists of n closed disjoint intervals with $n > N$. Suppose that

$$\Sigma_{MED}(A) = \mathbf{I}_1 \cup \mathbf{I}_2 \cup \cdots \cup \mathbf{I}_{n-1} \cup \mathbf{I}_n,$$

taking $\gamma_i \in (b_i, a_{i+1})$ for $i = 1, \ldots, N$, linking with Lemma 5, we conclude

$$0 \leqslant \dim S_{\gamma_1,\varepsilon,L} < \dim S_{\gamma_2,\varepsilon,L} < \cdots < \dim S_{\gamma_N,\varepsilon,L} \leqslant N.$$

The result above is available with either $\dim S_{\gamma_1,\varepsilon,L} = 0$ or $\dim S_{\gamma_N,\varepsilon,L} = N$. If $\dim S_{\gamma_1,\varepsilon,L} = 0$ then $S_{\gamma_1,\varepsilon,L} = R \times \{0\}$, $U_{\gamma_1,\varepsilon,L} = R \times R^N$, and invariant projection $P(t) \equiv 0$ for any $t \in R$.

Obviously, for all $\gamma < \gamma_1$, as $t \leqslant \tau$,

$$\begin{aligned}&\|T_\gamma(t,\tau)[I - P(\tau)]\xi\| \\ &= e^{(-\gamma+\gamma_1)(t-\tau)}\|T_{\gamma_1}(t,\tau)[I - P(\tau)]\xi\| \\ &\leqslant K e^{(\zeta-\gamma+\gamma_1)(t-\tau)} e^{\varepsilon(t,\tau)|t-\tau|\chi_{[-L,L]}(t-\tau)}\|\xi\|,\end{aligned}$$

hence $\gamma \in \rho_{MED}(A)$ and $(-\infty, \gamma_1] \subset \rho_{MED}(A)$. Actually, due to $\gamma_1 \in (b_1, a_2)$, $(-\infty, \gamma_1] \cap \Sigma_{MED}(A) \neq \emptyset$, which contradicts $(-\infty, \gamma_1] \subset \rho_{MED}(A)$.

Likewise, if $\dim S_{\gamma_N,\varepsilon,L} = N$, we can prove $[\gamma_N, +\infty) \subset \rho_{MED}(A)$. On the contrary, $[\gamma_N, +\infty) \cap \Sigma_{MED}(A) \neq \emptyset$. In short, the number n of closed intervals is no more than N.

Next we claim that $\dim W_i \geqslant 1$ for $i = 1, \ldots, n$. As $i = 1$, $W_1 = S_{\gamma_1,\varepsilon,L} \cap U_{\gamma_0,\varepsilon,L}$. If spectral interval $\mathbf{I}_1 = [a_1, b_1]$, then both $\gamma_0 \in (-\infty, a_1)$ and $\gamma_1 \in (b_1, a_2)$ belong to $\rho_{MED}(A)$. From Lemma 5, we conclude that

$$W_1 = S_{\gamma_1,\varepsilon,L} \cap U_{\gamma_0,\varepsilon,L} \supsetneq S_{\gamma_0,\varepsilon,L} \cap U_{\gamma_0,\varepsilon,L}$$

and $\dim W_1 \geqslant 1$. If $\mathbf{I}_1 = (-\infty, b_1]$ then $W_1 = S_{\gamma_1,\varepsilon,L}$. By contradiction, we assume that $\dim W_1 = 0$, i.e., $W_1 = S_{\gamma_1,\varepsilon,L} = R \times \{0\}$ and $P(t) \equiv 0$. It is apparent that $(-\infty, \gamma_1] \subset \rho_{MED}(A)$, which contradicts $\gamma_1 \in (b_1, a_2)$. Therefore, $\dim W_1 \geqslant 1$.

As $i > 1$, we obtain $[\gamma_{i-1}, \gamma_i] \cap \Sigma_{MED}(A) \neq \emptyset$. Undoubtedly,

$$W_i = S_{\gamma_i,\varepsilon,L} \cap U_{\gamma_{i-1},\varepsilon,L} \supsetneq S_{\gamma_{i-1},\varepsilon,L} \cap U_{\gamma_{i-1},\varepsilon,L}$$

and $\dim W_i \geqslant 1$.

Moreover, we deduce that $U_{\gamma_i,\varepsilon,L} = W_{i+1} + U_{\gamma_{i+1},\varepsilon,L}$ for $i = 0, 1, \ldots, n-1$. Due to $U_{\gamma_i,\varepsilon,L} = U_{\gamma_i,\varepsilon,L} \cap (S_{\gamma_{i+1},\varepsilon,L} + U_{\gamma_{i+1},\varepsilon,L}) = U_{\gamma_i,\varepsilon,L} \cap S_{\gamma_{i+1},\varepsilon,L} + U_{\gamma_{i+1},\varepsilon,L} = W_{i+1} + U_{\gamma_{i+1},\varepsilon,L}$, we conclude that

$$\begin{aligned} R \times R^N &= S_{\gamma_0,\varepsilon,L} + U_{\gamma_0,\varepsilon,L} \\ &= W_0 + W_1 + U_{\gamma_1,\varepsilon,L} \\ &= W_0 + W_1 + \cdots + W_n + U_{\gamma_n,\varepsilon,L} \\ &= W_0 + W_1 + \cdots + W_n + W_{n+1}. \end{aligned}$$

For any $0 \leqslant i < j \leqslant n+1$, it is obvious that

$$W_i \cap W_j \subset S_{\gamma_i,\varepsilon,L} \cap U_{\gamma_{j-1},\varepsilon,L} \subset S_{\gamma_i,\varepsilon,L} \cap U_{\gamma_i,\varepsilon,L} = R \times \{0\}$$

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and $W_i \cap W_j = R \times \{0\}$ for $i \neq j$. The statement of

$$W_0 \oplus W_1 \oplus \cdots \oplus W_{n+1} = R \times R^N$$

is proved. Note that the linear integral manifolds $W_0, \ldots, W_{n+1}$ are independent of the choice of γ_i owing to Lemma 4. □

EXAMPLE 1. Let us consider the quasi-periodic shifted system in R^2 with

$$\begin{aligned} \dot{x} &= [A(t) - \gamma I]x \\ &= \begin{pmatrix} \sin 2t + \sin \sqrt{3}t + \frac{1}{6} - \gamma & 0 \\ 0 & \sin 2t + \sin \sqrt{3}t - \frac{1}{4} - \gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned} \tag{3.4}$$

Denote vector norm $|v| := \sqrt{v_1^2 + v_2^2}$ for any $v \in R^2$. For given initial value $x(s) = v$, we assume that there exists invariant projector $P(s)$ satisfying $P(s)v = v_1$ and $[I - P(s)]v = v_2$, such that the evolution operators of (Equation 3.4) are

$$T_1(t, s)v_1 := T_\gamma(t, s) \circ P(s)v = \exp\left[\int_s^t (\sin 2u + \sin \sqrt{3}u + \frac{1}{6} - \gamma)du\right] v_1,$$

and

$$T_2(t, s)v_2 := T_\gamma(t, s) \circ [I - P(s)]v = \exp\left[\int_s^t (\sin 2u + \sin \sqrt{3}u - \frac{1}{4} - \gamma)du\right] v_2.$$

First of all, we consider the forward evolution with projector P as $t \geqslant s$. It is obvious that there exists properly large constant $L > 0$ such that as $t - s > L$,

$$\begin{aligned} &\frac{1}{t-s} \log \|T_\gamma(t, s) \circ P(s)\| \\ &\leqslant \frac{1}{t-s} \big[\textstyle\int_s^t (\sin 2u + \sin \sqrt{3}u + \frac{1}{6} - \gamma)du \big] \\ &\leqslant \frac{1}{t-s} \big[-\frac{1}{2} \cos 2u - \frac{1}{\sqrt{3}} \cos \sqrt{3}u \big] \big|_s^t + \frac{1}{6} - \gamma \\ &< \frac{1}{5} - \gamma. \end{aligned}$$

As $0 \leqslant t - s \leqslant L$, the norm satisfies

$$\begin{aligned} &\|T_\gamma(t, s) \circ P(s)\| \\ &\leqslant \exp \big[\textstyle\int_s^t (\sin 2u + \sin \sqrt{3}u + \frac{1}{6} - \gamma)du \big] \\ &\leqslant \exp \left\{ \big[-\frac{1}{2} \cos 2u - \frac{1}{\sqrt{3}} \cos \sqrt{3}u \big] \big|_s^t + (\frac{1}{6} - \gamma)(t - s) \right\}. \end{aligned}$$

The same method is applicable to the backward evolution with projection $I-P$ such that for $t-s<-L$,

$$\begin{aligned}&\frac{1}{t-s}\log\|T_\gamma(t,s)\circ[I-P(s)]\|\\&\leqslant\frac{1}{t-s}\big[\int_s^t(\sin 2u+\sin\sqrt{3}u-\frac{1}{4}-\gamma)du\big]\\&<-\frac{1}{5}-\gamma;\end{aligned}$$

and for $-L\leqslant t-s\leqslant 0$,

$$\begin{aligned}&\|T_\gamma(t,s)\circ[I-P(s)]\|\\&\leqslant\exp\big[\int_s^t(\sin 2u+\sin\sqrt{3}u-\frac{1}{4}-\gamma)du\big]\\&\leqslant\exp\left\{\big[-\frac{1}{2}\cos 2u-\frac{1}{\sqrt{3}}\cos\sqrt{3}u\big]\big|_s^t-(\frac{1}{4}+\gamma)(t-s)\right\}.\end{aligned}$$

Denote $G(u):=-\frac{1}{2}\cos 2u-\frac{1}{\sqrt{3}}\cos\sqrt{3}u$ and $G(t)-G(s):=\varepsilon(t,s)|t-s|$. Naturally, it can be obtained that

$$\|T_\gamma(t,s)\circ P(s)\|\leqslant e^{(-\gamma+\frac{1}{5})(t-s)}e^{\varepsilon(t,s)|t-s|\chi_{[s-L,s+L]}(t)},\quad t\geqslant s,\tag{3.5}$$

and

$$\|T_\gamma(t,s)\circ[I-P(s)]\|\leqslant e^{(-\gamma-\frac{1}{5})(t-s)}e^{\varepsilon(t,s)|t-s|\chi_{[s-L,s+L]}(t)},\quad t\leqslant s.\tag{3.6}$$

Therefore, there are invariant projection $P(t)$, $K=1$, $L>0$ and bounded function $\varepsilon(t,s)$ such that shifted system (Equation 3.4) is mean hyperbolic and $(-\infty,-\frac{1}{5})\cup(\frac{1}{5},+\infty)\subset\rho_{MED}(A)$.

- Here we claim that $\Sigma_{MED}(A)=[-\frac{1}{5},\frac{1}{5}]$.

From discussion above, we conclude that the mean dichotomy spectrum $\Sigma_{MED}(A)\subset[-\frac{1}{5},\frac{1}{5}]$. We only need to prove $[-\frac{1}{5},\frac{1}{5}]\subset\Sigma_{MED}(A)$. By Lemma 4, if the shifted system (Equation 3.4) has mean hyperbolicity with $\gamma=-\frac{1}{5}$, then there exists $\beta>0$ such that $(-\frac{1}{5}-\beta,-\frac{1}{5}+\beta)\subset\rho_{MED}(A)$ with the same projector. By contradiction, the bound of (Equation 3.5) is less than $e^{\frac{2}{5}(t-s)+\varepsilon^*L}$; on the other hand, the norm of (Equation 3.6) with backward evolution is no more than e^{ε^*L}, which cannot guarantee overall compression and expansion behaviours as $t\to\pm\infty$. Similarly, for any $\gamma\in[-\frac{1}{5},\frac{1}{5}]$, we conclude that the shifted system (Equation 3.4) has no mean hyperbolicity. Hence, $[-\frac{1}{5},\frac{1}{5}]\subset\Sigma_{MED}(A)$ and $\Sigma_{MED}(A)=[-\frac{1}{5},\frac{1}{5}]$.

- The dichotomy spectrum $\Sigma_{ED}(A)=[-\frac{9}{4},\frac{13}{6}]$ and the non-uniform dichotomy spectrum $\Sigma_{NED}(A)=[-\frac{9}{4},\frac{13}{6}]$.

Here, we recall that uniform and non-uniform hyperbolic are special cases for mean hyperbolic systems. The results that cannot be ignored are $\rho_{ED}(A)\subset\rho_{NED}(A)\subset\rho_{MED}(A)$ and $\Sigma_{MED}(A)\subset\Sigma_{NED}(A)\subset\Sigma_{ED}(A)$.

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Let us consider the norm

$$\|T_\gamma(t,s)\circ P(s)\| \leqslant e^{(-\gamma+\frac{13}{6})(t-s)},\ t \geqslant s,$$

and

$$\|T_\gamma(t,s)\circ [I-P(s)]\| \leqslant e^{(-\gamma-\frac{9}{4})(t-s)},\ t \leqslant s.$$

It is evident that the shifted system (Equation 3.4) has exponential dichotomy iff $\gamma \in (-\infty, -\frac{9}{4}) \cup (\frac{13}{6}, +\infty)$. In particular, the non-uniform hyperbolicity of (Equation 3.4) can be verified with $\varepsilon = 0$. Therefore, $\Sigma_{NED}(A) = \Sigma_{ED}(A) = [-\frac{9}{4}, \frac{13}{6}]$.

4. Reducibility for mean hyperbolic systems

In this section, the reducibility theory is illustrated for linear system (Equation 2.1) through mean dichotomy spectrum. The reducibility for mean hyperbolic system emphasizes that system (Equation 2.1) is averagely kinematically similar to a block diagonal system, of which every block has corresponding spectral interval and spectral manifold.

Definition 4. *Suppose that the linear system (Equation 2.1) is mean hyperbolic. The system (Equation 2.1) is called reducible, denote $A(t) \sim B(t)$ or (Equation 2.1)$\sim$(Equation 4.1), if it is averagely kinematically similar to system*

$$\dot{y} = B(t)y, \tag{4.1}$$

with the coefficient matrix

$$B(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix} \tag{4.2}$$

and $B \in L^1_{loc}(R, R^{N\times N}), N \in \mathbb{N}$. Average kinematic similarity demands that there exists an absolutely continuous function $G : R \to GL_N(R)$ with

$$\sup_{t\in R} \|G(t)\| < \infty, \quad \sup_{t\in R} \|G^{-1}(t)\| < \infty$$

satisfying the differential equation

$$\dot{G}(t) = A(t)G(t) - G(t)B(t). \tag{4.3}$$

The transformation $x = G(t)y$ is referred to as Lyapunov transformation.

Lemma 6 ([17]). *Let $P \in R^{N\times N}$ be a symmetric projection and $X : R \to GL_N(R)$ be an absolutely continuous matrix. We have the following conclusions:*

(a) *The mapping*

$$\hat{H}(t) = PX(t)^T X(t)P + QX(t)^T X(t)Q$$

is absolutely continuous and $\hat{H}(t)$ is a positive definite, symmetric matrix for every $t \in R$. There exists unique absolutely continuous function $H : R \to R^{N\times N}$ of *positive definite symmetric matrices $H(t)$ with*

$$H^2(t) = \hat{H}(t), \quad PH(t) = H(t)P.$$

(b) *For any $t \in R$, the function $G(t) := X(t)H^{-1}(t)$ is absolutely continuous and satisfying*

$$\begin{gathered} G(t)PG^{-1}(t) = X(t)PX^{-1}(t), \\ G(t)QG^{-1}(t) = X(t)QX^{-1}(t), \\ \|G(t)\| \leqslant \sqrt{2}, \\ \|G^{-1}(t)\| \leqslant [\|X(t)PX^{-1}(t)\|^2 + \|X(t)QX^{-1}(t)\|^2]^{\frac{1}{2}}. \end{gathered}$$

Remark 3. Differs from non-uniform dichotomy spectrum, due to non-uniform term $e^{\varepsilon|s|}$ relying on initial moment $s \in R$, the operator norm $\|G^{-1}(t)\|$ is unbounded. Although the mean hyperbolic system emphasizes more complex dynamic behaviours, such as non-hyperbolic phenomena during evolution process, the norm of $G^{-1}(t)$ is bounded by the aid of characteristic function, which can simplify the following calculation.

Theorem 2. *Suppose that the system (Equation 2.1) is mean hyperbolic of type $(K, L, \zeta, \varepsilon)$ with invariant projector $P(t) \neq 0, I$. We conclude that the linear system (Equation 2.1) is averagely kinematically similar to the decoupled system*

$$\dot{x} = B(t)x = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_2(t) \end{pmatrix} x \tag{4.4}$$

for locally integrable matrix functions $B_1 : R \to R^{N_1 \times N_1}$ and $B_2 : R \to R^{N_2 \times N_2}$ with $N_1 := \dim ImP$ and $N_2 := \dim KerP$. In addition, $\Sigma_{MED}(A) = \Sigma_{MED}(B)$ as $A(t) \sim B(t)$.

Proof. With regard to mean hyperbolic system (Equation 2.1), combining with Lemma 1, there exist fundamental matrix $X(t)$ and invariant projector

$$\hat{P}_0 = \begin{pmatrix} I_{N_1 \times N_1} & 0_{N_1 \times N_2} \\ 0_{N_2 \times N_1} & 0_{N_2 \times N_2} \end{pmatrix}, \quad 0 < N_1 < N,$$

such that inequalities (Equation 2.4)–(Equation 2.5) hold.

Denote $B(t) := \dot{H}(t)H^{-1}(t)$ with $H(t) = G^{-1}(t)X(t)$ and $B(t) = 0$ for which $\dot{H}(t)$ does not exist. Naturally, $H(t)$ is the fundamental matrix of $\dot{y} = B(t)y$. As $t \geqslant s$, we gain

$$\begin{aligned} &\|H(t)\hat{P}_0 H^{-1}(s)\| \\ &= \|G^{-1}(t)X(t)\hat{P}_0 X^{-1}(s)G(s)\| \\ &\leqslant \|G^{-1}(t)\| \cdot \|X(t)\hat{P}_0 X^{-1}(s)\| \cdot \|G(s)\| \\ &\leqslant 2K^2 e^{-\zeta(t-s)} e^{\varepsilon(t,s)|t-s|\chi_{[-L,L]}(t-s)}. \end{aligned}$$

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Similar argument can be applied to the case of $t \leqslant s$ and projector $I - \hat{P}_0$. Apparently, system $\dot{y} = B(t)y$ admits mean exponential dichotomy with the same projector $\hat{P}_0$.

Next we prove that $A(t) \sim B(t)$ and $B(t)$ has the block diagonal form. More importantly, $\Sigma_{MED}(A) = \Sigma_{MED}(B)$.

Let us consider

$$\begin{aligned} \dot{G}(t) &= [X(t)H^{-1}(t)]' \\ &= A(t)G(t) - G(t)H^{-1}(t)\dot{H}(t) \\ &= A(t)G(t) - G(t)B(t), \end{aligned}$$

which means that there is Lyapunov transformation $x = G(t)y$ such that $\dot{x} = A(t)x$ is averagely kinematically similar to $\dot{y} = B(t)y$. Based on the fact that $H(t), H^{-1}(t), \dot{H}(t)$ commute with invariant projector $\hat{P}_0$, we obtain that

$$\hat{P}_0\dot{H}(t)H^{-1}(t) = \dot{H}(t)H^{-1}(t)\hat{P}_0$$

for almost all $t \in R$. Therefore, due to $\hat{P}_0 B(t) = B(t)\hat{P}_0$, one can decompose $B : R \to R^{N\times N}$ into four functions with

$$B(t) = \begin{pmatrix} B_1(t) & B_3(t) \\ B_4(t) & B_2(t) \end{pmatrix}, \quad t \in R,$$

such that

$$\begin{pmatrix} B_1(t) & B_3(t) \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} B_1(t) & 0 \\ B_4(t) & 0 \end{pmatrix}, \quad t \in R,$$

and $B_3(t) = B_4(t) \equiv 0,\ t \in R$. Hence, mean hyperbolic system (Equation 2.1) is averagely kinematically similar to a decoupled system with diagonal matrix $B(t)$.

If we take $\gamma \in \rho_{MED}(A)$, there are positive constants $K_1, L_1, \zeta_1 > 0$ and bounded function $\varepsilon_1(t, s)$ such that for $t \geqslant s$

$$\begin{aligned} &\|X_\gamma(t)\hat{P}_0X_\gamma^{-1}(s)\| \\ &= e^{-\gamma(t-s)}\|X(t)\hat{P}_0X^{-1}(s)\| \\ &\leqslant K_1e^{-\zeta_1(t-s)}e^{\varepsilon_1(t,s)|t-s|\chi_{[-L_1,L_1]}(t-s)}, \end{aligned}$$

and

$$\begin{aligned} &\|H_\gamma(t)\hat{P}_0H_\gamma^{-1}(s)\| \\ &= e^{-\gamma(t-s)}\|H(t)\hat{P}_0H^{-1}(s)\| \\ &\leqslant e^{-\gamma(t-s)}\|G^{-1}(t)\| \cdot \|G(s)\| \cdot \|X(t)\hat{P}_0X^{-1}(s)\| \\ &\leqslant 2K_1^2e^{-\zeta_1(t-s)}e^{\varepsilon_1(t,s)|t-s|\chi_{[-L_1,L_1]}(t-s)}. \end{aligned}$$

As discussed above, the mean hyperbolicity with fundamental matrix $H_\gamma(t)$ can be verified with projector $I - \hat{P}_0$ and $t \leqslant s$. Hence $\gamma \in \rho_{MED}(B)$ and

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$\rho_{MED}(A) \subset \rho_{MED}(B)$. The same principle also works in reverse such that $\Sigma_{MED}(A) = \Sigma_{MED}(B)$. □

THEOREM 3. *Assume that linear system (Equation 2.1) admits mean exponential dichotomy of type $(K, L, \zeta, \varepsilon)$. The mean dichotomy spectrum $\Sigma_{MED}(A)$ of (Equation 2.1) is $\emptyset$ or R or the disjoint union of closed spectral intervals $\mathbf{I}_1, \ldots, \mathbf{I}_n$ with $1 \leqslant n \leqslant N$. Then there exists an average kinematic similar action $G : R \to R^{N\times N}$ between (Equation 2.1) and block diagonal system*

$$\dot{x} = \begin{pmatrix} B_0(t) & & & \\ & B_1(t) & & \\ & & \ddots & \\ & & & B_{n+1}(t) \end{pmatrix} x \tag{4.5}$$

with locally integrable functions $B_i : R \to R^{N_i \times N_i}$, $N_i = \dim W_i$, and $\Sigma_{MED}(B_0) = \emptyset$, $\Sigma_{MED}(B_1) = \mathbf{I}_1, \ldots, \Sigma_{MED}(B_n) = \mathbf{I}_n$, $\Sigma_{MED}(B_{n+1}) = \emptyset$.

Proof. Due to Lemma 4 and Theorem 1, the resolvent set $\rho_{MED}(A)$ is open and $\Sigma_{MED}(A)$ is the disjoint union of several closed intervals. Instead of talking about trivial case $\Sigma_{MED}(A) = \emptyset$ or $\Sigma_{MED}(A) = R$, we suppose that

$$\mathbf{I}_1 := [a_1, b_1] \text{ or } (-\infty, b_1],\ \mathbf{I}_k := [a_k, b_k]\ (k = 2, \ldots, n-1),\ \mathbf{I}_n := [a_n, b_n] \text{ or } [a_n, +\infty)$$

with $a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_n \leqslant b_n$.

Step 1. If $\mathbf{I}_1 = [a_1, b_1]$, there is $\gamma_0 < a_1$ such that $(-\infty, \gamma_0] \subset \rho_{MED}(A)$ and $W_0 := S_{\gamma_0,\varepsilon,L}$. It is obvious that

$$\dot{x} = [A(t) - \gamma_0 I]x$$

admits mean exponential dichotomy. In other words, there exist invariant projector $\hat{P}_0$, constants $K_0 > 0, L_0 > 0$, $\zeta_0 > 0$, and bounded function $0 \leqslant |\varepsilon_0(t,s)| \leqslant \varepsilon_0^*$ such that

$$\|X(t)\hat{P}_0 X^{-1}(s)\| \leqslant K_0 e^{(\gamma_0-\zeta_0)(t-s)+\varepsilon_0(t,s)|t-s|\chi_{[-L,L]}(t-s)}, \quad t \geqslant s, \tag{4.6}$$

$$\|X(t)[I - \hat{P}_0]X^{-1}(s)\| \leqslant K_0 e^{(\gamma_0+\zeta_0)(t-s)+\varepsilon_0(t,s)|t-s|\chi_{[-L,L]}(t-s)}, \quad t \leqslant s. \tag{4.7}$$

Claim: The system (Equation 2.1) is averagely kinematically similar to

$$\dot{y} = B(t)y = \begin{pmatrix} B_0(t) & 0 \\ 0 & B_{11}(t) \end{pmatrix} y, \tag{4.8}$$

with $B_0 : R \to R^{N_0 \times N_0}$ and $B_{11} : R \to R^{M_1 \times M_1}$, where $N_0 := \dim Im\hat{P}_0$ and $M_1 := \dim Ker\hat{P}_0$. Moreover, $\Sigma_{MED}(B_0) = \emptyset$ and $\Sigma_{MED}(B_{11}) = \Sigma_{MED}(A)$.

By Theorem 2, there exists Lyapunov transformation $x = G(t)y$ such that mean hyperbolic system (Equation 2.1) is averagely kinematically similar to decoupled

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system (Equation 4.8). Notice that systems (Equation 2.1) and (Equation 4.8) have the same invariant projection $\hat{P}_0$ and mean dichotomy spectrum $\Sigma_{MED}(B) = \Sigma_{MED}(A)$.

For any $\gamma > \gamma_0$, it is apparent that

$$\|X_\gamma(t)\hat{P}_0 X_\gamma^{-1}(s)\| \leqslant K_0 e^{(-\gamma+\gamma_0-\zeta_0)(t-s)+\varepsilon_0(t,s)|t-s|\chi_{[-L,L]}(t-s)}, \quad t \geqslant s. \tag{4.9}$$

Therefore, for any $\gamma \geqslant a_1 > \gamma_0$, $\dot{x} = [B_0(t) - \gamma I]x$ admits mean hyperbolicity with $-\gamma + \gamma_0 - \zeta_0 < 0$ and $\Sigma_{MED}(B_0) \subset (-\infty, a_1)$. However, $\Sigma_{MED}(B_0) \subset \Sigma_{MED}(A)$ and $\Sigma_{MED}(B_0) = \emptyset$.

If $\mathbf{I}_1 = (-\infty, b_1]$, we set $W_0 := S_{\gamma_0,\varepsilon,L} = R \times \{0\}$. An obvious fact is $\dim B_0 = \dim W_0 = 0$, which leads to $\Sigma_{MED}(B_0) = \emptyset$. The result $\Sigma_{MED}(B_{11}) = \Sigma_{MED}(A)$ is proved due to $\Sigma_{MED}(B) = \Sigma_{MED}(A)$.

Step 2. If we take $\gamma_1 \in (b_1, a_2)$, then $\dot{y} = (B(t) - \gamma_1 I)y$ is mean exponential dichotomic with an invariant projection $\hat{P}_1$. There are positive constants K_1, L_1, ζ_1 and bounded function $\varepsilon_1(t,s)$ such that

$$\|Y(t)\hat{P}_1 Y^{-1}(s)\| \leqslant K_1 e^{(\gamma_1-\zeta_1)(t-s)+\varepsilon_1(t,s)|t-s|\chi_{[-L_1,L_1]}(t-s)}, \quad t \geqslant s, \tag{4.10}$$

$$\|Y(t)[I - \hat{P}_1]Y^{-1}(s)\| \leqslant K_1 e^{(\gamma_1+\zeta_1)(t-s)+\varepsilon_1(t,s)|t-s|\chi_{[-L_1,L_1]}(t-s)}, \quad t \leqslant s, \tag{4.11}$$

with the fundamental matrix $Y(t)$ of linear system $\dot{y} = B(t)y$.

Linking $\Sigma_{MED}(B_{11}) = \Sigma_{MED}(A)$ with Theorem 2, there is diagonal matrix such that

$$\dot{y_1} = B_{11}(t)y_1$$

is averagely kinematically similar to

$$\dot{z_1} = B^*(t)z_1 = \begin{pmatrix} B_1(t) & 0 \\ 0 & B_{22}(t) \end{pmatrix} z_1$$

with Lyapunov transformation $y_1 = S_{11}(t)z_1$ and $\Sigma_{MED}(B^*) = \Sigma_{MED}(B_{11})$. Denote

$$S_1(t) = \begin{pmatrix} I_{N_1\times N_1} & 0 \\ 0 & S_{11}(t) \end{pmatrix} S_0(t).$$

Therefore, the linear system (Equation 2.1) is averagely kinematically similar to

$$\dot{z} = \begin{pmatrix} B_0(t) & 0 & 0 \\ 0 & B_1(t) & 0 \\ 0 & 0 & B_{22}(t) \end{pmatrix} z(t) \tag{4.12}$$

with transformation $x(t) = S_1(t)z(t)$.

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For any $\gamma \in [a_2, +\infty)$, the inequality

$$\|Z_\gamma(t)\hat{P}_1 Z_\gamma^{-1}(s)\| \leqslant K_1 e^{(-\gamma+\gamma_1-\zeta_1)(t-s)+\varepsilon_1(t,s)|t-s|\chi_{[-L_1,L_1]}(t-s)}, \quad t \geqslant s,$$

holds; in a similar way, the norm inequality

$$\|Z_\gamma(t)[I - \hat{P}_1]Z_\gamma^{-1}(s)\| \leqslant K_1 e^{(-\gamma+\gamma_1+\zeta_1)(t-s)+\varepsilon_1(t,s)|t-s|\chi_{[-L_1,L_1]}(t-s)}, \quad t \leqslant s,$$

is true for any $\gamma \in (-\infty, b_1]$. To summarize, we obtain $\Sigma_{MED}(B_0, B_1) \subset (-\infty, a_2)$ and $\Sigma_{MED}(B_{22}) \subset (b_1, +\infty)$. Hence $\Sigma_{MED}(B_1) = \mathbf{I}_1$.

By induction, there is Lyapunov transformation $x(t) = S^*(t)u(t)$ such that system (Equation 2.1) is averagely kinematically similar to

$$\dot{u} = \begin{pmatrix} B_0(t) & & & \\ & B_1(t) & & \\ & & \ddots & \\ & & & B_{n+1}(t) \end{pmatrix} u(t), \tag{4.13}$$

with $\Sigma_{MED}(B_0) = \emptyset$, $\Sigma_{MED}(B_i) = \mathbf{I}_i$ for $i = 1, \ldots, n-1$.

If $\mathbf{I}_n = [a_n, b_n]$, by taking $\gamma_n \in (b_n, +\infty)$ and using same arguments as above, we gain that $\Sigma_{MED}(B_0, \ldots, B_n) \subset (-\infty, b_n]$ and $\Sigma_{MED}(B_{n+1}) \subset (b_n, +\infty)$, which means that $\Sigma_{MED}(B_n) = [a_n, b_n]$ and $\Sigma_{MED}(B_{n+1}) = \emptyset$.

If $\mathbf{I}_n = [a_n, +\infty)$, then $\Sigma_{MED}(B_0, \ldots, B_n) \subset R$ and $\Sigma_{MED}(B_n) = [a_n, +\infty)$. Moreover, $W_{n+1} := U_{\gamma_n,\varepsilon,L} = R \times \{0\}$ and $\Sigma_{MED}(B_{n+1}) = \emptyset$.

Step 3. Next we prove that the order N_i of block $B_i(t)$ equals to $\dim W_i$ as $i = 0, \ldots, n+1$.

As $\mathbf{I}_1 = [a_1, b_1]$, we can choose $\gamma_0 \in (-\infty, a_1)$ such that $W_0 = S_{\gamma_0,\varepsilon,L}$ and

$$N_0 = \dim Im\hat{P}_0 = \dim S_{\gamma_0,\varepsilon,L} = \dim W_0.$$

In particular, if $\mathbf{I}_1 = (-\infty, b_1]$ then $W_0 = R \times \{0\}$ and $N_0 = \dim W_0$.

As $\gamma_1 \in (b_1, a_2)$, from discussion above, it is apparent that $N_0 + N_1 = \dim Im\hat{P}_1$, and

$$\dim Im\hat{P}_1 = \dim S_{\gamma_1,\varepsilon,L} = \dim(S_{\gamma_1,\varepsilon,L} \cap (S_{\gamma_0,\varepsilon,L} \oplus U_{\gamma_0,\varepsilon,L})) = \dim W_0 + \dim W_1,$$

thus $N_1 = \dim W_1$. Likewise, for the case of $\gamma_2 \in (b_2, a_3)$, $N_2 = \dim W_2$ depends on the fact that

$$N_0 + N_1 + N_2 = \dim Im\hat{P}_2 = \dim S_{\gamma_2,\varepsilon,L} = \dim S_{\gamma_1,\varepsilon,L} + \dim W_2.$$

By induction, we conclude $N_i = \dim W_i$ for $i = 0, \ldots, n$. In addition, for $\gamma_n \in (b_n, +\infty)$ or $\gamma_n \in (a_n, +\infty)$, one has $N_{n+1} = \dim W_{n+1}$ due to

$$N_0 + N_1 + \cdots + N_{n+1} = \dim W_0 + \dim W_1 + \cdots + \dim W_{n+1}.$$

The proof is completed. □

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