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On the boundary regularity of phase-fields for Willmore's energy

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We demonstrate that Radon measures which arise as the limit of the Modica-Mortola measures associated with phase-fields with uniformly bounded diffuse area and Willmore energy may be singular at the boundary of a domain and discuss implications for practical applications. We furthermore give partial regularity results for the phase-fields u_ε at the boundary in terms of boundary conditions and counterexamples without boundary conditions.

Keywords: Willmore energy; phase-field; boundary regularity

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1. Introduction

Phase-field approximations provide a convenient way of treating curvature energies numerically. Typically, the semi-linear phase-field problem is of a simpler form than the highly non-linear original problem and it is easier to develop a stable numerical implementation. A classical example of a curvature energy is the Willmore functional

$$\mathcal{W}(\Sigma) = \int_\Sigma H^2 \,\mathrm{d}\mathcal{H}^{n-1}$$

where $\Sigma \subset \mathbb{R}^n$ is a hypersurface, H denotes its mean curvature and $\mathcal{H}^k$ the k-dimensional Hausdorff measure. The same functional on plane curves is also sometimes referred to as Euler's elastica.

There are several distinct phase-field approximations of Willmore's energy [3]. The model we will use in the following is due to Bellettini and Paolini [2], based on a functional proposed by De Giorgi [5, conjecture 4].

Let $\Omega \Subset \mathbb{R}^n$ and W be the double-well potential $W(u) = 1/4\,(u^2-1)^2$. Then we consider the Modica-Mortola energy [9, 10]

$$S_\varepsilon\colon L^1(\Omega) \to \mathbb{R}, \quad S_\varepsilon(u) = \begin{cases} \frac{1}{c_0}\int_\Omega \frac{\varepsilon}{2}\,|\nabla u|^2 + \frac{1}{\varepsilon}\,W(u)\,\mathrm{d}x & u \in W^{1,2}(\Omega) \\ +\infty & \text{else} \end{cases}$$

as an approximation of the perimeter functional and

$$\mathcal{W}_\varepsilon\colon L^1(\Omega) \to \mathbb{R}, \quad \mathcal{W}_\varepsilon(u) = \begin{cases} \frac{1}{c_0\,\varepsilon}\int_\Omega \left(\varepsilon\,\Delta u - \frac{1}{\varepsilon}\,W'(u)\right)^2\,\mathrm{d}x & u \in W^{2,2}(\Omega) \\ +\infty & \text{else} \end{cases}$$

as an approximation of Willmore's energy, where $c_0 = \int_{-1}^1 \sqrt{2\,W(s)}\,\mathrm{d}s = 2\sqrt{2}/3$ is a normalizing constant. As proved in [12], the sum of the functionals satisfies

$$\left[\Gamma(L^1(\Omega)) - \lim_{\varepsilon\to 0}\,(\mathcal{W}_\varepsilon + \Lambda\,S_\varepsilon)\right]\,(\chi_E - \chi_{\Omega\setminus E}) \;=\; \mathcal{W}(\partial E) + \Lambda\,\mathcal{H}^{n-1}(\partial E)$$

for any $\Lambda > 0$ if $E \Subset \Omega$ and $\partial E \in C^2$ in low dimension $n = 2, 3$. Consider a general sequence u_ε such that

$$\limsup_{\varepsilon\to 0}(S_\varepsilon + \mathcal{W}_\varepsilon)(u_\varepsilon) < \infty.$$

Then the diffuse area measures

$$\mu_\varepsilon := \frac{1}{c_0}\left(\frac{\varepsilon}{2}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}\,W(u_\varepsilon)\right)\cdot\mathcal{L}^n$$

which localize the diffuse perimeter functional S_ε and the diffuse Willmore measures

$$\alpha_\varepsilon := \frac{1}{c_0\,\varepsilon}\left(\varepsilon\,\Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\right)\cdot\mathcal{L}^n$$

which localize the functionals $\mathcal{W}_\varepsilon$ have weak limits μ and α in the sense of Radon measures, at least for a suitable subsequence. Due to [12], μ is the mass measure of an integral $(n-1)$-varifold V in Ω with square integrable mean curvature and

$$|H_\mu|^2\cdot\mu \leqslant \alpha. \tag{1.1}$$

In this paper, we will show among other things that the relationship (1.1) is only valid *inside* Ω and that μ may be very irregular on $\partial\Omega$ if the boundary values of the phase-fields u_ε are not controlled. In particular, μ may cease to be $n-1$-dimensional and the mean curvature H_μ may not even be defined the boundary, even if $\mu(\partial\Omega) > 0$. The choice of boundary values corresponds to a modelling assumption. In [7], we have investigated thin elastic structures in a bounded container, where the natural boundary condition is

$$u_\varepsilon \equiv -1, \quad \partial_\nu u_\varepsilon \equiv 0 \quad \text{on } \partial\Omega \quad \text{or in simpler terms } u_\varepsilon \in -1 + W^{2,2}_0(\Omega) \tag{1.2}$$

to express that the structures are confined to Ω and only touch the boundary tangentially. Another interesting boundary condition is

$$\partial_\nu u_\varepsilon \equiv 0 \quad \text{on } \partial\Omega \tag{1.3}$$

which expresses that the level sets of u_ε can only meet $\partial\Omega$ at a right angle. This approximates the minimization problem explored in [**1**], where the Willmore functional is considered in the class of surfaces of a given (small) area which meet $\partial\Omega$ orthogonally. Another possible boundary condition is

$$u_\varepsilon \equiv 1 \quad \text{on } \Gamma_+, \qquad u_\varepsilon \equiv -1 \quad \text{on } \Gamma_-, \quad u_\varepsilon \quad \text{free on } \partial\Omega \setminus \Gamma_+ \cup \Gamma_- \tag{1.4}$$

which prescribes a phase transition inside Ω but leaves the particular nature of the transition free. It is clear that any regularity result for μ or the functions u_ε inside Ω can be extended to $\overline{\Omega}$ under the boundary conditions (1.2), since u_ε can be extended to the whole space $\mathbb{R}^n$ as a constant function without changing the energy

$$\mathcal{E}_\varepsilon(u_\varepsilon) := (\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon).$$

On the contrary, the regularity of u_ε and μ under the boundary values (1.3) or (1.4) is less obvious. Furthermore, not specifying boundary values can simplify proofs significantly when local results are considered, see for example [**6**, corollary 2.15]. In this paper, we extend regularity results for the phase-fields u_ε from [**6**, **7**]. Our main results are the following.

THEOREM 1.1. *Let $\Omega \Subset \mathbb{R}^n$ for $n = 2, 3$ and $u_\varepsilon \in W^{2,2}(\Omega)$ such that*

$$\sup_{\varepsilon>0} \mathcal{E}_\varepsilon(u_\varepsilon) < \infty.$$

Then the following hold true.

(1) *Assume that $u_\varepsilon \in C^0(\overline{\Omega})$ is uniformly bounded in $L^\infty(\partial\Omega)$. Then u_ε is uniformly bounded in $L^\infty(\Omega)$ if $n = 2$ and in $L^p(\Omega)$ for all $p < \infty$ if $n = 3$.*

(2) *Assume that $\partial\Omega \in C^2$ and $\partial_\nu u_\varepsilon \equiv 0$ on $\partial\Omega$ for all $\varepsilon > 0$. Then u_ε is uniformly bounded in $L^\infty(\Omega)$ and*

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leqslant \frac{C}{\varepsilon^\gamma}\,|x-y|^\gamma \qquad \forall\ x \in \overline{\Omega},\ y \in B_\varepsilon(x) \cap \overline{\Omega}$$

with $\gamma < 1$ if $n = 2$ and $\gamma \leqslant 1/2$ if $n = 3$. The constant C depends on n, γ, Ω and $\limsup_{\varepsilon\to 0} \mathcal{E}_\varepsilon(u_\varepsilon)$.

(3) *If either condition is given and $u_\varepsilon \to u$ in $L^1(\Omega)$, then $u_\varepsilon \to u$ in $L^p(\Omega)$ for all $1 \leqslant p < \infty$.*

Further results can be found in the main text. The proof is split over lemmas 2.1, 2.3 and 2.5. On the contrary, we have the following results in situations where phase-fields fail to be regular at the boundary.

THEOREM 1.2. *Let $\partial\Omega \in C^2$. Then the following hold true.*

(1) *There exists a sequence $u_\varepsilon \in W^{2,2}(\Omega)$ such that $(\mathcal{W}_\varepsilon + S_\varepsilon)(u_\varepsilon) \to 0$, but u_ε is not bounded in $L^\infty(\Omega)$.*

(2) *There exists a sequence u_ε such that such that $\alpha = 0$, $\mu = 0$ but the Hausdorff limit*

$$K := \lim_{\varepsilon\to 0} u_\varepsilon^{-1}(I) \qquad \emptyset \neq I \Subset (-1,1)$$

of level sets or their unions contains an open subset of $\partial\Omega$. Similar constructions give $K = \{x_0\}$ or $K = \gamma$ for a point $x_0 \in \partial\Omega$ and a closed curve $\gamma \subset \partial\Omega$.

(3) *Let $S > 0$ and $\emptyset \neq I \Subset (-1,1)$. Then there exists a point $x_0 \in \partial\Omega$ and a sequence $u_\varepsilon \in W^{2,2}(\Omega)$ such that $|u_\varepsilon| \leqslant 1$ in $\overline{\Omega}$, $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$, $\mu_\varepsilon(\Omega) \equiv S$, $K = \emptyset$ and $\mu = S \cdot \delta_{x_0}$.*

If Ω is convex, any point x_0 or closed curve γ in $\partial\Omega$ can be chosen and u_ε may be such that it is not uniformly bounded in $\Omega \cap U$ for all open sets U with $U \cap \partial\Omega \neq \emptyset$. All sequences have uniformly bounded energies $\mathcal{E}_\varepsilon(u_\varepsilon)$.

The second point contrasts the theorem with the situation *inside* Ω where $K \cap \Omega = \mathrm{spt}(\mu) \cap \Omega \cup \{x_1, \ldots, x_N\}$ holds, that is, K and $\mathrm{spt}(\mu)$ agree up to finitely many points $x_1, \ldots, x_N$. These points are atoms of the measure α [**6**]. This shows that for example the minimization problem for

$$\mathcal{F}_\varepsilon = \mathcal{W}_\varepsilon + \varepsilon^{-\sigma}(S_\varepsilon - S)^2$$

does not approximate the formal sharp interface analogue without boundary conditions or with partly free boundary conditions (1.4) if $\partial_{\text{free}}\Omega := \partial\Omega \setminus (\Gamma_+ \cup \Gamma_-) \neq \emptyset$. A minimizing sequence is given by the superposition of a phase-field making an optimal transition along a minimal surface spanning a suitable boundary curve inside $\partial_{\text{free}}\Omega$ and a second phase-field creating an atom of μ of the correct size at a single point $x \in \partial_{\text{free}}\Omega$. This can be realized with energy $\mathcal{W}_\varepsilon(u_\varepsilon) \to 0$ as $\varepsilon \to 0$.

The question under which boundary conditions other than (1.2) the measure μ can be expected to be regular at the boundary for either finite energy sequences or minimizing sequences remains open.

2. Positive results on boundary regularity

In this chapter, we describe partial regularity results for weakly controlled boundary values. We always assume that the dimension is $n \in \{2, 3\}$. Denote

$$\bar{\alpha} = \limsup_{\varepsilon\to 0} \mathcal{W}_\varepsilon(u_\varepsilon).$$

LEMMA 2.1. *Assume that u_ε is continuous on $\overline{\Omega}$ and there is $\theta \geqslant 1$ such that $|u_\varepsilon| \leqslant \theta$ on $\partial\Omega$ for all $\varepsilon > 0$. Then the following holds true.*

(1) *We have $\mu_\varepsilon(\{|u_\varepsilon| > \theta\}) \leqslant ((\alpha_\varepsilon(\Omega))/(4))\,\varepsilon^2$.*

(2) *For the set $\tilde{\Omega}_\varepsilon = \{x \in \Omega \mid B_{2\varepsilon}(x) \subset \Omega\}$, we can show that there exists C depending only on $\bar{\alpha}, \gamma$ and θ such that*

$$||u_\varepsilon||_{\infty,\tilde{\Omega}_\varepsilon} \leqslant C, \quad |u_\varepsilon(y) - u_\varepsilon(z)| \leqslant \frac{C_{\bar{\alpha},\theta,\gamma}}{\varepsilon^\gamma}\,|y-z|^\gamma$$

if there is $x \in \tilde{\Omega}_\varepsilon$ such that $y, z \in B_\varepsilon(x)$ and $\gamma \leqslant 1/2$ if $n = 3$, $\gamma < 1$ if $n = 2$.

Proof. This proof is an adaptation of the proof of Lemma [7, lemma 3.1] using a modified argument in the first step of the proof. We observe that for the proof of Lemma [7, lemma 3.1] to work, we needed that $B_{2\varepsilon}(x) \subset \Omega$ to employ the elliptic inequality

$$||\tilde{u}_\varepsilon||_{2,2,B_1(0)} \leqslant C\,(||\tilde{u}_\varepsilon||_{2,B_2(0)} + ||\Delta\tilde{u}_\varepsilon||_{2,B_2(0)})$$

and an estimate of $\int_{B_{2\varepsilon}(x)}((1)/(\varepsilon^n))W'(u_\varepsilon)^2\,\mathrm{d}x$. The fact that $B_{2\varepsilon}(x) \subset \Omega$ for the x we consider is automatic by the choice of $\tilde{\Omega}_\varepsilon$, the integral estimate can be obtained through integration by parts as follows. Choose $\theta' > \theta$ such that $\{|u_\varepsilon| > \theta'\}$ is a Caccioppoli set (which is true for almost all $\theta' > \theta$). As $|u_\varepsilon| \leqslant \theta < \theta'$ on $\partial\Omega$, the set $\{u_\varepsilon > \theta'\}$ does not touch the boundary $\partial\Omega$, so $\partial\{u_\varepsilon > \theta'\} \subset \{u_\varepsilon = \theta'\} \subset \Omega$. Thus

$$\begin{aligned} c_0\,\alpha_\varepsilon(\{|u_\varepsilon| > \theta'\}) &= \int_{\{|u_\varepsilon|>\theta'\}} \frac{1}{\varepsilon}\left(\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2 \mathrm{d}x \\ &= -\frac{2}{\varepsilon}\int_{\partial\{|u_\varepsilon|>\theta'\}} W'(u_\varepsilon)\,\partial_\nu u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1} \\ &\quad + \int_{\{|u_\varepsilon|>\theta'\}} \varepsilon\,(\Delta u_\varepsilon)^2 + \frac{2}{\varepsilon}\,W''(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x \\ &\geqslant \int_{\{|u_\varepsilon|>\theta'\}} \varepsilon\,(\Delta u_\varepsilon)^2 + \frac{4}{\varepsilon}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x \end{aligned}$$

because $W'(\theta') > 0$ (since $\theta' > \theta \geqslant 1$) and ∇u_ε is inward pointing on $\partial\{u_\varepsilon > \theta'\}$, the boundary integral is non-positive. This implies a uniform bound on

$$\int_{B_{2\varepsilon}(x)} \frac{1}{\varepsilon^n} W'(u_\varepsilon)^2\,\mathrm{d}x \leqslant 2^n\,\max\{W(0), W(\theta')\} + c_0\,\alpha_\varepsilon(\{u_\varepsilon > \theta'\})$$

for $n = 2, 3$ and some fixed $\theta' > \theta$. Using that $[W'(u)]^2 = (u^2-1)^2u^2 \geqslant 4\,(((u^2-1)^2)/(4)) = 4\,W(u)$ for $u \geqslant 1$, we also find that

$$c_0\,\alpha_\varepsilon(\Omega) \geqslant c_0\,4\varepsilon^{-2}\,\mu_\varepsilon(\{u_\varepsilon > \theta'\})$$

for all $\theta' > \theta$, which is preserved in the limit $\theta' \to \theta$ by the continuity of measures. The rest of the argument goes through as before. □

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REMARK 2.2. The same bound holds for example on $\widetilde\Omega_{\varepsilon^{1/2}} := \{x\in\Omega \mid B_{\varepsilon^{1/2}}(x)\subset\Omega\}$ without boundary values. In that situation, we employ the estimate from [12, proposition 3.6] to bound

$$\frac{1}{\varepsilon^3}\int_{\{|u_\varepsilon|>1\}} W'(u_\varepsilon)^2\,\mathrm{d}x \leqslant C.$$

Another situation with a similar improvement is that of prescribed Neumann boundary data.

LEMMA 2.3. *Assume that Ω has a Lipschitz boundary and $\partial_\nu u_\varepsilon = 0$ almost everywhere on $\partial\Omega$. Then the following hold true.*

(1) *There exists $C>0$ such that $\mu_\varepsilon(\{|u_\varepsilon|\geqslant 1\}) \leqslant C\,\varepsilon^2$.*

(2) *For the set $\tilde\Omega_\varepsilon = \{x\in\Omega \mid B_{2\varepsilon}(x)\subset\Omega\}$, we can show that there exists C depending only on $\bar\alpha$ and γ such that*

$$||u_\varepsilon||_{\infty,\tilde\Omega_\varepsilon} \leqslant C, \qquad |u_\varepsilon(y)-u_\varepsilon(z)| \leqslant \frac{C}{\varepsilon^\gamma}\,|y-z|^\gamma$$

if there is $x\in\tilde\Omega_\varepsilon$ such that $y,z\in B_\varepsilon(x)$. Here $\gamma\leqslant 1/2$ if $n=3$, $\gamma<1$ if $n=2$.

If $\partial\Omega\in C^2$ and $\partial_\nu u_\varepsilon = 0$ almost everywhere on $\partial\Omega$, then the second statement can be sharpened as follows:

(2') *For all $x\in\overline\Omega$ there exists a constant C depending only on $\bar\alpha,\gamma$ and $\partial\Omega$ such that*

$$|u_\varepsilon(x)|\leqslant C, \qquad |u_\varepsilon(y)-u_\varepsilon(z)| \leqslant \frac{C}{\varepsilon^\gamma}\,|x-y|^\gamma \qquad \forall\ y,z\in B_\varepsilon(x)\cap\overline\Omega.$$

The dependence of C on $\partial\Omega$ vanishes in the limit $\varepsilon\to 0$.

In particular, for regular boundaries, the Neumann condition implies the boundedness of solutions (in particular also on the boundary).

Proof. Since Ω is a Lipschitz domain, $W^{2,2}(\Omega)$ embeds into $C^{0,1/2}(\overline\Omega)$ in dimensions $n=2,3$ and $W'(u_\varepsilon)$ is a bounded function on $\partial\Omega$ for fixed $\varepsilon>0$. As before, we obtain

$$\begin{aligned}
\alpha_\varepsilon(\{|u_\varepsilon|>\theta'\}) &= \int_{\{|u_\varepsilon|>\theta'\}} \frac{1}{\varepsilon}\left(\varepsilon\,\Delta u_\varepsilon - \frac{1}{\varepsilon}\,W'(u_\varepsilon)\right)^2\mathrm{d}x\\
&= -\frac{2}{\varepsilon}\int_{\partial\Omega\cap\partial\{|u_\varepsilon|>\theta'\}} W'(u_\varepsilon)\,\partial_\nu u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad - \frac{2}{\varepsilon}\int_{\partial\{|u_\varepsilon|>\theta'\}\cap\Omega} W'(u_\varepsilon)\,\partial_\nu u_\varepsilon\,\mathrm{d}\mathcal{H}^{n-1}\\
&\quad + \int_{\{|u_\varepsilon|>\theta'\}} \varepsilon\,(\Delta u_\varepsilon)^2 + \frac{2}{\varepsilon}\,W''(u_\varepsilon)\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x
\end{aligned}$$

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$$\geqslant \int_{\{|u_\varepsilon|>\theta'\}} \varepsilon\,(\Delta u_\varepsilon)^2 + \frac{4}{\varepsilon}\,|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x$$

for any $\theta' > 1$ such that $\{|u_\varepsilon| > \theta'\}$ is a Caccioppoli set. Here the boundary integral can be split into two parts, one of which has a sign, while the other one vanishes due to the Neumann condition. This implies the boundedness on $\tilde{\Omega}_\varepsilon$ and the bound on the mass measures $\mu_\varepsilon(\{|u_\varepsilon| > \theta'\})$ as before. We can take $\theta' \to 1$ to prove the first part of the lemma.

Now assume that $\partial\Omega \in C^2$ and pick $x \in \partial\Omega$. The rest of the argument is a fairly standard 'straightening the boundary' argument with the feature that the boundary becomes flatter as $\varepsilon \to 0$. Without loss of generality, we assume that $x = 0$. We may now blow up to

$$\tilde{u}_\varepsilon : B_2(0) \cap (\Omega/\varepsilon) \to \mathbb{R}, \qquad \tilde{u}_\varepsilon(y) = u_\varepsilon(\varepsilon y).$$

We pick a C^2-diffeomorphism $\phi_\varepsilon : B_2(0) \to B_2(0)$ such that

(1) $\phi_\varepsilon(\Omega/\varepsilon \cap B_2(0)) = B_2^+(0)$,

(2) $\phi_\varepsilon \to \mathrm{id}_{B_2(0)}$ in $C^2(B_2(0), B_2(0))$ as the domain becomes increasingly flat,

(3) under ϕ_ε, the normal to $\partial\Omega/\varepsilon$ gets mapped to e_n on the boundary, that is, the orthogonality condition is preserved.

With this, we obtain a function

$$\tilde{w}_\varepsilon : B_2^+(0) \to \mathbb{R}, \qquad \tilde{w}_\varepsilon(y) = \tilde{u}_\varepsilon(\phi_\varepsilon^{-1}(y))$$

in flattened coordinates. Since ϕ_ε is C^2-smooth and $\tilde{u}_\varepsilon$ is $W^{2,2}$-smooth, also $\tilde{w}_\varepsilon$ is $W^{2,2}$-smooth on its domain and it is easy to calculate

$$\begin{aligned} \partial_i \tilde{u}_\varepsilon &= \partial_i(\tilde{w}_\varepsilon \circ \phi_\varepsilon) \\ &= \partial_i(\phi_\varepsilon)_j\ ((\partial_j \tilde{w}_\varepsilon) \circ \phi_\varepsilon) \\ \partial_{ij}\,\tilde{u}_\varepsilon &= \partial_{ij}(\phi_\varepsilon)_k\ ((\partial_k \tilde{w}_\varepsilon) \circ \phi_\varepsilon) + \partial_i(\phi_\varepsilon)_k\,\partial_j(\phi_\varepsilon)_l\ ((\partial_{kl}\tilde{w}_\varepsilon) \circ \phi_\varepsilon)\,. \end{aligned}$$

In shorter notation, this means that

$$\nabla \tilde{u}_\varepsilon = D\phi \cdot \nabla \tilde{w}_\varepsilon, \qquad \Delta \tilde{u}_\varepsilon = a_\varepsilon^{ij}\,\partial_{ij}\tilde{w}_\varepsilon + \langle \Delta\phi_\varepsilon, \nabla \tilde{w}_\varepsilon\rangle$$

with

$$a_\varepsilon^{ij} = \langle \partial_i \phi_\varepsilon, \partial_j \phi_\varepsilon\rangle.$$

The coefficients are C^1-differentiable – so the associated operator A_ε can be equivalently written in divergence form – and C^1-close to δ_{ij}. We observe that

$$\begin{aligned} (\Delta \tilde{u}_\varepsilon - W'(\tilde{u}_\varepsilon))\,(\phi_\varepsilon(y)) = \big(&\partial_i\left(a_\varepsilon^{ij}\,\partial_j \tilde{w}_\varepsilon\right) - (\partial_i\,a_\varepsilon^{ij})\partial_j \tilde{w}_\varepsilon \\ &+ \langle \Delta\phi_\varepsilon, \nabla\tilde{w}_\varepsilon\rangle - W'(\tilde{w}_\varepsilon)\big)\,(y). \end{aligned}$$

We extend $\tilde{w}_\varepsilon$ by even reflection to the whole ball $B_2(0)$, which preserves the $W^{2,2}$-smoothness since we preserved the property that $\partial_\nu \tilde{u}_\varepsilon = 0$ on the boundary when straightening the boundary. We observe that

$$\partial_i\,(a_\varepsilon^{ij}\partial_j\tilde w_\varepsilon) - \langle \operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon, \nabla\tilde w_\varepsilon\rangle =: f_\varepsilon \in L^2(B_2(0))$$

since

$$\begin{aligned}\int_{B_2(0)} W'(\tilde w_\varepsilon)^2\,\mathrm{d}y &= 2\int_{B_2^+(0)} W'(\tilde w_\varepsilon)^2(y)\,\mathrm{d}y\\ &= 2\int_{\Omega/\varepsilon\cap B_2(0)} W'(\tilde u_\varepsilon((z)))\,\det(D\phi_\varepsilon^{-1})(z)\,\mathrm{d}z\\ &\leqslant 2(1+c_\varepsilon)\int_{B_{2\varepsilon}(x)} \frac{1}{\varepsilon^n} W'(\tilde u_\varepsilon)\,\mathrm{d}z\\ &\leqslant C\end{aligned}$$

as already in lemma 2.1. The constants c_ε vanish as $\varepsilon\to 0$ and $\phi_\varepsilon \to \mathrm{id}$. The coefficients a_{ij} are uniformly elliptic and approach δ_{ij} uniformly as $\varepsilon\to 0$, so we can obtain an elliptic estimate on $||\nabla\tilde w_\varepsilon||_{L^2(B_{3/2})}$. Since the gradient term appears in an unusual way, we present the details. Let η be a radial, smooth cut-off function such that

$$0\leqslant\eta\leqslant 1,\quad \eta\equiv 1 \text{ on } B_{3/2},\quad \eta\equiv 0 \text{ on } \mathbb{R}^n\setminus B_2.$$

Then we find that

$$\begin{aligned}\frac{3}{4}\int_{B_2}|\nabla(\eta\tilde w_\varepsilon)|^2\,\mathrm{d}x &\leqslant \int_{B_2} a_\varepsilon^{ij}\partial_i(\eta\tilde w_\varepsilon)\,\partial_j(\eta\tilde w_\varepsilon)\,\mathrm{d}x\\ &= \int_{B_2} \eta\, a_\varepsilon^{ij}\partial_j\tilde w_\varepsilon\,\partial_i(\eta\tilde w_\varepsilon) + \tilde w_\varepsilon\, a_\varepsilon^{ij}\partial_j\eta\,\partial_i(\eta\tilde w_\varepsilon)\,\mathrm{d}x\\ &= \int_{B_2} -\eta\tilde w_\varepsilon\, a_\varepsilon^{ij}\partial_i\eta\,\partial_j\tilde w_\varepsilon - \eta^2\,\tilde w_\varepsilon\partial_i(a_\varepsilon^{ij}\,\partial_j\tilde w_\varepsilon) + \tilde w_\varepsilon\, a_\varepsilon^{ij}\partial_j\eta\,\partial_i(\eta\tilde w_\varepsilon)\,\mathrm{d}x\\ &= \int_{B_2} -\tilde w_\varepsilon\, a_\varepsilon^{ij}\partial_i\eta\,\partial_j(\eta\,\tilde w_\varepsilon) + \tilde w_\varepsilon^2\, a_\varepsilon^{ij}\,\partial_i\eta\,\partial_j\eta\\ &\quad - \eta^2\tilde w_\varepsilon\,[f_\varepsilon + \langle \operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon, \nabla\tilde w_\varepsilon\rangle] + \tilde w_\varepsilon\, a_\varepsilon^{ij}\partial_j\eta\,\partial_i(\eta\tilde w_\varepsilon)\,\mathrm{d}x\\ &= \int_{B_2} \tilde w_\varepsilon^2\, a_\varepsilon^{ij}\,\partial_i\eta\,\partial_j\eta - \eta^2\tilde w_\varepsilon\, f_\varepsilon + \eta\tilde w_\varepsilon\langle \operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon, \nabla(\eta\tilde w_\varepsilon)\rangle\\ &\quad + \eta\tilde w_\varepsilon^2\langle \operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon, \nabla\eta\rangle\,\mathrm{d}x\end{aligned}$$

using that a_ε^{ij} is uniformly elliptic with constant close to 1 and a fortunate cancellation. Young's inequality tells us that

$$\begin{aligned}\frac{3}{4}\int_{B_2}|\nabla(\eta\tilde w_\varepsilon)|^2\,\mathrm{d}x &\leqslant ||A_\varepsilon||_{L^\infty}\,||\nabla\eta||_{L^\infty}^2\,||\tilde w_\varepsilon||_{L^2}^2 + \frac{1}{2}\,||\tilde w_\varepsilon||_{L^2}^2\\ &\quad + \frac{1}{2}\,||f_\varepsilon||_{L^2}^2 + ||\operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon||_{L^\infty}^2\,||\tilde w_\varepsilon||_{L^2}^2\\ &\quad + \frac{1}{4}\,||\nabla(\eta\tilde w_\varepsilon)||_{L^2}^2 + ||\operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon||_{L^\infty}\,||\nabla\eta||_{L^\infty}\,||\tilde w_\varepsilon||_{L^2}^2\end{aligned}$$

thus in total that

$$\frac{1}{2}\int_{B_{3/2}} |\nabla \tilde{w}_\varepsilon|^2 \,\mathrm{d}x \leqslant \frac{1}{2}\int_{B_2} |\nabla(\tilde{w}_\varepsilon)|^2 \,\mathrm{d}x \leqslant C\left[||\tilde{w}_\varepsilon||^2_{L^2(B_2)} + ||f_\varepsilon||^2_{L^2(B_2)}\right]$$

with a constant C uniform in ε since $||\operatorname{div} A_\varepsilon - \Delta\phi_\varepsilon||_{L^\infty(B_2)} \to 0$ as $\varepsilon \to 0$. This gives us a uniform $W^{1,2}$-bound for all sufficiently small ε, where the necessary smallness depends only on $\mathcal{W}_\varepsilon(u_\varepsilon)$ and $\partial\Omega$. In a second step, this gives us a uniform bound on $||\tilde{w}_\varepsilon||_{W^{2,2}(B_1(0))}$, which gives us a uniform bound on $||\tilde{u}_\varepsilon||_{W^{2,2}(B_{3/2}(0)\cap\Omega/\varepsilon)}$ after transforming back. The rest follows by Sobolev embeddings as in [7, lemma 3.1]. □

REMARK 2.4. The case that Ω has finite perimeter and $\partial_\nu u_\varepsilon = 0$ almost everywhere on the reduced boundary is a generalization of the situation in which $\partial\Omega \in C^2$ and the level sets of u_ε meet $\partial\Omega$ at a 90° angle. Such conditions arise naturally when we search for surfaces of minimal perimeter bounding a prescribed volume and may be useful also for models containing Willmore's energy [1].

We give an improvement of the L^∞-bound up to the boundary which implies L^p-convergence for all finite p.

LEMMA 2.5. *Assume that there is $\theta \geqslant 1$ such that $|u_\varepsilon| \leqslant \theta$ on $\partial\Omega$ for all $\varepsilon > 0$. Then the following hold true.*

(1) *If $n = 2$, $\partial\Omega \in C^{1,1}$ and $\theta > 1$, then for every $\beta < 1$ there exists a constant C depending only on $\bar{\alpha}, \theta, \Omega$ and β such that $\sup_{x\in\Omega} |u_\varepsilon(x)| \leqslant \theta + C\varepsilon^\beta$ for all $\varepsilon > 0$.*
If $\theta = 1$, then for every $\beta < 1/2$ there exists a constant C depending only on $\bar{\alpha}, \Omega$ and β such that $\sup_{x\in\Omega} |u_\varepsilon(x)| \leqslant 1 + C\varepsilon^\beta$ for all $\varepsilon > 0$.

(2) *If $n = 3$ and $\partial\Omega \in C^{1,1}$, then for every $p < \infty$ there exists C depending only on $\bar{\mu}, \bar{\alpha}, \theta, p$ and Ω such that $||u_\varepsilon||_{p,\Omega} \leqslant C$. Furthermore, for every $\sigma > 0$ there exists C depending only on $\bar{\alpha}, \theta, \Omega$ and σ such that $||u_\varepsilon||_{\infty,\Omega} \leqslant C\,\varepsilon^{-\sigma}$.*

We conjecture that also in three dimensions, uniformly bounded boundary values lead to uniform interior bounds.

Proof. The proof is a modified version of that of [12, proposition 3.6]. We follow that proof closely, but use a different maximum principle.

Let $\theta' > \theta \geqslant 1$ such that $\{|u_\varepsilon| > \theta'\}$ has finite perimeter and define $w_\varepsilon \coloneqq (u_\varepsilon - \theta')_+$. Then $w_\varepsilon \in W_0^{1,2}(\Omega)$ and from the same integration by parts as before, we obtain that

$$||w_\varepsilon||^2_{1,2,\Omega} \leqslant \int_{\{u_\varepsilon > \theta'\}} W'(u_\varepsilon)^2 + |\nabla u_\varepsilon|^2 \leqslant \alpha_\varepsilon(\Omega)\,\varepsilon.$$

The function satisfies

$$\begin{aligned}\int_\Omega w_\varepsilon\,(-\Delta\phi)\,\mathrm{d}x &= \int_{\{u_\varepsilon>\theta'\}}(u_\varepsilon-\theta')\,(-\Delta\phi)\,\mathrm{d}x\\ &= -\int_{\partial\{u_\varepsilon>\theta'\}}(u_\varepsilon-\theta')\,\partial_\nu\phi\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\{u_\varepsilon>\theta'\}}\langle\nabla\phi,\nabla u_\varepsilon\rangle\,\mathrm{d}x\\ &= \int_{\partial\{u_\varepsilon>\theta'\}}\phi\,\partial_\nu u_\varepsilon-(u_\varepsilon-\theta')\,\partial_\nu\phi\,\mathrm{d}\mathcal{H}^{n-1}+\int_{\{u_\varepsilon>\theta'\}}\phi\,(-\Delta u_\varepsilon)\,\mathrm{d}x\\ &\leqslant \int_{\{u_\varepsilon>\theta'\}}\phi\,(-\Delta u_\varepsilon)\,\mathrm{d}x\end{aligned}$$

for $\phi\geqslant 0$. Again, this holds true because $\partial\{u_\varepsilon>\theta'\}\subset\{u_\varepsilon=\theta'\}$. Denoting $h_\varepsilon = -\varepsilon\,\Delta u_\varepsilon+((1)/(\varepsilon))W'(u_\varepsilon)$, we get

$$\begin{aligned}\int_{\{u_\varepsilon>\theta'\}}\phi\,(-\Delta u_\varepsilon)\,\mathrm{d}x &= \int_{\{u_\varepsilon>\theta'\}}\left(-\Delta u_\varepsilon+\frac{1}{\varepsilon^2}W'(u_\varepsilon)-\frac{1}{\varepsilon^2}\,W'(u_\varepsilon)\right)\,\phi\,\mathrm{d}x\\ &\leqslant \int_{\{u_\varepsilon>\theta'\}}\frac{1}{\varepsilon}\,\left(h_\varepsilon-\frac{1}{\varepsilon}\,W'(\theta')\right)_+\phi\,\mathrm{d}x,\end{aligned}$$

so $-\Delta w_\varepsilon\leqslant((1)/(\varepsilon))\,\chi_{\{u_\varepsilon>\theta'\}}(h_\varepsilon-((1)/(\varepsilon))\,W'(\theta'))_+$ in the distributional sense. When we consider the solution $\psi_\varepsilon\in W_0^{1,2}(\Omega)$ of the problem

$$-\Delta\psi_\varepsilon=\frac{1}{\varepsilon}\,\left(h_\varepsilon-\frac{1}{\varepsilon}\,W'(\theta')\right)_+\chi_{\{u_\varepsilon>\theta'\}},$$

the weak maximum principle [8, theorem 8.1] applied to $w_\varepsilon-\psi_\varepsilon$ implies that

$$u_\varepsilon\leqslant\theta+w_\varepsilon\leqslant\theta+\psi_\varepsilon. \tag{2.1}$$

We proceed to estimate

$$\begin{aligned}||\Delta\psi_\varepsilon||_{q,\Omega}^q &= \varepsilon^{-q}\int_{\{u_\varepsilon>\theta'\}}\left(h_\varepsilon-\frac{1}{\varepsilon}\,W'(\theta')\right)_+^q\,\mathrm{d}x\\ &\leqslant \varepsilon^{-q}\left(\int_{\{u_\varepsilon>\theta'\}}1\,\mathrm{d}x\right)^{1-q/2}\left(\int_\Omega h_\varepsilon^2\,\mathrm{d}x\right)^{q/2}\\ &\leqslant \varepsilon^{-q}\left(\frac{\varepsilon^3}{W'(\theta')^2}\int_{\{u_\varepsilon>\theta'\}}\frac{1}{\varepsilon^3}\,W'(u_\varepsilon)^2\,\mathrm{d}x\right)^{1-q/2}\left(\varepsilon\int_\Omega\frac{1}{\varepsilon}\,h_\varepsilon^2\,\mathrm{d}x\right)^{q/2}\\ &\leqslant c_{\bar\alpha,q}\,(W'(\theta'))^{q-2}\,\varepsilon^{-q+3(1-q/2)+q/2}\\ &= c_{\bar\alpha,q}\,(W'(\theta'))^{q-2}\,\varepsilon^{3-2q}\end{aligned}$$

for $1\leqslant q<2$. Thus $||\Delta\psi_\varepsilon||_{q,\Omega}\leqslant C_{\bar\alpha,q}\,(W'(\theta'))^{1-2/q}\,\varepsilon^{3/q-2}$, and by the elliptic estimate [8, lemma 9.17], we have

$$||\psi_\varepsilon||_{2,q,\Omega}\leqslant c_{\Omega,\bar\alpha,q}\,(W'(\theta'))^{1-2/q}\,\varepsilon^{3/q-2}.$$

Let us insert this estimate into (2.1). If $n = 3$, we take $q = 3/2$ and use that $W^{2,3/2}(\Omega)$ embeds into $L^p(\Omega)$ for all finite p. Thus (taking some $\theta' > 1$ if $\theta = 1$), we see that $u_\varepsilon \leqslant \theta' + \psi_\varepsilon$ where ψ_ε is uniformly bounded in $L^p(\Omega)$. We may use the same argument on the negative part of u_ε, so in total u_ε is uniformly bounded in $L^p(\Omega)$ for all $1 \leqslant p < \infty$ by domination through ψ_ε. Taking $q = 3/(2-\sigma) > 3/2$ proves the L^∞-estimate by the same comparison.

If $n = 2$, we have a Sobolev embedding $W^{2,q}(\Omega) \to L^\infty(\Omega)$ for all $q > 1$. Assuming that $\theta > 1$ and $\beta < 1$, we take $\theta' \to \theta$ to obtain

$$u_\varepsilon \leqslant \theta + w_\varepsilon \leqslant \theta + \psi_\varepsilon \leqslant \theta + C_{\Omega,\bar{\alpha},q}\,(W'(\theta))^{1-2/q}\,\varepsilon^{3/q-2}.$$

For $q = 3/(2+\beta)$, this gives $u_\varepsilon \leqslant 1 + C\,\varepsilon^\beta$. Here $q \in (1,2)$ is admissible since $\beta \in (0,1)$. If $\theta = 1$, we may take $0 < \beta < 1/2$, $q = (3-2\beta)/2 \in (1,2)$ and $1 + \varepsilon^\beta \leqslant \theta' \leqslant 1 + 2\varepsilon^\beta$ to obtain

$$|u_\varepsilon| \leqslant 1 + C_{\Omega,\bar{\alpha},q}\varepsilon^{\beta(1-2/q)+(3/q-2)} = 1 + C_{\Omega,\bar{\alpha},q}\varepsilon^\beta$$

with the approximation $W'(\theta') = O(\varepsilon^\beta)$. □

Corollary 2.6. *If $u_\varepsilon \to u$ in $L^1(\Omega)$ and either*

(1) *$u_\varepsilon \in C^0(\overline{\Omega})$ and there exists $\theta \geqslant 1$ such that $|u_\varepsilon| \leqslant \theta$ on $\partial\Omega$ for all $\varepsilon > 0$ or*

(2) *$\partial\Omega \in C^2$ and $\partial_\nu u_\varepsilon = 0$ a.e. on $\partial\Omega$,*

then $u_\varepsilon \to u$ in $L^p(\Omega)$ for all $1 \leqslant p < \infty$.

Proof. The sequence u_ε converges to u in $L^1(\Omega)$ and is bounded in $L^q(\Omega)$ for all $q < \infty$ (or even $L^\infty(\Omega)$). Hölder's inequality implies L^p-convergence. □

Remark 2.7. If $n = 2$, $\beta < 1/2$ and $|u_\varepsilon| \leqslant 1 + \varepsilon^\beta$ on $\partial\Omega$, then the proof still shows that

$$\sup_\Omega |u_\varepsilon| \leqslant 1 + C\,\varepsilon^\beta$$

for this particular β. The case $\beta = 1/2$ is still open at the boundary.

For a counterexample to uniform boundedness on Ω without boundary conditions, see example 3.1. Even with boundary values satisfying $|u_\varepsilon| \leqslant 1$ on $\partial\Omega \in C^2$, we shall construct a sequence u_ε for which uniform Hölder continuity fails at the boundary in example 3.3.

Proof of theorem 1.1. A slightly stronger version of the first statement of theorem 1.1 is proved in lemma 2.5, the second point can be found in lemma 2.3. The third statement is an immediate consequence and noted in corollary 2.6. □

3. Counterexamples to boundary regularity

The idea here is simple: namely, the energy $\mathcal{W}_\varepsilon$ can be seen to control the $W^{2,2}$-norm of blow ups of phase-fields onto ε-scale since those are asymptotic to bounded

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entire solutions of the stationary Allen-Cahn equation $-\Delta\tilde{u} + W'(\tilde{u}) = 0$ at (almost all) points away from the boundary. At the boundary on the contrary, the asymptotic behaviour corresponds to solutions of the same equation on half-space, whose behaviour is essentially governed by their boundary values. To make this precise, take $h \in C_c^\infty(\mathbb{R}^n)$ and $H \coloneqq \{x_n > 0\}$. The energy

$$\mathcal{F}\colon W^{1,2}_{loc}(H) \to \mathbb{R}\cup\{\infty\}, \quad \mathcal{F}(u) = \int_H \frac{1}{2}\,|\nabla u|^2 + W(u)\,\mathrm{d}x$$

has a minimizer $\tilde{u}$ in the affine space $(1+h) + W_0^{1,2}(H)$ by the direct method of the calculus of variations. Namely, take a sequence $u_k \in (1+h) + W_0^{1,2}(H)$ such that $\lim_{k\to\infty}\mathcal{F}(u_k) = \inf\mathcal{F}(u) \leqslant \mathcal{F}(h+1) < \infty$. By density, we may assume that $u_k \in (1+h) + C_c^2(H)$. Then

$$|\nabla u_k||_{L^2(H)} \leqslant C, \quad \text{and since } (u_k-1)^2(x) \leqslant (u_k-1)^2(u_k+1)^2(x) = 4\,W(u_k(x))$$

at all points $x \in H$ such that $u_k(x) \geqslant 0$, we see that the positive part of u_k is uniformly bounded in $(1+h) + W_0^{1,2}(H)$. Furthermore, when we consider the function

$$F : \mathbb{R} \to \mathbb{R}, \qquad F(1) = 0, \qquad F'(u) = \sqrt{2\,W(u)} > 0$$

and use that $F(u_k) \equiv 0$ outside a compact set, we deduce that

$$\begin{aligned}\int_H |F(u_k)|^{((n)/(n-1))}\,\mathrm{d}x \leqslant C\int_H |D(F\circ u_k)|\,\mathrm{d}x &= C\int_H |\sqrt{2\,W(u_k)}|\,|\nabla u_k|\,\mathrm{d}x \\ &\leqslant C\int_H \frac{\varepsilon}{2}\,|\nabla u_k|^2 + \frac{1}{\varepsilon}\,W(u_k)\,\mathrm{d}x.\end{aligned}$$

The constant is the same as in the Sobolev inequality on $\mathbb{R}^n$ since the extension of $u_k \in C^2(H)$ to $\mathbb{R}^n$ by even reflection is $W^{1,2}$-smooth. This shows that the sets where the functions u_k are negative have uniformly bounded measures. In combination with the fact that $\int_H W(u_k)\,\mathrm{d}x < \infty$, this implies a uniform bound on the L^2-norm of the negative part of u_k and finally on $u_k - 1 - h$ in $W_0^{1,2}(H)$. Thus the sequence $u_k - 1$ is bounded in $W^{1,2}(H)$ and there exists $\tilde{u}$ such that $u_k - 1 \overset{*}{\rightharpoonup} \tilde{u} - 1$ (up to a subsequence). Since the affine space is convex and strongly closed, it is weakly = weakly* closed and $\tilde{u} \in 1 + h + W_0^{1,2}(H)$. For any $R > 0$, we can use the compact embedding $W^{1,2}(B_R^+) \to L^4(B_R^+)$ to deduce that

$$\begin{aligned}\int_{B_R^+} \frac{1}{2}\,|\nabla\tilde{u}|^2 + W(\tilde{u})\,\mathrm{d}x &\leqslant \liminf_{k\to\infty}\int_{B_R^+} \frac{1}{2}\,|\nabla u_k|^2 + W(u_k)\,\mathrm{d}x \\ &\leqslant \liminf_{k\to\infty}\int_H \frac{1}{2}\,|\nabla u_k|^2 + W(u_k)\,\mathrm{d}x.\end{aligned}$$

Letting $R\to\infty$ shows that $\tilde{u}$ is, in fact, a minimizer of $\mathcal{F}$. If $h \geqslant 0$, then

$$1 + (\tilde{u}-1)_+ \in 1 + h + W_0^{1,2}(H), \qquad \mathcal{F}\left(1 + (\tilde{u}-1)_+\right) \leqslant \mathcal{F}(\tilde{u})$$

with strict inequality unless $\tilde{u} = 1 + (\tilde{u}-1)_+$. Since we assume $\tilde{u}$ to be a minimizer, we find that $\tilde{u} \geqslant 1$ almost everywhere. The same argument shows that

$\tilde{u} \leqslant 1 + ||h||_\infty$ almost everywhere. Calculating the Euler-Lagrange equation of $\mathcal{F}$, we see that $\tilde{u}$ is a weak solution of

$$-\Delta\tilde{u} + W'(\tilde{u}) = 0.$$

On the convex set

$$C_h \coloneqq \{u \in W^{1,2}(H) \mid u = 1 + h \text{ on } \partial H, u \geqslant 1\}$$

the operator

$$A \colon C_h \to W^{-1,2}(H), \quad A(u) = -\Delta u + W'(u)$$

is well-defined (since $n \leqslant 3$ and W' has cubical growth) and strongly monotone, so the equation $Au = 0$ has a unique solution $\tilde{u} \in C_h$ which coincides with the minimizer $\tilde{u}$ of $\mathcal{F}$ in $1 + h + W_0^{1,2}(H)$). A bootstrapping argument via elliptic regularity theory shows that $\tilde{u} \in C^\infty_{loc}(\overline{H})$. Since the trace operator $\mathrm{tr} : W^{1,2}(H) \to L^2(\partial H)$ is continuous [4, lemma 9.9], we have that

$$||h||^2_{2,\partial H} \;=\; ||\tilde{u} - 1||^2_{2,\partial H} \;\leqslant\; ||\tilde{u} - 1||^2_{1,2,H}/2 \;\leqslant\; \mathcal{F}(\tilde{u}) \;\leqslant\; \mathcal{F}(1 + h).$$

In this way, we can fully control the mass density $\tilde{\mu} = 1/2\,|\nabla\tilde{u}|^2 + W(\tilde{u})$ created by $\tilde{u}$ in terms of its boundary values. For later purposes, we have to obtain suitable decay estimates for the functions $\tilde{u}$ depending on h. In the first step, we show that the limit $\lim_{|x|\to\infty} \tilde{u}(x) = 1$ exists. Assume the contrary. Then there exist $\theta > 1$ and a sequence $x_k \in H$ such that

$$|x_k| \to \infty, \qquad \tilde{u}(x_k) \geqslant \theta.$$

Taking a suitable subsequence, we may assume that the balls $B_1(x_k)$ are disjoint and $|x_k| \geqslant R + 2$ is so large that h is supported in $B_R(0)$. If $B_2(x_k) \subset H$, we may proceed as in lemma 2.5 to deduce uniform Hölder continuity on the balls $B_1(x_k)$ from the L^∞-bound to $\tilde{u}$ and the fact that $\tilde{u}$ solves $\Delta\tilde{u} = W'(\tilde{u})$. This means that there exists $r > 0$ such that $\tilde{u} \geqslant (1 + \theta)/2$ on $B_r(x_k)$. Otherwise, the same argument still goes through after extending $\tilde{u}$ by a standard reflection principle and the fact that the boundary values are constant on $\partial H \cap B_2(x_k)$. The geometry of H gives us $\mathcal{L}^n(B_r(x_k) \cap H) \geqslant \omega_n\, r^n/2$. So we deduce that

$$\mathcal{F}(\tilde{u}) \geqslant \sum_{k=0}^{\infty} \int_{B_r(x_k)} W(\,(1+\theta)/2)\,\mathrm{d}x \geqslant \sum_{k=0}^{\infty} W((1+\theta)/2)\,\omega_n\, r^n/2 = \infty$$

in contradiction to the definition of $\tilde{u}$. Now we can estimate the decay of $\tilde{u}$ in a more precise fashion. Since $h \in C_c(\partial H)$, there is $C_h > 0$ such that $h \leqslant C_h\, e^{-|x|}$ on ∂H. To simplify the following calculations, we assume that $C_h = 1$. Then we claim that $1 \leqslant u \leqslant 1 + e^{-|x|}$ for all $x \in \mathbb{R}^n$. Assume the contrary and observe that

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$\psi(x) = 1 + e^{-|x|}$ satisfies

$$\Delta\psi(x) = \left(1 + \frac{1-n}{|x|}\right) e^{-|x|}, \qquad W'(\psi(x)) = \left(2 + 3\,e^{-|x|} + e^{-2\,|x|}\right) e^{-|x|},$$

so, in particular, $\Delta\psi(x) \leqslant W'(\psi(x))$ for all $x \in \mathbb{R}^n$. Since $\tilde{u} = h \leqslant \psi$ on ∂H by assumption and $\lim_{|x|\to\infty} \tilde{u}(x) = 1$, there must be a point $x_0 \in H$ such that

$$(\psi - u)(x_0) = \min_H(\psi - u) < 0,$$

but then

$$\Delta(\psi - u)(x_0) \leqslant W'(\psi(x_0)) - W'(u(x_0)) < 0$$

so $\psi - u$ cannot be minimal at x_0. This proves the claim. It follows that

$$\int_{H\setminus B_R^+} W(\tilde{u})\,\mathrm{d}x \leqslant 2\int_R^\infty e^{-2r}\,r^{n-1}\,\mathrm{d}r = P_n(R)\,e^{-2R}$$

where P_n is a polynomial of degree n depending on the dimension. To estimate the second part of the energy functional, we use the gradient bound

$$|\nabla u(x)| \leqslant n\sqrt{n}\,\sup_{\partial Q}|u| + \frac{1}{2}\,\sup_Q|\Delta u|$$

from [8, section 3.4] where Q is a cube of side length $d = 1$ with a corner at x. Applied to our problem, for $x \in \partial B_R^+$ we can find a cube Q satisfying $\bar{Q} \cap \overline{B_R^+} = \{x\}$ such that

$$|\nabla\tilde{u}(x)| \;=\; |\nabla(\tilde{u} - 1)|(x) \leqslant n\sqrt{n}\,\sup_{\partial Q}|\tilde{u} - 1| + \frac{1}{2}\,\sup_Q|W'(\tilde{u})| \;\leqslant\; (n\sqrt{n} + 5/2)\,e^{-|x|}.$$

Thus, we also have

$$\int_{H\setminus B_R^+} \frac{1}{2}\,|\nabla\tilde{u}|^2\,\mathrm{d}x \;\leqslant\; \left(n\sqrt{n} + 5/2\right)^2\,P_n(R)\,\mathrm{e}^{-2R}$$

Finally, we remark that the same type of estimate obviously holds for $\Delta\tilde{u} = W'(\tilde{u}) \in L^2(H)$. Having given the general construction for suitable functions of zero $\mathcal{W}_1$ curvature energy, we are finally ready to apply these results to obtain counterexamples. For simplicity, we construct the counterexamples first on the half-space H and transfer them to bounded Ω later on.

Example 3.1 (Counterexample to Boundedness). Fix $h \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leqslant h \leqslant e^{-|x|}$, $h \not\equiv 0$ and set $h_\theta = \theta\,h$. Every function of this type induces a minimizer $\tilde{u}_\theta$. We may take a sequence $\theta_\varepsilon \to \infty$ such that $\varepsilon^{n-1}/\theta_\varepsilon^4 \to 0$ and set $u_\varepsilon(x) = \tilde{u}_{\theta_\varepsilon}(x/\varepsilon)$. Clearly, u_ε becomes unbounded as $\varepsilon \to 0$, but

(1) $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$ and

(2) $S_\varepsilon(u_\varepsilon) = \varepsilon^{n-1}\,\mathcal{F}(\tilde{u}_{\theta_\varepsilon}) \leqslant C\,\varepsilon^{n-1}\,\mathcal{F}(h_{\theta_\varepsilon}) \to 0.$

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So the sequence u_ε induces limiting measures $\mu = \alpha = 0$, but fails to be uniformly bounded.

The next example is a technically more demanding version of this one where the energy scaling is chosen so that we create an atom of size $S > 0$ at the origin.

EXAMPLE 3.2 (Counterexample to Boundary Regularity of μ). Take $h_\theta, \tilde u_\theta$ as above. Then the map

$$f\colon [0,\infty) \to \mathbb{R}, \quad f(\theta) = \mathcal{F}(\tilde u_\theta) = \inf\{\mathcal{F}(u) \mid u \in 1 + h_\theta + W^{1,2}_0(H)\}$$

is continuous. To see this, take pairs θ_1, θ_2 and the corresponding minimizers $\tilde u_1$, $\tilde u_2$ and observe that

$$\tilde u_{1,2} = \frac{\theta_2}{\theta_1}\,[\tilde u_1 - 1] + 1 \;\in\; 1 + h_{\theta_2} + W^{1,2}_0(H).$$

Since

$$W(1+\alpha u) = ((1+\alpha u)^2 - 1)^2/4 = (2\alpha u + \alpha^2 u^2)^2/4 \leqslant \max\{\alpha^2, \alpha^4\} W(1+u)$$

we have

$$\begin{aligned} f(\theta_2) = \mathcal{F}(\tilde u_2) \leqslant \mathcal{F}(\tilde u_{1,2}) &\leqslant \max\left\{\left(\frac{\theta_2}{\theta_1}\right)^2, \left(\frac{\theta_2}{\theta_1}\right)^4\right\} \mathcal{F}(\tilde u_1) \\ &= \max\left\{\left(\frac{\theta_2}{\theta_1}\right)^2, \left(\frac{\theta_2}{\theta_1}\right)^4\right\} f(\theta_1). \end{aligned}$$

Reversing the roles of θ_1 and θ_2 shows that f is continuous. Now let $S > 0$. Due to the continuity of f in θ and the trace inequality

$$\theta^2 ||h||^2_{2,\partial H} = ||h_\theta||^2_{2,\partial H} \leqslant \mathcal{F}(\tilde u_\theta)$$

we can pick a sequence $\theta_\varepsilon \to \infty$ at most polynomially in $1/\varepsilon$ such that $\mathcal{F}(\tilde u_{\theta_\varepsilon}) = S\,\varepsilon^{1-n}$. As before, set $u_\varepsilon(x) = \tilde u_{\theta_\varepsilon}(x/\varepsilon)$ and observe that $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$, $S_\varepsilon(u_\varepsilon) \equiv S$. It remains to show that $\mu = S\,\delta_0$, i.e. that the limiting measure is concentrated in one point. The functions $\tilde u_\theta$ actually, tend to shift more of their mass towards the origin as $\theta \to \infty$ since the steepness (and overall height) is best concentrated on a ball of small radius for a low energy.

The same application of the maximum principle as before shows that $\tilde u_\theta \leqslant \tilde w_\theta := 1 + \theta(\tilde u_1 - 1)$ since

$$\Delta(\tilde w_\theta - \tilde u_\theta) = \theta\,\Delta\tilde u_1 - \Delta\tilde u_\theta = \theta\,W'(\tilde u_1) - W'(\tilde u_\theta) \leqslant W'(\tilde w_\theta) - W'(\tilde u_\theta)$$

is monotone in $\tilde w_\theta$, $\tilde u_\theta$ and the boundary values satisfy $\tilde u_\theta = \tilde w_\theta$ on ∂H and $\lim_{|x|\to\infty} \tilde u_\theta = \lim_{|x|\to\infty} \tilde w_\theta = 1$. Like above, we now obtain that

$$\int_{H\setminus B_R^+} \frac{1}{2}\,|\nabla \tilde u_\varepsilon|^2 + W(\tilde u_\varepsilon)\,\mathrm{d}x \leqslant \max\{\theta_\varepsilon^2, \theta_\varepsilon^4\}\,P_n(R)\,e^{-2R}.$$

Thus we can choose a sequence $R_\varepsilon \to \infty$ such that $\theta_\varepsilon^4\,P_n(R_\varepsilon)\,e^{-2R_\varepsilon} \to 0$ and $\varepsilon\,R_\varepsilon \to 0$ since θ_ε grows only polynomially in $1/\varepsilon$ and the exponential term dominates (take e.g. $R_\varepsilon = \varepsilon^{-1/2}$). Thus for all $R > 0$

$$\mu_\varepsilon(B_R(0)) = \varepsilon^{1-n}\int_{B^+_{R/\varepsilon}} |\nabla \tilde u_{\theta_\varepsilon}|^2 + W(\tilde u_{\theta_\varepsilon})\,\mathrm{d}x \geqslant \varepsilon^{1-n}\int_{B^+_{R_\varepsilon}} |\nabla \tilde u_{\theta_\varepsilon}|^2 + W(\tilde u_{\theta_\varepsilon})\,\mathrm{d}x \to S$$

and hence $\mu(B_R(0)) \geqslant S$. Taking $R \to 0$ shows that $\mu(\{0\}) = \mu(\overline{H}) = S$, i.e. $\mu = S\,\delta_0$.

Functions as described above can appear as minimizers of functionals like $\mathcal{W}_\varepsilon + \varepsilon^{-1}\,(S_\varepsilon - S)^2$ which are used to search for minimizers of Willmore's energy with prescribed surface area – even as functions with energy zero. The same is true for functionals including the topological penalization term discussed below.

By construction, the previous example shows that the inclusion $\mathrm{spt}(\mu) \subset \lim_{\varepsilon\to 0} u_\varepsilon^{-1}(I)$ need not be true for any $I \Subset (-1,1)$ since $u_\varepsilon \geqslant 1$ and thus $K = \emptyset$. We use a similar construction to demonstrate that the reverse inclusion need not hold, either.

Example 3.3 (Counterexample to Hausdorff Convergence). Using the same arguments as above, if $0 \leqslant h \leqslant 2$, we can find a solution $\tilde u \in (1-h) + W^{1,2}_0(H) \cap C^\infty_{loc}(\overline{H})$ of

$$-\Delta \tilde u + W'(\tilde u) = 0 \quad \text{in } H, \qquad \bar u = 1 - h \quad \text{on } \partial H$$

satisfying $-1 \leqslant \tilde u \leqslant 1$, $\lim_{|x|\to\infty} \tilde u(x) = 1$ and $\mathcal{F}(\tilde u) \leqslant \mathcal{F}(1+h) < \infty$. Decay estimates are harder to obtain here since W' is not monotone inside $[-1,1]$, but we will not need them, either. If we take h such that $h(0) = 2$, $h \in C^\infty_c(B_1)$, we can use continuity up to the boundary to deduce that $\tilde u^{-1}(\rho) \cap B_1^+ \neq \emptyset$ for all $\rho \in (-1,1)$. So when we set $u_\varepsilon(x) = \tilde u(x/\varepsilon)$, we see that

(1) $\mu_\varepsilon(H) = \varepsilon^{n-1}\,\tilde\mu(H) = \varepsilon^{n-1}\,\mathcal{F}(\tilde u) \to 0$,

(2) $\mathcal{W}_\varepsilon(u_\varepsilon) \equiv 0$ and

(3) $0 \in \lim_{\varepsilon\to 0} u_\varepsilon^{-1}(I)$ in the Hausdorff sense for all $\emptyset \neq I \Subset (-1,1)$.

Example 3.4 (Counterexample to Uniform Hölder Continuity). If we take h like in the previous example and replace it by $h^\omega(x) = h(\omega x)$ we observe that the associated minimizers satisfy

$$\mathcal{F}(\tilde u^\omega) \leqslant \mathcal{F}(h^\omega) \leqslant \mathcal{F}(h)$$

for all $\omega \geqslant 1$ since the gradient term stays invariant in two dimensions and decreases in three, while the integral of the double-well potential decreases in both cases for any fixed h. Thus, if we take any sequence $\omega_\varepsilon \to \infty$ and define $u_\varepsilon(x) = \tilde u^{\omega_\varepsilon}(x/\varepsilon)$, we get the same results as before. As the function becomes steeper and steeper on the boundary faster than ε, uniform Hölder continuity up to the boundary cannot hold, even for uniformly bounded boundary values.

EXAMPLE 3.5 (Counterexample to Boundary Regularity of μ with $-1 < u_\varepsilon < 1$). We can refine the examples to show that growth of u_ε on $\partial\Omega$ is not the only reason that μ might develop atoms on $\partial\Omega$, but that this is, in fact, possible with $|u_\varepsilon| \leqslant 1$. This happens when we prescribe highly oscillating boundary values on ∂H. Let $h \in C_c^\infty(\partial H)$, then for any $u \in H^1(H)$ with $u|_{\partial H} = h$ we have

$$\int_H |\nabla u|^2 \, dx \geqslant [h]^2_{H^{1/2}(\partial H)} = c_{n-1} \int_{\partial H \times \partial H} \frac{|h(x) - h(y)|^2}{|x-y|^{n+1}} \, dx \, dy.$$

for a constant $c_{n-1} > 0$ depending on the dimension $n - 1 \in \{1, 2\}$ [**11**, proposition 3.8]. For any $S' > 0$ and $\delta > 0$ we can construct $h \in C^\infty(H)$ such that

(1) $0 \leqslant h \leqslant \delta$,

(2) $\mathrm{supp}(h) \subset B_1(0)$ and

(3) $[h]^2_{H^{1/2}} = S'$

as the convolution of the characteristic function $\chi = \chi_{B_{1/2}(0)}$ with a standard mollifier η_r for sufficiently small $r > 0$. The function $h = \delta \cdot \chi * \eta_r$ is C^∞-smooth and lies between 0 and δ by standard results, but since the singular limit $\delta \cdot \chi$ as $r \to 0$ does not lie in $H^{1/2}(\mathbb{R}^{n-1})$ [**13**], the norm has to become as large as $r \to 0$. As the energy depends continuously on the parameter r, the problem can be solved for any $S' > 0$, potentially choosing δ smaller if S' is small. Now we construct a solution of the stationary Allen-Cahn equation with the boundary values $1 - h$ as before, but for a modified potential

$$\overline{W}(s) = \begin{cases} W(1-2\delta) & s \leqslant 1 - 2\delta \\ W(s) & s \geqslant 1 - 2\delta \end{cases}.$$

An energy minimizer will never dip below $1 - 2\delta$ then, and consequently never below $1 - \delta$ by the maximum principle if δ is chosen so small that W' is monotone on $[1 - 2\delta, \infty)$. The rest of the proof goes through as before considering $u_\varepsilon(x) = u_{h_\varepsilon}(x/\varepsilon)$ where h_ε is chosen for $\delta = \varepsilon$ and $S' = S \cdot \varepsilon^{1-n}$. The energy contribution of the double-well potential thus becomes negligible and μ approaches $S \cdot \delta_0$ as it would for the pure Dirichlet energy [**13**, remark 3.9]. We will not give the details.

The boundary values need to be constructed with slightly more care since we cannot just have vertical growth and the $H^{1/2}$-norm behaves badly under spacial scaling. This is compensated in the boundary construction by having a larger number of faster oscillations. When we have constructed h with a large enough half-norm, we can always reduce it by scaling with a constant < 1.

For the sake of simplicity, we chose to construct the examples on half-space due to its scaling invariance. Let us sketch how they can be transferred to C^2-domains. If $\Omega \Subset \mathbb{R}^n$ and $\partial\Omega \in C^2$ there exists $x_0 \in \partial\Omega$ such that $|x_0| = \max_{x \in \partial\Omega} |x|$. At x_0, both principal curvatures of $\partial\Omega$ are strictly positive, so in a ball around x_0, up to

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a rigid motion, we may write

$$\Omega \cap B_r(x_0) = \{x \in B_r(x_0) \mid x_n > \phi(\hat{x})\}$$

where $\hat{x} = (x^1, \ldots, x^{n-1})$ and ϕ is a strictly convex C^2-function satisfying $\phi(0) = 0$, $\nabla\phi(0) = 0$ and $\Omega \subset H$. If Ω is convex in the first place, this is possible at every point $x_0 \in \partial\Omega$.

Thus, the function $u_\varepsilon(x) = \tilde{u}(x/\varepsilon)$ is well-defined on Ω for any of the functions $\tilde{u}$ constructed above. If ε is chosen small enough, the difference between H and Ω/ε becomes negligible for any given $\tilde{u}$ and we can still construct counterexamples to boundedness, local Hölder-continuity, the relationship between $\mathrm{spt}(\mu)$ and the Hausdorff limit of the level sets and to the regularity of μ this way.

Example 3.6 (Singularity along a curve). Let Ω be a convex C^2-domain and $\gamma \subset \partial\Omega$ a smooth curve in $\partial\Omega$. For $\varepsilon > 0$, choose $n_\varepsilon \sim \varepsilon^{-1/2}$ points $x_{i,\varepsilon}$ on γ such that

$$|x_{i,\varepsilon} - x_{j,\varepsilon}| \geqslant \sqrt{\varepsilon} \quad \forall\, i \neq j \qquad \text{and} \qquad \max_{x\in\gamma} \min_{1\leqslant i\leqslant n_\varepsilon} |x - x_{i,\varepsilon}| \leqslant \sqrt{\varepsilon}.$$

Take a function $h \in C_c^\infty(\mathbb{R}^{n-1})$ such that $0 \leqslant h \leqslant \delta$ such that W' is monotone increasing on $[1-\delta, 1]$ and note that u constructed as before approaches 1 exponentially fast by the same argument as in example 3.2 and like before

$$\int_{H\setminus B_R^+} \frac{1}{2}|\nabla u|^2 \,\mathrm{d}x \leqslant \mathrm{e}^{-R}\, P(R)$$

where P is a polynomial. Now we define $u_{i,\varepsilon}(x) = u(O_i(x - x_{i,\varepsilon}))$ where O_i is any rotation mapping the tangent space $T_{x_{i,\varepsilon}}\partial\Omega$ to $\{x_n = 0\}$ and the inner normal to $\partial\Omega$ at $x_{i,\varepsilon}$ to e_n. Since Ω is convex, the functions are well-defined. Finally, we set

$$u_\varepsilon(x) = \sum_{i=1}^{n_\varepsilon} u_{i,\varepsilon}(x)$$

and observe that due to the exponential decay of u away from the origin, the decay of the gradient integral away from the origin in combination with Hölder's inequality and the minimal distance of $\sqrt{\varepsilon}$ between distinct points $x_{i,\varepsilon}$ and $x_{j,\varepsilon}$, we have

$$S_\varepsilon(u_\varepsilon) \leqslant n_\varepsilon\, \varepsilon^{n-1} \cdot S_1(u) + n_\varepsilon \cdot \Big[P(\varepsilon^{-1/2}) + C\Big]\; e^{-1/\sqrt{\varepsilon}} \to 0.$$

On the contrary, the increasing density of points $x_{i,\varepsilon}$ on the curve shows that

$$\lim_{\varepsilon\to 0}\{1-\delta < u_\varepsilon < 1-\delta/2\} \subset \lim_{\varepsilon\to 0}\{x_{1,\varepsilon}, \ldots, x_{n_\varepsilon,\varepsilon}\} = \gamma$$

in the sense of Hausdorff convergence. The same construction can be applied to obtain any closed set $K \subset \partial\Omega$ as a Hausdorff limit and with a slight modification to obtain a limiting measure μ which is supported by a smooth curve. The only difference is the different scaling of n_ε, which may force a larger distance of $\varepsilon^{1/4}$ between points in the first modification. In the second extension, the precise scaling of n_ε must be taken into account.

Finally, we collect the examples in the main result.

Proof of theorem 1.2. The first statement is demonstrated in example 3.1. The second part is discussed in examples 3.3 and 3.6. The statement that the Hausdorff limit of certain level sets can contain an open subset of $\partial\Omega$ is not demonstrated, but follows as sketched at the end of example 3.6. Two different examples for the third claim are given in examples 3.2 and 3.5. □

We restricted our analysis to convex boundary points since then $u_\varepsilon = \tilde{u}_\theta(x/\varepsilon)$ is well-defined for all small $\varepsilon > 0$, whereas at other points, half-space does not provide enough information to fill an entire neighbourhood of x_0. We believe that the same pathologies can arise at general boundary points.

References

1 R. Alessandroni and E. Kuwert. Local solutions to a free boundary problem for the Willmore functional. *Calc. Var. PDE* **55** (2016), Article:24.

2 G. Bellettini and M. Paolini. Approssimazione variazionale di funzionali con curvatura. *Seminario di Analisi Matematica, Dipartimento di Matematica dell'Università di Bologna.* (1993).

3 E. Bretin, S. Masnou and E. Oudet. Phase-field approximations of the Willmore functional and flow. *Numerische Mathematik* **131** (2015), 115–171.

4 H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations* (New York: Universitext. Springer, 2011).

5 E. De Giorgi. Some remarks on Γ-convergence and least squares method. In *Composite media and homogenization theory (Trieste, 1990)*, pp. 135–142 (Boston, MA, Boston, MA: Birkhäuser Boston, 1991).

6 P. W. Dondl and S. Wojtowytsch. Uniform convergence of phase-fields for Willmore's energy. *Calc. Var. PDE* **56** (2017).

7 P. W. Dondl, A. Lemenant and S. Wojtowytsch. Phase field models for thin elastic structures with topological constraint. *Arch. Ration. Mech. Anal.* **223** (2017), 693–736.

8 D. Gilbarg and N. S. Trudinger. Elliptic partial differential equations of second order, volume 224 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, 2nd edn (Berlin: Springer-Verlag, 1983).

9 L. Modica. The gradient theory of phase transitions and the minimal interface criterion. *Arch Ration Mech Anal* **98** (1987), 123–142.

10 L. Modica and S. Mortola. Un esempio di Γ-convergenza. *Boll. Un. Mat. Ital. B (5)* **14** (1977), 285–299.

11 E. D. Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573, 04.

12 M. Röger and R. Schätzle. On a modified conjecture of De Giorgi. *Math. Z.* **254** (2006), 675–714.

13 O. Savin and E. Valdinoci. Γ-convergence for nonlocal phase transitions. Ann. Inst. H. Poincaré. *Anal. Non Linéaire*, **29** (2012), 479–500.

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