Proceedings of the Royal Society of Edinburgh, **143A**, 669–688, *2013*

Analytic obstructions to isochronicity in codimension 1*

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(MS received 23 December 2011; accepted 31 July 2012)

In this paper we study the analytical obstructions preventing the germ of a generic analytic family of elliptic ordinary differential equations from being isochronous. The formal obstructions are purely arithmetical and can be set in terms of commutators. The analytical obstructions arise out of the non-convergence of the normalizing coordinates in the Poincaré–Dulac normalization process. The temporal part of the invariant is the obstruction to isochronicity in a particular context.

1. Introduction

The word *isochronous* means literally *at equal intervals of time*. This term applies to oscillators for which the frequency is independent of the amplitude. It formerly referred to the timekeeping principle of pendulums studied by Galileo in the 16th century. He claimed that a simple pendulum is isochronous; the period is approximately independent of the amplitude or width of the swing. In 1673 Huygens (see [14]) proved that Galileo's claim on isochronicity (or *isochrony*) was accurate only for small swings.

The term isochronous applies to a particular class of ordinary differential equations for which the *return time* to an analytic transversal is locally constant. The subject enjoys great popularity among mathematicians and the study of such systems has been extensively developed. Much of the work done thus far comes from the Spanish and Italian schools (see, for example, [1, 12, 13, 21]).

In this paper we study the analytical obstructions that prevent the unfolding (deformation) of a germ of holomorphic elliptic ordinary differential equations from being isochronous. Consider a germ of a planar analytic family of ordinary differential equations depending on one real parameter η and of the form

$$\left.\begin{aligned} \dot{x} &= \alpha(\eta)x - \beta(\eta)y + \sum_{j+k\geqslant 2} b_{jk}(\eta)x^j y^k, \\ \dot{y} &= \beta(\eta)x + \alpha(\eta)y + \sum_{j+k\geqslant 2} c_{jk}(\eta)x^j y^k. \end{aligned}\right\}$$

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https://doi.org/10.1017/S0308210511000023 Published online by Cambridge University Press

Hence, the origin $(x, y) = (0, 0)$ is a singular point. We will assume that α, β, b_{jk}, c_{jk} are real analytic and $\beta(0) \neq 0$. System (1.1) is elliptic or *monodromic*, i.e. the pair of eigenvalues are complex conjugate.

An isolated equilibrium of (1.1) is called a *focus* if there exists an open neighbourhood containing the singularity where all the orbits spiral either in forward or backward time. The focus is *strong* if the real part of the eigenvalues is non-vanishing. It is *weak* otherwise. The singularity is called a *centre* if there exists a punctured neighbourhood of the singular point filled with periodic orbits. An isolated singularity of (1.1) is called isochronous if every periodic orbit has the same minimal period.

DEFINITION 1.1. Let f_η be a germ of a real analytic family of functions. A monodromic singularity of (1.1) is isochronous if there exists a germ of an analytic change of coordinates, fibred over the parameter space, bringing the system into an equation that in polar coordinates (r, θ) takes the form

$$\begin{aligned} \dot{r} &= f_\eta(r, \theta), \\ \dot{\theta} &= \beta(\eta). \end{aligned} \tag{1.2}$$

It is well known that strong foci and non-degenerate centres have analytic isochronous sections (i.e. an analytic curve that meets each orbit contained in a neighbourhood of the singularity at equal minimal time intervals). In [6], a simple *holomorphic* transformation has been introduced, applicable to quite a large class of dynamical systems. The transformation yields autonomous systems that are isochronous (see [7] for further examples). This justifies the notion that isochronous systems are not rare in nature.

However, the case of a weak focus is radically different. The transformation of definition 1.1 is generally non-convergent and a different treatment must be set. For this, we assume in the following that (1.1) unfolds a weak focus, that is $\alpha(0) = 0$. We suppose also that the system is *generic*: $\alpha'(0) \neq 0$. Genericity and the implicit function theorem allow us to take $\varepsilon = \alpha/\beta$ as a new parameter and to redefine β as a function of ε. The eigenvalues become the complex conjugate pair $\beta(\varepsilon)(\varepsilon \pm \mathrm{i})$. We will write β instead of $\beta(\varepsilon)$.

The foliation of (1.1) is described locally by the unfolding of the Poincaré first-return map of the positive Ox-axis, $\mathcal{P}_\varepsilon \colon (\mathbb{R}^+, 0) \to (\mathbb{R}^+, 0)$, also called (Poincaré) monodromy. It is known that the germ of this map is analytic and can be extended to an analytic diffeomorphism

$$\mathcal{P}_\varepsilon \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0). \tag{1.3}$$

Isolated roots of the *displacement function* $\mathcal{P}_\varepsilon(x) - x$ correspond to limit cycles of the vector field. From the general theory, the normal form of the displacement function always starts with an odd-power term x^{2k+1}, $k \geqslant 1$. The coefficient $\ell_k(\varepsilon)$ of this term is called the k-th Lyapunov constant. The system of differential equations is of *order* k provided $\ell_j(0) = 0$ for all $j = 1, \dots, k-1$ but $\ell_k(0) \neq 0$. In this case the unfolding undergoes a generic *Andronov–Hopf bifurcation of order* k. Evidently, $k-1$ additional parameters are required to describe this bifurcation; k limit cycles appear and merge with the origin as the parameters tend to the *bifurcation value* (typically

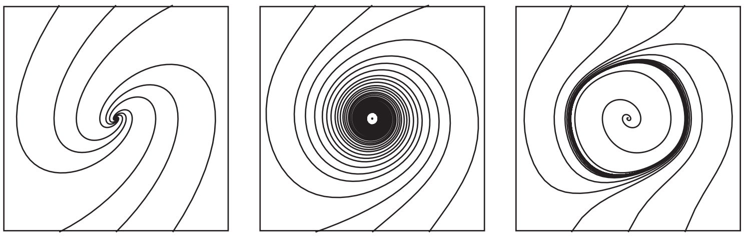

Figure 1. Supercritical Hopf bifurcation.

0). The definition of the order is independent of the choice of the coordinate system; the order is a geometric invariant of the germ of the analytic family. By definition, the order of an integrable field is equal to $+\infty$ (see [23]). For example, a germ of a parameter-dependent real analytic family unfolds a centre at zero, if and only if each Lyapunov constant vanishes at the bifurcation value. That is why centres are also called *weak foci of infinite order*.

In this paper we will assume that (1.1) is of order 1. The Lyapunov first constant can be then explicitly computed in terms of the coefficients:

$$\ell_1 = 3b_{30} + b_{12} + c_{21} + 3c_{03} + \frac{1}{\beta}[b_{11}(b_{20} + b_{02}) - c_{11}(c_{20} + c_{02}) - 2b_{20}c_{20} + 2b_{02}c_{02}].$$

In this case the system exhibits a generic Andronov–Hopf bifurcation of order 1 (or simply a Hopf bifurcation): the generic coalescence of a focus with a limit cycle (see figure 1). The Hopf bifurcation is *subcritical* if the cycle is present on negative values of ε. It is *supercritical* otherwise. Whether a Hopf bifurcation is subcritical or supercritical can be found from the sign of the first Lyapunov coefficient. An $\ell_1(0)$ of positive sign indicates a subcritical Hopf bifurcation and an $\ell_1(0)$ of negative sign corresponds to a supercritical Hopf bifurcation.

In order to understand the analytical obstructions that prevent the change of coordinates of definition 1.1 from being convergent, we compare the *time part* of the system with the time part of an (isochronous) polynomial formal normal form. A formal normal form for a germ of the family (1.1) is a germ of a 1-parameter family of elliptic polynomial differential equations of degree 5 containing only resonant monomials. This normal form has been generically denominated the *model family*. We can bring the family (1.1) into its model via a formal local transformation in a neighbourhood of zero. All the spaces of leaves of the foliation of the model family glue trivially, so the model is too poor to encode all the rich dynamics of an arbitrary analytic family with a Hopf bifurcation. There exists in general no analytic family of changes of coordinates (and time scalings in the case of orbital equivalence) to the model family. However, in the *Glutsyuk point of view*, there exist local analytic families of changes of coordinates (and time scalings in the case of orbital equivalence) to the model family over two canonical sectors of the parameter space. The modulus represents the obstruction to glue the different local charts into a global change of coordinates over a full neighbourhood of the origin of coordinates containing the fixed points of the monodromy.

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Martinet and Ramis [18] characterize a planar vector field under orbital equivalence by identifying the divergence of the normalizing formal power series with the non-triviality of a collection of transition diffeomorphisms between consecutive sectorial spaces of leaves. (Algebraically, the divergence of the normalizing coordinate is identified with a cochain in the ring of summable power series.) In the case of a saddle node, these invariants coincide with the Ecalle–Voronin invariant of the holonomy of the strong separatrix. In the case of a saddle point, these invariants coincide with the Ecalle–Voronin invariants of the holonomy of any separatrix. Meshcheryakova and Voronin added the first return time needed to identify classes of saddle nodes under conjugacy (vector fields). A different approach uses the geometry of the leaves in the neighbourhood of the saddle node, which is described in terms of asymptotic homology; see [22]. Both approaches have been generalized (unfolded) to the case of singularities in a neighbourhood of the order-1 Hopf bifurcation. Indeed, in [2, 4, 5] we have proved that the invariants of analytic classification under orbital equivalence of (1.1) coincide with the unfolding of the Ecalle–Voronin invariants of the monodromy; see, for example, [2, 3]. Furthermore, we have identified a complete modulus of analytic classification under *weak conjugacy* via an uncoupling of the system into a time part and an orbital part. The temporal invariants are related to the time part of the vector field.

The *formal obstructions* preventing the system from being isochronous have been described before (see [1, 10, 12, 13]). They can be set in terms of commutators between the vector field and a normal form. These criteria are purely arithmetical and do not give conditions under which the normalizing coordinate of definition 1.1 converges. One way to study the analytic behaviour of the normalizing chart is through identification of the *temporal part of the modulus*. Indeed, since the singular points of (1.1) are hyperbolic on values $\varepsilon \neq 0$, it is possible to *temporally normalize* the family (1.1) via an analytic change of coordinates on those values of the parameter. However, there exists in general a mismatch between the normalizing coordinate defined on $\varepsilon \in \mathbb{R}_-$ and the one defined on $\varepsilon \in \mathbb{R}_+$. The temporal part of the Glutsyuk modulus measures such an incompatibility and, in general, it measures the set of analytical obstructions to temporal normalizability over extended sectorial domains of the (complexified) parameter space.

2. The isochronicity problem

A cross-section through the origin $\Sigma \subset \mathbb{R}^2$ for the foliation of (1.1) is a simple real analytic transversal arc without contact and with the origin as an endpoint (see [13]). Given $p \in \Sigma$ and ε small, let $\phi_\varepsilon^t(p)$ be the flow of (1.1) with initial condition $\phi_\varepsilon^0(p) = p$. Since (1.1) is monodromic, there exists a germ of an analytic function $\tau_\Sigma(\varepsilon, \cdot)\colon \Sigma \to \mathbb{R}^+$ such that $\phi_\varepsilon^{\tau_\Sigma(\varepsilon,p)}(p) \in \Sigma$ and $\phi_\varepsilon^t(p) \notin \Sigma$ for every $0 < t < \tau_\Sigma(\varepsilon, p)$.

Definition 2.1. The Poincaré monodromy (1.3) of (1.1) is defined by

$$p \mapsto \phi_\varepsilon^{\tau_\Sigma(\varepsilon,p)}(p).$$

The germ of the function $\tau_\Sigma(\varepsilon, p)$ is called the period associated with the section Σ. If $\tau_\Sigma(\varepsilon, \cdot)$ is (locally) constant, then the singular point of (1.1) is called isochronous and Σ is an isochronous section.

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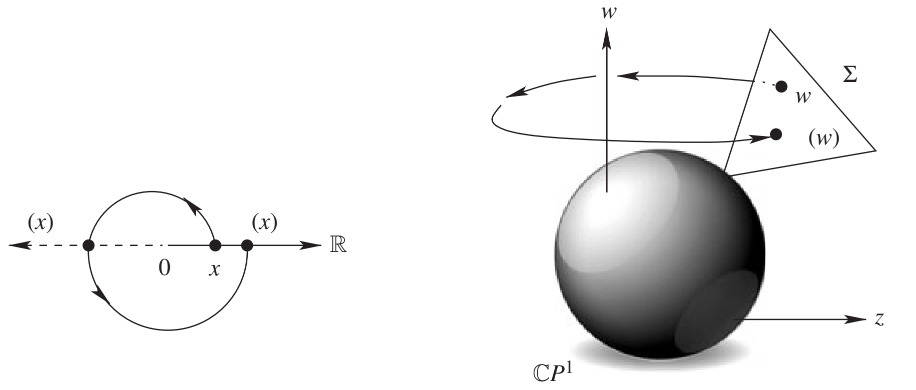

Figure 2. The complexification of the real line and its blow-up.

The period depends strongly on Σ if the system is not a centre (in the latter case the period is independent of Σ). Note that Σ is isochronous if either ϕ_ε^t (the flow for positive real time) or ϕ_ε^{-t} (the reversed flow) has the following property: there exists a family of real constants $T_\varepsilon > 0$ (analytic in ε) such that for all $p \in \Sigma$ the two following conditions are met.

(i) $\phi_\varepsilon^{nT_\varepsilon}(p) \in \Sigma$ for every $n \in \mathbb{N}$,

(ii) $\phi_\varepsilon^t(p) \notin \Sigma$ for every $t > 0$, $t \neq nT_\varepsilon$.

The family of constants T_ε is the period of Σ. If the latter is isochronous, then every curve $s \mapsto \phi_\varepsilon^s(\Sigma)$, with $0 < s < T_\varepsilon$, is an isochronous section of the system at the singular point (see [21]). Hence, in that case, the system has infinitely many isochronous sections. A special case covered by definition 1.1 occurs when the system can be linearized near the singularity (by the Poincaré linearization theorem). Thus, every strong focus or every non-degenerate centre of an analytic system has isochronous sections. (In the case of a centre, every section through the origin Σ is isochronous.) As pointed out before, the case of a weak focus is radically different. In general, the task of finding isochronous sections for holomorphic deformations of weak foci is involved and different techniques are needed.

First, we complexify the time t, coordinates (x, y) and parameter ε. (We will restrict the values of the (complex) parameter ε to special sectorial domains where the singular points of the system are hyperbolic; see below.) Once the coordinates (x, y) have been complexified, we set variables $z = x + \mathrm{i}y$, $w = x - \mathrm{i}y$ and express the complexified family of vector fields in these new coordinates. By the Hadamard–Perron theorem for holomorphic flows (see [16]), there exists a real analytic change of coordinates bringing the complexified system into the form

$$\left.\begin{aligned} \dot z &= z\beta\Bigl(\varepsilon + \mathrm{i} + \sum_{j+k\geqslant 2} a_{jk}(\varepsilon) z^j w^k\Bigr), \\ \dot w &= w\beta\Bigl(\varepsilon - \mathrm{i} + \sum_{j+k\geqslant 2} \overline{a_{jk}(\bar\varepsilon)} w^j z^k\Bigr), \end{aligned}\right\}$$

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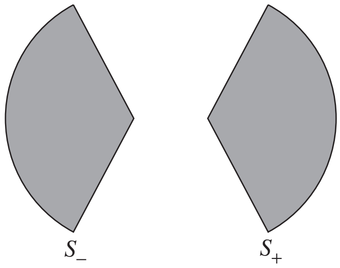

Figure 3. Sectorial domains in parameter space.

with coefficients a_{jk} depending analytically on ε. Here, the map $z \mapsto \bar{z}$ is the usual complex conjugation. In parallel, the complexification embeds the monodromy (1.3) in a complex real analytic family $\mathcal{P}_\varepsilon\colon \Sigma \to \Sigma$ called, by analogy, the Poincaré monodromy of (2.1), defined on the cross-section $\Sigma\colon \{z = w\}$, which is the complexification of the real Ox-axis. The (complex) monodromy is defined as the second iterate of the holonomy self-map (or semi-monodromy) $\mathcal{Q}_\varepsilon$ along the equator $\mathbb{R}P^1$ of the exceptional divisor $\mathbb{C}P^1$ after standard blow-up [3]; see figure 2. The semi-monodromy unfolds the germ of a codimension-1 real analytic diffeomorphism that, after elimination of the quadratic term by a normal form argument, has the form $\mathcal{Q}_0(w) = -w \mp \frac{1}{2}w^3 + aw^5 + \cdots$. We will assume that the unfolding $\mathcal{Q}_\varepsilon(w) = \mathcal{Q}(w, \varepsilon)$ is generic, i.e. $(\partial^2 \mathcal{Q}/\partial w \partial \varepsilon)(0, 0) \neq 0$.

Let ρ be a positive real number. In the following, we will restrict the complexified parameter to two different sectorial domains of the parameter space containing negative and positive segments of the real line and denoted

$$S_- = \{\varepsilon \in \mathbb{C}\colon\ \arg(\varepsilon) \in (\tfrac{1}{2}\pi + \delta, \tfrac{3}{2}\pi - \delta),\ |\varepsilon| < \rho\},$$
$$S_+ = \{\varepsilon \in \mathbb{C}\colon\ \arg(\varepsilon) \in (-\tfrac{1}{2}\pi + \delta, \tfrac{1}{2}\pi - \delta),\ |\varepsilon| < \rho\},$$

where $\delta \in (0, \pi/2)$; see figure 3. It is possible to choose $\rho > 0$ sufficiently small such that the singular points of (2.1) are hyperbolic and then linearizable independently over S_- and S_+. In this case, we say we are in the *Glutsyuk point of view* of the dynamics. It is known that it is not possible, in general, to embed the two linearizing charts (corresponding to S_- and S_+, respectively) in a holomorphic change of coordinates defined over a full neighbourhood of the origin (with the values of ε in the union of the two sectorial domains; see, for example, [8, 15]).

2.1. Time and orbital parts: model family

The isochronicity problem is closely related to a more general problem: the temporal normalizability of the family of analytic systems (see [4]). The analytical obstructions that prevent the system from being isochronous can be identified with the non-triviality of a functional component of the modulus of analytic classification under weak conjugacy plus an additional vanishing condition. Indeed, the modulus has two functional parts: the orbital part and the temporal part. We will prove that two isochronous families (2.1) with the same period T_ε must belong to the same temporal class and that their *formal temporal invariant* vanishes identically (see definition 2.5).

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2.1.1. *Weak conjugacy*

Two germs of complex analytic families (2.1) are *weakly analytically (respectively formally) conjugate* if, for every fixed value of ε, there exists the germ of a holomorphic (respectively formal) change of coordinates $\Psi_\varepsilon(z, w)$ and a holomorphic reparametrization $\varepsilon' = \kappa(\varepsilon)$ bringing the first ε-system into the second ε'-system. The term weak means that the equivalence Ψ_ε depends analytically (respectively formally) on $\varepsilon \neq 0$ and continuously on ε at $\varepsilon = 0$.

Two germs of complex analytic families of diffeomorphisms (1.3) are weakly analytically (respectively formally) conjugate if, for every fixed value of ε, there exists a germ of holomorphic (respectively formal) conjugacy $\boldsymbol{h}_\varepsilon$ between the two families of diffeomorphisms and a holomorphic reparametrization κ. The conjugacy $\boldsymbol{h}_\varepsilon$ depends analytically (respectively formally) on $\varepsilon \neq 0$ and continuously on ε at $\varepsilon = 0$.

A conjugacy between two germs of analytic families of vector fields (respectively diffeomorphisms) is called *real* if the reparametrization κ is real and the change of coordinates Ψ_ε (respectively the conjugacy $\boldsymbol{h}_\varepsilon$) preserves the real plane (respectively the real line) on real values of the parameter. A conjugacy between two families of differential equations must preserve the time parametrization along the orbits of the vector fields (i.e. no time scaling is allowed). On the contrary, if two systems are *orbitally equivalent* then there is a change of coordinates sending the leaves of one of them into leaves of a non-vanishing multiple (the time scaling) of the other system. Two families of vector fields can be (weakly) conjugate only if they are (weakly) orbitally equivalent.

In the Glutsyuk point of view, the modulus under conjugacy of an analytic family is constructed by adding the temporal part to the modulus of orbital equivalence (see [5]) on values $\varepsilon \in S_- \cup S_+$, as explained below. Furthermore, it is known (see, for example, [3,5]) that two generic systems (2.1) are (weakly) analytically orbitally equivalent if and only if the germs of their monodromies (1.3) are (weakly) analytically conjugate.

Proposition 2.2 (see [4]). *There exists a germ of real analytic change of coordinates bringing (2.1) into a prepared form*

$$\left.\begin{aligned} \dot z &= z\boldsymbol{t}_\varepsilon(u)(\mathrm{i} + \boldsymbol{A}_\varepsilon(z, w)),\\ \dot w &= w\boldsymbol{t}_\varepsilon(u)(-\mathrm{i} + \boldsymbol{A}_\varepsilon(z, w)),\end{aligned}\right\}$$

where $\boldsymbol{A}_\varepsilon = \varepsilon + \cdots$ is analytic in (z, w) and ε. The term $u = zw$ is the resonant monomial. The germ of function $\boldsymbol{t}_\varepsilon = \boldsymbol{t}_\varepsilon(u)$ is called the time part of (2.2). (Note that the time part is a function of the resonant monomial.) In these coordinates, the invariant manifold $\boldsymbol{m}$ of the orbital part

$$\boldsymbol{X}_\varepsilon = z(\mathrm{i} + \boldsymbol{A}_\varepsilon)\frac{\partial}{\partial z} + w(-\mathrm{i} + \boldsymbol{A}_\varepsilon)\frac{\partial}{\partial w} \tag{2.3}$$

is given by $\boldsymbol{m} = \{\varepsilon + su = 0\}$ and the monodromy of $\boldsymbol{X}_\varepsilon$ has the form

$$\mathcal{P}_\varepsilon(w) = w + w(\varepsilon + sw^2)(2\pi + \mathcal{O}(w) + \mathcal{O}(\varepsilon)), \tag{2.4}$$

where $\mathcal{O}(\varepsilon) = \varepsilon((2\pi)^2)/(2!) + \varepsilon^2((2\pi)^3)/(3!) + \cdots$*. In particular,* $\mathcal{P}'_\varepsilon(0) = \exp(2\pi\varepsilon)$ *and the parameter* ε *is called canonical. It is an invariant. The formal invariant* $\mathsf{B}(\varepsilon)$ *is defined implicitly through* $\mathcal{P}'_\varepsilon(\pm\sqrt{-s\varepsilon}) = \exp(-4\pi\varepsilon(1 - s\mathsf{B}(\varepsilon)\varepsilon))$*.*

It is clear that the time part $\boldsymbol{t}_\varepsilon$ can be computed along with any solution (z, w) of (2.2) through the formula

$$\boldsymbol{t}_\varepsilon(u) = \frac{1}{2\mathrm{i}} \frac{\dot{z}w - z\dot{w}}{zw} \tag{2.5}$$

for every fixed value of the parameter.

DEFINITION 2.3. A germ of family (2.2) is isochronous if there exists a germ of real analytic change of coordinates, depending holomorphically on ε, bringing the time part $\boldsymbol{t}_\varepsilon$ into the constant β in (1.2) (called the period) and bringing the orbital part $\boldsymbol{X}_\varepsilon$ into a (generically) different orbital part X_ε.

The orbital part X_ε also has the form (2.3). It is easy to see that the orbital parts of monodromic families are isochronous in the following sense.

COROLLARY 2.4. *Let* (z, w) *be an integral curve of an orbital part* X_ε *of the form (2.3). That is,* $(\dot{z}, \dot{w})^T = \mathsf{X}_\varepsilon(z, w)$*. Then,*

$$\frac{1}{2\mathrm{i}} \frac{\dot{z}w - z\dot{w}}{zw} = 1,$$

i.e. the orbital part is isochronous with speed 1 *and hence with period* 2π*.*

Therefore, the obstructions preventing the family (2.2) from being isochronous are closely and solely related to its time part. Furthermore, by [3, lemma 2.4] the orbital part (2.3) is formally equivalent to

$$z(\mathrm{i} + (\varepsilon + su)(1 + \mathsf{B}(\varepsilon)u)) \frac{\partial}{\partial z} + w(-\mathrm{i} + (\varepsilon + su)(1 + \mathsf{B}(\varepsilon)u)) \frac{\partial}{\partial w}. \tag{2.6}$$

The constant $\mathsf{B}(\varepsilon)$ in this form is called the *orbital formal invariant*, defined implicitly in proposition 2.2. It unfolds the orbital formal invariant $\mathsf{B}(0)$. The latter is computed through successive elimination of non-resonant monomials of the nonlinear part via Poincaré–Dulac normalization (see [3]).

By ellipticity, the monodromy (2.4) has two analytic return times naturally associated with it. The first is the *return time near the origin* and is denoted τ_0. The other is the *return time near the invariant manifold* $\boldsymbol{m}$ and is denoted $\tau_{\boldsymbol{m}}$. These times are only formal invariants of the family under (weak) conjugacy (i.e. they can change if time scalings are allowed along the leaves of the foliation of (2.2)). It is easy to prove via (2.6) that $\tau_0(\varepsilon) = 2\pi/\beta$ and we can scale variables such that

$$\tau_{\boldsymbol{m}}(\varepsilon) = \frac{\tau_0(\varepsilon)}{1 - s\varepsilon\mathsf{C}(\varepsilon)}$$

for a real constant $\mathsf{C}(\varepsilon)$ that is completely determined by $\tau_{\boldsymbol{m}}$. Note that

$$\mathsf{C}(\varepsilon) = \frac{1}{s\varepsilon}\left(1 - \frac{\tau_0}{\tau_{\boldsymbol{m}}}\right) \tag{2.7}$$

and hence $\mathsf{C} = 0$ if and only if $\tau_0 = \tau_{\boldsymbol{m}}$ on every $\varepsilon \neq 0$. The constant $\mathsf{C}(\varepsilon)$ depends analytically on $\varepsilon \neq 0$ and admits a continuous limit at $\varepsilon = 0$; see [20].

Definition 2.5. The constant $\mathsf{C}(\varepsilon)$ is the formal temporal invariant of (2.2).

The time part (2.5) is, generically, not a germ of an analytic function. However, it proves analytic in the case of elliptic singularities (see [4]). Furthermore, the preparation of proposition 2.2 brings the time part into

$$\boldsymbol{t}_\varepsilon(u) = \beta(1 + \mathsf{C}(\varepsilon)u)(1 + \mathcal{O}(u(\varepsilon + su))). \tag{2.8}$$

This argument leads us to compare (2.2) with the normal form

$$\begin{aligned} \dot{z} &= z\beta(1 + \mathsf{C}(\varepsilon)u)(\mathrm{i} + (\varepsilon + su)(1 + \mathsf{B}(\varepsilon)u)), \\ \dot{w} &= w\beta(1 + \mathsf{C}(\varepsilon)u)(-\mathrm{i} + (\varepsilon + su)(1 + \mathsf{B}(\varepsilon)u)), \end{aligned} \tag{2.9}$$

which is called the *model family* of (2.2). The sign $s = \pm 1$ coincides with the sign of $\ell(0)$ (the Lyapunov constant of (1.1) at $\varepsilon = 0$). By analogy, the germ of analytic map $\mathbb{t}_\varepsilon = \beta(1 + \mathsf{C}(\varepsilon)u)$ is called the time part of the normal form. The germ of vector field (2.6), denoted $\mathbb{X}_\varepsilon$, is the orbital part of the normal form.

In order to define the normal form of the Poincaré monodromy we compute the speed of the orbits along the *skew*-resonant monomial $v = \sqrt{u}$. The formal normal form of the Poincaré monodromy of $\boldsymbol{X}_\varepsilon$ is thus the time-2π flow of the equation $\dot{v} = v(\varepsilon + sv^2)(1 + \mathsf{B}(\varepsilon)v^2)$. Hence, the monodromy and its formal normal form have the same multipliers.

2.2. Orbital part of the Glutsyuk invariant

In [5] we studied the obstructions that prevent the orbital part of the system (2.2) from being equivalent to (2.6), through identification of a complete modulus under orbital equivalence. On values of $\varepsilon \neq 0$, the invariant is constructed via comparison of the *orbit space* of the monodromy (2.4) and the orbit space of its formal normal form. In the orbital case, the normal form of the monodromy corresponds to the time-2π map of the formal vector field

$$\frac{w(\varepsilon + sw^2)}{1 + \mathsf{A}(\varepsilon)w^2}\frac{\partial}{\partial w}$$

for values of ε in $S_- \cup S_+$, where

$$\mathsf{A}(\varepsilon) = \frac{\mathsf{B}(\varepsilon)}{s\varepsilon\mathsf{B}(\varepsilon) - 1}. \tag{2.10}$$

In the Glutsyuk point of view, there are orbits connecting the fixed points of the monodromy in a small neighbourhood of zero. Indeed, the fixed points are hyperbolic and thus linearizable. In particular, the monodromy is normalizable. The respective normalizing coordinates over S_- and S_+ are denoted φ_- and φ_+. The orbit space of $\mathcal{P}_\varepsilon$ is obtained by taking three closed curves $\{\ell_0, \ell_+, \ell_-\}$ around the fixed points, and then taking their images $\{\mathcal{P}_\varepsilon(\ell_0), \mathcal{P}_\varepsilon(\ell_+), \mathcal{P}_\varepsilon(\ell_-)\}$. Since the fixed points are hyperbolic, the closed regions $\{C_0, C_+, C_-\}$ between the curves and their images are isomorphic to three closed annuli.

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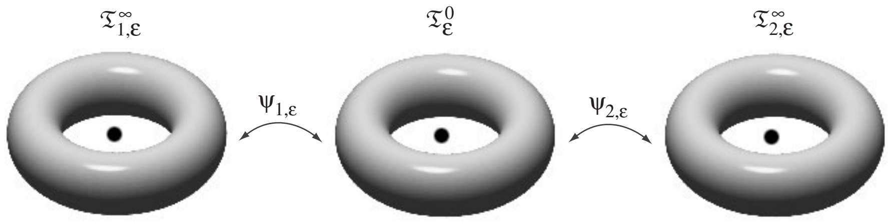

Figure 4. Orbit space of the monodromy.

We identify $\ell_\# \sim \mathcal{P}_\varepsilon(\ell_\#)$, where $\# \in \{0,+,-\}$. Then, the quotient $C_\#/\sim$ is a conformal torus; see figure 4. The orbit space is conformally equivalent to the union of three tori $\mathfrak{T}^0_\varepsilon$, $\mathfrak{T}^\infty_{1,\varepsilon}$, $\mathfrak{T}^\infty_{2,\varepsilon}$ plus the three singular points, such that: each orbit has at most one point in each torus, each orbit is either a fixed point or is represented in a torus and some orbits may have representatives in two different tori. Let us denote

$$\mathcal{L}_C(w) = Cw \tag{2.11}$$

the linear map, for any $C \in \mathbb{C}$. The orbital part of the Glutsyuk modulus consists in the identification of orbits via (almost) intrinsic coordinates on the tori. Indeed, by Abel's theorem (see [11]) each torus $\mathfrak{T}$ is a quotient $\mathfrak{T} = \mathbb{C}^*/\mathcal{L}_C$ for some $C \in \mathbb{C}^*$. Then, a natural coordinate on $\mathfrak{T}$ is the projection of a coordinate on $\mathbb{C}^* = \mathbb{CP}^1 \backslash \{0,\infty\}$. The identification of orbits in two different tori induces germs of families of analytic diffeomorphisms

$$\psi_{j,\varepsilon} \colon \mathbb{C}^* \mapsto \mathbb{C}^* \tag{2.12}$$

for $j \in \{1,2\}$, such that $\psi_{j,\varepsilon} \circ \mathcal{L}_{C_1} = \mathcal{L}_{C_2} \circ \psi_{j,\varepsilon}$ if $\psi_{j,\varepsilon}$ represents a map from $\mathfrak{T}_1 = \mathbb{C}^*/\mathcal{L}_{C_1}$ to $\mathfrak{T}_2 = \mathbb{C}^*/\mathcal{L}_{C_2}$.

It is known that the modulus is represented by only one germ ψ_ε. The orbital part of the modulus measures the obstruction to match together in a holomorphic chart the two (generically) different normalizing charts φ_- and φ_+. We refer the reader to [5] for further details on the orbital part of the invariant and symmetries.

3. Temporal normalization and isochronicity

In [4], we ask the question of whether there exists a weak conjugacy bringing the family (2.2), with the time part (2.8), into a family with the time part $\mathbb{t}_\varepsilon = \beta(1 + \mathsf{C}(\varepsilon)u)$ and the orbital part X_ε; see definition 2.3. Inasmuch as $\boldsymbol{t}_\varepsilon \boldsymbol{X}_\varepsilon$ and $\mathbb{t}_\varepsilon \mathsf{X}_\varepsilon$ would be weakly orbitally equivalent we can suppose, in particular, that the conjugacy fixes the orbital part and $\boldsymbol{X}_\varepsilon = \mathsf{X}_\varepsilon$. (Such conjugacies are called *symmetries of the foliation*.) Any such symmetry brings the term $1 + \mathcal{O}(u(\varepsilon + su))$ of (2.8) to 1. Furthermore, it is known that any such conjugacy must be given by the ξ_ε-flow of the vector field $\mathbb{t}_\varepsilon \boldsymbol{X}_\varepsilon$ for some real function ξ_ε (see [20]). The latter is a solution of the cohomological equation

$$\boldsymbol{X}_\varepsilon \cdot \xi_\varepsilon = \frac{1}{\boldsymbol{t}_\varepsilon} - \frac{1}{\mathbb{t}_\varepsilon}. \tag{3.1}$$

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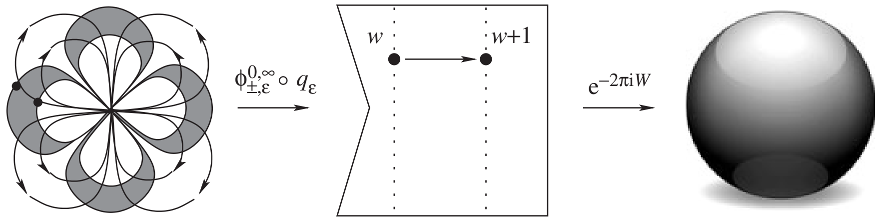

Figure 5. On the left, the 4 domains of sectorial trivialization.

DEFINITION 3.1. The family $\mathbb{t}_\varepsilon \boldsymbol{X}_\varepsilon$ is the temporal normal form of the family $\boldsymbol{t}_\varepsilon \boldsymbol{X}_\varepsilon$. If (3.1) admits a solution ξ_ε then $\boldsymbol{t}_\varepsilon \boldsymbol{X}_\varepsilon$ is called temporally normalizable. The orbital symmetry is called a temporal normalizing coordinate.

In contrast to the orbital case (foliation), the obstructions to solving (3.1) give rise to the *temporal part* of the modulus. Hence, the temporal part of the invariant of a prepared system (2.1) is related to its monodromy. If this temporal part happens to be trivial, the vector field is temporally normalizable. There are, in general, formal and analytical obstructions preventing the system from being temporally normalizable.

3.1. Temporal functional part of the invariant

In the Glutsyuk point of view, i.e. when the parameter is taken over the two sectors S_- and S_+, the fixed points of the monodromy (2.4) are hyperbolic. In particular, the system $\boldsymbol{t}_\varepsilon \boldsymbol{X}_\varepsilon$ is temporally normalizable over those sectors. That is, there are two normalizing coordinates (sectorial conjugacies) defined, respectively, on values $\varepsilon \in S_-$ and $\varepsilon \in S_+$ bringing the system $\boldsymbol{t}_\varepsilon \boldsymbol{X}_\varepsilon$ into its temporal formal normal form $\mathbb{t}_\varepsilon \boldsymbol{X}_\varepsilon$. In general, the two conjugacies do not coincide. The temporal part of the Glutsyuk invariant measures the obstructions for these normalizing coordinates to match together and embed in a single analytic conjugacy. This is an exceptional situation rather than the general rule (Stokes phenomenon).

In order to find solutions of (3.1) we need to work in the blow-up space. In this space the monodromy is constructed as an iterate of the holonomy of the cross-section $\Sigma\colon \{z = w\}$. The (complex) desingularization of the origin $(z, w) = (0, 0)$ endows the exceptional divisor with complex coordinates (Z, w) and (z, W) around the singular points $(Z, w) = (0, 0)$ and $(z, W) = (0, 0)$, respectively; see figure 2. The variables (z, w) are then retrieved through the blow-down map (c_1, c_2) where $c_1\colon (Z, w) \mapsto (Zw, w) = (z, w)$ and $c_2\colon (z, W) \mapsto (z, zW) = (z, w)$, respectively (see [3, 16]).

We denote by $\boldsymbol{T}_\varepsilon \mathcal{X}_\varepsilon$ and $\mathbb{T}_\varepsilon \mathcal{X}_\varepsilon$ the pullback of the vector fields $\boldsymbol{t}_\varepsilon \boldsymbol{X}_\varepsilon$ and $\mathbb{t}_\varepsilon \boldsymbol{X}_\varepsilon$, respectively, by the map c_1. (The functions $\boldsymbol{T}_\varepsilon$, $\mathbb{T}_\varepsilon$ are the blow-up of the time parts of the fields, and $\mathcal{X}_\varepsilon$ is the blow-up of the orbital part.) Let (z, w) be an integral curve of the field $\boldsymbol{t}_\varepsilon \boldsymbol{X}_\varepsilon$. According to (2.5) (in terms of the chart (Z, w)) we have

$$\boldsymbol{t}_\varepsilon(u) = \frac{1}{2\mathrm{i}} \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{z}{w} \right) \right) \left(\frac{z}{w} \right)^{-1} = \frac{1}{2\mathrm{i}} \frac{\dot{Z}}{Z} = \boldsymbol{T}_\varepsilon(Z, w)$$

or, equivalently, $\dot{Z} = 2\mathrm{i}\boldsymbol{T}_\varepsilon(Z, w)Z$ in (Z, w) coordinates (c_1 complex chart). On the other hand, a time form $\mathrm{d}t$ of the vector field $\mathcal{X}_\varepsilon$ is a 1-form such that $i_{\mathcal{X}_\varepsilon}\,\mathrm{d}t = 1$,

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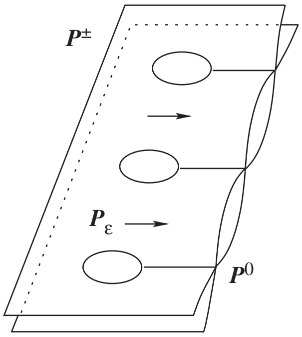

Figure 6. A global portrait of $\mathcal{R}_\varepsilon$.

where $i_{\mathcal{X}_\varepsilon}$ is the interior product on 1-forms: $i_{\mathcal{X}_\varepsilon}\,\mathrm{d}t = \mathrm{d}t(\mathcal{X}_\varepsilon)$. By the equation above, this form can be taken as

$$\mathrm{d}t = \frac{\mathrm{d}Z}{2\mathrm{i}\boldsymbol{T}_\varepsilon(Z,w)Z}.$$

Then, the function ξ_ε in (3.1) must solve the integral equation

$$(\xi_\varepsilon \circ \mathcal{P}_\varepsilon)(w) - \xi_\varepsilon(w) = \int_{\gamma^2(w)} \left(\frac{1}{\boldsymbol{T}_\varepsilon} - \frac{1}{\mathbb{T}_\varepsilon}\right) \mathrm{d}t, \tag{3.2}$$

where $\mathcal{P}_\varepsilon$ is the monodromy of the system, γ^2 is the monodromy path given by the double circuit around the singular point $(Z,w) = (0,0)$, obtained by lifting the equator of the divisor. (This circuit is not a simple Jordan curve, but a double curve instead.) Define

$$\kappa_\varepsilon(w) = \int_{\gamma(w)} \left(\frac{1}{\boldsymbol{T}_\varepsilon} - \frac{1}{\mathbb{T}_\varepsilon}\right) \mathrm{d}t, \tag{3.3}$$

where γ is the semi-monodromy path given by the simple circuit around the singular point $(Z,w) = (0,0)$ obtained by lifting the equator of the divisor. (Corollary 3.2 of [3] proves that the integral in (3.2) and $\kappa(\varepsilon)$ are well-defined. Furthermore, the integral in (3.2) is the second iterate of $\kappa(\varepsilon)$.) Let $(Z(t), w(t))$ be an integral curve of the field $\boldsymbol{T}_\varepsilon\mathcal{X}_\varepsilon$ with initial condition $(1, w_0) \in \varSigma$. It is known (see [19]) that the function

$$\int_0^t \left(\frac{1}{\boldsymbol{T}_\varepsilon} - \frac{1}{\mathbb{T}_\varepsilon}\right)\bigg|_{(Z(s),w(s))} \mathrm{d}s, \tag{3.4}$$

defined on a neighbourhood of the origin minus the axis $Z = 0$, is uniformly bounded. Therefore, $\kappa_\varepsilon(w)$ is holomorphic in w over a full neighbourhood of zero.

In [4], we proved that the solution ξ_ε of (3.2) is obtained through an intermediary step consisting of straightening the monodromy into translation by one; see figure 5. Indeed, ξ_ε is the composite function

$$\xi_\varepsilon = \varXi^{0,\infty}_{\pm,\varepsilon} \circ \varPhi^{0,\infty}_{\pm,\varepsilon} \circ q_\varepsilon, \tag{3.5}$$

where q_ε is a $\mathrm{i}\pi/\varepsilon$-periodic multi-valued map with target space a 2-folded Riemann surface $\mathcal{R}_\varepsilon$ deprived of a countable number of holes [2, 5, 17, 19]; see figure 6. The

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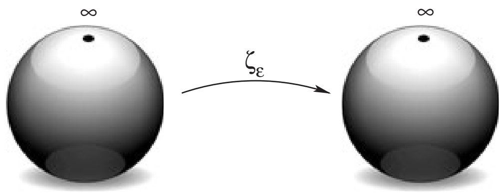

Figure 7. Domain of the temporal part of the invariant.

maps $\Phi^{0,\infty}_{\pm,\varepsilon} : Q^{0,\infty}_{\pm} \to \mathbb{C}$ are germs of holomorphic sectorial trivializations conjugating the *monodromy* $\boldsymbol{P}_\varepsilon = q_\varepsilon \circ \mathcal{P}_\varepsilon \circ q_\varepsilon^{-1}$ with the translation by one $\mathcal{W} \mapsto \mathcal{W}+1$, and known as *real Fatou–Glutsyuk* coordinates (see [5]). The domain $Q^{0,\infty}_{\pm} \subset \mathcal{R}_\varepsilon$ of the Fatou coordinate is called a *translation domain*. The subscripts $\pm$ on the coordinate and its domain make reference to the sheets of $\mathcal{R}_\varepsilon$: the one on top corresponding to the image through q_ε of a neighbourhood of the non-zero fixed point $+\sqrt{-s\varepsilon}$ of the monodromy, the one below corresponding to the image of a neighbourhood of the other non-zero fixed point $-\sqrt{-s\varepsilon}$; see (2.4). The superscripts 0, ∞ indicate whether the translation domain contains a neighbourhood of $P^0 = q_\varepsilon(0)$ or a neighbourhood of $P^\pm = q_\varepsilon(\pm\sqrt{-s\varepsilon})$.

The maps $\Xi^{0,\infty}_{\pm,\varepsilon} : \Phi^{0,\infty}_{\pm,\varepsilon}(Q^{0,\infty}_{\pm}) \to \mathbb{C}$ are (unique up to constant) germs of holomorphic functions characterized by the following property:

$$\Xi^{0,\infty}_{\varepsilon,\pm}(\mathcal{W}+1) - \Xi^{0,\infty}_{\varepsilon,\pm}(\mathcal{W}) = \kappa_\varepsilon \circ q_\varepsilon^{-1} \circ \Phi^{0,\infty}_{\pm,\varepsilon}(\mathcal{W}) + \kappa_\varepsilon \circ q_\varepsilon^{-1} \circ \Phi^{0,\infty}_{\mp,\varepsilon}(\mathcal{W} + \tfrac{1}{2})$$

(see [4]). These analytic maps are endowed with particular symmetries. Furthermore, it is known that the coordinate (3.5) is real and well defined; its values are independent of the sub- and superscripts $\pm$, 0, ∞, whence the notation (see [4]). The solution (3.5) is also unique provided it unfolds the identity as $\varepsilon \to 0$.

Let $\delta^{0,\infty}_{\pm,\varepsilon}$ be the complex coordinate induced by $\Xi^{0,\infty}_{\pm,\varepsilon}$ on the Riemann sphere $\mathbb{C}P^1$ via the exponential map $\mathcal{E} : \mathcal{W} \mapsto \exp(-2\pi\mathrm{i}\mathcal{W})$, i.e. $\delta = \Xi \circ \mathcal{E}^{-1}$. Let us define germs of holomorphic functions ζ_ε on the Riemann sphere as follows. If $\varepsilon \in S_-$ then we set

$$\zeta^{\pm\pm}_\varepsilon = \delta^0_{\pm,\varepsilon} - \delta^\infty_{\pm,\varepsilon} \circ (\psi^{\pm\pm}_\varepsilon)^{-1},$$

while if $\varepsilon \in S_+$, then we write

$$\zeta^{\pm\pm}_\varepsilon = \delta^\infty_{\pm,\varepsilon} - \delta^0_{\pm,\varepsilon} \circ (\psi^{\pm\pm}_\varepsilon)^{-1},$$

where $\psi_\varepsilon = \{\psi^{\pm\pm}_\varepsilon\}$ is the orbital part of the Glutsyuk invariant (2.12). In either case, the collection above will be denoted ζ_ε. The chart ζ_ε is by no means a diffeomorphism. However, it depends analytically on $\varepsilon \neq 0$ over the union $S_- \cup S_+$ and continuously on ε at $\varepsilon = 0$. The domain of the functions ζ_ε corresponds to open annuli containing the real equator $\mathbb{R}P^1$ of the Riemann sphere; see figure 7.

Definition 3.2. The temporal part of the Glutsyuk modulus consists of the coefficient $\beta = \beta(\varepsilon)$, the formal temporal invariant $\mathsf{C}(\varepsilon)$ and the family of equivalence classes of ζ_ε with respect to the equivalence relation

$$\zeta_\varepsilon \sim \hat{\zeta}_\varepsilon \iff \exists d(\varepsilon) \in \mathbb{C} : \zeta_\varepsilon \circ \mathcal{L}_{d(\varepsilon)} = \hat{\zeta}_\varepsilon.$$

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The symbol ζ_ε denotes any chosen component $\zeta_\varepsilon^{\pm\pm}$ of the functional part and $\mathcal{L}_{d(\varepsilon)}$ is the linear map (2.11). The constant $d(\varepsilon)$ is not necessarily real on real ε, but it must satisfy the identity $d(\varepsilon)\overline{d(\bar{\varepsilon})} = 1$ (thus taking its values on the unit circle on ε real). Besides, it depends analytically on $\varepsilon \neq 0$ over the union $S_- \cup S_+$ and continuously on ε at $\varepsilon = 0$.

The non triviality of the functional part ζ_ε is a measure of the obstructions to glue together the two different temporal normalizing coordinates (induced, respectively, on values $\varepsilon \in S_-$ and $\varepsilon \in S_+$) in a uniform analytic conjugacy; see definition 3.1. The space of temporal moduli is also a huge functional space of infinite dimension (see [15]). As in the orbital case, temporally normalizable systems unfolding a weak focus are quite exceptional. However, monodromic isochronous dynamical systems (i.e. germs of elliptic families of differential equations for which (2.5) is holomorphically conjugate to a constant on every integral solution) other than weak foci are generically not rare. For example, it has been observed that a certain class of Hamiltonian systems are very often isochronous; see [7, ch. 2]. Evidently, *integrable* monodromic singularities (centres) are always isochronous.

3.2. Symmetries of the temporal part

(The justification for the symmetries presented in this section can be found in [4].) The temporal part is the germ of a non-ramified function that is invariant under rotations of angle $-2\pi\mathrm{i}$ in the source space, i.e.

$$\zeta_\varepsilon \circ \mathcal{L}_{\exp(-2\pi\mathrm{i})} = \zeta_\varepsilon.$$

Moreover, it is possible to choose representatives of the temporal Glutsyuk invariant such that

$$\begin{aligned} \zeta_\varepsilon^{++} \circ \mathcal{L}_{-1} &= \zeta_\varepsilon^{--}, \\ \zeta_\varepsilon^{-+} \circ \mathcal{L}_{-1} &= \zeta_\varepsilon^{+-} \end{aligned} \tag{3.6}$$

for every $\varepsilon \in S_- \cup S_+$. Define complex numbers

$$\alpha_0(\varepsilon) = \frac{2\pi\mathrm{i}}{\varepsilon}, \qquad \alpha_\infty(\varepsilon) = -\frac{\pi\mathrm{i}(1 - s\mathsf{A}(\varepsilon)\varepsilon)}{\varepsilon},$$

where $\mathsf{A}(\varepsilon)$ is the formal invariant (2.10). There exists a representative of the temporal part of the modulus that satisfies, in addition to (3.6), the identities

$$\varepsilon \in S_- : \begin{cases} \zeta_\varepsilon^{++} \circ \mathcal{L}_{\exp(\mathrm{i}\pi\alpha_0)} = \zeta_\varepsilon^{+-}, \\ \zeta_\varepsilon^{--} \circ \mathcal{L}_{\exp(\mathrm{i}\pi\alpha_0)} = \zeta_\varepsilon^{-+}, \end{cases} \qquad \varepsilon \in S_+ : \begin{cases} \zeta_\varepsilon^{++} \circ \mathcal{L}_{\exp(2\pi\mathrm{i}\alpha_\infty)} = \zeta_\varepsilon^{-+}, \\ \zeta_\varepsilon^{--} \circ \mathcal{L}_{\exp(2\pi\mathrm{i}\alpha_\infty)} = \zeta_\varepsilon^{+-}. \end{cases} \tag{3.7}$$

The temporal part ζ_ε is invariant under reflection with respect to the real equator of $\mathbb{C}P^1$. This comes from a symmetry on the coefficients of its asymptotic expansion in the annulus containing $\mathbb{R}P^1$.

THEOREM 3.3. *Let* $\boldsymbol{w} = \mathcal{E}(\mathcal{W})$ *be the coordinate induced in the Riemann sphere as described in* § *2.2. (*$\mathcal{W}$ *is the image of the Fatou coordinate.) There exists a* *representative of the temporal part of the modulus with asymptotic expansion in the Laurent series*

$$\zeta_\varepsilon(\boldsymbol{w}) = \sum_{-\infty}^{+\infty} a_n(\varepsilon)\boldsymbol{w}^n$$

such that if $\varepsilon \in S_-$*, then* $a_n(\varepsilon) = \overline{a_{-n}(\bar\varepsilon)}$ *for every* $n \in \mathbb{Z}$*, and if* $\varepsilon \in S_+$*, then* $a_n(\varepsilon) = (-1)^n\overline{a_{-n}(\bar\varepsilon)}$ *for every integer* n*.*

Proof. We start by fixing coordinates $\delta^{0,\infty}_{\pm,\varepsilon}$. Let

$$\zeta^{\pm\pm}_\varepsilon(\boldsymbol{w}) = \sum_{-\infty}^{+\infty} a^{\pm\pm}_n(\varepsilon)\boldsymbol{w}^n$$

be the asymptotic expansion of each component of the temporal part. It is known (see [4, (4.17)]) that the coefficients are two-by-two related through

$$\begin{aligned} a^{++}_n(\varepsilon) &= \overline{a^{+-}_n(\bar\varepsilon)}, \\ a^{--}_n(\varepsilon) &= \overline{a^{-+}_n(\bar\varepsilon)}. \end{aligned} \tag{3.8}$$

Let $\varepsilon \in S_-$. We will check the case $+-$; the rest of the identities are completely analogous. Combining the first identity in (3.8) with $\zeta^{++}_\varepsilon \circ \mathcal{L}_{\exp(\mathrm{i}\pi\alpha_0)} = \zeta^{+-}_\varepsilon$ in (3.7) yields the relation

$$\overline{a^{+-}_n(\bar\varepsilon)}\mathrm{e}^{n\pi\mathrm{i}\alpha_0} = a^{+-}_n(\varepsilon) \tag{3.9}$$

on every $\varepsilon \in S_-$ and $n \in \mathbb{Z}$. According to definition 3.2, the composition of ζ^{+-}_ε with the linear map $\mathcal{L}_{\exp(\pi\mathrm{i}(\alpha_0/2))}$ in the source space defines a new representative of the temporal part. (Indeed, the constant $\exp(\pi\mathrm{i}(\alpha_0/2))$ is not a value of the real circle on real ε.) However, it is easily seen that the new representative $\zeta^{+-}_\varepsilon \circ \mathcal{L}_{\exp(\pi\mathrm{i}(\alpha_0/2))}$ satisfies the relation

$$\sum_{-\infty}^{+\infty} \overline{a^{+-}_{-n}(\bar\varepsilon)}\mathrm{e}^{n\pi\mathrm{i}(\alpha_0/2)}\boldsymbol{w}^n = \sum_{-\infty}^{+\infty} a^{+-}_n(\varepsilon)\mathrm{e}^{n\pi\mathrm{i}(\alpha_0/2)}\boldsymbol{w}^n$$

and this yields the required property.

The case $\varepsilon \in S_+$ requires an additional step. It is clear that the identities on the right-hand side in (3.7) and (3.8) imply simultaneously that

$$\overline{a^{+-}_{-n}(\bar\varepsilon)}\mathrm{e}^{2\pi\mathrm{i}\alpha_\infty} = a^{-+}_n(\varepsilon) \quad \text{and} \quad \overline{a^{--}_{-n}(\bar\varepsilon)}\mathrm{e}^{2\pi\mathrm{i}\alpha_\infty} = a^{++}_n(\varepsilon)$$

for each $n \in \mathbb{Z}$. Subsequent *non-circular* corrections $\zeta^{+-}_\varepsilon \mapsto \zeta^{+-}_\varepsilon \circ \mathcal{L}_{\exp(\pi\mathrm{i}\alpha_\infty)}$, $\zeta^{-+}_\varepsilon \mapsto \zeta^{-+}_\varepsilon \circ \mathcal{L}_{\exp(\pi\mathrm{i}\alpha_\infty)}$ and $\zeta^{--}_\varepsilon \mapsto \zeta^{--}_\varepsilon \circ \mathcal{L}_{\exp(\pi\mathrm{i}\bar\alpha_\infty)}$, $\zeta^{++}_\varepsilon \mapsto \zeta^{++}_\varepsilon \circ \mathcal{L}_{\exp(\pi\mathrm{i}\bar\alpha_\infty)}$ yield new representatives of the temporal part, for which

$$a^{++}_n(\varepsilon) = \overline{a^{--}_{-n}(\bar\varepsilon)} \quad \text{and} \quad a^{+-}_n(\varepsilon) = \overline{a^{-+}_{-n}(\bar\varepsilon)},$$

respectively. Since this representative also satisfies (3.6), the desired conclusion follows. □

Corollary 3.4. *A representative of the temporal part of the Glutsyuk modulus is completely determined by only one of its components* $\zeta^{\pm\pm}_\varepsilon$*.*

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This justifies the use of the notation ζ_ε above, instead of writing explicitly the superscripts $\pm\pm$.

As announced in the title of this section, the normalizability of the system carries the information about its isochronicity. Indeed, the triviality of the functional part ζ_ε defines two types of temporally normalizable systems.

LEMMA 3.5 (see [4]). *Consider two germs of generic families of real analytic vector fields (2.2) with the same speed $\beta(\varepsilon)$, same sign s and same formal temporal invariant $\mathsf{C}(\varepsilon)$. These families are weakly analytically conjugate by real conjugacy, if and only if the orbital and temporal (functional) parts of their moduli coincide.*

THEOREM 3.6. *The family (2.2) is isochronous if and only if the temporal part ζ_ε is trivial and the formal temporal invariant $\mathsf{C}(\varepsilon)$ vanishes identically.*

Proof. If the family is isochronous then there exists a real analytic conjugacy (depending holomorphically on the parameter) bringing (2.2) into a family with the time part β (thus with trivial temporal part of the modulus) and orbital part X_ε. After applying a symmetry of the foliation, we can suppose that X_ε coincides with the orbital part of (2.2). Inasmuch as the orbital part is isochronous with period 2π (see corollary 2.4), the formal temporal invariant must vanish identically. By lemma 3.5, the temporal part of the modulus of the resulting family is trivial.

Conversely, suppose that the temporal part of the functional modulus of (2.2) is trivial. Then, there is no obstruction to solving the cohomological equation (3.1). The solution defines a real analytic germ of weak conjugacy (depending analytically on $\varepsilon \neq 0$ and continuously on ε at $\varepsilon = 0$), fixing the orbital part (2.3) and bringing the time part of the family into $\mathbb{t}_\varepsilon = \beta(1+\mathsf{C}(\varepsilon)u)$. Inasmuch as the formal temporal invariant $\mathsf{C}(\varepsilon)$ is identically null, the family is isochronous. □

4. Examples via the Darboux criterion

Let $p \in \mathbb{C}[z, w]$ be a degree-m, square-free polynomial in complex coordinates (z, w). An algebraic curve $\mathcal{C} = \{p = 0\} \subset \mathbb{C}^2$ of degree m is *invariant* under a degree-r polynomial vector field F on the affine plane $\mathbb{C}^2$ if the Lie derivative $F \cdot p$ takes the form $F \cdot p = pq$, where $q \in \mathbb{C}[z, w]$ is the *cofactor* of p and $\deg(q) \leqslant r - 1$. It is well known that if any such vector field F has n different irreducible invariant curves $\mathcal{C}_1, \dots, \mathcal{C}_n$, then it admits a (multivalued) first integral of the form $H = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$, provided

$$\sum_1^n \alpha_j q_j = 0.$$

The p_is are irreducible polynomials in (z, w) determining the respective curves $\mathcal{C}_i$. The exponents $\alpha_i \in \mathbb{C}$ are not all equal to zero. The vector field F is said to be *Darboux integrable*. Yet, if

$$\sum_1^n \alpha_j q_j = \operatorname{div}(F),$$

where $\operatorname{div}(F)$ is the divergence of the vector field F, then H is still a (reciprocal) first integral provided that exactly two factors $p_j(z, w) = z + o(z, w)$ and $p_k(z, w) = w + o(z, w)$ vanish at the origin and α_j, α_k are not both integers greater than 1

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(see [9]). The topological type of an elliptic singularity is completely determined by the existence of first integrals.

PROPOSITION 4.1 (see [16]). *A monodromic integrable (either formally or analytically) singularity is a centre.*

EXAMPLE 4.2. Let a, b, c be real constants. The origin of coordinates in $\mathbb{R}^2$ is a centre for the planar system

$$\begin{aligned}\dot{x} &= -y + (x^2 + y^2)(ax + (b - c)y),\\ \dot{y} &= x + (x^2 + y^2)((b + c)x - ay).\end{aligned} \tag{4.1}$$

Indeed, the function $H = p_1 p_2$ is a first integral of (4.1), where $p_1(x, y) = x + \mathrm{i}y$ and $p_2(x, y) = x - \mathrm{i}y$. The latter are invariant lines of (4.1):

$$\frac{\partial p_j}{\partial x}\dot{x} + \frac{\partial p_j}{\partial y}\dot{y} = p_j q_j,$$

where $q_1(x, y) = a(x^2 - y^2) + 2bxy + \mathrm{i}(1 - 2axy + b(x^2 - y^2) + c(x^2 + y^2))$ and $q_2(x, y) = a(x^2 - y^2) + 2bxy - \mathrm{i}(1 - 2axy + b(x^2 - y^2) + c(x^2 + y^2))$. It is easy to see that $q_1 + q_2 = \mathrm{div}(F)$, where F is the right-hand side of (4.1).

In some cases, a similar criterion allows one to decide whether the vector field is orbitally normalizable or not.

THEOREM 4.3 (see [9]). *Suppose that the vector field F has two factors of the form $p_j(z, w) = z + o(z, w)$ and $p_k(z, w) = w + o(z, w)$ for $j, k \in \{1, \dots, n\}$. Let $\alpha_1, \dots, \alpha_n$ be a collection of complex numbers such that*

$$\sum_{1}^{n} \alpha_j q_j = \mathrm{div}(F).$$

If α_j, α_k are integers greater than 1, then the vector field F is orbitally normalizable.

EXAMPLE 4.4 (see [21]). Consider the Liénard differential equation with damping in the real coordinate $x = x(t)$: $\ddot{x} - 4x^2\dot{x} + x + x^5 = 0$. The coordinate $y = -\dot{x}$ brings this equation into a planar polynomial equivalent system

$$\begin{aligned}\dot{x} &= -y,\\ \dot{y} &= x + 4x^2 y + x^5.\end{aligned} \tag{4.2}$$

The system is of order 1 with the Lyapunov first constant $\ell_1 = 4$. Thus, the origin is a weak focus with eigenvalues $\pm\mathrm{i}$. Moreover, the system is isochronous. Indeed, the change of coordinates $(x, y) \mapsto (X, Y - X^3)$ brings (4.2) into

$$\begin{aligned}\dot{X} &= -Y + X^3,\\ \dot{Y} &= X + X^2 Y,\end{aligned} \tag{4.3}$$

with the (planar) time part

$$\boldsymbol{t}(X, Y) = \frac{X\dot{Y} - \dot{X}Y}{X^2 + Y^2} = 1.$$

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One further step consists of establishing whether (4.3) is normalizable or not. Complexification of the latter systems yields the isochronous form

$$\begin{aligned}\dot{z} &= \mathrm{i}z + \tfrac{1}{4}z(z+w)^2,\\ \dot{w} &= -\mathrm{i}w + \tfrac{1}{4}w(z+w)^2.\end{aligned} \tag{4.4}$$

(This system is already in the form (2.1).) Let F be the right-hand side of (4.4) and set $p_1(z,w) = z$ and $p_2(z,w) = w$ with respective cofactors $q_1(z,w) = \mathrm{i} + \frac{1}{4}(z+w)^2$ and $q_2(z,w) = -\mathrm{i} + \frac{1}{4}(z+w)^2$. Then, it is easily seen that

$$F \cdot p_i = \frac{\partial p_i}{\partial z}\dot{z} + \frac{\partial p_i}{\partial w}\dot{w} = p_i q_i$$

and

$$\alpha_1 q_1 + \alpha_2 q_2 = (z+w)^2 = \operatorname{div}(F),$$

with $\alpha_1 = \alpha_2 = 2$. By theorem 4.3, system (4.2) is orbitally normalizable and therefore normalizable.

Example 4.5. More generally, let γ be any real number. The system

$$\begin{aligned}\dot{x} &= -y - x^2 + \gamma x^3,\\ \dot{y} &= x + 2xy + 2x^3 + \gamma x^2 y - \gamma x^4\end{aligned} \tag{4.5}$$

has the Lyapunov first constant equal to 4γ and the eigenvalues at the origin are $\pm\mathrm{i}$. Thus, the origin is a weak isochronous focus of order 1 if $\gamma \neq 0$. This becomes clear if we take the change of coordinates $(x,y) \mapsto (X, Y - X^2)$ and subsequently complexify variables, so as to bring (4.5) into the form

$$\begin{aligned}\dot{z} &= \mathrm{i}z + \tfrac{1}{4}\gamma z(z+w)^2,\\ \dot{y} &= -\mathrm{i}w + \tfrac{1}{4}\gamma w(z+w)^2,\end{aligned} \tag{4.6}$$

with the time part $\boldsymbol{t}(z,w) = (\dot{z}w - z\dot{w})/zw = 1$. The divergence of the right-hand side is equal to $\gamma(z+w)^2 = \alpha_1 q_1 + \alpha_2 q_2$, with $\alpha_1 = \alpha_2 = 2$ and $q_1(z,w) = \mathrm{i} + \frac{1}{4}\gamma(z+w)^2$, $q_2(z,w) = -\mathrm{i} + \frac{1}{4}\gamma(z+w)^2$. By theorem 4.3, the system (4.5) is normalizable.

5. Final remarks

As far as the author is aware, this paper relates for the first time (in the category of monodromic analytic systems) two different properties that can be detected in the invariant of analytic classification: temporal normalizability and isochronicity. The former property has been described in [4]. In that article, we basically identify the analytical obstructions preventing the family from being temporally normalizable and investigate the inherent symmetries on the modulus. In this paper, we establish the existence of two classes of temporally normalizable systems: those with vanishing formal temporal invariant, and those for which $\mathsf{C}(\varepsilon)$ is not identically zero on values $\varepsilon \in S_+ \cup S_-$ (codimension 1). Theorem 3.6 states that monodromic systems of the first category are isochronous. The invariant of analytic classification is, therefore, different for both classes of differential equations. Furthermore, the

https://doi.org/10.1017/S0308210511000023 Published online by Cambridge University Press

parametrization along the orbits of the foliation of a family with vanishing formal temporal invariant admits a geometrical interpretation in the phase space. This is not always the case for families of the second class. This distinction is important, because it tells us something about the rareness of isochronous weak foci; the latter are scarce in the class of normalizable elliptic singularities. The reference [4] does not take this into account.

In this paper we identify the symmetries on the modulus in *almost intrinsic* coordinates and give a geometrical interpretation of the invariant in terms of its asymptotic Laurent expansion (see theorem 3.3).

Acknowledgements

This work was partly supported by CONICYT (Chile), Fondecyt no. 11121207, and by Dirección de Investigación y Desarrollo Universidad Austral de Chile. The author thanks Alexander Brudnyi of University of Calgary for supporting this work. The author also thanks Isaac García, Jaume Giné and Maité Grau for helpful discussions (2008) and expresses cordial gratitude to an unknown referee.

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(*Issued 2 August 2013*)

https://doi.org/10.1017/S0308210511000023 Published online by Cambridge University Press