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A functional equation related to a repairable system subjected to priority rules

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We analyse the survival time of a general duplex system sustained by an auxiliary cold standby unit and subjected to priority rules. The duplex system is attended by two general repairmen R_p and R_h. Repairman R_p has priority in repairing failed units with regard to repairman R_h provided that both repairmen are jointly idle. Otherwise, the priority is overruled. The auxiliary unit has its own repair facility. The duplex system has overall, break-in priority (often called pre-emptive priority) in operation and in standby with regard to the auxiliary unit. The analysis of the survival time is based on advanced complex function theory (sectionally holomorphic functions). The main problem is to convert a functional equation into a (parameter dependent) Sokhotski–Plemelj problem.

Key words: duplex system, priority rule, survival function, stopping time, security interval, sectionally holomorphic function.

1 Introduction

Standby provides a powerful tool to increase the reliability and quality of operational systems, e.g. [2, 6, 8, 19]. A frequently employed standby mode is the so-called "cold" standby. The notion of cold standby signifies that a backup unit is kept in reserve, with a zero failure rate, until the repairable online unit fails, e.g. [26]. An alternative cold standby mode occurs in the management of robot-safety device systems to prolong the lifetime of the safety unit, i.e. upon failure of the robot, the safety device is shut-off and kept in cold standby until the repair of the robot has been completed, e.g. [24]. The involvement of cold standby redundancy in satellite systems has been cited by Kim *et al.* [9]. Standby systems are frequently endowed with priority rules. For instance, the external power supply station of a technical plant has usually overall (pre-emptive) priority in operation with regard to an internal (local) power generator in standby, i.e. the local generator is only deployed if the external station is down. Engineering systems characterized by cold standby and subjected to priority rules have received considerable attention in the

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previous Literature, e.g. [17]. A comprehensive review has been compiled by Leung *et al.* [10]. However, most of the systems are dealing with a repair priority rule and with a single repairman, e.g. [11, 25]. Alternatively, a repairable duplex system characterized by an overall (pre-emptive) priority in operation with regard to a cold standby unit and with general distributions for failure and repair has been introduced by Vanderperre and Makhanov [23].

A basic duplex system endowed with cold standby and with a single repairman, henceforth called the **G**-system, has been cited by Gnedenko and Ushakov [8, p. 275]. The **G**-system consists of an active unit, the online unit, sustained by an identical unit in cold standby attended by a single repairman. The **G**-system is down if both units are down. Otherwise, the **G**-system is up. The **G**-system acts as a closed queuing system evolving in time, i.e. a failed unit goes immediately into repair provided that the repairman is idle. Otherwise, the failed unit has to queue for repair. On the other hand, a repaired unit lines up in cold standby if the remaining unit is still available. Otherwise, the repaired unit becomes instantaneously operative. Any repair is assumed to be perfect.

As a modification, we first consider a **G**-system attended by two general heterogeneous repairmen R_p and R_h, henceforth called the **P**-system. The **P**-system is endowed with the following priority rule. Repairman R_p has priority in repairing failed units with regard to repairman R_h provided that both repairmen are idle. Otherwise, the priority rule is over-ruled. The **P**-system is up, if at least one unit (called a **p**-unit) is up. Otherwise, the **P**-system is down. Apart from a tangible variant of the **G**-system, e.g. [20], we now introduce the so-called **T**-system. The **T**-system consists of the **P**-system sustained by an additional auxiliary unit in cold standby, henceforth called the **s**-unit. Each **p**-unit has overall (pre-emptive) priority both in operation and in standby with regard to the **s**-unit. Thus, the **s**-unit is only deployed if the **P**-system is down. The **T**-system is up if at least one unit is up. Otherwise, the **T**-system is down. Finally, we assume that the **s**-unit has its own repairman R_s. The various states of the **T**-system are described in Section 2, Figures 2–6.

A practical example of an **s**-unit is the so-called ram air turbine (RAT). The device consists of a small propeller that, upon request, drops out of the bottom of an aircraft (cf. the landing gear) converting kinetic energy, induced by the airstream, into electrical power. Thus, the RAT is actually a small wind turbine! Note that this auxiliary power device can provide almost all vital components with the required amount of power needed to monitor the plane's flight control in case of emergency. So, the RAT increases the reliability of the aircraft. However, note that the device is only deployed if the global (internal) power generator system (usually a multiple standby system) is down. Therefore, the RAT is a non-priority unit designed to operate in the exceptional case of (internal) loss of power.

In order to derive the survival function of the **T**-system, we employ a stochastic process describing the various states of the **T**-system and endowed with time-dependent transition measures satisfying coupled partial differential equations. The solution procedure of the equations is based on a refined application of the theory of sectionally holomorphic functions, e.g. [7, 12] combined with the notion of dual transforms, [21]. The main problem is to convert a functional equation into a Sokhotski–Plemelj problem.

Furthermore, we introduce a security interval $[0, \tau)$ related to a security level $0 < \delta < 1$ and satisfying a suitable risk criterion. The security interval ensures a survival of the **T**-system up to time τ with probability δ.

Finally, as an example, we consider the case of Coxian repair time distributions. Some graphs are displaying the survival function jointly with the security interval corresponding to a security level of 90%.

2 Formulations, stochastic process, survival function

We now focus on the survival function of the **T**-system. In order to introduce a precise definition of the survival time, we employ a stochastic process $\{N_t, t \geqslant 0\}$ with (discrete) statespace $\{A, B, C, C_s, D\}$ where D is an absorbing state, characterized by the following exhaustive set of mutually independent events.

$\{N_t = A\}$: "All units of the **T**-system are up at time t."

$\{N_t = B\}$: "The **P**-system is up, repairman R_p is busy and the **s**-unit is in cold standby at time t."

$\{N_t = C\}$: "The **P**-system is up, repairman R_h is busy and the **s**-unit is in cold standby at time t."

$\{N_t = C_s\}$: "The **P**-system is down and the **s**-unit is operative at time t."

$\{N_t = D\}$: "The **T**-system is down at time t."

Note that the absorbing property of state D signifies that the process $\{N_t\}$, once entered state D at some random time θ, cannot escape state D anymore. Therefore, taking our priority rule into account, we may assume that a failure of the **s**-unit is catastrophic, i.e. terminates the lifetime of the **T**-system. The inclusion of state D into the state space of the process $\{N_t\}$ invokes the introduction of a so-called stopping time, e.g. [3, Ch. 1, pp. 1–3; 5, p. 190]. Consequently, we first define the non-Markovian process $\{N_t\}$ on a filtered probability space $\{\Omega, \mathcal{A}, P, \mathfrak{F}\}$ where the history $\mathfrak{F} := \mathfrak{F}_t, t \geqslant 0$ satisfies the Dellacherie-conditions,

- $\mathfrak{F}_0$ contains the P-null sets of $\mathcal{A}$,
- $\forall t \geqslant 0,\ \mathfrak{F}_t = \bigcap_{u<t} \mathfrak{F}_u$ i.e. the family $\mathfrak{F}$ is right-continuous.

Consider the $\mathfrak{F}$-stopping time (Markov time)

$$\theta := \inf\{t > 0 :\ N_t = D | N_0 = A\}.$$

We assume that the **T**-system starts functioning at some time origin $t = 0$ in state A, i.e. let $N_0 = A$ with probability one. Thus, from $t = 0$ onwards, θ is the survival time (lifetime) of the **T**-system. The corresponding survival function is denoted by $\mathfrak{R}(t)$. Clearly,

$$\mathfrak{R}(t) = Pr\{\theta > t\},\ t \geqslant 0.$$

It should be noted that θ does not depend on the repair time of the **s**-unit. Therefore, the state space of the process $\{N_t\}$ is sufficient (exhaustive) to describe the random behaviour of the **T**-system during the survival time θ. Figure 1 displays the transitions of N_t related to failures and repairs. An upward (downward) arrow corresponds to a repair (failure) of a unit.

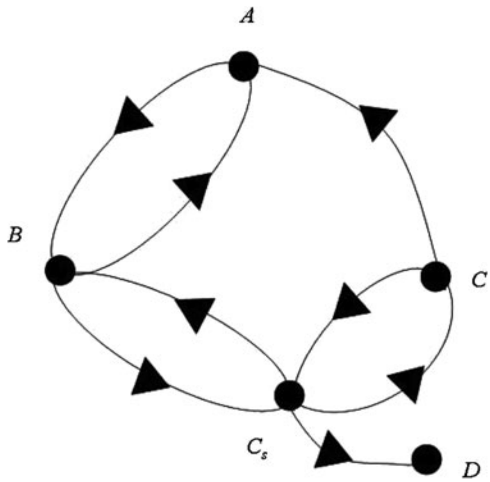

FIGURE 1. Transition diagram related to failures and repairs.

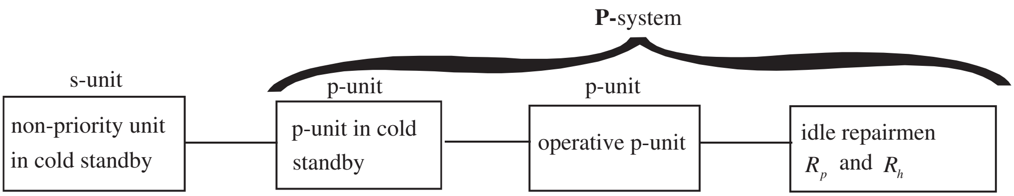

FIGURE 2. Functional block-diagram of the **T**-system operating in state A.

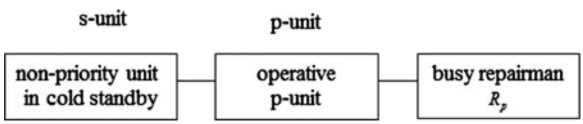

FIGURE 3. Functional block-diagram of the **T**-system operating in state B.

The various states of the **T**-system are described by functional block-diagrams in Figures 2–6.

Along with the survival function of the **T**-system, we now introduce a security interval $[0, \tau)$, where

$$\tau := \sup \left\{ t \geqslant 0 \ : \ \Re(t^-) \geqslant \delta \right\}$$

for some $0 < \delta < 1$, which is called the security level. In practice, δ is usually large. For instance, $\delta = 0.9$. Therefore, we require that the **T**-system satisfies the risk criterion $\lim_{t \uparrow \tau} \Re(t) \geqslant \delta \gg 0$. Note that the security interval, corresponding to the security level δ, ensures a continuous operation (survival) of the **T**-system up to time τ with probability δ.

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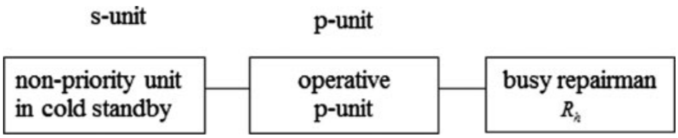

FIGURE 4. Functional block-diagram of the **T**-system operating in state C.

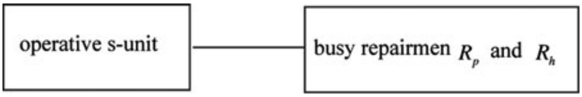

FIGURE 5. Functional block-diagram of the **T**-system operating in state C_s.

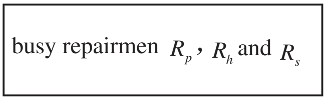

FIGURE 6. The **T**-system down state D.

3 Assumptions, definitions, properties

3.1 Assumptions

Consider the **T**-system satisfying the following assumptions. Each operative **p**-unit has a failure-free time f with distribution $F(\cdot), F(0) = 0$, a constant repair rate μ if the repair is carried out by repairman R_p and a repair time r with distribution $R(\cdot), R(0) = 0$ if the repair is performed by repairman R_h. The **s**-unit has a zero failure rate in standby (cold standby) and a constant failure rate λ_s in the operative state. We recall that the **s**-unit is only deployed if the **P**-system is down. Therefore, θ is independent of the repair time r_s of the **s**-unit. Consequently, the repair time distribution of r_s needs no specification. All underlying random variables are supposed to be independent and any repair is perfect.

3.2 Definitions and properties

Characteristic functions (and their duals) are formulated in terms of a *complex* transform variable.

For instance,

$$\mathbf{E}e^{i\omega r} = \int_0^\infty e^{i\omega x}\, dR(x), \ \operatorname{Im}\omega \geqslant 0.$$

Note that

$$\mathbf{E}e^{-i\omega r} = \int_{-\infty}^0 e^{i\omega x}\, d\{(1 - R((-x)^-)\}, \ \operatorname{Im}\omega \leqslant 0.$$

The corresponding Fourier–Stieljes transforms are called *dual* transforms. Without loss of generality (see Remarks 8.1), we may assume that F and R have density functions of bounded variation on $[0, \infty)$ with finite mean. A (vector) Markov characterization of the

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non-Markovian process $\{N_t, t \geqslant 0\}$, with absorbing state D, is piecewise and conditionally defined by the following:

$\{(N_t, X_t)\}$ if $N_t = A$, where X_t denotes the remaining failure-free time of the **p**-unit being operative at time t,

$\{(N_t, X_t)\}$ if $N_t = B$,

$\{(N_t, Y_t)\}$ if $N_t = C_s$, where Y_t denotes the remaining repair time of failed **p**-unit under progressive R-repair at time t.

$\{(N_t,\ X_t,\ Y_t)\}$ if $N_t = C$.

$\{N_t\}$ if $N_t = D$, the absorbing state.

The state space of the underlying Markov process is given by

$\{(A, x)\} \bigcup \{(B, x)\} \bigcup \{(C_s, y)\} \bigcup \{(C, x, y)\} \bigcup \{D\}, x \geqslant 0,\ y \geqslant 0.$

For $K = A, B, C, C_s, D$ let $p_K(t) := Pr\{N_t = K\},\ t \geqslant 0$, where $\sum_K p_K(t) = 1$.

Finally, we introduce the measures

$$\begin{aligned}
p_A(t,\ x)dx &:= Pr\{N_t = A,\ x - \Delta x < X_t \leqslant x\},\\
p_B(t,\ x)dx &:= Pr\{N_t = B,\ x - \Delta x < X_t \leqslant x\},\\
p_{C_s}(t,\ y)dy &:= Pr\{N_t = C_s,\ y - \Delta y < Y_t \leqslant y\},\\
p_C(t,\ x,\ y)dxdy &:= Pr\{N_t = C,\ x - \Delta x < X_t \leqslant x, y - \Delta y < Y_t \leqslant y\}.
\end{aligned}$$

Note that, for instance,

$$p_C(t) = \int_0^\infty \int_0^\infty p_C(t, x, y)dxdy.$$

The indicator (function) of an event $\{N_t = K\} \in \mathcal{A}$ is denoted by $\mathbf{1}\{N_t = K\}$. The complex plane and the real line are respectively denoted by $\mathbf{C}$ and $\mathbf{R}$ with obvious superscript notations such as $\mathbf{C}^+$ and $\mathbf{C}^-$. For instance,

$$\mathbf{C}^+ := \{\omega \in \mathbf{C} : \operatorname{Im} \omega > 0\}.$$

The Laplace transform of any locally integrable and bounded function on $[0, \infty)$ is denoted by the corresponding character marked with an asterisk. For instance,

$$p^*(z) := \int_0^\infty e^{-zt} p(t)dt,\ \operatorname{Re} z > 0.$$

Moreover, if $p(t)$ is of bounded variation on $[0, \infty)$, the product rule for Lebesgue–Stieltjes integrals, e.g. [3, Appendix], entails that

$$zp^*(z) := \int_{0-}^\infty e^{-zt} dp(t),\ \operatorname{Re} z > 0.$$

Let $\alpha(\tau),\ \tau \in \mathbf{R}$ be a bounded and continuous function. $\alpha(\cdot)$ is called Γ–integrable if

$$\lim_{\substack{T \to \infty \\ \varepsilon \downarrow 0}} \int_{\Gamma_{T,\varepsilon}} \alpha(\tau)\frac{d\tau}{\tau - u},\ u \in \mathbf{R}$$

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exists, where $\Gamma_{T,\varepsilon} := (-T, u-\varepsilon] \bigcup [u+\varepsilon, T)$. The corresponding integral, denoted by

$$\frac{1}{2\pi i} \int_{\Gamma} \alpha(\tau) \frac{d\tau}{\tau - u}$$

is called a Cauchy principal value in double sense. A function $\alpha(\tau)$, $\tau \in \mathbf{R}$ is Lipschitz-continuous (**L**-continuous) on **R** if $\forall \tau_1, \tau_2 \in \mathbf{R}$ there exists a constant c such that

$$|\alpha(\tau_2) - \alpha(\tau_1)| \leqslant c|\tau_2 - \tau_1|.$$

The function $\alpha(\tau)$, $\tau \in \mathbf{R}$ is called **L**-continuous at infinity if

$$|\alpha(\tau)| = O\left(\frac{1}{|\tau|}\right), \ |\tau| \to \infty.$$

Note that the **L**-continuity of $\alpha(\cdot)$ on **R** and at infinity is sufficient for the existence of the Cauchy-type integral

$$\frac{1}{2\pi i} \int_{\Gamma} \alpha(\tau) \frac{d\tau}{\tau - \omega}, \ \omega \in \mathbf{C}.$$

4 Differential equations

In order to derive a set of differential equations, we observe the behaviour of the **T**-system in some time interval $[t, t+\varDelta]$, $\varDelta \downarrow 0$. Applying a general birth and death technique, e.g. [22] and taking the absorbing state D into account, yields the balance equations

$$\begin{aligned}
p_A(t+\varDelta, x-\varDelta) &= p_A(t,x) + \mu p_B(t,x)\varDelta + p_C(t,x,0)\varDelta + o(\varDelta),\\
p_B(t+\varDelta, x-\varDelta) &= p_B(t,x)(1-\mu\varDelta) + (p_A(t,0) + p_{C_s}(t,0))\frac{dF}{dx}\varDelta + o(\varDelta),\\
p_C(t+\varDelta, x-\varDelta, y-\varDelta) &= p_C(t,x,y) + \mu p_{C_s}(t,y)\frac{dF}{dx}\varDelta + o(\varDelta),\\
p_{C_s}(t+\varDelta, y-\varDelta) &= p_{C_s}(t,y)(1-(\mu+\lambda_s)\varDelta) + p_B(t,0)\frac{dR}{dy}\varDelta +\\
&\quad p_C(t,0,y)\varDelta + o(\varDelta),\\
p_D(t+\varDelta) &= p_D(t) + \lambda_s p_{C_s}(t)\varDelta + o(\varDelta),
\end{aligned}$$

where the notation $o(\varDelta)$, $\varDelta \downarrow 0$ stands for any function $\mathcal{K}(\cdot)$ such that

$$\lim_{\varDelta \downarrow 0} \frac{\mathcal{K}(\varDelta)}{\varDelta} = 0.$$

Taking the definition of *directional* derivative into account, for instance,

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) p_C(t,x,y) := \lim_{\varDelta \downarrow 0} \frac{p_C(t+\varDelta, x-\varDelta, y-\varDelta) - p_C(t,x,y)}{\varDelta},$$

entails that for $t > 0$, $x > 0$, $y > 0$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) p_A(t,\, x) = \mu p_B(t, x) + p_C(t, x, 0), \tag{4.1}$$

$$\left(\mu + \frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) p_B(t,\, x) = [p_A(t,\, 0) + p_{C_s}(t, 0)]\frac{dF}{dx}, \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right) p_C(t,\, x,\, y) = \mu p_{C_s}(t, y)\frac{dF}{dx}, \tag{4.3}$$

$$\left(\lambda_s + \mu + \frac{\partial}{\partial t} - \frac{\partial}{\partial y}\right) p_{C_s}(t,\, y) = p_C(t,\, 0,\, y) + p_B(t, 0)\frac{dR}{dy}, \tag{4.4}$$

$$\frac{d}{dt} p_D(t) = \lambda_s p_{C_s}(t). \tag{4.5}$$

Note that the initial condition $N_0 = A$, $X_0 = f$ with probability one, entails that $p_A(0, x) = dF/dx$.

Moreover, $Pr\{\theta \leqslant t\} = p_D(t)$. Finally, observe that the equations (4.1)–(4.5) are consistent with the probability law $\sum_K p_K(t) = 1$ and that $p_A(0) = 1$.

5 Functional equation

First, we remark that our set of differential equations is well adapted to a transformation by means of Laplace–Fourier transforms of the underlying transition functions. As a matter of fact, the transition functions are bounded on their appropriate regions and locally integrable with respect to t. Consequently, each Laplace transform exists for $\operatorname{Re} z > 0$. Moreover, the obvious integrability of the density functions and the transition functions with regard to x, y also implies the integrability of the corresponding partial derivatives. Applying a Laplace–Fourier transform technique to the equations and taking the initial condition into account yields the equations

$$(z + i\omega)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t = A\})dt + p_A^*(z, 0) =$$

$$\mu\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t = B\})dt + \int_0^\infty e^{i\omega x} p_C^*(z, x, 0)dx + \mathbf{E}e^{i\omega f}, \tag{5.1}$$

$$(z + \mu + i\omega)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t = B\})dt + p_B^*(z, 0) =$$

$$(p_A^*(z, 0) + p_{C_s}^*(z, 0))\mathbf{E}e^{i\omega f}, \tag{5.2}$$

$$(z + i\omega + i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}e^{i\eta Y_t}\mathbf{1}\{N_t = C\})dt + \int_0^\infty e^{i\omega x} p_C^*(z, x, 0)dx+$$

$$\int_0^\infty e^{i\eta y} p_C^*(z,0,y)dy = \mu\, \mathbf{E} e^{i\omega f} \int_0^\infty e^{-zt} \mathbf{E}(e^{i\eta Y_t} \mathbf{1}\{N_t = C_s\})dt, \tag{5.3}$$

$$(z+\lambda_s+\mu+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}\mathbf{1}\{N_t = C_s\})dt + p_{C_s}^*(z,0) =$$

$$\int_0^\infty e^{i\eta y} p_C^*(z,0,y)dy + p_B^*(z,0)\mathbf{E}e^{i\eta r}, \tag{5.4}$$

$$z p_D^*(z) = \mathbf{E}e^{-z\theta} = \lambda_s p_{C_s}^*(z). \tag{5.5}$$

Substituting $\omega = i(\mu + z)$ into equation (5.2), yields

$$p_B^*(z,0) = p^*(z)\mathbf{E}e^{-(z+\mu)f},$$

where

$$p^*(z) := p_A^*(z,\, 0) + p_{C_s}^*(z,0).$$

Hence,

$$p_B^*(z) = p^*(z)\frac{1-\mathbf{E}e^{-(z+\mu)f}}{z+\mu}. \tag{5.6}$$

Adding equations (5.1)–(5.4) yields the functional equation

$$(z+i\omega)\left[\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t = A\})dt + \int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}\mathbf{1}\{N_t = B\})dt\right]$$

$$+p_B^*(z,0)(1-\mathbf{E}e^{i\eta r}) + (p_A^*(z,0)+p_{C_s}^*(z,\,0))(1-\mathbf{E}e^{i\omega f})+$$

$$(z+i\omega+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\omega X_t}e^{i\eta Y_t}\mathbf{1}\{N_t = C\})dt+$$

$$(\lambda_s+\mu(1-\mathbf{E}e^{i\omega f})+z+i\eta)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\eta Y_t}\mathbf{1}\{N_t = C_s\})dt = \mathbf{E}e^{i\omega f}, \tag{5.7}$$

valid for $\operatorname{Re} z > 0,\ \operatorname{Im}\omega \geqslant 0,\ \operatorname{Im}\eta \geqslant 0$.

6 Determination of $\mathbf{E}e^{-z\theta}$

Substituting $\omega = iz, \eta = 0$ into the function equation (5.7), yields the basic relation

$$p^*(z)(1-\mathbf{E}e^{-zf}) + (z+\lambda_s+\mu(1-\mathbf{E}e^{-zf}))p_{C_s}^*(z) = \mathbf{E}e^{-zf}. \tag{6.1}$$

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Hence, by equation (5.5),

$$\mathbf{E}e^{-z\theta} = \lambda_s \frac{\mathbf{E}e^{-zf} - p^*(z)(1 - \mathbf{E}e^{-zf})}{z + \lambda_s + \mu(1 - \mathbf{E}e^{-zf})}. \tag{6.2}$$

Note that the Laplace transforms of the survival function is uniquely determined by the relation

$$\mathfrak{R}^*(z) = \int_0^\infty e^{-zt} Pr\{\theta > t\}\, dt = \frac{1 - \mathbf{E}e^{-z\theta}}{z}. \tag{6.3}$$

Some algebra reveals that

$$\frac{1 - \mathbf{E}e^{-z\theta}}{z} = \frac{1 + \frac{1 - \mathbf{E}e^{-zf}}{z}[\mu + \lambda_s(1 + p^*(z))]}{z + \lambda_s + \mu(1 - \mathbf{E}e^{-zf})}. \tag{6.4}$$

7 A Sokhotski–Plemelj problem

In order to determine $p^*(z)$, we first transform the functional equation (5.7) into a preliminary equation by substituting $\omega = -\tau + iz$, $\eta = \tau$, $\operatorname{Re} z = \varepsilon > 0$, $\tau \in \mathbf{R}$.

Some algebra entails that

$$\mathfrak{p}_z^-(\tau)\int_0^\infty e^{-zt}\mathbf{E}(e^{i\tau Y_t}\mathbf{1}\{N_t = C_s\})dt - \Big\{ i\tau\Big[\int_0^\infty e^{-zt}\mathbf{E}(e^{-i(\tau - iz)X_t}\mathbf{1}\{N_t = A\})dt +$$

$$\int_0^\infty e^{-zt}\mathbf{E}(e^{-i(\tau - iz)X_t}\mathbf{1}\{N_t = B\})dt\Big] - p^*(z)(1 - \mathbf{E}e^{-i(\tau - iz)f} + \mathbf{E}e^{-(z+\mu)f}) + \mathbf{E}e^{-i(\tau - iz)f}\Big\}$$

$$= p^*(z)\mathbf{E}e^{i\tau r}\mathbf{E}e^{-(z+\mu)f}, \tag{7.1}$$

where

$$\mathfrak{p}_z^-(\omega) := i\omega + z + \lambda_s + \mu(\mathbf{1} - \mathbf{E}e^{-i(\omega - iz)f}),\ \operatorname{Im}\omega \leqslant 0. \tag{7.2}$$

Next, we need the following property.

Lemma 7.1 *The function* $\mathfrak{p}_z^-(\omega)$, *$Im\,\omega \leqslant 0$ is zero-free in* $\mathbf{C}^- \bigcup \mathbf{R}$.

Proof Clearly, $\mathfrak{p}_z^-(\omega) = (i\omega + z + \lambda_s + \mu)(1 - \varepsilon_z^-(\omega))$, where

$$\varepsilon_z^-(\omega) := \frac{\mu}{i\omega + z + \lambda_s + \mu}\mathbf{E}e^{-i(\omega - iz)f}.$$

Consider the region D^- with boundary $[-T, T] \bigcup C_T$ where C_T denotes the semi-circle with radius T as depicted in Figure 7. Note that for all $\omega \in \mathbf{C}^- \bigcup \mathbf{R}$,

$$|\varepsilon_z^-(\omega)| < 1.$$

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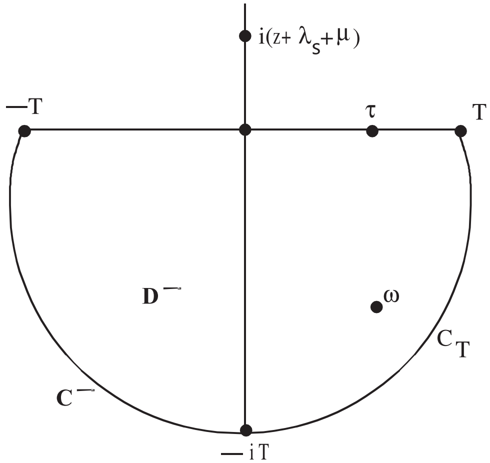

FIGURE 7. Region D^- with boundary $[-T, T]\bigcup C_T$.

Applying Rouchés theorem, e.g. [14], to the functions 1 and $\varepsilon_z^-(\omega)$, being analytic in $\mathbf{C}^-$, shows that $\mathfrak{p}_z^-(\omega)$ has no zeros inside D^-. However, since T is arbitrary large and since

$$1-\epsilon_z^-(\omega)\to 1,\ |\omega|\to\infty, -\pi\leqslant \arg\omega\leqslant 0,$$

we may conclude that $\mathfrak{p}_z^-(\omega)$ is zero-free in $\mathbf{C}^-$. In addition, we have for $\tau\in\mathbf{R}$,

$$|\mathfrak{p}_z^-(\tau)|\geqslant |i\tau+z+\lambda_s+\mu||1-|\varepsilon_z^-(\tau)||>0.$$

Hence, $\mathfrak{p}_z^-(\omega)$ has no zeros in $\mathbf{C}^-\bigcup\mathbf{R}$. □

Remark 7.1 *Lemma 7.1 allows to transform equation (7.1) into a relevant boundary value equation on the real line. Dividing equation (7.1) by the factor $\mathfrak{p}_z^-(\tau)$ entails that*

$$\psi_z^+(\tau)-\psi_z^-(\tau)=p^*(z)\boldsymbol{E}e^{-(z+\mu)f}\varphi_z(\tau),\ \tau\in\boldsymbol{R} \tag{7.3}$$

where

$$\psi_z^+(\omega):=\int_0^\infty e^{-zt}\boldsymbol{E}(e^{i\omega Y_t}\boldsymbol{1}\{N_t=C_s\})dt,\ Im\,\omega\geqslant 0. \tag{7.4}$$

$$\psi_z^-(\omega):=\Bigg\{i\omega\Bigg[\int_0^\infty e^{-zt}\boldsymbol{E}(e^{-i(\omega-iz)X_t}\boldsymbol{1}\{N_t=A\})dt+$$

$$\int_0^\infty e^{-zt}\boldsymbol{E}(e^{-i(\omega-iz)X_t}\boldsymbol{1}\{N_t=B\})dt\Bigg]-$$

$$p^*(z)(1-\boldsymbol{E}(e^{-i(\omega-iz)f}+\boldsymbol{E}(e^{-(z+\mu)f})+$$

$$\boldsymbol{E}\, e^{-i(\omega - iz)f}\} / \mathfrak{p}_z^-(\omega), \;\; Im\,\omega \leqslant 0.$$

$$\varphi_z(\tau) := \frac{\boldsymbol{E} e^{i\tau r}}{\mathfrak{p}_z^-(\tau)}, \;\; \tau \in R.$$

*Equation (7.3) constitutes a z-dependent Sokhotski–Plemelj problem on **R**, solvable by the theory of sectionally holomorphic functions, e.g. [7, Ch. 1, pp. 1–44] combined with the notion of dual transforms, [21]. For direct reference, we state the following particular definition:*

Definition 7.1 *A bounded function $\mathfrak{L}(\omega)$ is called sectionally holomorphic in **C** cut along the real line **R** if it satisfies the following properties:*

- *$\mathfrak{L}(\omega)$ is analytic in $\boldsymbol{C}^+$ and $\boldsymbol{C}^-$,*
- *$\mathfrak{L}(\omega)$ has distinct boundary values $\mathfrak{L}^+(u)$ and $\mathfrak{L}^-(u)$, $u \in \boldsymbol{R}$,*

where

$$\mathfrak{L}^+(u) := \lim_{\substack{\omega \to u \\ \omega \in \boldsymbol{C}^+}} \mathfrak{L}(\omega),$$

$$\mathfrak{L}^-(u) := \lim_{\substack{\omega \to u \\ \omega \in \boldsymbol{C}^-}} \mathfrak{L}(\omega).$$

The function $\mathfrak{L}(\omega)$ is called regular if $\mathfrak{L}(\infty) = 0$. We will show that the function

$$\frac{1}{2\pi i} \int_\Gamma \varphi_z(\tau) \frac{d\tau}{\tau - \omega}$$

*is sectionally holomorphic in **C** cut along **R** and present the solution of the Sokhotski–Plemelj problem generated by equation (7.3). First, we need the following property.*

Property 7.1 *The function $\varphi_z(\tau)$ is **L**-continuous on **R** and at infinity.*

Proof Clearly, $\left|\frac{d}{d\tau}\mathbf{E}e^{i\tau r}\right| \leqslant \mathbf{E}r$, where $\mathbf{E}r$ denotes the mean of r, i.e. $\mathbf{E}r = \int_0^\infty t dR(t)$. Hence, by the mean value theorem for derivatives, e.g. [1, p. 110], there exists a constant $c = \mathbf{E}r$ such that for all $\tau_1, \tau_2 \in \mathbf{R}$

$$|\mathbf{E}e^{i\tau_1 r} - \mathbf{E}e^{i\tau_2 r}| \leqslant c|\tau_1 - \tau_2|.$$

Thus, $\mathbf{E}e^{i\tau r}$ is **L**-continuous on **R**. In a similar way we can show that $1/\mathfrak{p}_z^-(\tau)$ is also **L**-continuous on **R**. Hence, the function $\varphi_z(\tau)$, being a product of bounded **L**-continuous functions, is **L**-continuous on **R**. Finally, note that

$$|\mathfrak{p}_z^-(\tau)|^{-1} = O(|\tau|^{-1}) \text{ if } |\tau| \to \infty.$$

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Hence, the function $\varphi_z(\tau)$ is also **L**-continuous at infinity. □

Corollary 7.1 *The boundary value equation (7.3) has a unique solution*

$$p^*(z)\boldsymbol{E}e^{-(z+\mu)f}\,\frac{1}{2\pi i}\int_\Gamma \varphi_z(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\boldsymbol{C}.$$

The corresponding Cauchy-type integral generates a regular sectionally holomorphic function in $\boldsymbol{C}$ cut along the real line. The proof follows from Property 7.1 and the straightforward mathematical tools compiled in the Appendix.

8 Determination of $p^*(z)$

By Corollary 7.1, we have

$$\psi_z^+(\omega) = p^*(z)\mathbf{E}e^{-(z+\mu)f}\,\frac{1}{2\pi i}\int_\Gamma \varphi_z(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}^+. \tag{8.1}$$

Note that equation (8.1) is only valid for $\operatorname{Im}\omega > 0$. However, equation (7.4) and the continuity of $\psi_z^+(\omega)$ at $\omega = 0$ entails that

$$p^*_{C_s}(z) = p^*(z)\mathbf{E}e^{-(z+\mu)f}\,\gamma(z), \tag{8.2}$$

where

$$\gamma(z) := \lim_{\substack{\omega\to 0\\ \omega\in\mathbf{C}^+}} \frac{1}{2\pi i}\int_\Gamma \varphi_z(\tau)\frac{d\tau}{\tau-\omega}.$$

Note that, see Appendix, equation (A.1)

$$\gamma(z) = \frac{1}{2}\varphi_z(0) + \frac{1}{2\pi i}\int_\Gamma \varphi_z(\tau)\frac{d\tau}{\tau}.$$

From equations (6.1) and (8.2), we finally obtain

$$p^*(z) = \frac{\mathbf{E}e^{-zf}}{1-\mathbf{E}e^{-zf} + (z+\lambda_s+\mu(1-\mathbf{E}e^{-zf}))\mathbf{E}e^{-(z+\mu)f}\gamma(z)}. \tag{8.3}$$

Remark 8.1 *It should be noted that the kernel $\varphi_z(\cdot)$ preserves all the relevant properties to ensure the existence of the Cauchy-type integral*

$$\frac{1}{2\pi i}\int_\Gamma \varphi_z(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\boldsymbol{C}$$

for an arbitrary repair time distribution. In fact, the generality of Lemma 7.1 ensures that the order relation

$$|\varphi_z(\tau)| = O(|\tau|^{-1}),\ |\tau|\to\infty$$

also holds for any R with finite mean $\boldsymbol{E}r$. The requirements of finite moments is extremely

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mild. As a matter of fact, the current probability distributions employed to model repair times, e.g. [2] even have moments of any order. Consequently, our initial assumption concerning the existence of density functions is totally superfluous to ensure the existence of $p^*(\cdot)$.

9 Application example: Coxian distribution

Next, we derive an algorithm to compute $p^*(z)$. Let

$$\mathbf{E}e^{i\omega r} = A_m(\omega)/B_n(\omega),\ 0 \leqslant m < n,\ \operatorname{Im}\omega \geqslant 0,$$

where $A_m(\omega)$, $B_n(\omega)$ are polynomials of degree m, n. Cox[4] has shown that this exclusive family of probability distributions is surprisingly large. In addition, a particular family of discrete Coxian distributions, e.g. [16], is quite useful to model the reliability of engineering systems. Clearly,

$$\frac{1}{2\pi i}\int_\Gamma \varphi_z(\tau)\frac{d\tau}{\tau-\zeta} = \frac{1}{2\pi i}\int_\Gamma K_z(\tau)\frac{d\tau}{\tau-\zeta},\ \zeta \in \mathbf{C}^+,$$

where

$$K_z(\tau) := \frac{A_m(\tau)}{B_n(\tau)}\frac{1}{\mathfrak{p}_z^-(\tau)},\ \ \tau \in \mathbf{R}.$$

The polynomial equation $B_n(\omega) = 0$, has n roots ω_j; $j = 1, 2, \ldots, n$ (counted according to multiplicity) located in $\mathbf{C}^-$. Consequently, the function $K_z(\tau)$ can be extended to a meromorphic function in $\mathbf{C}$. Whence by continuity,

$$\lim_{\substack{\zeta\to 0\\ \zeta\in\mathbf{C}^+}} \frac{1}{2\pi i}\int_\Gamma K_z(\tau)\frac{d\tau}{\tau-\zeta} = \frac{1}{2\pi i}\int_\mathcal{L} K_z(\tau)\frac{d\tau}{\tau} = \gamma(z),$$

where $\mathcal{L}$ denotes the real line deformed at the origin of $\mathbf{C}$ by an open semi-disk lying entirely in $\mathbf{C}^-$ with radius smaller than $\min\{|\omega_j|,\ j = 1, \ldots, n\}$. Evaluating

$$\frac{1}{2\pi i}\int_\mathcal{L} K_z(\tau)\frac{d\tau}{\tau}$$

by means of the Cauchy residue theorem, e.g. [1, p. 460], reveals that

$$\gamma(z) = -\sum_{j=1}^{n} \operatorname{Res}_{\omega=\omega_j}\left\{K_z(\omega)\omega^{-1}\right\}, \tag{9.1}$$

where the minus sign is due to clockwise integration along a closed contour surrounding the poles ω_j; $j = 1, 2, \ldots, n$. As an application, we consider the so-called hyper-exponential distribution with negative weight:

$$R(t) := p_1(1 - e^{-\lambda_1 t}) + p_2(1 - e^{-\lambda_2 t}),$$

where $p_1 > 0$, $p_1 + p_2 = 1$, $\lambda_1 p_1 + \lambda_2 p_2 = 0$ where without loss of generality $0 < \lambda_1 < \lambda_2$. Observe that we allow p_2 to be negative. However, since $R(\cdot)$ is supposed to be a probability

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distribution, we must have $p_1 > 0$. Note that $(1 - R(\cdot))^{-1}$ is log-convex. Hence, r has an increasing repair rate. Moreover, $R(\cdot)$ is uni-model, i.e. $R(\cdot)$ has a single maximum

$$t_{\max} = (\log \lambda_2 - \log \lambda_1)/(\lambda_2 - \lambda_1).$$

Lastly, $R'(0) = 0$ and $R'(\cdot)$ is strongly decreasing in a neighbourhood of infinity. Hence, $R(\cdot)$ belongs to an important family of Coxian distributions with tractable engineering properties. For instance, $R(\cdot)$ is suitable to model repair times. Note that

$$\mathbf{E}e^{i\tau r} = \frac{-\lambda_1\lambda_2}{(\tau + i\lambda_1)(\tau + i\lambda_2)}.$$

Hence, by equation (9.1),

$$\gamma(z) = \frac{1}{\lambda_2 - \lambda_1}(\alpha(z) - \beta(z)), \tag{9.2}$$

where

$$\alpha(z) := \frac{\lambda_1}{z + \lambda_1 + \lambda_s + \mu(1 - \mathbf{E}e^{-(\lambda_1+z)f})}; \; \beta(z) := \frac{\lambda_2}{z + \lambda_2 + \lambda_s + \mu(1 - \mathbf{E}e^{-(\lambda_2+z)f})}.$$

Finally, as an application, we consider the Erlang distribution

$$F(t) = 1 - e^{-\lambda t}\sum_{k=0}^{M-1}\frac{(\lambda t)^k}{k!}, \; M \geqslant 1$$

to model the failure process of the **p**-unit. Note that $F(\cdot)$ has an increasing failure rate, Birolini [2, p. 423], and that $\mathbf{E}e^{-zf} = (\lambda/(\lambda+z))^M$. As a numerical example, we consider the case $\lambda = 1$, $\lambda_s = 2$, $\mu = 3$, $\lambda_1 = 1, \lambda_2 = 2$, $M = 2$. Note that the condition $p_1\lambda_1 + p_2\lambda_2 = 0$, $p_1 + p_2 = 1$ implies $p_1 = 2$, $p_2 = -1$.

By equations (6.4), (8.3) and (9.2), we obtain $\mathfrak{R}^*(z) = N(z)/D(z)$, where

$$\begin{aligned} N(z) :&= 43521 + 135999z + 182008z^2 + 137230z^3 + 64379z^4 + 19529z^5 + 3838z^6 + 472z^7 \\ &\quad + 33z^8 + z^9, \\ D(z) :&= 582 + 44541z + 136705z^2 + 182242z^3 + 137266z^4 + 64381z^5 + 19529z^6 + 3838z^7 \\ &\quad + 472z^8 + 33z^9 + z^{10}. \end{aligned}$$

The equation $D(z) = 0$ has the following roots:

$$\begin{aligned} &(-6.97696, -5.26265, -4.22192 - 0.990541i, -4.22192 + 0.990541i, -4.15178, \\ &\; -2.73415, -2.14942, -2., -1.26759, -0.0136262). \end{aligned}$$

Clearly, $\mathfrak{R}(t)$ is continuous on $(0, \infty)$ and of bounded variation on $[0, \infty)$. Note that $\mathfrak{R}(0) = 1$. Hence, by the inversion theorem

$$\mathfrak{R}(t) = \lim_{T\to\infty}\int_{-iT}^{iT} e^{zt}\frac{N(z)}{D(z)}dz, \; t > 0.$$

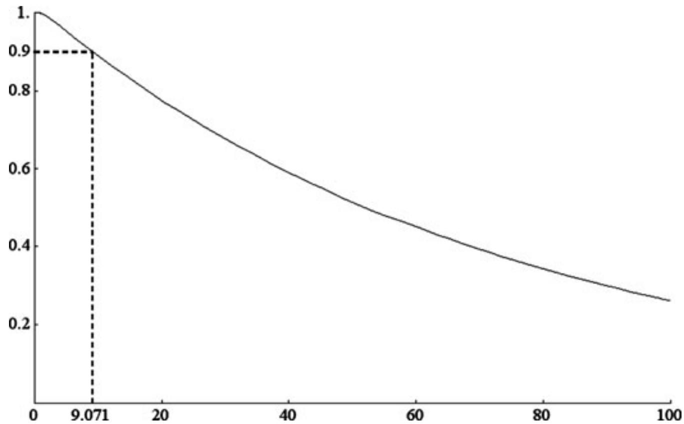

FIGURE 8. Graph of $\mathfrak{R}(t)$ with the security interval $[0, \sigma]$, $\sigma = 9.071$, corresponding to the security level $\delta = 0.9$.

An application of the residue theorem for Laplace transforms, e.g. [1, p. 438, Theorem 16.39], reveals that

$$\begin{aligned}\mathfrak{R}(t) = & -0.000510512e^{-6.97696t} + 0.0180696e^{-5.26265t} + 0.000350361e^{-4.15178t} \\ & -0.0248004e^{-2.73415t} - 0.0758886e^{-2.14942t} + 0.0909091e^{-2t} - 0.0345735e^{-1.26759t} \\ & +1.0192e^{-0.0136262t} + 0.00724082e^{-4.22192t}\cos 0.990541t \\ & +0.0449386e^{-4.22192t}\sin 0.990541t.\end{aligned}$$

Figure 8 displays the graph of $\mathfrak{R}(t)$, $0 \leqslant t \leqslant 100$, together with the security interval $[0, \sigma]$, $\sigma = 9.071$, corresponding to the security level $\delta = 0.9$. The interval ensures the survival of the **T**-system up to time σ with probability 90%.

10 Conclusions

The Laplace transform of the survival function related to the **T**-system can be derived by solving a set of coupled partial differential equations corresponding to a stochastic process with an absorbing barrier. The important case of Coxian distributions shows how to obtain computational results for the survival function by a numerical analysis based on the inversion formula for Laplace transforms. Therefore, we may conclude that our proposed differential equations approach provides a tangible contribution to statistical reliability engineering and its ramifications.

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Appendix A

For direct reference, we propose to state some particular properties of sectionally holomorphic functions and their ramifications for the solution of some boundary value problems on the real line. See Gakhov [7, pp. 1–360], Lu [12, pp. 1–73], Roos [15, pp. 118–242] for proofs and details. Let $\varphi(\tau)$ be a function satisfying the Hölder (Lipschitz) condition on $\mathbf{R}$ and at infinity. In addition, let

$$\mathcal{L}^{+}(u) := \lim_{\substack{\omega\to u\\ \omega\in\mathbf{C}^{+}}} \frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-\omega},\ u\in\mathbf{R},$$

$$\mathcal{L}^{-}(u) := \lim_{\substack{\omega\to u\\ \omega\in\mathbf{C}^{-}}} \frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-\omega},\ u\in\mathbf{R}.$$

We have

$$\mathcal{L}^{+}(u) = \frac{1}{2}\varphi(u) + \frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-u}. \tag{A 1}$$

$$\mathcal{L}^{-}(u) = -\frac{1}{2}\varphi(u) + \frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-u}. \tag{A 2}$$

Hence, for $u\in\mathbf{R}$

$$\mathcal{L}^{+}(u) - \mathcal{L}^{-}(u) = \varphi(u), \tag{A 3}$$

$$\frac{\mathcal{L}^{+}(u) + \mathcal{L}^{-}(u)}{2} = \frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-u}. \tag{A 4}$$

The relations (A.1)–(A.4) are called the Sokhotski–Plemelj formulas on the real line. The functions $\mathcal{L}^{+}(u)$, $\mathcal{L}^{-}(u)$ are continuous on $\mathbf{R}$ and infinity. The function $\varphi(\tau)$ has a unique decomposition and the resulting boundary value equation (A.3) has a unique regular solution

$$\frac{1}{2\pi i}\int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-\omega},$$

valid for all $\omega\in\mathbf{C}$ and the Cauchy-type integral generates a regular sectionally holomorphic function in $\mathbf{C}$ cut along the real line. Furthermore,

$$\mathcal{L}^{+}(\omega) = \int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}^{+},$$

$$\mathcal{L}^{-}(\omega) = \int_{\Gamma}\varphi(\tau)\frac{d\tau}{\tau-\omega},\ \omega\in\mathbf{C}^{-}.$$