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# SCALE-FREE PERCOLATION IN CONTINUOUS SPACE: QUENCHED DEGREE AND CLUSTERING COEFFICIENT

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### Abstract

Spatial random graphs capture several important properties of real-world networks. We prove quenched results for the continuous-space version of scale-free percolation introduced in [14]. This is an undirected inhomogeneous random graph whose vertices are given by a Poisson point process in $\mathbb{R}^d$. Each vertex is equipped with a random weight, and the probability that two vertices are connected by an edge depends on their weights and on their distance. Under suitable conditions on the parameters of the model, we show that, for almost all realizations of the point process, the degree distributions of all the nodes of the graph follow a power law with the same tail at infinity. We also show that the averaged clustering coefficient of the graph is *self-averaging*. In particular, it is almost surely equal to the annealed clustering coefficient of one point, which is a strictly positive quantity.

*keywords:* Random graph; scale-free percolation; degree distribution; clustering coefficient; small world; Poisson point process; self-averaging

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## 1. Introduction

Random graphs are a powerful tool for modeling real-world large networks such as the internet [1], telecommunication networks [21], social networks [33], neural networks [28], transportation networks [25], financial systems [10], and many more (see e.g. [32] and [39] for overviews). The idea is to overcome the intractability of a given network, for example because of its size, by mimicking its most interesting features with a probabilistic model. In particular, three main characteristics have been highlighted in the recent literature that are often observed in real life (see e.g. [8], [22], and [23]).

- *Scale-free property.* A graph is said to exhibit the scale-free property if the degree (i.e. the number of neighbors) of its vertices follows a power law. This results in the presence of hubs, i.e. nodes with a very high degree.

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- *Small-world property.* When sampling two vertices at random, their graph distance (the minimum number of edges to cross to go from one vertex to the other) is typically of logarithmic order on the total number of nodes. If the nodes of the graph are embedded in a metric space, for example in $\mathbb{R}^d$, this translates into saying that two nodes at Euclidean distance $D$ are with high probability at graph distance $O(\log D)$.
- *High clustering coefficient.* When two vertices share a common neighbor, there is a 'high' probability that they are linked, too.

These properties are of great importance when studying, for example, the spread of information or infections (see e.g. [35], [31], [11], and [36]) or other related random processes on the network (such as first passage percolation [27], the Ising model [3], or random walks [22]). Unfortunately, classical models of random graphs do not exhibit these three properties at once. Just to give a few examples, the Erdős–Rényi model has only the small-world property, the Chung–Lu [9], Norros–Reittu [34], and preferential attachment [2] models are scale-free and small-world, but all have vanishing clustering coefficient as the number of nodes tends to infinity. As a rule of thumb, one can think that regular, nearest-neighbor lattices have high clustering but long distances, while classic random graphs have low clustering and small distances. The Watts–Strogatz network [40] was one of the first attempts to 'artificially' build a random graph that boasts high clustering and the small-world property, but it lacks the scale-free property.

Spatial random graphs, i.e. graphs whose vertices are embedded in some metric space, offer a possible solution to the shortcomings of classical models. While in the physics literature they have been quite extensively studied (see [4] for a broad review of the topic), very few models have been analyzed in a fully rigorous mathematical fashion. Among those which exhibit or are conjectured to exhibit the three properties mentioned above, we mention the hyperbolic random graph [20], the spatial preferential attachment model [23], the geometric inhomogeneous random graph [7], the age-dependent random connection model [19], and scale-free percolation.

Scale-free percolation was introduced by Deijfen, van der Hofstad, and Hooghiemstra [13], and can be considered a combination of long-range percolation (see e.g. [5] and [6]) and inhomogeneous random graphs such as the Norros–Reittu model. The vertices of the graph lie on the $\mathbb{Z}^d$ lattice and random weights $\{W_x\}_{x\in\mathbb{Z}^d}$ are assigned independently to each of them. The distribution of the weights follows a power law: we have $P(W > w) = w^{-(\tau-1)}L(w)$ for some $\tau > 1$ and an $L$ function slowly varying at infinity. Finally, any two vertices $x, y \in \mathbb{Z}^d$ are linked by an unoriented edge with probability

$$p_{x,y} = 1 - \mathrm{e}^{-\lambda W_x W_y / \|x-y\|^{\alpha}},$$

where $\lambda, \alpha > 0$ are two parameters of the model and $\|\cdot\|$ is the Euclidean distance. Deprez, Hazra, and Wüthrich [15] argued that scale-free percolation is a suitable model, for example, for the interbank network presented in [37]. Deprez and Wüthrich [14] introduced the continuous-space counterpart of scale-free percolation, where the vertices of the graph are sampled according to a homogeneous Poisson point process of intensity $\nu > 0$ in $\mathbb{R}^d$. On the one hand, this additional source of randomness makes the model more flexible and possibly more suitable for applications to real-world networks. On the other hand, technical difficulties emerge, especially when dealing with fixed configurations of the point process. The probability of linking two vertices is the same as in [13], but Deprez and Wüthrich prefer to restrict to a Pareto distribution for the weight distribution: $P(W > w) = w^{-(\tau-1)}$ for $w \geq 1$ (note that in the present paper we have chosen to stick to the notation of [13]). We point out that the

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results appearing in [14] are *annealed*, that is, obtained after integration against the underlying Poisson point process.

Both the original scale-free percolation model of [13] and the continuum scale-free percolation of [14] share the following features. If $\alpha \le d$ or $\gamma := \alpha(\tau - 1)/d \le 1$, then the nodes of the graph have almost surely infinite degree. If instead $\min\{\alpha, (\tau-1)\alpha\} > d$, then the degree of the vertices follows a power law of index $\gamma$ (hence the graph has the *scale-free* property). Now let $\lambda_c$ be the percolation threshold of the graph, that is, the value such that for $\lambda < \lambda_c$ all the connected components of the graph are finite and for $\lambda > \lambda_c$ there exists almost surely a (unique) infinite connected component. Assume $\min\{\alpha, (\tau-1)\alpha\} > d$. Then, again for both models, the following holds: for $d \ge 2$, if $\gamma \in (1, 2)$ then $\lambda_c = 0$, while if $\gamma > 2$ then $\lambda_c \in (0, \infty)$; for $d = 1$, if $\gamma \in (1, 2)$ then $\lambda_c = 0$, if $\gamma > 2$ and $\alpha \in (1, 2]$ then $\lambda_c \in (0, \infty)$, and if $\min\{\alpha, (\tau-1)\alpha\} > 2$ then $\lambda_c = \infty$. Once the percolation properties have been established, one can talk about graph distances (for the discrete-space model the results on percolation and distances of [13] have been complemented in [15]). The graph distance between two nodes $x$ and $y$ belonging to the same connected component of the graph is the minimum number of edges that one has to cross to go from $x$ to $y$. Assume again $\min\{\alpha, (\tau-1)\alpha\} > d$. Again for both models, when the vertex degrees have infinite variance (corresponding to $\gamma \in (1, 2)$), the graph distance between two points belonging to the infinite component of the graph grows like the $\log\log$ of their Euclidean distance (in this case the graph is said to exhibit the *ultra-small-world* property). When the vertex degrees have finite variance ($\gamma > 2$) and $\lambda > \lambda_c$, two cases are possible: if $\alpha \in (d, 2d)$, then the graph distance of two points at Euclidean distance $D$ grows roughly as $(\log D)^{\Delta}$ for some constant $\Delta > 0$ still not precisely known (*small-world* property); if $\alpha > 2d$ the graph distance is bounded from below by a constant times the Euclidean distance. We point out that the regime where the degrees have infinite variance ($\gamma \in (1, 2)$ in our case) is usually the relevant one for applications (see e.g. [1] and [24]).

### 1.1. Our contribution

In the present paper we aim to prove *quenched* results for scale-free percolation in continuous space, that is, statements that hold for almost every realization $\eta$ of the underlying Poisson point process. This is one of the main differences from [14]: taking the annealed measure therein allows the authors to calculate relevant quantities explicitly. Unfortunately, in most cases, annealed properties give little information about a *given* configuration $\eta$. Say, for example, that the degree distribution has an exponentially decaying tail for half of the realizations of the Poisson point process, and a heavy tail for the other half. The annealed result would still state that the degree distribution is heavy-tailed, yet half of the simulations would behave otherwise! Besides their mathematical interest, quenched results are therefore also important because they guarantee that a simulation of the graph will always exhibit a given feature. Another difference from [14] is that we assume more general distributions for the weights, not restricting to Pareto.

We start by analyzing the tail of the degree distribution. The regime of parameters that ensures infinite degree for all the vertices slightly improves those of [13] and [14]; see Theorem 2.1 and Remark 2.2. More interesting is the case of finite degrees. In [14], under the condition of weights that follow a Pareto distribution, it was possible to calculate the whole distribution of the annealed degree of a vertex just by integrating over the Poisson measure. While there is clearly no chance that this works for all the vertices of the graph for a fixed configuration $\eta$, we show that the behavior of the tail of the degree of *all* vertices is the same (hence the graph is *scale-free*). More precisely, we show (see Theorem 2.2) that for almost every $\eta$ the

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following holds: for all $x \in \eta$ there exists a slowly varying function $\ell(\cdot) = \ell(\cdot, \eta, x)$ such that the probability that $D_x$, the degree of $x$, is bigger than some $s > 0$ is equal to $\ell(s)s^{-\gamma}$, where $\gamma := \alpha(\tau - 1)/d$. Our proof follows the strategy of [13] rather than that of [14], but a major difference emerges already when calculating the expectation of the degree of a vertex given the value $w > 0$ of its weight. While in [13] it is shown by direct computation that the expectation is equal up to a constant to $\xi w^{d/\alpha}$ for an explicit $\xi > 0$, in our case there is a correction of the order $w^{d/(2\alpha)}$ that accounts for the fluctuations of the Poisson point process (see Proposition 3.3). We apply concentration inequalities in combination with the so-called Campbell theorem to achieve this result. With some further effort we prove that the result holds for all the points of $\eta$ at once.

We then move to the clustering coefficient of the graph, which was not studied in either [13] or [14]. For a given graph, the *local* clustering coefficient $\mathrm{CC}(x)$ of a node $x$ is given by $\Delta_x$, the number of triangles with a vertex in $x$ (i.e. triplets of edges of the form $(x, y), (y, z), (z, x)$), divided by $D_x(D_x - 1)/2$. This second quantity represents the number of 'possible' triangles with a vertex in $x$, also called open triangles. The *averaged* clustering coefficient of a finite graph is the average of $\mathrm{CC}(x)$ over all its vertices (there is also a notion of *global* clustering coefficient that we do not analyze here). For an infinite graph like ours, we define the averaged clustering coefficient as the limit (if it exists) for $n \to \infty$ of $\mathrm{CC}_n$, where $\mathrm{CC}_n$ is the average of the local clustering coefficients of the vertices inside the $d$-dimensional box of side-length $n$ centered at the origin. We show that $\mathrm{CC}_n$ is *self-averaging* (a similar property has been recently proved in [38] for the hyperbolic random graph when the number of nodes goes to infinity; see also [18]), so that its limit exists and is almost surely equal to the expectation of the local clustering coefficient of the origin 0 obtained under the Palm measure associated with the Poisson point process with a point added at 0 (see Theorem 2.3). We also show that this quantity is strictly positive (*high clustering coefficient*). The proof consists in dividing the box of size $n$ into mesoscopic boxes of side-length $m < n$. We then approximate the clustering coefficient of each $m$-box by a truncated clustering coefficient that does not take into account either vertices that are close to the border of the boxes or vertices touched by edges that connect different $m$-boxes. In so doing, we obtain independent truncated clustering coefficients in each mesoscopic box and we can use the law of large numbers as $n \to \infty$. In decorrelating the clustering coefficient of each $m$-box, we must control the correlation of the local clustering coefficient of vertices lying in different $m$-boxes. To do so, we use a second moment approach in combination with the Slivnyak–Mecke theorem. It is interesting to note that the positivity of the clustering coefficient does not depend on the local connectivity properties of the graph or on $\nu$; see Remark 2.4. We believe that the mesoscopic boxes approach might also be applied to the original scale-free percolation model of [13] to prove positivity of the clustering coefficient.

Positivity of the clustering coefficient was proved for a related model, the geometric inhomogeneous random graph (GIRG), in [7]; see [27] for a comparison with scale-free percolation. We briefly stress the differences from our work. First of all, the GIRG is a finite-volume model, so the clustering coefficient is always well-defined and no limiting procedure is needed. Secondly, the methods of proof used in [7] (namely, Le Cam's theorem and an Azuma-type inequality with error event) do not apply in our setting, since the problem of dealing with an infinite number of random variables cannot be overcome. Furthermore, these techniques guarantee only results that hold with high probability, and not almost surely. Finally, in addition to showing that the clustering coefficient is positive, we also show a self-averaging property and an exact formula for the clustering coefficient.

We point out that, at least under the hypothesis of Pareto weights, the results on percolation and graph distances presented in the Introduction have annealed probability equal to 1; see [14, Theorems 3.2 and 3.6]. Hence they also hold in the quenched sense (see Remark 2.1 for a comparison between our parameters and those of [14]). It follows that under the right range of parameters, scale-free percolation in continuous space fulfills, for almost every realization of the Poisson point process $\eta$, all the three properties presented in the Introduction (scale-free, small-world, and positive clustering coefficient), almost surely. This suggests that the model is a good candidate for modeling real spatial networks. We believe, for example, that it encloses the main features of the cattle trading network in France (see [16]), which was the original motivation for our work. The nodes of the graph are given by farms, assembling centers and cattle markets in the country (for a total of about 200 000 holdings), so that the hypothesis of vertices placed according to a Poisson point process reflects the geographic irregularities in a more realistic way than $\mathbb{Z}^d$. An undirected edge is placed between two holdings if there has been at least a transaction between the two during a given time window. It has been observed that the degree of the nodes on yearly aggregated data follows a power law of index $\gamma \simeq 1.3$ which is quite stable over different years ($\pm 0.1$). Note that this falls into the interesting regime of infinite variance of the degrees. Likewise, the network exhibits a high clustering coefficient and a small diameter (about 15, which is close to the logarithm of the total number of nodes). Comprehension of the topology of this network might be of paramount importance when studying the possible outbreak of an infection in the cattle population.

### 1.2. Structure of the paper

We will introduce the model and some notation in Section 2.1. We will present the main results on the degree distribution (Theorem 2.1 and Theorem 2.2) and on the clustering coefficient (Theorem 2.3) in Sections 2.2 and 2.3. We deal with the proof of the infinite-degree case in Section 3.1 and of the finite-degree case in Section 3.2. Section 4 is dedicated to the proof of the positivity of the clustering coefficient. In Appendix A we recall the Campbell and Slivnyak–Mecke theorems.

## 2. Model and main results

### 2.1. The model

Scale-free percolation in continuous space is a random variable on the space of all simple spatial undirected random graphs with vertices in $\mathbb{R}^d$. To construct an instance of the graph $G = (V, E)$ we proceed in three steps.

- We sample the nodes $V$ of the graph according to a homogeneous Poisson point process in $\mathbb{R}^d$ of intensity $\nu > 0$. We let $\mathrm{P}$ denote its law and let $\mathrm{E}$ denote the expectation with respect to $\mathrm{P}$. A configuration of the point process will be denoted by $\eta \in (\mathbb{R}^d)^{\mathbb{N}}$.
- We assign to each vertex $x \in \eta$ a random weight $W_x > 0$. The weights are independent and identically distributed (i.i.d.) with law $P$. The distribution function $F$ associated with $P$ is regularly varying at infinity with exponent $\tau - 1$, that is,

$$1 - F(w) = P(W > w) = w^{-(\tau-1)} L(w),$$

where $L(\cdot)$ is a function that is slowly varying at $+\infty$.

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- We finally draw the edges $E$ of the random graph via a percolation process. For every pair of points $x, y \in \eta$ with weights $W_x$, $W_y$, the undirected edge $(x, y)$ is present with probability
$$1 - \mathrm{e}^{-W_x W_y / \|x-y\|^{\alpha}}, \tag{1}$$
where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$ and $\alpha > 0$ is a parameter. When two vertices $x, y$ are connected by an edge we write $x \leftrightarrow y$ (if they are not connected we write $x \not\leftrightarrow y$). For the event that a vertex $x$ is connected to some other point in a region $A \subseteq \mathbb{R}^d$, we write $x \leftrightarrow A$.

**Remark 2.1.** In [14] an additional parameter $\lambda > 0$ appears in the definition of the linking probabilities: $p_{x,y} = 1 - \exp\{-\lambda W_x W_y \|x - y\|^{-\alpha}\}$. In this paper we prefer to fix $\lambda = 1$ without loss of generality. Indeed, having parameters $\lambda = \tilde{\lambda}$ and $\nu = \tilde{\nu}$ in the model presented in [14] is completely equivalent to choosing $\nu = \tilde{\nu}\tilde{\lambda}^{d/\alpha}$ in our case.

For a given realization $\eta$ of the Poisson point process we will often perform the last two steps at once under the *quenched* law $P^\eta$, with $E^\eta$ denoting the associated expectation. This is the law of the graph once we have fixed $\eta$. We write $\mathbb{P}$ for the *annealed* law of the graph, i.e. $\mathbb{P} = \mathrm{P} \times P^\eta$, and $\mathbb{E}$ for the corresponding expectation. For $x \in \mathbb{R}^d$, we let $\mathrm{P}_x$ indicate the law of the Poisson point process conditioned on having a point in $x$; analogously, for $x, y \in \mathbb{R}^d$, the measure $\mathrm{P}_{x,y}$ is obtained by conditioning on having one point in $x$ and one in $y$. We will not enter into the details of ($n$-fold) Palm measures, but we point out that in both cases the conditioning does not influence the rest of the Poisson point process; see e.g. [12, Chapter 13]. Consequently we will let $\mathbb{P}_x = \mathrm{P}_x \times P^\eta = \mathrm{P} \times P^{\eta\cup\{x\}}$ be the law of the graph conditioned on having a point at $x \in \mathbb{R}^d$ and $\mathbb{E}_x$ the associated expectation. Analogously we will write $\mathbb{P}_{x,y} = \mathrm{P}_{x,y} \times P^\eta = \mathrm{P} \times P^{\eta\cup\{x\}\cup\{y\}}$ and $\mathbb{E}_{x,y}$ for the expectation with respect to $\mathbb{P}_{x,y}$.

We let 0 denote the origin of $\mathbb{R}^d$. We let $B_n$ be the $d$-dimensional box of side-length $n > 0$ centered at 0. We let $V_n = V_n(\eta)$ be the set of points of $\eta$ that are in $B_n$ and let $N_n = N_n(\eta)$ be their number. Finally, $\mathcal{B}_r(x)$ denotes the Euclidean ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$.

### 2.2. Results on the degree

For each $x \in \eta$ we let $D_x := \#\{y \in \eta : \, y \leftrightarrow x\}$ be the degree of a vertex $x$ in the random graph, i.e. the random variable counting the number of neighbors of $x$. We start by showing that for certain values of the parameters the graph is almost surely degenerate, in the sense that all of its vertices have infinite degree.

**Theorem 2.1.** *Suppose one of the following conditions is satisfied:*

(a) $\alpha \le d$,

(b) $E[W^{d/\alpha}] = \infty$.

*Then, for* P*-almost every* $\eta$, $P^\eta$*-almost surely all points in* $\eta$ *have infinite degree.*

**Remark 2.2.** Note that condition (b) in Theorem 2.1 is implied in particular by

(b′) the weight distribution satisfies
$$1 - F(w) \ge c\, w^{-(\tau-1)}$$
for some $c > 0$ and $\tau > 1$ such that $\gamma := \alpha(\tau - 1)/d \le 1$.

This stronger condition is the one appearing in [13].

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We point out that the proof of Theorem 2.1 follows immediately from the analogous result of [14] when the weights have a Pareto distribution. We will provide an alternative proof that, besides covering more general distribution functions $F$, serves as a warm-up for the kind of techniques we will use more intensively later on, based on Campbell's theorem.

We move to the analysis of the regime where the degree of the nodes is almost surely finite. In this case, we show that for almost all $\eta$ the graph is scale-free and the degree of all points follows a power law with the same exponent.

**Theorem 2.2.** *Suppose that $\alpha > d$ and $\gamma := \alpha(\tau - 1)/d > 1$. Then, for* P*-almost every $\eta$, we have that $s^\gamma P^\eta(D_x > s)$ is slowly varying for each $x \in \eta$. That is, there exists a slowly varying function $\ell(\cdot) = \ell(\cdot, \eta, x)$ such that*

$$P^\eta(D_x > s) = s^{-\gamma}\ell(s). \tag{2}$$

**Remark 2.3.** The slowly varying function in (2) clearly has to depend on $x$ and $\eta$. However, it is not clear whether all the $\ell$ have the same asymptotic behavior. That is, with our proof it is not possible to say whether, for P-almost every $\eta$ and $\eta'$, and for all $x \in \eta$ and $x' \in \eta'$, we have $\ell(s, \eta, x)/\ell(s, \eta', x') \to 1$ as $s \to \infty$.

### 2.3. Results on the clustering coefficient

The average clustering coefficient is usually defined for finite graphs as the average of the local clustering coefficients. More precisely, for a finite undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a node $x \in \mathcal{V}$, we define the local clustering coefficient at $x$ as

$$\mathrm{CC}(x) := \frac{2\Delta_x}{D_x(D_x - 1)},$$

where $\Delta_x$ is the number of closed triangles in the graph that have $x$ as one of their vertices (a closed triangle is a triplet of edges in $\mathcal{E}$ of the type $\{(y, z), (z, w), (w, y)\}$), with the convention that $\mathrm{CC}(x)$ is 0 if $D_x$ is equal to 0 or 1. The quantity $D_x(D_x - 1)/2$ can be interpreted as the number of open triangles in $x$ (an open triangle in a vertex $z$ is a pair of edges of the kind $\{(y, z), (z, w)\}$, without any requirement for the presence of the third edge that would close the triangle), so $\mathrm{CC}(x)$ takes values in [0,1]. The averaged clustering coefficient of the graph $\mathcal{G}$ is

$$\mathrm{CC}_{\mathrm{av}}(\mathcal{G}) := \frac{1}{|\mathcal{V}|}\sum_{x\in\mathcal{V}} \mathrm{CC}(x).$$

Clearly also $\mathrm{CC}_{\mathrm{av}}(\mathcal{G})$ takes values in [0,1]. It is not completely obvious what the definition of averaged clustering coefficient should be for an infinite graph. We say that an infinite spatial graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ embedded in $\mathbb{R}^d$ with all vertices having finite degree has averaged clustering coefficient $\mathrm{CC}_{\mathrm{av}}(\mathcal{G})$ if the following limit exists:

$$\lim_{n\to\infty} \frac{1}{|\mathcal{V} \cap B_n|} \sum_{x\in\mathcal{V}\cap B_n} \mathrm{CC}(x) =: \mathrm{CC}_{\mathrm{av}}(\mathcal{G})$$

(we work with integers for simplicity). It is possible to construct infinite graphs for which this limit does not exist. We also note that this definition is in principle different from considering the limit of the clustering coefficients of the subgraphs of $\mathcal{G}$ obtained by considering only vertices in $B_n$, since $\mathrm{CC}(x)$ also takes into account the edges connecting points that lie out of $B_n$.

Going back to our model, we will let

$$\mathrm{CC}_n := \frac{1}{N_n} \sum_{x \in V_n} \mathrm{CC}(x)$$

be the random variable describing the averaged clustering coefficient inside the box of side $n$. Note that under the assumptions $\alpha > d$ and $\gamma > 1$, for $P$-almost every $\eta$, all the nodes $x \in \eta$ have finite degree, $P^\eta$-almost surely (see Theorem 2.2). Thus, the $\mathrm{CC}(x)$ are well-defined and finite, except perhaps on a set of measure 0. Our main result of this section states that the averaged clustering coefficient of the scale-free percolation in continuous space exists, is *self-averaging*, and is strictly positive.

**Theorem 2.3.** *Suppose that $\alpha > d$ and $\gamma := \alpha(\tau - 1)/d > 1$. For* P*-almost all configurations $\eta$, the average clustering coefficient exists $P^\eta$-almost surely and does not depend on $\eta$. It is given by*

$$\mathrm{CC}_{\mathrm{av}}(G) := \lim_{n\to\infty} \mathrm{CC}_n = \mathbb{E}_0[\mathrm{CC}(0)] > 0.$$

**Remark 2.4.** Going through the proof of Theorem 2.3, it is plausible that the positivity of the clustering coefficient does not depend on the local properties of the graph or on $\nu$, as one might have thought. For example, we can set to 0 the probability of connecting two points that are at distance smaller than some $R > 0$ and leave (1) for points at distance larger than $R$. Then we would still have that the associated $n$-truncated clustering coefficient $\tilde{\mathrm{CC}}_n$ converges almost surely to $\mathbb{E}_0[\tilde{\mathrm{CC}}(0)]$, with $\tilde{\mathrm{CC}}(0)$ the corresponding local clustering coefficient. At the same time, $\mathbb{E}_0[\tilde{\mathrm{CC}}(0)]$ would still be strictly positive for any $R > 0$ by arguments similar to those in the proof of Theorem 2.3.

## 3. Degree

In both the infinite- and finite-degree cases, for convenience we will first prove the properties of the degree of the vertices via an auxiliary random variable under $P^\eta$. We call it $D_0$ with a slight abuse of notation, since P-almost surely the origin 0 does not belong to $\eta$. It is convenient, though, to imagine having a further point in the origin and treating it like all the other points of the graph, so that when we talk about $D_0$ we can think of $P^\eta$ as being replaced by $P^{\eta\cup\{0\}}$. In a second moment we will transpose the properties of $D_0$ to the degree of the other vertices.

### 3.1. Infinite degree

In this section we prove Theorem 2.1. We start with the following proposition about $D_0$.

**Proposition 3.1.** *If conditions* (a) *or* (b) *in Theorem 2.1 are satisfied, then for* P*-almost all $\eta$ we have $P^\eta(D_0 = \infty) = 1$.*

*Proof of Proposition 3.1.* The statement will be proved by showing that, for each value $w > 0$ of the weight in 0 and for P-almost every $\eta$, the following sum is infinite:

$$\sum_{x\in\eta} P^\eta(0 \leftrightarrow x \mid W_0 = w) = \sum_{x\in\eta} E[1 - \mathrm{e}^{-wW\|x\|^{-\alpha}}],$$

where the expectation on the right-hand side is taken with respect to $W$. Indeed, if we condition on the value of $W_0$, the presence of each edge $(0,x)$ becomes independent from the others. We

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can therefore use the second Borel–Cantelli lemma to imply that there are infinitely many points in the configuration $\eta$ connected to 0. By Campbell's theorem (see Theorem A.1), the above sum is infinite for almost every $\eta$ if the following integral is

$$\int_{\mathbb{R}^d} E[1-\mathrm{e}^{-wW\|y\|^{-\alpha}}]\,\mathrm{d}y.$$

However, in view of the inequality $1-\mathrm{e}^{-u}\geq (u\wedge 1)/2$ for $u\geq 0$, it is enough to show the divergence of the integral

$$\int_{\mathbb{R}^d} E[wW\|y\|^{-\alpha}\wedge 1]\,\mathrm{d}y \geq \int_{\mathbb{R}^d} E[\mathbb{1}_{\{\|y\|^{\alpha}>wW\}}wW\|y\|^{-\alpha}]\,\mathrm{d}y. \tag{3}$$

By first applying Tonelli's theorem and then passing to polar coordinates, we can rewrite the right-hand side of (3) as

$$E\bigg[wW\int_{y\colon\, \|y\|>(wW)^{1/\alpha}} \|y\|^{-\alpha}\,\mathrm{d}y\bigg] = E\bigg[wW\sigma(S_{d-1})\int_{(wW)^{1/\alpha}}^{\infty} r^{-\alpha+d-1}\,\mathrm{d}r\bigg],$$

where $\sigma(S_{d-1})$ denotes the surface area of the $(d-1)$-sphere of radius 1. The last integral is divergent for $\alpha\leq d$, so case (a) is proved.

When instead $\alpha>d$, we note that the integral inside the expectation is finite and we can continue the manipulation, obtaining

$$E\bigg[wW\sigma(S_{d-1})\frac{(wW)^{(d-\alpha)/\alpha}}{\alpha-d}\bigg] = cw^{d/\alpha}E[W^{d/\alpha}]$$

for some constant $c>0$. The last expression is again infinite under the assumptions of case (b). □

*Proof of Theorem 2.1.* In view of the Proposition 3.1 we know that the set of $\eta$ for which $\sum_{x\in\eta} P^{\eta}(0\leftrightarrow x \mid W_0=w)=\infty$ has P-measure 1. Take an $\eta$ in this set and any point $y\in\eta$. We have

$$\sum_{x\in\eta\setminus\{y\}} P^{\eta}(y\leftrightarrow x\mid W_y=w) = \sum_{x\in\eta\setminus\{y\}} P^{\eta}(0\leftrightarrow x\mid W_0=w)\frac{E[1-\mathrm{e}^{-wW\|x-y\|^{-\alpha}}]}{E[1-\mathrm{e}^{-wW\|x\|^{-\alpha}}]}=\infty,$$

since the fraction goes to 1 as $\|x\|$ goes to infinity. By Borel–Cantelli this shows that $y$ also has infinite degree $P^{\eta}$-almost surely. Since there is a countable number of points in $\eta$, we are done. □

### 3.2. Polynomial degree

As in the previous section we will first prove a statement about the random variable $D_0$. The proof of Theorem 2.2 will be inferred as a consequence.

**Proposition 3.2.** *Suppose that $\alpha>d$ and $\gamma:=\alpha(\tau-1)/d>1$. Then, for* P*-almost every $\eta$, there exists a slowly varying function $\ell(\cdot)=\ell(\cdot,\eta)$ such that*

$$P^{\eta}(D_0>s)=s^{-\gamma}\ell(s). \tag{4}$$

The strategy for demonstrating Proposition 3.2 follows the lines of the proof of Theorem 2.2 in [13], which in turn follows [41]. Analogously to Proposition 2.3 of [13], we first analyze

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the properties of the expectation of the degree of a point conditional to its weight $w > 0$. One of the main problems when dealing with vertices that are randomly distributed in space is the following: while in scale-free percolation on the lattice one can show by direct computations that the expectation of the degree is equal to a constant times $w^{d/\alpha}$, in the continuous-space case one has to finely control the fluctuations of the degree due to the irregularity of the point process. We show that these fluctuations are of order smaller than $w^{d/2\alpha} \log w$.

**Proposition 3.3.** *Suppose that* $\alpha > d$ *and* $\gamma := \alpha(\tau - 1)/d > 1$. *Then, for* P*-almost every realization* $\eta$*, we have*

$$E^{\eta}[D_0 \mid W_0 = w] = \nu c_0\, w^{d/\alpha} + \mathrm{O}(w^{d/(2\alpha)} \log w), \tag{5}$$

*with* $c_0 := v_d \Gamma(1 - d/\alpha) E[W^{d/\alpha}]$. *Here* $v_d$ *indicates the volume of the d-dimensional unitary ball,* $\Gamma(\cdot)$ *is the gamma function, and the constant in* $\mathrm{O}(\cdot)$ *is uniformly bounded in* $\eta$ *from below and above.*

*Proof of Proposition 3.3.* We let $Z_w = Z_w(\eta) := E^{\eta}[D_0 \mid W_0 = w]$ and note that $Z_w$ is a random variable that depends only on the Poisson process. Since $D_0 = \sum_{x\in\eta} \mathbb{1}_{\{0\leftrightarrow x\}}$, we can use Campbell's theorem (see (27) in Theorem A.1) applied to the function $f(x) := E[1 - \mathrm{e}^{-wW/\|x\|^{\alpha}}]$, to explicitly compute

$$\mathrm{E}[Z_w] = \nu \int_{\mathbb{R}^d} E[1 - \mathrm{e}^{-wW\|x\|^{-\alpha}}]\, \mathrm{d}x = w^{d/\alpha} E[W^{d/\alpha}] \nu \int_{\mathbb{R}^d} (1 - \mathrm{e}^{-\|y\|^{-\alpha}})\, \mathrm{d}y = \nu c_0 w^{d/\alpha}. \tag{6}$$

For the second equality we used Fubini and then made the change of variables $y = x(wW)^{-1/\alpha}$. The constant $c_0$ is the one appearing in the statement of the theorem; it is obtained by passing to polar coordinates and then using integration by parts. We can instead use (28) in order to calculate the variance V:

$$\mathrm{V}(Z_w) = \nu \int_{\mathbb{R}^d} E[1 - \mathrm{e}^{-wW\|x\|^{-\alpha}}]^2\, \mathrm{d}x = \nu c_1 w^{d/\alpha}. \tag{7}$$

To see the last equality it is sufficient to make the change of variables $y = xw^{-1/\alpha}$. We also note that $c_1 \le c_0$ because the expectation inside the integral is smaller than 1, so the variance has to be smaller than the expectation.

We will show that for P-almost every $\eta$ we have

$$-1 \le \liminf_{w\to\infty} \frac{Z_w - \mathrm{E}[Z_w]}{\sqrt{\mathrm{V}(Z_w)} \log w} \le \limsup_{w\to\infty} \frac{Z_w - \mathrm{E}[Z_w]}{\sqrt{\mathrm{V}(Z_w)} \log w} \le 1,$$

which implies (5). We will only show the upper bound, the lower bound being very similar. Let $\theta > 0$. We use the exponential Chebyshev inequality to bound

$$\begin{aligned} \mathrm{P}(Z_w - \mathrm{E}[Z_w] \ge \sqrt{\mathrm{V}(Z_w)} \log w) &= \mathrm{P}(\mathrm{e}^{\theta Z_w} \ge \mathrm{e}^{\theta(\sqrt{\mathrm{V}(Z_w)} \log w + \mathrm{E}[Z_w])}) \\ &\le \exp\{-\theta(\sqrt{\mathrm{V}(Z_w)} \log w + \mathrm{E}[Z_w]) + \log \mathrm{E}[\mathrm{e}^{\theta Z_w}]\}. \end{aligned} \tag{8}$$

Campbell's theorem in its exponential form (see (26)) applied to the function $f(x) := \theta E[1 - \mathrm{e}^{-wW\|x\|^{-\alpha}}]$ gives

$$\log \mathrm{E}[\mathrm{e}^{\theta Z_w}] = \nu \int_{\mathbb{R}^d} (\exp\{\theta E[1 - \mathrm{e}^{-wW\|x\|^{-\alpha}}]\} - 1)\, \mathrm{d}x.$$

If we restrict to values of $\theta$ smaller than 1, a third-order Taylor expansion shows that

$$\begin{aligned}&\exp\{\theta E[1-\mathrm{e}^{-wW\|x\|^{-\alpha}}]\}-1\\&\quad\le \theta E[1-\mathrm{e}^{-wW\|x\|^{-\alpha}}]+\frac{\theta^2}{2}E[1-\mathrm{e}^{-wW\|x\|^{-\alpha}}]^2+\mathrm{O}(\theta^3 E[1-\mathrm{e}^{-wW\|x\|^{-\alpha}}]),\end{aligned}$$

where we used the trivial bound $E[1-\mathrm{e}^{-wW\|x\|^{-\alpha}}]\le 1$ for the last term. Integrating over $x$ in the above inequality and in view of the expressions for the expectation (6) and the variance (7) of $Z_w$, we obtain

$$\log \mathrm{E}[\mathrm{e}^{\theta Z_w}]\le \theta \mathrm{E}[Z_w]+\frac{\theta^2}{2}\mathrm{V}(Z_w)+\mathrm{O}(\theta^3\mathrm{E}[Z_w]).$$

Inserting this estimate back into (8) yields, for all $\theta\in[0,1]$,

$$\mathrm{P}(Z_w-\mathrm{E}[Z_w]\ge \sqrt{\mathrm{V}(Z_w)}\log w)\le \exp\left\{-\theta\sqrt{\mathrm{V}(Z_w)}\log w+\frac{\theta^2}{2}\mathrm{V}(Z_w)+\mathrm{O}(\theta^3\mathrm{E}[Z_w])\right\}.$$

In particular, by choosing $\theta=\log w/\sqrt{\mathrm{V}(Z_w)}$, we obtain for $w$ large enough

$$\mathrm{P}(Z_w-\mathrm{E}[Z_w]\ge \sqrt{\mathrm{V}(Z_w)}\log w)\le \mathrm{e}^{-\log^2 w/4}. \tag{9}$$

If we consider the sequence $w_n=n$, we see that the quantity on the right-hand side of (9) is summable in $n$ and Borel–Cantelli tells us that

$$\limsup_{n\to\infty}\frac{Z_n-\mathrm{E}[Z_n]}{\sqrt{\mathrm{V}(Z_n)}\log n}\le 1.$$

The random variable $Z_w$ is increasing in $w$, so that $Z_{\lfloor w\rfloor}\le Z_w\le Z_{\lfloor w\rfloor+1}$. Moreover, in view of (6) and (7), we see that the sequences $\mathrm{E}(Z_n)$, $\sqrt{\mathrm{V}(Z_n)}$ and $\log n$ all satisfy

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=1.$$

We can thus conclude that

$$\begin{aligned}\limsup_{n\to\infty}\frac{Z_w-\mathrm{E}(Z_w)}{\sqrt{\mathrm{V}(Z_w)}\log w}&\le \limsup_{n\to\infty}\frac{Z_{\lfloor w\rfloor+1}-\mathrm{E}(Z_{\lfloor w\rfloor})}{\sqrt{\mathrm{V}(Z_{\lfloor w\rfloor})}\log\lfloor w\rfloor}\\&=\limsup_{n\to\infty}\frac{Z_{\lfloor w\rfloor+1}-\mathrm{E}(Z_{\lfloor w\rfloor+1})}{\sqrt{\mathrm{V}(Z_{\lfloor w\rfloor+1})}\log(\lfloor w\rfloor+1)}\\&\le 1.\end{aligned}$$

□

*Proof of Proposition 3.2.* We let $Y_w$ be the random variable describing the value of $D_0$ conditioned on the event $\{W_0=w\}$:

$$Y_w\sim(D_0\mid W_0=w).$$

We split the integral

$$P^\eta(D_0>s)=\int_0^{m(s)}P^\eta(Y_w>s)\,\mathrm{d}F(w)+\int_{m(s)}^{\infty}P^\eta(Y_w>s)\,\mathrm{d}F(w)=:A_1(s)+A_2(s), \tag{10}$$

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where

$$m(s) := \left(\frac{s - \sqrt{s}\log^2 s}{\nu c_0}\right)^{\alpha/d}, \tag{11}$$

$c_0$ being the same constant appearing in (5). We point out that this definition is slightly different from the equivalent appearing in [13, eq. (2.11)] in order to control the fluctuations of the expected degree of 0. Since $P^\eta(D_0 > s)$ is a monotone function, (4) follows if we can prove that

$$\lim_{t\to\infty} \frac{P^\eta(D_0 > st)}{P^\eta(D_0 > t)} = s^{-\gamma} \tag{12}$$

on a dense set of points (see [17, Section VIII.8]). We claim that $A_1(s) = A_1(\eta, s)$ does not contribute to the regular variation of $P^\eta(D_0 > s)$ for P-almost all $\eta$, that is,

$$\lim_{s\to\infty} s^a A_1(s) = 0 \quad \text{for all } a > 0\text{, P-almost surely.} \tag{13}$$

We will show how to get (13) at the end of the proof. Thanks to (13), we see that in order to prove (12) it is enough to verify that, for $s \in (0, \infty)$,

$$\lim_{t\to\infty} \frac{A_2(st)}{A_2(t)} = s^{-\gamma}. \tag{14}$$

On the one hand, for all $t > 0$ we clearly have

$$A_2(t) \le 1 - F(m(t)).$$

On the other hand, for all $\varepsilon > 0$ we have

$$\begin{aligned} A_2(t) &\ge 1 - F((1+\varepsilon)m(t)) - \int_{(1+\varepsilon)m(t)}^{\infty} P^\eta(Y_w \le t)\,\mathrm{d}F(w) \\ &\ge (1 - F((1+\varepsilon)m(t))(1 + \mathrm{o}_t(1)). \end{aligned}$$

For the last line, we used the following fact. By (5), $E^\eta[Y_w] > t(1 + \varepsilon d/2\alpha)$ for all $w \ge (1+\varepsilon)m(t)$ when $t$ is sufficiently large (depending on $\eta$); this allows us to use the Chebyshev inequality and bound, uniformly in $w > (1+\varepsilon)m(t)$,

$$P^\eta(Y_w \le t) \le \frac{V^\eta(Y_w)}{(E^\eta[Y_w] - t)^2} \le \frac{E^\eta[Y_w]}{(E^\eta[Y_w] - t)^2} \le \frac{C}{\varepsilon^2 t} = \mathrm{o}_t(1),$$

where $V^\eta$ represents the variance with respect to $P^\eta$ and the second inequality follows from the fact that $Y_w$ is a sum of independent indicator functions.

Putting it all together and using the fact that $\lim_{t\to\infty} m(st)/m(t) = s^{\alpha/d}$, we obtain

$$\lim_{t\to\infty} \frac{A_2(st)}{A_2(t)} = \lim_{t\to\infty} \frac{1 - F(m(st))}{1 - F(m(t))} = s^{-\alpha(\tau-1)/d}$$

as we wanted to prove.

We now only need to show (13). We start by upper-bounding $A_1(s)$ by

$$\begin{aligned} &P^\eta(Y_{m(s)} > s) \\ &\quad = P^\eta\left(\sum_{y\in\eta} \mathbb{1}_{\{y\leftrightarrow 0\}} - E^\eta[\mathbb{1}_{\{y\leftrightarrow 0\}} \mid W_0 = m(s)] > s - E^\eta[Y_{m(s)}] \mid W_0 = m(s)\right). \end{aligned} \tag{15}$$

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Taking $s$ sufficiently large and thanks to Proposition 3.3, we can make $s - E^\eta[Y_{m(s)}]$ positive, allowing us to apply Bernstein's inequality. We obtain

$$A_1(s) \le \exp\left\{-\frac{1}{2}\frac{(s - E^\eta[Y_{m(s)}])^2}{E^\eta[Y_{m(s)}] + (s - E^\eta[Y_{m(s)}])/3}\right\},$$

where for the first term in the denominator of the exponent we used the inequality

$$E^\eta[(\mathbb{1}_{\{y\leftrightarrow 0\}} - E^\eta[\mathbb{1}_{\{y\leftrightarrow 0\}} \mid W_0 = m(s)])^2 \mid W_0 = m(s)] \le E^\eta[\mathbb{1}_{\{y\leftrightarrow 0\}} \mid W_0 = m(s)].$$

Since

$$E^\eta[Y_{m(s)}] = s - \sqrt{s}\log^2 s + \mathrm{O}(\sqrt{s}\log s),$$

we get $A_1(s) \le \mathrm{e}^{-\log^4 s/4}$ and (13) is proved. □

*Proof of Theorem 2.2.* Take any $x \in \eta$. We first prove a lower bound on $P^\eta(D_x > s)$. First of all we claim that

$$P^\eta(D_x > s \mid W_x = w) \ge P^\eta(D_0 > s + 1 \mid W_0 = c(x, \eta)\cdot w),$$

where $c(x, \eta) := (1 + \|x\|/\|y\|)^{-\alpha}$ and $y \in \eta$ is the closest point to the origin in $\eta$. In order to prove the claim we write

$$D_x = \sum_{z\in\eta,\, z\neq x} \mathbb{1}_{\{x\leftrightarrow z\}} \quad \text{and} \quad D_0 \le \sum_{z\in\eta,\, z\neq x} \mathbb{1}_{\{0\leftrightarrow z\}} + 1.$$

Since the indicator functions in the first sum are mutually independent under $P^\eta(\cdot \mid W_x = w)$ and those in the second sum are mutually independent under $P^\eta(\cdot \mid W_0 = c(x, \eta)\cdot w)$, it will be sufficient to show that, for all $z \in \eta \setminus \{x\}$,

$$P^\eta(\mathbb{1}_{\{x\leftrightarrow z\}} = 1 \mid W_x = w) \ge P^\eta(\mathbb{1}_{\{0\leftrightarrow z\}} = 1 \mid W_0 = c(x, \eta)\cdot w),$$

since this ensures that $D_x \ge D_0 - 1$ stochastically. We calculate

$$\begin{aligned} P^\eta(\mathbb{1}_{\{x\leftrightarrow z\}} = 1 \mid W_x = w) &= E[1 - \mathrm{e}^{-wW_z\|z-x\|^{-\alpha}}] \\ &\ge E[1 - \mathrm{e}^{-c(x,\eta)\cdot wW_z\|z\|^{-\alpha}}] \\ &= P^\eta(\mathbb{1}_{\{0\leftrightarrow z\}} = 1 \mid W_0 = c(x, \eta)\cdot w), \end{aligned}$$

where for the inequality we used the fact that $\|z - x\|/\|z\| \le (\|y\| + \|x\|)/\|y\|$.

Thanks to the claim we can now bound

$$\begin{aligned} P^\eta(D_x > s) &= \int_0^\infty P^\eta(D_x > s \mid W_x = w)\,\mathrm{d}F(w) \\ &\ge \int_0^\infty P^\eta(D_0 > s + 1 \mid W_0 = c(x, \eta)\cdot w)\,\mathrm{d}F(w). \end{aligned}$$

If the weights simply follow a Pareto distribution with exponent $\tau$, then with a change of variables $u = c(x, \eta)\cdot w$ we would obtain $P^\eta(D_x > s) \ge c(x, \eta)^{\tau-1}P^\eta(D_0 > s + 1)$, and we would be done thanks to Proposition 3.2. In the general case we have to proceed as in the proof of Proposition 3.2 by splitting the integral on the right-hand side of the last display into the integral between 0 and $m(s)/c(x, \eta)$ plus the integral between $m(s)/c(x, \eta)$ and $+\infty$, where $m(s)$ is defined in (11). We call the first part $\tilde{A}_1(s)$ and the second part $\tilde{A}_2(s)$ similarly to (10). Following step by step what we did in the proof of Proposition 3.2 (see (15) and below), we note that

$$\tilde{A}_1(s) \le P^\eta(D_0 > s+1 \mid W_0 = m(s)) \le \mathrm{e}^{-(\log s)^4/4},$$

which does not contribute to the regular variation of the sum. Finally (see (14) and the argument below) we have that $\lim_{t\to\infty} \tilde{A}_2(st)/\tilde{A}_2(t) = s^{-\gamma}$, since $\tilde{A}_2(t)$ is upper-bounded by $1 - F(m(t)/c(x, \eta))$ and lower-bounded by $1 - F((1+\varepsilon)m(t)/c(x, \eta))$ minus a negligible term. We therefore conclude that there exists a slowly varying function $\ell_1(\cdot) = \ell_1(\cdot, x, \eta)$ such that

$$P^\eta(D_x > s) \ge \ell_1(s) s^{-\gamma}.$$

An upper bound $P^\eta(D_x > s) \le \ell_2(s) s^{-\gamma}$ for some other slowly varying function $\ell_2 = \ell_2(x, \eta)$ can be obtained in a completely similar way. These two bounds yield the desired result. □

## 4. Clustering coefficient

The idea for the proof of Theorem 2.3 consists in approximating $\mathrm{CC}_n$ with the sum of independent random variables in order to use the standard law of large numbers.

First of all we divide $\mathbb{R}^d$ into disjoint mesoscopic boxes of side-length $m > 0$, one of which is centered at the origin (the superposition of the sides of the boxes is of no importance). For a point $x \in \mathbb{R}^d$ we let $Q_m(x)$ be the unique $m$-box containing $x$. We also fix $\delta > 0$ small and divide each of the boxes $Q_m$ as $Q_m = \overline{Q}_m \cup \partial Q_m$, where

$$\overline{Q}_m = \overline{Q}_{m,\delta} := \{x \in Q_m \colon \|x - y\| > \delta m, \quad \text{for all } y \in Q_m^c\}$$

are the interior points of the box and $\partial Q_m = \partial Q_{m,\delta} := Q_m \setminus \overline{Q}_m$ is the $\delta$-frame of the box. For a realization of our graph $G = (V, E)$ and a point $x \in V$, we define

$$\widehat{\mathrm{CC}}^m(x) = \widehat{\mathrm{CC}}^{m,\delta}(x) := \begin{cases} 0 & \text{if } x \in \partial Q_m(x), \\ 0 & \text{if } x \leftrightarrow Q_m^c(x), \\ \mathrm{CC}(x) & \text{otherwise.} \end{cases}$$

For $n \ge m$ we finally define the $(m, \delta)$-truncated clustering coefficient as

$$\widehat{\mathrm{CC}}_n^m = \widehat{\mathrm{CC}}_n^{m,\delta} := \frac{1}{\nu n^d} \sum_{x \in V_n} \widehat{\mathrm{CC}}^m(x).$$

The idea is now to approximate $\mathrm{CC}_n$ by $\widehat{\mathrm{CC}}_n^m$, to show that $\widehat{\mathrm{CC}}_n^m$ converges thanks to the law of large numbers to $\mathbb{E}[\widehat{\mathrm{CC}}_m^m]$ and that this value is close to the desired $\mathbb{E}_0[\mathrm{CC}(0)]$. This is formalized in the next three statements, which are valid under the hypothesis of Theorem 2.3.

**Proposition 4.1.** *$\mathbb{P}$-almost surely we have*

$$\limsup_{n\to\infty} |\mathrm{CC}_n - \widehat{\mathrm{CC}}_n^m| \le c_1 \delta + c_2 (\delta m)^{d-\alpha},$$

*for some constants $c_1$, $c_2 > 0$ that may depend on d, α, E[W], and ν, but that do not depend on either m or δ.*

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**Lemma 4.1.** $\mathbb{P}$*-almost surely we have*

$$\lim_{n\to\infty} \widehat{\mathrm{CC}}_n^m = \mathbb{E}[\widehat{\mathrm{CC}}_m^m].$$

**Lemma 4.2.** *There exist constants* $c_1, c_2 > 0$ *that may depend on d, α, E[W], and ν, but that do not depend on either m or δ, such that*

$$|\mathbb{E}[\widehat{\mathrm{CC}}_m^m] - \mathbb{E}_0[\mathrm{CC}(0)]| \le c_1\delta + c_2(\delta m)^{d-\alpha}.$$

The proofs of Lemmas 4.1 and 4.2 are pretty straightforward and are collected in Section 4.1. The proof of Proposition 4.1 is much more involved and is the object of Section 4.2. We are now able to prove Theorem 2.3.

*Proof of Theorem 2.3.* The convergence of $\mathrm{CC}_n$ to $\mathbb{E}_0[\mathrm{CC}(0)]$ can be obtained by putting together the results of Proposition 4.1 and Lemmas 4.1 and 4.2, and letting first $m \to \infty$ and then $\delta \to 0$.

We now need only prove that $\mathbb{E}_0[\mathrm{CC}(0)] > 0$. Write $\mathcal{B}$ for $\mathcal{B}_1(0)$ and consider the following events:

$$\begin{aligned} \mathcal{E}_1 &= \{\#(\eta\cap\mathcal{B}) = 2\}, \\ \mathcal{E}_2 &= \{\text{the points in } (\eta\cup\{0\})\cap\mathcal{B} \text{ form a clique}\}, \\ \mathcal{E}_3 &= \{0 \text{ has no neighbors outside } \mathcal{B}\}. \end{aligned}$$

Under the event $\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3$, the local clustering coefficient of the origin is 1. Therefore

$$\mathbb{E}_0[\mathrm{CC}(0)] \ge \mathbb{P}_0(\mathrm{CC}(0) = 1) \ge \mathbb{P}_0(\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3).$$

Given the value of $W_0$, the events $\mathcal{E}_1\cap\mathcal{E}_2$ and $\mathcal{E}_3$ are independent. Hence

$$\mathbb{P}_0(\mathcal{E}_1\cap\mathcal{E}_2\cap\mathcal{E}_3) \ge \int_1^\infty \mathbb{P}_0(\mathcal{E}_1\cap\mathcal{E}_2 \mid W_0 = w)\mathbb{P}_0(\mathcal{E}_3 \mid W_0 = w)\,\mathrm{d}F(w).$$

In order to bound from below the second of the probabilities in the integral, we note that

$$\mathbb{1}_{\{0\not\leftrightarrow\mathcal{B}^c\}} = \prod_{x\in\eta\cap\mathcal{B}^c} \mathbb{1}_{\{0\not\leftrightarrow x\}}$$

and we subsequently apply Jensen's inequality and Campbell's theorem, thus obtaining

$$\begin{aligned} \mathbb{P}_0(\mathcal{E}_3 \mid W_0 = w) &= \mathrm{E}_0\Bigg[\prod_{x\in\eta\cap\mathcal{B}^c} E[\mathrm{e}^{-wW_x/\|x\|^\alpha}]\Bigg] \\ &\ge \exp\Bigg\{-\mathrm{E}_0\Bigg[\sum_{x\in\eta\cap\mathcal{B}^c} \frac{wE[W_y]}{\|y\|^\alpha}\Bigg]\Bigg\} \\ &= \mathrm{e}^{-cE[W]w\nu} \end{aligned}$$

for some $c > 0$ that does not depend on $w$, where $\mathrm{E}_0$ is the expectation with respect to $\mathrm{P}_0$. Since $\mathbb{P}_0(\mathcal{E}_1\cap\mathcal{E}_2 \mid W_0 = w)$ is uniformly bounded from below by a constant when $w \ge 1$, and since $P(W \ge 1) > 0$, we obtain $\mathbb{E}_0[\mathrm{CC}(0)] > 0$. □

### 4.1. Proofs of the lemmas

*Proof of Lemma 4.1.* We write $n = mq + r$, with $q \in \mathbb{N}$ and $0 \le r < m$. We let $Q(1), \ldots, Q(q^d)$ be the $q^d$-boxes of side-length $m$ fully contained in $B_n$ and $H = \big(\bigcup_j Q(j)\big)^c$. We also define

$$X(j) := (1/\nu m^d) \sum_{x \in Q(j)\cap\eta} \widehat{\mathrm{CC}}^m(x) \quad \text{for } j = 1, \ldots, q^d.$$

Then

$$\widehat{\mathrm{CC}}^m_n = \left(\frac{mq}{n}\right)^d \frac{1}{q^d} \sum_{j=1}^{q^d} X(j) + \frac{1}{\nu n^d} \sum_{x \in H\cap V_n} \widehat{\mathrm{CC}}^m(x).$$

Since the $X(j)$ are i.i.d. random variables (note that the overlap of the $Q$-boxes does not represent a problem: as their intersection is a set of Lebesgue measure 0, with probability 1 there will not be any point lying in their intersection) with finite $\mathbb{P}$-expectation, as $n$ goes to infinity, the first summand on the right-hand side converges $\mathbb{P}$-almost surely to $\mathbb{E}[X(1)] = \mathbb{E}[\widehat{\mathrm{CC}}^m_m]$ by translation invariance. The second summand converges almost surely to 0 since $\widehat{\mathrm{CC}}^m(x) \le 1$ and the number of points in $H \cap V_n$ follows a Poisson distribution with parameter $n^d - (mq)^d = \mathrm{O}(n^{d-1})$. □

*Proof of Lemma 4.2.* By the Slivnyak–Mecke theorem (see Theorem A.2) with $n = 1$ we have

$$\mathbb{E}\left[\frac{1}{\nu m^d} \sum_{x \in V_m} \mathrm{CC}(x)\right] = \frac{1}{\nu m^d} \int_{B_m} \mathbb{E}_x[\mathrm{CC}(x)]\nu \,\mathrm{d}x = \mathbb{E}_0[\mathrm{CC}(0)],$$

where we used translation invariance for the last equality. On the other hand,

$$\left|\mathbb{E}[\widehat{\mathrm{CC}}^m_m] - \mathbb{E}\left[\frac{1}{\nu m^d} \sum_{x \in V_m} \mathrm{CC}(x)\right]\right| = \frac{\mathbb{E}[W_m + U_m]}{\nu m^d},$$

where

$$W_m := \#\{x \in \partial B_m(x) \cap \eta\} \quad \text{and} \quad U_m := \#\{x \in \overline{B}_m(x) \cap \eta \colon x \leftrightarrow (B_m(x))^c\}.$$

Since $W_m$ follows a Poisson law of parameter $(1-(1-2\delta)^d)\nu m^d$, $\mathbb{E}[W_m]/(\nu m^d)$ is smaller than $2d\delta$. The fact that $\mathbb{E}[U_m]/(\nu m^d)$ is bounded from above by $c(\delta m)^{d-\alpha}$ will be proved in Proposition 4.2. Putting it all together we obtain

$$|\mathbb{E}[\widehat{\mathrm{CC}}^m_m - \mathbb{E}_0[\mathrm{CC}(0)]]| \le c_1\delta + c_2(\delta m)^{d-\alpha}.$$ □

### 4.2 Proof of Proposition 4.1

We begin with an elementary lemma.

**Lemma 4.3.** *For* P*-almost all* $\eta$ *there exists* $\bar{n} \in \mathbb{N}$ *such that, for all* $n \ge \bar{n}$,

$$N_n(\eta) \in [\nu n^d - \sqrt{\nu n^d} \log n,\ \nu n^d + \sqrt{\nu n^d} \log n].$$

*Proof.* For a Poisson random variable $X$ of parameter $\mu > 0$, the bound $P(|X - \mu| > \varepsilon) < 2\exp\{-\varepsilon^2/(4\mu)\}$ holds for all $0 < \varepsilon < \mu$; see e.g. [29, Theorem 5.4]. Since under P the number

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of points falling in $B_n$ follows a Poisson distribution of parameter $\nu n^d$, the conclusion follows by a simple Borel–Cantelli argument (in fact it is possible to improve the statement further). □

*Proof of Proposition 4.1.* For simplicity we will consider a sequence of $n$ of the form $n = qm$, where $q \in \mathbb{N}$. It is in fact highly plausible that for all other $n$ of the form $n = qm + r$ with $r \in [0, m)$, what happens in the area $B_n \setminus B_{qm}$ is negligible in the limit $n \to \infty$. We bound

$$|\mathrm{CC}_n - \widehat{\mathrm{CC}}_n^m| = \left|\frac{1}{N_n} - \frac{1}{\nu n^d}\right| \sum_{x \in V_n} \mathrm{CC}(x) + \frac{1}{\nu n^d}\left(\sum_{x \in V_n} \mathrm{CC}(x) - \widehat{\mathrm{CC}}^m(x)\right)$$
$$\leq \left|1 - \frac{N_n}{\nu n^d}\right| + \frac{1}{\nu n^d}(W_n + U_n), \tag{16}$$

where

$$W_n = W_n(m) := \#\{x \in V_n : x \in \partial Q_m(x)\}$$

and

$$U_n = U_n(m) := \#\{x \in V_n : x \in \overline{Q}_m(x),\ x \leftrightarrow (Q_m(x))^c\}.$$

The first summand in (16) tends $\mathbb{P}$-almost surely to 0 as $n \to \infty$ by Lemma 4.3. We then note that under $\mathbb{P}$ the random variable $W_n$ has a Poisson distribution of parameter $(1 - (1 - 2\delta)^d)\nu n^d$, which can be dominated by a Poisson distribution of parameter $2d\delta\nu n^d$. Reasoning as in the the proof of Lemma 4.3, it is possible to show that, for all $n$ large enough, $W_n$ is smaller than $4d\delta\nu n^d$, for example, so $\limsup_{n\to\infty} n^{-d} W_n \leq c_1\delta$.

It remains to show that $\mathbb{P}$-almost surely.

$$\limsup_{n\to\infty} \frac{1}{\nu n^d} U_n \leq c_2(\delta m)^{d-\alpha}. \tag{17}$$

Assume for now that there exists a constant $c > 0$ such that the following estimates hold:

$$\mathbb{E}[U_n] \leq c\, n^d (\delta m)^{d-\alpha}, \qquad \mathbb{V}(U_n) \leq c\, n^{3d-\alpha},$$

where $\mathbb{V}$ denotes the variance with respect to $\mathbb{P}$. The proof of these two bounds will be postponed to Proposition 4.2 below. By the Chebyshev inequality,

$$\mathbb{P}(U_n > 2c\, n^d (\delta m)^{d-\alpha}) \leq \frac{\mathbb{V}(U_n)}{c^2 n^{2d} (\delta m)^{2(d-\alpha)}} \leq \tilde{c}\, (\delta m)^{2(\alpha-d)} n^{d-\alpha},$$

for some $\tilde{c} > 0$. We would like to finish the proof via Borel–Cantelli, but the right-hand side of the last display is not summable in $n$. Nevertheless $U_n$ is an increasing sequence, so we can proceed as follows. Recall that $n = qm$. By taking the sequence $n_k := 2^k m$, we have

$$\mathbb{P}(U_{n_k} > 2c\, {n_k}^d (\delta m)^{d-\alpha}) \leq \tilde{c}\, (\delta m)^{2(\alpha-d)} (m 2^k)^{d-\alpha},$$

which is summable in $k$. Hence there exists almost surely a $\bar{k}$ such that $U_{n_k} \leq 2c\, {n_k}^d (\delta m)^{d-\alpha}$ for all $k \geq \bar{k}$. Now take any $n > 2^{\bar{k}} m$ and let $K$ be the integer such that $2^K m \leq n < 2^{K+1} m$. By monotonicity we get

$$U_n \leq U_{2^{K+1} m} \leq 2c (2^{K+1} m)^d (\delta m)^{d-\alpha} \leq 2^{d+1} c\, (\delta m)^{d-\alpha} n^d,$$

and (17) follows. □

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**Proposition 4.2.** *There exists a constant $c > 0$ such that*

$$\mathbb{E}[U_n] \le c\, n^d (\delta m)^{d-\alpha}, \tag{18}$$

$$\mathbb{V}(U_n) \le c\, n^{3d-\alpha}, \tag{19}$$

*where $\mathbb{V}$ denotes the variance with respect to $\mathbb{P}$.*

*Proof of Proposition 4.2.* Let $Q(1), \ldots, Q(q^d)$ be the boxes of side-length $m$ contained in $B_n$. For the expectation we have

$$\begin{aligned}\mathbb{E}[U_n] &= \mathbb{E}\left[\sum_{j=1}^{q^d} \sum_{x\in \overline{Q}(j)\cap\eta} \mathbb{1}_{\{x\leftrightarrow (Q_m(x))^c\}}\right] \\ &= q^d \mathbb{E}\left[\mathbb{E}\left[\sum_{x\in V_{m(1-2\delta)}} \mathbb{1}_{\{x\leftrightarrow B_m^c\}} \,\middle|\, V_m\right]\right] \\ &\le q^d \mathbb{E}[N_{m(1-2\delta)} \mathbb{P}_0(0\leftrightarrow B^c_{\delta m})].\end{aligned} \tag{20}$$

In the second equality we used the translation invariance of the model and then the tower property of expectation by conditioning on the position of the points of the Poisson process inside $B_m$. For the inequality we upper-bounded the probability that a point $x \in V_{m(1-2\delta)}$ is connected to the exterior of $B_m$ by the probability that, under the Palm measure, the origin is connected to some point in $B^c_{\delta m}$ (since for each $x \in V_{m(1-2\delta)}$ all the points in $B_m^c$ are outside a box of side $\delta m$ centered at $x$). We estimate this probability as follows:

$$\begin{aligned}\mathbb{P}_0(0\leftrightarrow B^c_{\delta m}) &= 1 - \mathbb{E}_0\left[\prod_{y\in V\setminus V_{\delta m}} E[\mathrm{e}^{-W_0 W_y/\|y\|^\alpha} \mid W_0]\right] \\ &\le 1 - \exp\left\{-\mathbb{E}_0\left[\sum_{y\in V\setminus V_{\delta m}} \frac{W_0 E[W_y]}{\|y\|^\alpha}\right]\right\} \\ &\le \int_{y\in B^c_{\delta m}} \frac{E[W]^2}{\|y\|^\alpha}\,\mathrm{d}y \\ &= c'(\delta m)^{d-\alpha}\end{aligned} \tag{21}$$

for some $c' > 0$ that does not depend on either $m$ or $\delta$. In the first line we used the fact that, conditioned on the weight of point 0, the presence of connections between 0 and the other points of the Poisson point process becomes independent. For the second line we applied Jensen's inequality twice. For the third line we first used the inequality $1 - \mathrm{e}^{-x} \le x$ for $x > 0$, then we applied Campbell's theorem (see Theorem A). Finally we used polar coordinates in order to evaluate the integral. Plugging this back into (20), we obtain (18).

We move to the variance of $U_n$. For $x \in \eta$ we set

$$A(x) := \mathbb{1}_{\{x\in \overline{Q}_m(x),\, x\leftrightarrow (Q_m(x))^c\}}$$

so that $U_n = \sum_{x\in V_n} A(x)$. Note that by the Slivnyak–Mecke theorem (see Theorem A.2) we have

$$\mathbb{E}[U_n] = \nu \int_{B_n} \mathbb{E}_x[A(x)]\,\mathrm{d}x.$$

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On the other hand we can write

$$\mathbb{V}(U_n) = \mathbb{E}[U_n^2] - \mathbb{E}[U_n]^2 = \mathbb{E}[U_n] + \mathbb{E}\left[\sum_{x \neq y \in V_n} A(x)A(y)\right] - \mathbb{E}[U_n]^2. \tag{22}$$

Now applying Theorem A.2 to the function

$$f(x, y, \eta) := \mathbb{1}_{\{x,y \in B_n\}} E^{\eta \cup \{x\} \cup \{y\}}[A(x)A(y)],$$

we can evaluate the second summand on the right-hand side:

$$\begin{aligned}\mathbb{E}\left[\sum_{x \neq y \in V_n} A(x)A(y)\right] &= \nu^2 \int_{B_n} \int_{B_n} \mathbb{E}_{x,y}[A(x)A(y)]\, \mathrm{d}x\, \mathrm{d}y \\ &= \nu^2 \int_{B_n} \int_{B_n} C(x, y)\, \mathrm{d}x\, \mathrm{d}y + \mathbb{E}[U_n]^2 + \nu^2 R, \end{aligned} \tag{23}$$

where

$$C(x, y) = \mathbb{E}_{x,y}[A(x)A(y)] - \mathbb{E}_{x,y}[A(x)]\mathbb{E}_{x,y}[A(y)]$$

and $R$ is the rest given by

$$R = \int_{B_n} \int_{B_n} \mathbb{E}_{x,y}[A(x)]\mathbb{E}_{x,y}[A(y)] - \mathbb{E}_x[A(x)]\mathbb{E}_y[A(y)]\, \mathrm{d}x\, \mathrm{d}y.$$

Note that for $y \in Q_m(x)$ we have $\mathbb{E}_{x,y}[A(x)] = \mathbb{E}_x[A(x)]$. For $y \notin Q_m(x)$,

$$\mathbb{E}_{x,y}[A(x)] \leq \mathbb{E}_x[A(x)] + E[1 - \exp\{-W_x W_y \|x - y\|^{-\alpha}\}] \leq \mathbb{E}_x[A(x)] + \tilde{c}_1 \|x - y\|^{-\alpha}$$

for some $\tilde{c}_1 > 0$. For the first inequality we used the fact that for $A(x) = 1$, the point $x$ has to be connected to $y$ (which gives the second addendum) or to any other point outside $Q_m(x)$ (first addendum), and in this second case the presence of $y$ in the point process plays no role. This, together with the bound

$$\int_{B_n \setminus Q_m(x)} \|x - y\|^{-\alpha}\, \mathrm{d}y \leq \tilde{c}_2 (\delta m)^{d-\alpha} \quad \text{for some } \tilde{c}_2 > 0,$$

yields

$$R \leq \tilde{c}_3 (\mathbb{E}[U_n](\delta m)^{d-\alpha} + n^d (\delta m)^{-2\alpha+d}) \leq \tilde{c}_4 n^d (\delta m)^{2(d-\alpha)}.$$

We now only need to bound the correlation $C(x, y)$. We introduce the event

$$\mathcal{E}(x, y) := \{\text{neither } x \text{ nor } y \text{ have neighbors at distance larger than } \|x - y\|/2\}.$$

The random variables $A(x)$ and $A(y)$ are independent under the event $\mathcal{E}(x, y)$. Therefore

$$\begin{aligned} C(x, y) &\leq \mathbb{E}_{x,y}[A(x)|\mathcal{E}(x, y)]\mathbb{E}_{x,y}[A(y)|\mathcal{E}(x, y)]\mathbb{P}(\mathcal{E}(x, y)) \\ &\quad + \mathbb{P}_{x,y}(\mathcal{E}(x, y)^c) - \mathbb{E}_{x,y}[A(x)]\mathbb{E}_{x,y}[A(y)] \\ &\leq \mathbb{E}_{x,y}[A(x)]\mathbb{E}_{x,y}[A(y)]\left(\frac{1}{\mathbb{P}_{x,y}(\mathcal{E}(x, y))} - 1\right) + \mathbb{P}_{x,y}(\mathcal{E}(x, y)^c) \\ &\leq \mathbb{P}_{x,y}(\mathcal{E}(x, y)^c)\left(\frac{1}{\mathbb{P}_{x,y}(\mathcal{E}(x, y))} + 1\right), \end{aligned} \tag{24}$$

where in the second line we used the inequality $\mathbb{E}_{x,y}[W|Z] \le \mathbb{E}_{x,y}[W]/\mathbb{P}_{x,y}(Z)$ and in the last line we just bounded $\mathbb{E}_{x,y}[A(x)]$ and $\mathbb{E}_{x,y}[A(y)]$ by 1. We also observe that

$$\begin{aligned}\mathbb{P}_{x,y}(\mathcal{E}(x,y)^c) &\le \mathbb{P}_{x,y}(x \leftrightarrow \mathcal{B}_{\|x-y\|/2}(x)) + \mathbb{P}_{x,y}(y \leftrightarrow \mathcal{B}_{\|x-y\|/2}(y)) \\ &\le \mathbb{P}_{x,y}(x \leftrightarrow y) + 2\mathbb{P}_0(0 \leftrightarrow \mathcal{B}_{\|x-y\|/2}(0)) \\ &\le \tilde{c}_5 \, \|x-y\|^{d-\alpha} \end{aligned} \tag{25}$$

for some $\tilde{c}_5 > 0$, where the last inequality can be obtained as in (21). From (24) and (25) it follows that whenever we consider $x, y \in \mathbb{R}^d$ such that, for example, $\tilde{c}_5 \, \|x-y\|^{d-\alpha} \le 1/2$, we get $C(x,y) \le \tilde{c}_6 \, \|x-y\|^{d-\alpha}$. Hence, setting $r := (2\tilde{c}_5)^{1/(\alpha-d)}$, we can bound

$$\int_{B_n}\int_{B_n} C(x,y)\,\mathrm{d}x\,\mathrm{d}y \le \int_{B_n}\left[\tilde{c}_6 r^d + \int_{y\notin\mathcal{B}_r(x)} \tilde{c}_5\|x-y\|^{d-\alpha}\,\mathrm{d}y\right]\mathrm{d}x \le c\,n^{3d-\alpha}$$

for some constant $c > 0$. The proof is finished by putting together (22), (18), (23), and this last estimate. □

## Appendix A. Campbell and Slivnyak–Mecke theorems

In this section we present versions of Campbell's theorem and of the so-called extended Slivnyak–Mecke theorem in the simple case of a homogeneous Poisson point process. For a version of Theorem A.1 similar to the one presented here, see [26, Section 3.2]. For a simple proof of Theorem A.2, see [30, Theorem 13.3]. More complete and general versions of these theorems in the framework of Palm theory can be found in [12, Chapter 13].

**Theorem A.1.** (Campbell theorem, homogeneous case.) *Let $\eta$ be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\nu$ under measure* P, *with* E *the associated expectation. Let $f:\mathbb{R}^d \to \mathbb{R}$ be a measurable function. The random sum $S = \sum_{x\in\eta} f(x)$ is absolutely convergent with probability one if and only if*

$$\int_{\mathbb{R}^d} |f(y)| \wedge 1 \,\mathrm{d}y < \infty.$$

*If the previous integral is finite, then we have*

$$\mathrm{E}[\mathrm{e}^{\theta S}] = \exp\left(\nu \int_{\mathbb{R}^d} (\mathrm{e}^{\theta f(y)} - 1)\,\mathrm{d}y\right) \tag{26}$$

*for any complex $\theta$ such that the integral on the right-hand side converges. Moreover,*

$$\mathrm{E}[S] = \nu \int_{\mathbb{R}^d} f(y)\,\mathrm{d}y, \tag{27}$$

*and if the last integral is finite, then*

$$\mathrm{V}(S) = \nu \int_{\mathbb{R}^d} f(y)^2\,\mathrm{d}y, \tag{28}$$

*where V is the (possibly infinite) variance with respect to P.*

**Theorem A.2.** (Extended Slivnyak–Mecke theorem.) *Let $\eta$ be a homogeneous Poisson point process on $\mathbb{R}^d$ with intensity $\nu$ under measure P, with E the associated expectation. For $n \in \mathbb{N}$*

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*and for any positive function* $f\colon (\mathbb{R}^d)^n \times (\mathbb{R}^d)^{\mathbb{N}} \to [0, \infty)$*,*

$$\mathrm{E}\left[\sum_{x_1,\ldots,x_n\in\eta}^{\neq} f(x_1, \ldots, x_n; \eta \setminus \{x_1, \ldots, x_n\})\right] = \nu^n \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \mathrm{E}[f(x_1, \ldots, x_n; \eta)]\, \mathrm{d}x_1 \ldots \mathrm{d}x_n,$$

*where the* $\neq$ *sign over the sum means that* $x_1, \ldots, x_n$ *are pairwise distinct.*

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## References

[1] ALBERT, R., JEONG, H. AND BARABÁSI, A.-L. (1999). Internet: diameter of the world-wide web. *Nature* **401**, 130.
[2] BARABÁSI, A.-L. AND ALBERT, R. (1999). Emergence of scaling in random networks. *Science* **286**, 509–512.
[3] BARRAT, A. AND WEIGT, M. (2000). On the properties of small-world network models. *Eur. Phys. J. B* **13**, 547–560.
[4] BARTHÉLEMY, M. (2011). Spatial networks. *Phys. Rep.* **499**, 1–101.
[5] BERGER, N. (2002). Transience, recurrence and critical behavior for long-range percolation. *Commun. Math. Phys.* **226**, 531–558.
[6] BISKUP, M. (2004). On the scaling of the chemical distance in long-range percolation models. *Ann. Prob.* **32**, 2938–2977.
[7] BRINGMANN, K., KEUSCH, R. AND LENGLER, J. (2019). Geometric inhomogeneous random graphs. *Theoret. Comput. Sci.* **760**, 35–54.
[8] CANDELLERO, E. AND FOUNTOULAKIS, N. (2016). Clustering and the hyperbolic geometry of complex networks. *Internet Math.* **12**, 2–53.
[9] CHUNG, F. AND LU, L. (2002). The average distances in random graphs with given expected degrees. *Proc. Nat. Acad. Sci. USA* **99**, 15879–15882.
[10] CONT, R., MOUSSA, A. AND BASTOS E SANTOS, E. (2010). Network structure and systemic risk in banking systems. *SSRN Electron. J.* http://dx.doi.org/10.2139/ssrn.1733528.
[11] COUPECHOUX, E. AND LELARGE, M. (2014). How clustering affects epidemics in random networks. *Adv. Appl. Prob.* **46**, 985–1008.
[12] DALEY, D. J. AND VERE-JONES, D. (2008). *An Introduction to the Theory of Point Processes*, vol. II: *General Theory and Structure*, 2nd edn (Probability and its Applications). Springer, New York.
[13] DEIJFEN, M., VAN DER HOFSTAD, R. AND HOOGHIEMSTRA, G. (2013). Scale-free percolation. *Ann. Inst. H. Poincaré Prob. Statist.* **49**, 817–838.
[14] DEPREZ, P. AND WÜTHRICH, M. V. (2019). Scale-free percolation in continuum space. *Commun. Math. Statist.* **7**, 269–308.
[15] DEPREZ, P., HAZRA, R. S. AND WÜTHRICH, M. V. (2015). Inhomogeneous long-range percolation for real-life network modeling. *Risks* **3**, 1–23.
[16] DUTTA, B. L., EZANNO, P. AND VERGU, E. (2014). Characteristics of the spatio-temporal network of cattle movements in France over a 5-year period. *Preventive Veterinary Medicine* **117**, 79–94.
[17] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, vol. 2, 2nd edn. John Wiley, New York.
[18] FOUNTOULAKIS, N., VAN DER HOORN, P., MÜLLER, T. AND SCHEPERS, M. (2020). Clustering in a hyperbolic model of complex networks. Available at arXiv:2003.05525.
[19] GRACAR, P., GRAUER, A., LÜCHTRATH, L. AND MÖRTERS, P. (2019). The age-dependent random connection model. *Queueing Systems* **93**, 309–331.
[20] GUGELMANN, L., PANAGIOTOU, K. AND PETER, U. (2012). Random hyperbolic graphs: degree sequence and clustering. In *International Colloquium on Automata, Languages, and Programming* (Lecture Notes Comput. Sci. **7392**), pp. 573–585. Springer.
[21] HAENGGI, M., ANDREWS, J. G., BACCELLI, F., DOUSSE, O. AND FRANCESCHETTI, M. (2009). Stochastic geometry and random graphs for the analysis and design of wireless networks. *IEEE J. Sel. Areas Commun.* **27**, 1029–1046.

[22] Heydenreich, M., Hulshof, T. and Jorritsma, J. (2017). Structures in supercritical scale-free percolation. *Ann. Appl. Prob.* **27**, 2569–2604.
[23] Jacob, E. and Mörters, P. (2015). Spatial preferential attachment networks: power laws and clustering coefficients. *Ann. Appl. Prob.* **25**, 632–662.
[24] Jeong, H., Tombor, B., Albert, R., Oltvai, Z. N. and Barabási, A.-L. (2000). The large-scale organization of metabolic networks. *Nature* **407**, 651.
[25] Kaluza, P., Kölzsch, A., Gastner, M. T. and Blasius, B. (2010). The complex network of global cargo ship movements. *J. R. Soc. Interface* **7**, 1093–1103.
[26] Kingman, J. F. C. (1992). *Poisson Processes*, vol. 3. Clarendon Press.
[27] Komjáthy, J. and Lodewijks, B. (2020). Explosion in weighted hyperbolic random graphs and geometric inhomogeneous random graphs. *Stochastic Process. Appl*. **130**, 1309–1367.
[28] Lago-Fernández, L. F., Huerta, R., Corbacho, F. and Sigüenza, J. A. (2000). Fast response and temporal coherent oscillations in small-world networks. *Phys. Rev. Lett.* **84**, 2758.
[29] Mitzenmacher, M. and Upfal, E. (2017). *Probability and Computing: Randomization and Probabilistic Techniques in Algorithms and Data Analysis*. Cambridge University Press.
[30] Moller, J. and Waagepetersen, R. P. (2003). *Statistical Inference and Simulation for Spatial Point Processes*. Chapman & Hall/CRC.
[31] Moore, C. and Newman, M. E. (2000). Epidemics and percolation in small-world networks. *Phys. Rev. E* **61**, 5678.
[32] Newman, M. E. (2003). The structure and function of complex networks. *SIAM Rev.* **45**, 167–256.
[33] Newman, M. E., Watts, D. J. and Strogatz, S. H. (2002). Random graph models of social networks. *Proc. Nat. Acad. Sci. USA* **99**, 2566–2572.
[34] Norros, I. and Reittu, H. (2006). On a conditionally Poissonian graph process. *Adv. Appl. Prob.* **38**, 59–75.
[35] Pastor-Satorras, R. and Vespignani, A. (2001). Epidemic spreading in scale-free networks. *Phys. Rev. Lett.* **86**, 3200.
[36] Riley, S., Eames, K., Isham, V., Mollison, D. and Trapman, P. (2015). Five challenges for spatial epidemic models. *Epidemics* **10**, 68–71.
[37] Soramäki, K., Bech, M. L., Arnold, J., Glass, R. J. and Beyeler, W. E. (2007). The topology of interbank payment flows. *Physica A* **379**, 317–333.
[38] Stegehuis, C., van Der Hofstad, R. and van Leeuwaarden, J. S. (2019). Scale-free network clustering in hyperbolic and other random graphs. *J. Phys. A* **52**, 295101.
[39] Van Der Hofstad, R. (2016). *Random Graphs and Complex Networks*, vol. 1. Cambridge University Press.
[40] Watts, D. J. and Strogatz, S. H. (1998). Collective dynamics of 'small-world' networks. *Nature* **393**, 440.
[41] Yukich, J. (2006). Ultra-small scale-free geometric networks. *J. Appl. Prob.* **43**, 665–677.