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Bifurcation and standing wave solutions for a quasilinear Schrödinger equation

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We use bifurcation and topological methods to investigate the existence/nonexistence and the multiplicity of positive solutions of the following quasilinear Schrödinger equation

$$\begin{cases} -\Delta u-\kappa\Delta\left(u^2\right)u=\beta u-\lambda\Phi\left(u^2\right)u & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega \end{cases}$$

involving sublinear/linear/superlinear nonlinearities at zero or infinity with/without signum condition. In particular, we study the changes in the structure of positive solution with κ as the varying parameter.

Keywords: Bifurcation; positive solutions; Quasilinear Schrödinger equation

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1. Introduction

The following time-dependent quasilinear Schrödinger equation

$$\begin{cases} i\frac{\partial}{\partial t}z=-\Delta z+V(x)z-\kappa\Delta\left(z^2\right)z+\lambda\Phi\left(z^2\right)z & \text{in } \Omega,\\ \Phi(x,t)=0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

has been derived as models of several physical phenomena, where Ω is a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and $N>2$, $\lambda>0$ is a real parameter, $\kappa\neq 0$ is a real constant, $V(x)$ $(x\in\Omega)$ is a given potential and Φ is a real function. For example, the superfluid film equation in plasma physics [**35**, **36**, **38**] and the selfchanneling of a high-power ultra short laser in matter, see [**10**–**12**, **20**, **55**] and the reference therein. Equation (1.1) also appears in plasma physics and fluid mechanics [**41**, **45**, **49**], in the theory of Heisenberg ferromagnets and magnons [**7**, **34**, **37**, **51**, **60**], in dissipative quantum mechanics [**30**] and in condensed matter theory [**44**].

To obtain standing wave solutions of problem (1.1), we set $z(x,t):=e^{-i\lambda t}u(x)$ with $\lambda>0$. Then problem (1.1) with $V(x)\equiv 0$ is reduced to the following elliptic equation

$$\begin{cases} -\Delta u - \kappa\Delta(u^2)u = \lambda(u - \Phi(u^2)u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

In the mathematical literature, very few results are known about the existence of standing wave solutions for equations of the form of problem (1.1). The authors of [**15**, **42**, **43**, **48**] studied the existence of standing wave solutions for this kind of problems by variational method. Following the idea of [**14**, **15**, **42**], we make a change of variables for any $\kappa > 0$:

$$dv = \sqrt{1 + 2\kappa u^2}du, v = l_\kappa(u) = \frac{1}{2\sqrt{2\kappa}}(\sqrt{2\kappa}u\sqrt{1+2\kappa u^2} + \ln(\sqrt{2\kappa}u + \sqrt{1+2\kappa u^2})).$$

Clearly, l_κ is strictly monotone since $l'_\kappa(u) = \sqrt{1+2\kappa u^2}$ and has an inverse function: $u = h_\kappa(v)$. Then we can transform (see lemma 2.2) problem (1.2) into the following semilinear elliptic problem by the change of $u = h_\kappa(v)$

$$\begin{cases} -\Delta v = \frac{\lambda}{\sqrt{1+2\kappa h_\kappa^2(v)}}g(h_\kappa(v)) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $g(s) = s - \Phi(s^2)s$. Mainly by variational method, the authors of [**15**, **42**] obtained the existence of positive solutions of problem (1.3) with $\kappa = 1/2$ and $\kappa = 1$, respectively. Note that this strategy may be invalid for $\kappa < 0$ because l_κ may not be strictly monotone.

Putting $z(x,t) := e^{-i\beta t}u(x)$, $\beta > 0$, $V(x) \equiv 0$ and $\kappa = 0$, we get the following elliptic problem with two parameters

$$\begin{cases} -\Delta u = \beta u - \lambda F(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $F(s) = \Phi(s^2)s$. Clearly, to find standing wave solutions of the form $z(x,t) := e^{-i\beta t}u(x)$ of problem (1.1) with $V(x) \equiv 0$ and $\kappa = 0$ is equivalent to find solutions of problem (1.4).

The main aim of this paper is to investigate the existence/nonexistence and the multiplicity of positive solutions of problem (1.3) involving a sublinear/linear/superlinear growth nonlinearity at zero or infinity by using bifurcation and topological methods. We shall also study the existence of positive solution of problem (1.4) with asymptotic nonlinearity via several-parameter bifurcation theorem due to Fitzpztrick, Massabò and Pejsachowicz [**26**]. By a solution of problem (1.3). we understand that it is a C^2 function which satisfies problem (1.3) point-wise, that is, it is a classical solution.

Now, we are in the position to state the following hypotheses on the nonlinearity g.

(G1) The function $g : \mathbb{R}_+ := [0, +\infty) \to \mathbb{R}_+$ is continuous.

(G2) $g(s)s > 0$ for $s > 0$.

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(G3) There exist $g_0, g_\infty \in [0, +\infty]$ such that

$$g_0 = \lim_{s\to 0^+} \frac{g(s)}{s}, \; g_\infty = \lim_{s\to +\infty} \frac{g(s)}{s^3}.$$

(G4) There exists one positive constant α such that $g(\alpha) = 0$, $g(s)s > 0$ for $s \in (0, \alpha) \cup (\alpha, +\infty)$.

(G5) There exists a constant $\delta > 0$ such that

$$\lim_{s\to \alpha^-} \frac{g(s)}{l_\kappa(\alpha) - l_\kappa(s)} = \delta.$$

(G6)

$$\lim_{s\to +\infty} \frac{g(s)}{s^{2l+1}} = C,$$

for some $l \in (1, (N+2)/(N-2))$, where C is a positive constant.

Let

$$f(t) = \frac{1}{\sqrt{1 + 2\kappa h_\kappa^2(t)}} g(h_\kappa(t)).$$

We call f is linear growth at infinity if $g_\infty \in (0, +\infty)$ because it implies (see lemma 2.3) that

$$\lim_{t\to +\infty} \frac{f(t)}{t} = \frac{g_\infty}{\kappa} \in (0, +\infty).$$

Similarly, we call f is sublinear growth at infinity if $g_\infty = 0$ and superlinear growth at infinity if $g_\infty = +\infty$. The meaning of growth at zero is understood as usual because of

$$\lim_{t\to 0^+} \frac{f(t)}{t} = g_0.$$

Let λ_1 denote the first eigenvalue of $-\Delta$ with 0-Dirichlet boundary condition. It is well known that λ_1 is simple, isolated and the associated eigenfunction has one sign in Ω.

If κ is a fixed positive constant, we can establish the following two theorems.

THEOREM 1.1. *Assume that κ is a positive constant, and (G1)–(G3) and (G6) hold.*

(a) *If g_0, $g_\infty \in (0, +\infty)$ satisfying $\kappa g_0 \neq g_\infty$, then there exist four positive constants μ_1, μ_1', μ_2 and μ_2' with $\mu_1' \leqslant \mu_1$ and $\mu_2' \geqslant \mu_2$ such that problem (1.3) has at least one positive solution for all $\lambda \in (\mu_1, \mu_2)$ and has no positive solution for all $\lambda \in (0, \mu_1') \cup (\mu_2', +\infty)$.*

(b) *If $g_0 \in (0, +\infty)$ and $g_\infty = 0$, then there exist two positive constants μ_3 and μ_3' with $\mu_3' \leqslant \mu_3$ such that problem (1.3) has at least one positive solution for all $\lambda \in (\mu_3, +\infty)$ and has no positive solution for all $\lambda \in (0, \mu_3')$.*

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(c) *If* $g_0 \in (0, +\infty)$ *and* $g_\infty = +\infty$, *then there exist two positive constants* μ_4 *and* μ_4' *with* $\mu_4' \geqslant \mu_4$ *such that problem (1.3) has at least one positive solution for all* $\lambda \in (0, \mu_4)$ *and has no positive solution for all* $\lambda \in (\mu_4', +\infty)$.

(d) *If* $g_0 = 0$ *and* $g_\infty \in (0, +\infty)$, *then there exist two positive constants* μ_5 *and* μ_5' *with* $\mu_5' \leqslant \mu_5$ *such that problem (1.3) has at least one positive solution for all* $\lambda \in (\mu_5, +\infty)$ *and has no positive solution for all* $\lambda \in (0, \mu_5')$.

(e) *If* $g_0 = 0$ *and* $g_\infty = 0$, *then there exist three positive constants* μ_6, μ_6' *and* μ_7 *with* $\mu_6' \leqslant \mu_6 \leqslant \mu_7$ *such that problem (1.3) has at least two positive solutions for all* $\lambda \in (\mu_7, +\infty)$, *one positive solution for all* $\lambda \in [\mu_6, \mu_7]$ *and has no positive solution for all* $\lambda \in (0, \mu_6')$.

(f) *If* $g_0 = 0$ *(or* $+\infty$*) and* $g_\infty = +\infty$ *(or* 0*), then for any* $\lambda \in (0, +\infty)$, *problem (1.3) has at least one positive solution.*

(g) *If* $g_0 = +\infty$ *and* $g_\infty \in (0, \infty)$, *then there exist two positive constants* μ_8 *and* μ_8' *with* $\mu_8' \geqslant \mu_8$ *such that problem (1.3) has at least one positive solution for all* $\lambda \in (0, \mu_8)$ *and has no positive solution for all* $\lambda \in (\mu_8', +\infty)$.

(h) *If* $g_0 = +\infty$ *and* $g_\infty = +\infty$, *then there exist three positive constants* μ_9, μ_{10} *and* μ_{10}' *with* $\mu_9 \leqslant \mu_{10} \leqslant \mu_{10}'$ *such that problem (1.3) has at least two positive solutions for all* $\lambda \in (0, \mu_9)$, *has at least one positive solution for all* $\lambda \in [\mu_9, \mu_{10}]$ *and has no positive solution for all* $\lambda \in (\mu_{10}', +\infty)$.

THEOREM 1.2. *Let* κ *be a positive constant, and (G1), (G3)–(G6) hold.*

(i) *If* g_0, $g_\infty \in (0, +\infty)$ *with* $\kappa g_0 \neq g_\infty$, *then there exists* $\mu_{11} > 0$ *such that problem (1.3) has at least two positive solutions for all* $\lambda \in (\max\{(\lambda_1\kappa)/g_\infty, \lambda_1/g_0\}, +\infty)$, *has at least one positive solution for all* $\lambda \in (\min\{(\lambda_1\kappa)/g_\infty, \lambda_1/g_0\}, \max\{(\lambda_1\kappa)/g_\infty, \lambda_1/g_0\}]$ *and has no positive solution for all* $\lambda \in (0, \mu_{11})$.

(ii) *If* g_0, $g_\infty \in (0, +\infty)$ *with* $\kappa g_0 = g_\infty$, *then there exists* $\mu_{12} > 0$ *such that problem (1.3) has at least two positive solutions for all* $\lambda \in (\lambda_1/g_0, +\infty)$, *and has no positive solution for all* $\lambda \in (0, \mu_{12})$.

(iii) *If* $g_0 \in (0, +\infty)$ *and* $g_\infty = +\infty$, *then problem (1.3) has at least two positive solutions for all* $\lambda \in (\lambda_1/g_0, +\infty)$, *has at least one positive solution for all* $\lambda \in (0, \lambda_1/g_0]$.

(iv) *If* $g_0 = +\infty$ *and* $g_\infty \in (0, +\infty)$, *then problem (1.3) has at least two positive solutions for all* $\lambda \in ((\lambda_1\kappa)/g_\infty, +\infty)$, *has at least one positive solution for all* $\lambda \in (0, (\lambda_1\kappa)/g_\infty]$.

(v) *If* $g_0 = +\infty$ *and* $g_\infty = +\infty$, *then problem (1.3) has at least two positive solutions for all* $\lambda \in (0, +\infty)$.

To study the effect of the second term in problem (1.2), we now consider κ as the varying parameter and λ is fixed positive constant, which is less conventional.

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Without loss of generality, we can assume that $\lambda = 1$. The next two theorems present the changes in the structure of positive solution if κ is a varying parameter.

THEOREM 1.3. *Assume that $\lambda = 1$, and (G1)–(G3) and (G6) hold.*

(a) *If g_0, $g_\infty \in (0, +\infty)$ satisfying $\lambda_1 < g_0$, then problem (1.3) has at least one positive solution for all $\kappa \in (g_\infty/\lambda_1, +\infty)$.*

(b) *If g_0, $g_\infty \in (0, +\infty)$ satisfying $\lambda_1 > g_0$, then problem (1.3) has at least one positive solution for all $\kappa \in (0, g_\infty/\lambda_1)$.*

(c) *If $g_0 = 0$ and $g_\infty \in (0, +\infty)$, then problem (1.3) has at least one positive solution for all $\kappa \in (0, g_\infty/\lambda_1)$.*

(d) *If $g_0 = +\infty$ and $g_\infty \in (0, \infty)$, then problem (1.3) has at least one positive solution for all $\kappa \in (g_\infty/\lambda_1, +\infty)$.*

THEOREM 1.4. *Let $\lambda = 1$ and (G1), (G3)–(G6) hold.*

(i) *If g_0, $g_\infty \in (0, +\infty)$ with $\lambda_1 < g_0$, then problem (1.3) has at least two positive solutions for all $\kappa \in (0, g_\infty/\lambda_1)$, has at least one positive solution for all $\kappa \in (g_\infty/\lambda_1, +\infty)$.*

(ii) *If g_0, $g_\infty \in (0, +\infty)$ with $\lambda_1 > g_0$, then problem (1.3) has at least one positive solution for all $\kappa \in (0, g_\infty/\lambda_1)$.*

(iii) *If $g_0 = +\infty$ and $g_\infty \in (0, +\infty)$, then problem (1.3) has at least two positive solutions for all $\kappa \in (0, g_\infty/\lambda_1)$, has at least one positive solution for all $\kappa \in [g_\infty/\lambda_1, +\infty)$.*

To prove theorems 1.1–1.4, we consider the following semilinear elliptic problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is some given continuous nonlinearity. Such problems arise in a variety of fields. For example, in the theory of thermal ignition of gases [**27**, **31**], in quantum field theory and mechanics [**9**, **13**, **59**] and in the theory of gravitational equilibrium of stars [**31**, **39**]. We refer to the books [**3**, **25**, **29**] and their references for the classical results of problem (1.5). Problem (1.5) with sublinear/linear/superlinear nonlinearities at zero or infinity has been extensively studied. See for example [**1**, **2**, **4**, **6**, **21**, **22**, **40**, **52**]. Amann [**1**] and Rabinowitz [**52**, **53**] studied the global bifurcation phenomena from the trivial solution or infinity. In [**2**], by using of Rabinowitz's bifurcation theorem, Ambrosetti and Hess studied the global behaviour of the component of positive solutions of problem (1.5) involving an asymptotically nonlinearity with/without signum condition. In [**40**], Lions established the existence and the multiplicity results of problem (1.5) by topological degree arguments and variational techniques. In [**5**], Ambrosetti et al. studied the existence of branch of positive solutions for the asymptotically equidiffusive problem, which extends the corresponding ones of [**2**].

Here, we will use bifurcation and topological methods to study the existence/nonexistence and the multiplicity of positive solutions of problem (1.5) involving sublinear/linear/superlinear nonlinearities at zero or infinity with/without signum condition. Furthermore, we can prove theorems 1.1 and 1.2 via the relation of f and g. In addition, we can get theorems 1.3 and theorem 1.4 from theorem 1.1 and theorem 1.2.

As for problem (1.4), we assume that $F : \mathbb{R}_+ \to \mathbb{R}$ is continuous, and there exist $F_0, F_\infty \in (0, +\infty)$ such that $F_0 \neq F_\infty$ and

$$F_0 = \lim_{s\to 0^+} \frac{F(s)}{s},\ F_\infty = \lim_{s\to +\infty} \frac{F(s)}{s}.$$

Without loss of generality, we assume that $F_0 > F_\infty$. Moreover, we also require the signum condition: $0 < F(s)/s < \beta/\lambda$ for any $s > 0$ and any given $\beta,\ \lambda > 0$. Let

$$D = \left\{(\lambda, \beta) \in \mathbb{R}^2 : \lambda > 0, \lambda_1 + \lambda F_\infty \leqslant \beta \leqslant \lambda_1 + \lambda F_0\right\}.$$

By an abstract several-parameter bifurcation theorem of [**26**], we shall establish the following result, which is also one of our main results.

Theorem 1.5. *For any $(\lambda, \beta) \in D$, problem (1.4) has at least one positive solution.*

Analogous to theorems 1.1 and 1.2, we also can consider the various cases of $F_0 \notin (0, +\infty)$ or $F_\infty \notin (0, +\infty)$ with/without signum condition. We leave them to the interested readers.

The rest of this paper is arranged as follows. Some preliminaries are proved in § 2. In § 3, we first study the existence/nonexistence and the multiplicity of positive solutions of problem (1.5) involving a sublinear/linear/superlinear growth nonlinearity at zero or infinity with the signum condition (G2); then we give the proof of Theorem 1.1 and 1.3. The proof of Theorem 1.2 and theorem 1.4 is given in § 4. In the last Section, we give the proof of Theorem 1.5 and some corollaries involving the existence/nonexistence and the multiplicity of positive solutions of problem (1.2).

2. Preliminaries

Let l_κ and h_κ be defined as in the introduction. Then we have

$$h_\kappa'(v) = \frac{1}{l_\kappa'(u)} = \frac{1}{\sqrt{1 + 2\kappa u^2}}.$$

From the definition of l_κ, we can easily see that h_κ is odd, C^∞, $h_\kappa(0) = 0$ and $|h_\kappa'(t)| \leqslant 1$ for all $t \in \mathbb{R}$.

Lemma 2.1. *$h_\kappa(t)/t \to 1$ as $t \to 0$ and $h_\kappa(t)/\sqrt{t} \to (2/\kappa)^{1/4}$ as $t \to +\infty$.*

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Proof. We observe that

$$\lim_{t\to 0^+}\frac{l_\kappa(t)}{t}=\lim_{t\to +0}\frac{\frac{1}{2\sqrt{2\kappa}}\left(\sqrt{2\kappa}t\sqrt{1+2\kappa t^2}+\ln\left(\sqrt{2\kappa}t+\sqrt{1+2\kappa t^2}\right)\right)}{t}=1$$

and

$$\lim_{|t|\to +\infty}\frac{l_\kappa(t)}{t|t|}=\lim_{|t|\to +\infty}\frac{\frac{1}{2\sqrt{2\kappa}}\left(\sqrt{2\kappa}t\sqrt{1+2\kappa t^2}+\ln\left(\sqrt{2\kappa}t+\sqrt{1+2\kappa t^2}\right)\right)}{t|t|}=\sqrt{\frac{\kappa}{2}}.$$

It follows the desired conclusions immediately. □

LEMMA 2.2. *v is a classical solution of problem (1.3) if and only if $u=h_\kappa(v)$ is a classical solution of problem (1.2).*

Proof. Let v be a classical solution of problem (1.3). Then one has that $\nabla u = h'_\kappa(v)\nabla v$ and

$$\Delta u = h''_\kappa(v)|\nabla v|^2 + h'_\kappa(v)\Delta v.$$

It follows that

$$\Delta v = l''_\kappa(u)|\nabla u|^2 + l'_\kappa(u)\Delta u.$$

Since $l'_\kappa(t)=\sqrt{1+2\kappa t^2}$, one has that

$$-\Delta u - \kappa\left(2u|\nabla u|^2 + u^2\Delta u\right) = \lambda g(u).$$

The fact of $\Delta(u^2)u = 2u|\nabla u|^2 + u^2\Delta u$ shows that u satisfies problem (1.2). □

LEMMA 2.3. *One has that*

$$f_0 = g_0,\ \ f_\infty = \frac{g_\infty}{\kappa}$$

for any $\kappa > 0$.

Proof. Note that

$$\lim_{t\to 0^+}\frac{f(t)}{t}=\lim_{t\to 0^+}\frac{\frac{1}{\sqrt{1+2\kappa h_\kappa^2(t)}}g\left(h_\kappa(t)\right)}{t}=\lim_{t\to 0^+}\frac{g\left(h_\kappa(t)\right)}{h_\kappa(t)}\frac{h_\kappa(t)}{t}\frac{1}{\sqrt{1+2\kappa h_\kappa^2(t)}}.$$

Clearly, one has that $\lim_{t\to 0^+} h_\kappa(t) = 0$. This fact combining lemma 2.1 implies that

$$\lim_{t\to 0^+}\frac{f(t)}{t} = g_0.$$

To show $f_\infty = g_\infty/\kappa$, we claim that

$$\lim_{t\to +\infty}\frac{1}{h_\kappa^2(t)} = 0. \tag{2.1}$$

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Indeed, lemma 2.1 shows that

$$\lim_{t\to+\infty}\frac{1}{h_\kappa^2(t)}=\lim_{t\to+\infty}\frac{t}{h_\kappa^2(t)}\frac{1}{t}=\sqrt{\frac{\kappa}{2}}\lim_{t\to+\infty}\frac{1}{t}=0.$$

Thus, we have that

$$\lim_{t\to+\infty}\frac{f(t)}{t}=\lim_{t\to+\infty}\frac{\frac{1}{\sqrt{1+2\kappa h_\kappa^2(t)}}g\left(h_\kappa(t)\right)}{t}=\lim_{t\to+\infty}\frac{g\left(h_\kappa(t)\right)}{h_\kappa^3(t)}\frac{h_\kappa^2(t)}{t}\frac{h_\kappa(t)}{\sqrt{1+2\kappa h_\kappa^2(t)}}. \tag{2.2}$$

From lemma 2.1, we can obtain that

$$\lim_{t\to+\infty}h_\kappa(t)=\lim_{t\to+\infty}\frac{h_\kappa(t)}{\sqrt{t}}\sqrt{t}=\left(\frac{2}{\kappa}\right)^{1/4}\lim_{t\to+\infty}\sqrt{t}=+\infty. \tag{2.3}$$

It follows from (2.1)–(2.3) and lemma 2.1 that

$$\begin{aligned}\lim_{t\to+\infty}\frac{f(t)}{t}&=\lim_{t\to+\infty}\frac{g\left(h_\kappa(t)\right)}{h_\kappa^3(t)}\frac{h_\kappa^2(t)}{t}\frac{h_\kappa(t)}{\sqrt{1+2\kappa h_\kappa^2(t)}}\\&=g_\infty\sqrt{\frac{2}{\kappa}}\lim_{t\to+\infty}\frac{1}{\sqrt{\frac{1}{h_\kappa^2(t)}+2\kappa}}=\frac{g_\infty}{\kappa}.\end{aligned}$$

This completes the proof. □

Under the assumptions of (G1)–(G3) and (G6), if $\kappa=0$ and g_0, $g_\infty\in(0,+\infty)$, it follows from theorem 3.6 of the next section that problem (1.3) has at least one positive solution for all $\lambda\in(\tau_3,+\infty)$. However, if $\kappa\neq 0$ and g_0, $g_\infty\in(0,+\infty)$ satisfying $\kappa g_0\neq g_\infty$, theorem 1.1 (a) shows that there is no positive solution of problem (1.3) for all $\lambda\in(\mu_2',+\infty)$. This almost the opposite difference is mainly due to the fact of $f_\infty=g_\infty/\kappa$. Similarly, we can derive many differences of our results with the Laplace equation, which all illustrate the effect of the second term problem (1.3).

3. Positive solutions with the signum condition

In this section, we always assume that f satisfies the signum condition: $f(s)>0$ for $s>0$; and there exist $f_0,f_\infty\in[0,+\infty]$ such that

$$f_0=\lim_{s\to0^+}\frac{f(s)}{s},\ f_\infty=\lim_{s\to+\infty}\frac{f(s)}{s}.$$

Moreover, we also assume that f satisfies the growth restriction:

(G)

$$\lim_{s\to+\infty}\frac{f(s)}{s^l}=C$$

for some $l\in(1,(N+2)/(N-2))$, where C is a positive constant.

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We first have the following nonexistence results.

LEMMA 3.1. *Assume that there exists a positive constant $\rho > 0$ such that*

$$\frac{f(s)}{s} \geqslant \rho$$

for any $s > 0$. Then there exists $\zeta_* > 0$ such that problem (1.5) has no positive solution for any $\lambda \in (\zeta_*, +\infty)$.

Proof. Let φ_1 be a positive eigenfunction associated with λ_1. If u is a positive solution of problem (1.5), multiply the first equation of problem (1.5) by φ_1, and obtain after integrations by parts

$$\lambda_1 \int_\Omega u\varphi_1 \, \mathrm{d}x = \lambda \int_\Omega \frac{f(u)}{u} u\varphi_1 \, \mathrm{d}x \geqslant \lambda\rho \int_\Omega u\varphi_1 \, \mathrm{d}x.$$

It follows that $\lambda \leqslant \lambda_1/\rho$. □ □

LEMMA 3.2. *Assume that there exists a positive constant $\varrho > 0$ such that*

$$\frac{f(s)}{s} \leqslant \varrho$$

for any $s > 0$. Then there exists $\eta_* > 0$ such that problem (1.5) has no positive solution for any $\lambda \in (0, \eta_*)$.

Proof. If u is a positive solution of problem (1.5), similar to that of lemma 3.1, we can obtain $\lambda \geqslant \lambda_1/\varrho$. □

Let

$$E = \left\{u \in C^1\left(\overline{\Omega}\right) : u = 0 \quad \text{on } \partial\Omega\right\}$$

with the usual norm

$$\|u\| = \max_{\overline{\Omega}} |u| + \max_{\overline{\Omega}} |\nabla u| .$$

Set

$$\mathbb{P} := \left\{u \in E : u > 0 \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial \omega} < 0 \quad \text{on } \partial\Omega\right\},$$

where ω is the outward pointing normal to $\partial\Omega$.

By standard elliptic regularity theory (see [**29**, theorem 8.16, theorem 8.34]), we know that any solution in E of problem (1.5) belongs to $C^{1,\alpha}(\overline{\Omega})$ with some $\alpha \in (0, 1)$ under the condition of (G). Furthermore, by the Lagrange mean theorem, we can easily verify that it is also a classical solution of problem (1.5).

The following theorem is the result of the existence of positive solutions of problem (1.5) with linear nonlinearities at zero and infinity.

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THEOREM 3.3. *If f_0, $f_\infty \in (0, +\infty)$ with $f_\infty \neq f_0$, then problem (1.5) has at least one positive solution for every* $\lambda \in (\min\{\lambda_1/f_0, \lambda_1/f_\infty\}, \max\{\lambda_1/f_\infty, \lambda_1/f_0\})$.

Proof. Let $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ be such that

$$f(s) = f_0 s + \xi(s)$$

with

$$\lim_{s \to 0^+} \frac{\xi(s)}{s} = 0.$$

Let us consider

$$\begin{cases} -\Delta u = \lambda f_0 u + \lambda \xi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

as a bifurcation problem from the trivial solution axis.

Applying theorem 2.12 of [**52**] to problem (3.1), there exists a continuum $\mathscr{C}$ of nontrivial solutions of problem (3.1) emanating from $(\lambda_1/f_0, 0)$ such that $\mathscr{C} \subset ((\mathbb{R} \times \mathbb{P}) \cup \{(\lambda_1/f_0, 0)\})$, meets ∞ in $\mathbb{R} \times E$.

It is sufficient to show that $\mathscr{C}$ joins $(\lambda_1/f_0, 0)$ to $(\lambda_1/f_\infty, +\infty)$. Let $(\lambda_n, u_n) \in \mathscr{C}$ where $u_n \not\equiv 0$ satisfies $\lambda_n + \|u_n\| \to +\infty$. Lemma 3.2 implies that $\lambda_n > 0$ for all $n \in \mathbb{N}$. It follows from lemma 3.1 that there exists a constant M such that $\lambda_n \in (0, M]$ for any $n \in \mathbb{N}$. Therefore, we get that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty.$$

Set $\overline{u}_n = u_n/\|u_n\|$. Since $\overline{u}_n$ is bounded in E, after taking a subsequence if necessary, we have that $\overline{u}_n \rightharpoonup \overline{u}$ for some $\overline{u} \in E$. Then by an argument similar to that of [**19**, theorem 5.1], we obtain that

$$-\Delta \overline{u} = \overline{\mu} f_\infty \overline{u},$$

where $\overline{\mu} = \lim_{n \to +\infty} \lambda_n$, choosing a subsequence and relabelling it if necessary. It is clear that $\|\overline{u}\| = 1$ and $\overline{u} \in \overline{\mathscr{C}} \subseteq \mathscr{C}$ since $\mathscr{C}$ is closed in $\mathbb{R} \times E$. So one has that $\overline{\mu} = \lambda_1/f_\infty$. Therefore, $\mathscr{C}$ joins $(\lambda_1/f_0, 0)$ to $(\lambda_1/f_\infty, +\infty)$. □

REMARK 3.4. It is easy to verify that $(\lambda_1/f_0, 0)$ is the unique bifurcation point of positive solutions of problem (1.5) from the trivial solution axis. Moreover, note that the conclusion of theorem 3.3 also obtained in [**2**] with the more strong condition on f and more complicated argument.

REMARK 3.5. In view of theorem 3.3, we can see that if $f_0, f_\infty \in (0, +\infty)$ then there exist four positive constants τ_1, τ_1', τ_2 and τ_2' with $\tau_1 \geqslant \tau_1'$ and $\tau_2 \leqslant \tau_2'$ such that problem (1.5) has at least one positive solution for all $\lambda \in (\tau_1, \tau_2)$ and has no positive solution for all $\lambda \in (0, \tau_1') \cup (\tau_2', +\infty)$.

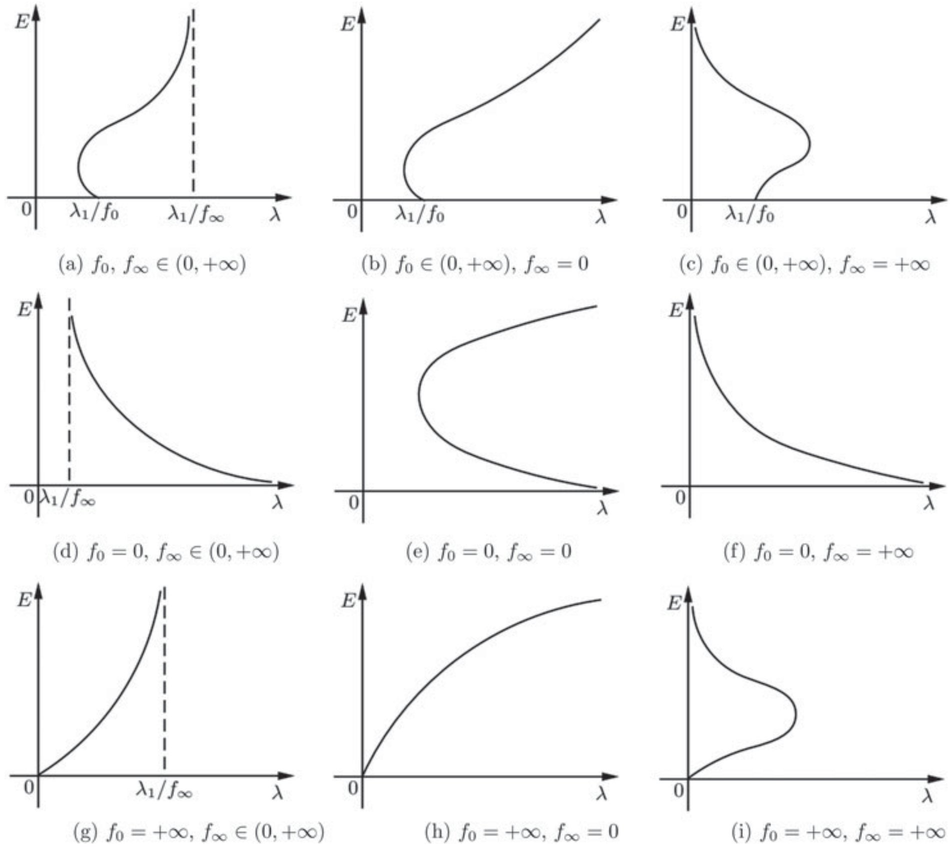

Fig. 1 - B/W online, B/W in print

Figure 1. Bifurcation diagrams of theorems 3.3–3.20.(a) f_0, $f_\infty \in (0, +\infty)$, (b) $f_0 \in (0, +\infty)$, $f_\infty = 0$, (c) $f_0 \in (0, +\infty)$, $f_\infty = +\infty$, (d) $f_0 = 0$, $f_\infty \in (0, +\infty)$, (e) $f_0 = 0$, $f_\infty = 0$, (f) $f_0 = 0$, $f_\infty = +\infty$, (g) $f_0 = +\infty$, $f_\infty \in (0, +\infty)$, (h) $f_0 = +\infty$, $f_\infty = 0$, (i) $f_0 = +\infty$, $f_\infty = +\infty$.

Proof. Clearly, $f_0, f_\infty \in (0, +\infty)$ implies that there exist two positive constants M_1 and M_2 such that

$$M_1 \leqslant \frac{f(s)}{s} \leqslant M_2 \quad \text{for any } s > 0.$$

If (λ, u) is a positive solution pair of problem (1.5), it follows from lemmas 3.1 and 3.2 that $\lambda \geqslant \lambda_1/M_2 := \tau_1'$ and $\lambda \leqslant \lambda_1/M_1 := \tau_2'$. So, problem (1.5) has no positive solution for all $\lambda \in (0, \tau_1') \cup (\tau_2', +\infty)$. The existence of τ_1 and τ_2 can be seen from the global structure of $\mathscr{C}$, see (a) of figure 1. □

Theorem 3.6. *If $f_0 \in (0, +\infty)$ and $f_\infty = 0$, then problem (1.5) has at least one positive solutions for every $\lambda \in (\lambda_1/f_0, +\infty)$.*

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Proof. Considering the proof of Theorem 3.3, we only need to show that $\mathscr{C}$ joins $(\lambda_1/f_0, 0)$ to $(+\infty, +\infty)$. Lemma 3.2 implies that $\lambda = 0$ does not the blow-up point of $\mathscr{C}$.

Firstly, we show that $\mathscr{C}$ is unbounded in the direction of E. Suppose on the contrary that $\mathscr{C}$ is bounded in the direction of E. So $\mathscr{C}$ is unbounded in the direction of λ, that is to say there exist $(\lambda_n, u_n) \in \mathscr{C}$ and a positive constant M such that $\lambda_n \to +\infty$ as $n \to +\infty$ and $\|u_n\| \leqslant M$ for any $n \in \mathbb{N}$. It follows that $f(u_n)/u_n \geqslant \delta$ for some positive constant δ and all $n \in \mathbb{N}$. Lemma 3.1 implies that $u_n \equiv 0$ for n large enough, which is absurd.

To complete the proof, it suffices to show that the unique blow-up point of $\mathscr{C}$ is $(+\infty, 0)$. Suppose, by contradiction, that there exists $\widehat{\lambda} > 0$ such that $(\widehat{\lambda}, 0)$ is a blow-up point of $\mathscr{C}$. Then there exists a sequence $\{(\lambda_n, u_n)\}$ such that $\lim_{n\to+\infty} \lambda_n = \widehat{\lambda}$ and $\lim_{n\to+\infty} \|u_n\| = +\infty$.

To deduce a contradiction, we consider the following auxiliary problem

$$\begin{cases} -\Delta u = \psi & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

for a given $\psi \in L^{r/(r-1)}(\Omega)$, where $r \in (1, 2^*)$ with $2^* = (2N)/(N-2)$. We have known that for every given $\psi \in L^{r/(r-1)}(\Omega)$ there is a unique solution u to problem (3.2) (see [**24**]), which is denoted by $\Psi(\psi)$. It is well known that $\Psi : L^\infty(\Omega) \to E$ is completely continuous and linear (see [**29**]). The definition of f implies that $u \in C^{1+\delta}(\overline{\Omega})$ with some constant $\delta \in (0, 1)$ for every weak solution u of problem (1.5). Now problem (1.5) can be equivalently written as

$$u = \Psi(\lambda f(u(x))). \tag{3.3}$$

Now, let

$$\widetilde{f}(u) = \max_{0 \leqslant |s| \leqslant u} |f(s)|,$$

then $\widetilde{f}$ is nondecreasing with respect to u. Define

$$\overline{f}(u) = \max_{u/2 \leqslant |s| \leqslant u} |f(s)|.$$

Then we can see that

$$\lim_{u\to+\infty} \frac{\overline{f}(u)}{u} = 0 \quad \text{and} \quad \widetilde{f}(u) \leqslant \widetilde{f}\left(\frac{u}{2}\right) + \overline{f}(u).$$

It follows that

$$\limsup_{u\to+\infty} \frac{\widetilde{f}(u)}{u} \leqslant \limsup_{u\to+\infty} \frac{\widetilde{f}(u/2)}{u} = \limsup_{u/2\to+\infty} \frac{\widetilde{f}(u/2)}{2(u/2)}.$$

So we have that

$$\lim_{u\to+\infty} \frac{\widetilde{f}(u)}{u} = 0. \tag{3.4}$$

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Further, it follows from (3.4) that

$$\frac{f(u)}{\|u\|} \leqslant \frac{\widetilde{f}\,(|u|)}{\|u\|} \leqslant \frac{\widetilde{f}\,(\|u\|_\infty)}{\|u\|} \leqslant \frac{\widetilde{f}\,(\|u\|)}{\|u\|} \to 0 \quad \text{as } \|u\| \to +\infty. \tag{3.5}$$

Let $w_n = u_n/\|u_n\|$. It follows from (3.3) that $w_n \to 0$ in E. This contradicts the fact of $\|w_n\| = 1$. □

REMARK 3.7. Clearly, theorem 3.6 and lemma 3.2 imply that if $f_0 \in (0, +\infty)$ and $f_\infty = 0$ then there exist two positive constants τ_3 and τ_3' with $\tau_3' \leqslant \tau_3$ such that problem (1.5) has at least one positive solution for all $\lambda \in (\tau_3, +\infty)$ and has no positive solution for all $\lambda \in (0, \tau_3')$, see (b) of figure 1.

By the conclusion of theorem 3.9 and the similar argument of theorem 1.4 of [**40**], we can immediately get the following corollary.

COROLLARY 3.8. *Besides the assumptions of theorem 3.6, we also assume that f is local Lipschitz continuous. If $\tau_3 < \lambda_1/f_0$, then for $\lambda \in (\tau_3, \lambda_1/f_0)$, there exist at least two positive solutions of problem (1.5) which are ordered. In addition, there exists at least one solution of problem (1.5) with $\lambda = \tau_3$.*

THEOREM 3.9. *If $f_0 \in (0, +\infty)$ and $f_\infty = +\infty$, then problem (1.5) has at least one positive solution for every $\lambda \in (0, \lambda_1/f_0)$.*

Proof. In view of theorem 3.3, we only need to show that $\mathscr{C}$ joins $(\lambda_1/f_0, 0)$ to $(0, +\infty)$. Lemma 3.1 implies that $\mathscr{C}$ is bounded in the direction of λ. For any $(\lambda, u) \in \mathscr{C}$ with $\lambda > 0$, by virtue of [**28**, theorem 1.1], [**29**, theorem 8.33] and the condition of (G), we have that $\|u\| \leqslant M$ for some positive constant M depending on f, N, λ and Ω. So $(0, +\infty)$ is the unique blow-up point of $\mathscr{C}$. □

REMARK 3.10. Under the assumptions of theorem 3.9, in view of lemma 3.1, we can see that there exist two positive constants τ_4 and τ_4' with $\tau_4 \leqslant \tau_4'$ such that problem (1.5) has at least one positive solution for all $\lambda \in (0, \tau_4)$ and has no positive solution for all $\lambda \in (\tau_4', +\infty)$, see (c) of figure 1.

By using the conclusion of theorem 3.9 and the similar argument of theorem 1.2 of [**40**], we can get the following corollary.

COROLLARY 3.11. *Besides the assumptions of theorem 3.9, we also assume that f is local Lipschitz continuous. If $\tau_4 > \lambda_1/f_0$, then for $\lambda \in (\lambda_1/f_0, \tau_4)$, there exist at least two positive solutions of problem (1.5) which are ordered. In addition, there exists at least one solution of problem (1.5) with $\lambda = \tau_4$.*

In [**40**], Lions conjectured that the convexity of the domain and the assumption $(7')$ are not needed in theorem 1.2. Theorem 3.9 and corollary 3.11 give the confirmation answer to this conjecture.

THEOREM 3.12. *If $f_0 = 0$ and $f_\infty \in (0, +\infty)$, then problem (1.5) has at least one positive solution for every $\lambda \in (\lambda_1/f_\infty, +\infty)$.*

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Proof. If (λ, u) is any solution of problem (1.5) with $\|u\| \not\equiv 0$, dividing problem (1.5) by $\|u\|^2$ and setting $w = u/\|u\|^2$ yield

$$\begin{cases} -\Delta w = \lambda \frac{f_\infty u + \eta(u)}{\|u\|^2} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where $\eta(s) = f(s) - f_\infty s$. Since $f_\infty \in (0, +\infty)$, one has that

$$\lim_{s \to +\infty} \frac{\eta(s)}{s} = 0.$$

Define

$$\widetilde{\eta}(w) = \begin{cases} \|w\|^2 \eta\left(\frac{w}{\|w\|^2}\right) & \text{if } w \neq 0, \\ 0 & \text{if } w = 0. \end{cases}$$

Clearly, problem (3.6) is equivalent to

$$\begin{cases} -\Delta w = \lambda\left(f_\infty w + \widetilde{\eta}(w)\right) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

It is obvious that $(\lambda, 0)$ is always the solution of problem (3.7). Similar to (3.5), we can show that

$$\frac{\widetilde{\eta}(w)}{\|w\|} \to 0 \quad \text{as } \|w\| \to 0.$$

By theorem 2.12 of [**52**], we obtain a continuum $\mathscr{D}$ of nontrivial solutions of problem (3.7) emanating from $(\lambda_1/f_\infty, 0)$ such that $\mathscr{D} \subset ((\mathbb{R} \times \mathbb{P}) \cup \{(\lambda_1/f_\infty, 0)\})$, meets ∞ in $\mathbb{R} \times E$.

Now applying the inversion $w \to w/\|w\|^2 = u$, we obtain a continuum $\mathscr{C}$ of nontrivial solutions of problem (1.5) emanating from $(\lambda_1/f_\infty, +\infty)$ such that either it is unbounded in the direction of λ or meets $\{(\lambda, 0) : \lambda \in \mathbb{R}_+\}$. By an argument similar as in theorem 3.6, we can show that the latter case is impossible. □

Remark 3.13. Under the assumptions of theorem 3.12, we note that there exist two positive constant τ_5 and τ_5' with $\tau_5' \leqslant \tau_5$ such that problem (1.5) has at least one positive solution for all $\lambda \in (\tau_5, +\infty)$ and has no positive solution for all $\lambda \in (0, \tau_5')$, see (d) of figure 1.

Theorem 3.14. *If $f_0 = 0$ and $f_\infty = 0$, then there exists $\lambda_* > 0$ such that for any $\lambda \in (\lambda_*, +\infty)$, problem (1.5) has at least two positive solutions.*

Proof. Define

$$f^n(s) = \begin{cases} \frac{1}{n}s, & s \in \left[0, \frac{1}{n}\right], \\ \left(f\left(\frac{2}{n}\right) - \frac{1}{n^2}\right) n \left(s - \frac{1}{n}\right) + \frac{2}{n^2}, & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ f(s), & s \in \left[\frac{2}{n}, +\infty\right). \end{cases}$$

Now, consider the following problem

$$\begin{cases} -\Delta u = \lambda f^n(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Clearly, we can see that $\lim_{n\to+\infty} f^n(s) = f(s)$, $f_0^n = 1/n$ and $f_\infty^n = f_\infty = 0$. Theorem 3.6 implies that there exists a sequence unbounded continua $\mathscr{C}_n$ emanating from $(n\lambda_1, 0)$ and joining to $(+\infty, +\infty) := z^*$, and $(+\infty, 0)$ is the unique blow-up point of $\mathscr{C}_n$.

Let $\mathscr{C} = \limsup_{n\to+\infty} \mathscr{C}_n$. For any $(\lambda, u) \in \mathscr{C}$, the definition of superior limit (see [**61**]) shows that there exists a sequence $(\lambda_n, u_n) \in \mathscr{C}_n$ such that $(\lambda_n, u_n) \to (\lambda, u)$ as $n \to +\infty$. Then a continuity argument shows that u is a solution of problem (1.5).

By proposition 2 of [**16**], for each $\epsilon > 0$ there exists an N_0 such that for all $n > N_0$, $\mathscr{C}_n \subset V_\epsilon(\mathscr{C})$ with $V_\epsilon(\mathscr{C})$ denoting the ϵ-neighbourhood of $\mathscr{C}$. It follows that

$$(n\lambda_1, +\infty) \subseteq \operatorname{Proj}(\mathscr{C}_n) \subseteq \operatorname{Proj}(V_\epsilon(\mathscr{C})),$$

where $\operatorname{Proj}(\mathscr{C}_n)$ denotes the projection of $\mathscr{C}_n$ on $\mathbb{R}$. So, we have that $(n\lambda_1 + \epsilon, +\infty) \subseteq \operatorname{Proj}(\mathscr{C})$. Therefore, we have $\mathscr{C} \setminus \{\infty\} \neq \emptyset$.

Let

$$S = \{(+\infty, u) : 0 < \|u\| < +\infty\}.$$

For any fixed $n \in \mathbb{N}$, we claim that $\mathscr{C}_n \cap S = \emptyset$. Otherwise, there exists a sequence $(\lambda_m, u_m) \in \mathscr{C}_n$ such that $(\lambda_m, u_m) \to (+\infty, u_*) \in S$ with $0 < \|u_*\| < +\infty$. It follows that $\|u_m\| \leqslant M_n$ for some constant $M_n > 0$. It implies that $f^n(u_m)/u_m \geqslant \delta_n$ for some positive constant δ_n and all $m \in \mathbb{N}$. Lemma 3.1 implies that $u_m \equiv 0$ for m large enough, which contradicts the fact of $\|u_*\| > 0$. It follows that $(\cup_{n=1}^{+\infty}\mathscr{C}_n) \cap S = \cup_{n=1}^{+\infty}(\mathscr{C}_n \cap S) = \emptyset$. Since $\mathscr{C} \subseteq (\cup_{n=1}^{+\infty}\mathscr{C}_n)$, one has that $\mathscr{C} \cap S = \emptyset$. Furthermore, set

$$S' := \{(\lambda, +\infty) : 0 \leqslant \lambda < +\infty\}.$$

Since $(+\infty, 0)$ is the unique blow-up point of $\mathscr{C}_n$, we have that $\mathscr{C}_n \cap S' = \emptyset$. Then reasoning as the above, we have that $\mathscr{C} \cap S' = \emptyset$. Hence, $\mathscr{C} \cap (S \cup S') = \emptyset$. Taking $z_* = (+\infty, 0)$, clearly $z_* \in \liminf_{n\to+\infty} \mathscr{C}_n$ with $\|z_*\|_{\mathbb{R}\times E} = +\infty$. Therefore, we obtain that $\mathscr{C} \cap \{\infty\} = \{z_*, z^*\}$.

The compactness of Ψ implies that $(\cup_{n=1}^{+\infty}\mathscr{C}_n) \cap \overline{B}_R$ is pre-compact. Lemma 3.1 of [**18**] implies that $\mathscr{C} = \limsup_{n\to+\infty} \mathscr{C}_n$ is connected. We claim that $\mathscr{C} \cap ([0, +\infty) \times \{0\}) = \emptyset$. Otherwise, there exists a sequence $\{(\mu_n, u_n)\}$ such that $\lim\limits_{n\to+\infty} \mu_n = \mu_M$

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and $\lim_{n\to+\infty} \|u_n\| = 0$ as $n \to +\infty$. Let $w_n = u_n / \|u_n\|$ and w_n should be the solutions to the following problem

$$w = \mu_n \Psi\left(\frac{f(u_n)}{u_n} w\right).$$

By the compactness of Ψ, we obtain that for some convenient subsequence $w_n \to w_0$ as $n \to +\infty$. Letting $n \to +\infty$, we obtain that $w_0 \equiv 0$. This contradicts $\|w_0\| = 1$.

See (e) of figure 1 for the global structure of $\mathscr{C}$. Since $\mathscr{C}$ is connected with $\mathscr{C} \setminus \{\infty\} \neq \emptyset$, $\mathscr{C} \cap ([0, +\infty) \times \{0\}) = \emptyset$ and $\mathscr{C} \cap \{\infty\} = \{z_*, z^*\}$, there exist at least two positive solutions on $\mathscr{C}$ when λ is sufficiently large. □

REMARK 3.15. From theorem 3.14 and lemma 3.2, we can also see that there exist two positive constants τ_6 and τ_6' with $\tau_6 \geqslant \tau_6'$ such that problem (1.5) has at least one positive solution for all $\lambda \in [\tau_6, \lambda_*]$ and has no positive solution for all $\lambda \in (0, \tau_6')$.

In [**8**, **40**, **54**], the authors got some existence results with $\overline{\lim}_{s\to 0} f(s)/s \leqslant 0$ and $f_\infty = 0$. Note that we do not need Ω that is star shaped, which is essential in [**8**].

THEOREM 3.16. *If $f_0 = 0$ and $f_\infty = +\infty$, then problem (1.5) has at least one positive solutions for any $\lambda \in (0, +\infty)$.*

Proof. Using an argument similar to that of theorem 3.14, in view of theorem 3.9, we can easily get the results of this theorem. See (f) of figure 1 for the global bifurcation diagram. □

THEOREM 3.17. *If $f_0 = +\infty$ and $f_\infty \in (0, +\infty)$, then for any $\lambda \in (0, \lambda_1/f_\infty)$, problem (1.5) has at least one positive solution.*

Proof. From the proof of Theorem 3.12, we know that there exists a continuum $\mathscr{D}$ of nontrivial solutions of problem (1.5) emanating from $(\lambda_1/f_\infty, +\infty)$ such that either it is unbounded in the direction of λ or meets $\{(\lambda, 0) : \lambda \in \mathbb{R}_+\}$. Lemma 3.1 implies that the former case is impossible. So there exists $(\lambda_n, u_n) \in \mathscr{D}$ such that $(\lambda_n, u_n) \to (\lambda_*, 0)$ as $n \to +\infty$ with $u_n \neq 0$. In view of $f_0 = +\infty$, if $\lambda_* > 0$, we obtain

$$\lambda_n \frac{f(u_n)}{u_n} > \rho\lambda_1$$

for some $\rho > 1$ and n large enough. Multiply the first equation of problem (1.5) by φ_1, after integrations by parts, we obtain

$$\lambda_1 \int_\Omega u_n\varphi_1 \,\mathrm{d}x = \lambda_n \int_\Omega \frac{f(u_n)}{u_n} u_n\varphi_1 \,\mathrm{d}x > \lambda_1\rho \int_\Omega u_n\varphi_1 \,dx,$$

which is a contradiction. So we conclude that $\lambda_* = 0$. □

REMARK 3.18. Similarly to remark 3.10, if $f_0 = +\infty$ and $f_\infty \in (0, +\infty)$, there exist two positive constant τ_7 and τ_7' with $\tau_7 \leqslant \tau_7'$ such that problem (1.5) has at least one

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positive solution for all $\lambda \in (0, \tau_7)$ and has no positive solution for all $\lambda \in (\tau_7', +\infty)$, see (g) of figure 1.

THEOREM 3.19. *If $f_0 = +\infty$ and $f_\infty = 0$, then for any $\lambda \in (0, +\infty)$, problem (1.5) has at least one positive solution.*

Proof. Define

$$f^n(s) = \begin{cases} ns, & s \in \left[0, \frac{1}{n}\right], \\ n\left(f\left(\frac{2}{n}\right) - 1\right)\left(s - \frac{1}{n}\right) + 1, & s \in \left(\frac{1}{n}, \frac{2}{n}\right), \\ f(s), & s \in \left[\frac{2}{n}, +\infty\right) \end{cases}$$

and consider problem (3.8) again. Clearly, we can see that $\lim_{n\to+\infty} f^n(s) = f(s)$, $f_0^n = n$ and $f_\infty^n = f_\infty = 0$. From the proof of Theorem 3.6, we know that there exists a sequence unbounded continua $\mathscr{C}_n$ emanating from $(\lambda_1/n, 0)$ and joining to $(+\infty, +\infty)$.

Taking $z^* = (0, 0)$, clearly $z^* \in \liminf_{n\to+\infty} \mathscr{C}_n$. Lemma 2.5 of [**16**] implies that $\mathscr{C} = \limsup_{n\to+\infty} \mathscr{C}_n$ is unbounded and connected such that $z^* \in \mathscr{C}$ and $(+\infty, +\infty) \in \mathscr{C}$. Likely to theorem 3.17, we can show that $\mathscr{C} \cap ((0, +\infty) \times \{0\}) = \emptyset$. See (h) of figure 1 for the global structure of $\mathscr{C}$. □

THEOREM 3.20. *If $f_0 = +\infty$ and $f_\infty = +\infty$, then there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, problem (1.5) has at least two positive solutions.*

Proof. By an argument similar to that of theorem 3.19 and the conclusions of theorem 3.9, we can obtain the desired conclusion, see (i) of figure 1. □

REMARK 3.21. By theorem 3.20 and lemma 3.1, we can see that there exist two positive constants τ_8 and τ_8' with $\tau_8 \leqslant \tau_8'$ such that problem (1.5) has at least one positive solution for all $\lambda \in [\lambda^*, \tau_8]$ and has no positive solution for all $\lambda \in (\tau_8', +\infty)$.

REMARK 3.22. In some particular case, theorems 3.3– 3.20 can be optimal, see [**32**, **33**, **46**, **47**, **56**–**58**] and their references.

Proof of Theorem 1.1. The definition of h implies that $h_\kappa(t) > 0$ for $t > 0$. This fact combining with the assumption of (G2) shows that f satisfies the signum condition. Lemma 2.3 shows that

$$f_0 = g_0, \ f_\infty = \frac{g_\infty}{\kappa}.$$

It is not difficult to show that (G6) implies f satisfying the subcritical growth condition (G). Then the desired conclusions can be deduced from theorems 3.3– 3.20 and remarks 3.5–3.21 immediately. □

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Proof of Theorem 1.3. (a) For any $\kappa \in (g_\infty/\lambda_1, +\infty)$, since $\lambda_1 < g_0$, it is easy to verify that

$$\frac{\lambda_1}{g_0} < 1 < \frac{\lambda_1 \kappa}{g_\infty}.$$

From theorem 3.3 we can easily see that $\mu_1 \leqslant \lambda_1/g_0$ and $\mu_2 \geqslant \lambda_1\kappa/g_\infty$. Then the desired existence can be derived from theorem 1.1 (a) immediately.

(b) The proof is similar to that of (a).

(c) For any $\kappa \in (0, g_\infty/\lambda_1)$, we get that

$$1 > \frac{\lambda_1 \kappa}{g_\infty}.$$

Theorem 3.12 shows that $\mu_5 \leqslant \lambda_1\kappa/g_\infty$. So theorem 1.1 (d) implies the desired conclusion.

(d) For any $\kappa \in (g_\infty/\lambda_1, +\infty)$, we have that

$$1 < \frac{\lambda_1 \kappa}{g_\infty}.$$

Theorem 3.19 shows that $\mu_8 \geqslant \lambda_1\kappa/g_\infty$. Hence theorem 1.1 (g) implies that problem (1.3) has at least one positive solution.

□

4. Positive solutions without the signum condition

In this section, we study the problem (1.5) without the signum condition and give the proof of Theorems 1.2 and 1.4. From now on, we assume that there exists one positive constant β such that $f(\beta) = 0$, $f(s)s > 0$ for $s \in (0, \beta) \cup (\beta, +\infty)$ and there exists a constant $\gamma > 0$ such that

$$\lim_{s \to \beta^-} \frac{f(s)}{\beta - s} = \gamma.$$

THEOREM 4.1. *Assume that f_0, $f_\infty \in (0, +\infty)$ with $f_0 \neq f_\infty$. Then*

(i) *if $\lambda \in (\min\{\lambda_1/f_\infty, \lambda_1/f_0\}, \max\{\lambda_1/f_\infty, \lambda_1/f_0\}]$, problem (1.5) has at least one positive solution;*

(ii) *if $\lambda \in (\max\{\lambda_1/f_\infty, \lambda_1/f_0\}, +\infty)$, problem (1.5) has at least two positive solutions.*

Proof. Firstly, we define

$$\widetilde{f}(s) = \begin{cases} f(s) & \text{if } 0 \leqslant s \leqslant \beta, \\ 0 & \text{otherwise} \end{cases}$$

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and consider the following problem

$$\begin{cases} -\Delta u = \lambda \widetilde{f}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

Applying theorem 2.12 of [**52**] to problem (4.1), there exists a continuum $\mathscr{C}$ of non-trivial solutions of problem (4.1) emanating from $(\lambda_1/f_0, 0)$ such that $\mathscr{C} \subset ((\mathbb{R} \times \mathbb{P}) \cup \{(\lambda_1/f_0, 0)\})$, meets ∞ in $\mathbb{R} \times E$.

The strong maximum principle implies that $u \leqslant \beta$ for any $(\lambda, u) \in \mathscr{C}$. It follows that u is also a solution of problem (1.5) for any $(\lambda, u) \in \mathscr{C}$. Next, we show that the projection of $\mathscr{C}$ on $\mathbb{R}$ is unbounded. It is sufficient to show that the set $\{(\lambda, u) \in \mathscr{C} : \lambda \in (0, d]\}$ is bounded for any fixed $d \in (0, +\infty)$. Arguing by contradiction, if there exists $(\lambda_n, u_n) \in \mathscr{C}$, $n \in \mathbb{N}$, such that $\lambda_n \to \mu \leqslant d$, $u_n \to +\infty$ as $n \to +\infty$. Let $w_n = u_n / \|u_n\|$. Then we have that

$$w_n = \Psi\left(\lambda_n \frac{f\left(u_n(x)\right)}{\|u_n\|}\right).$$

Clearly, we have that

$$f(u_n) \leqslant \max_{[0,\beta]} |f(s)|.$$

It means that

$$\lambda_n \frac{f(u_n)}{\|u_n\|} \to 0$$

as $n \to +\infty$. By the compactness of Ψ, we obtain that for some convenient subsequence $w_n \to 0$ as $n \to +\infty$, which contradicts the fact of $\|w_0\| = 1$. This together with the fact that $\mathscr{C}$ joins $(\lambda_1/f_0, 0)$ to infinity yields that

$$(\lambda_1/f_0, +\infty) \subseteq \operatorname{Proj}(\mathscr{C}).$$

Next we study the bifurcation phenomenon of problem (1.5) from infinity. Consider

$$\begin{cases} -\Delta u = \lambda f_\infty u + \lambda \eta(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

as a bifurcation problem from infinity. Applying theorem 2.28 of [**53**] to problem (4.2), there exists a continuum $\mathscr{D}$ of solutions of problem (1.5) meeting $(\lambda_1/f_\infty, \infty)$ and satisfying at least one of the alternatives of theorem 1.6 of [**53**]. The strong maximum principle implies that $\mathscr{D} \subset ((\mathbb{R} \times \mathbb{P}) \cup \{(\lambda_1/f_\infty, +\infty)\})$. In addition, it is not difficult to verify that $(\lambda_1/f_\infty, \infty)$ is the unique bifurcation point of positive solutions of problem (1.5) from ∞.

Next, we shall show that these two components are disjoint. Let

$$Y = \left\{u \in C\left(\overline{\Omega}\right) : u = 0 \quad \text{on } \partial\Omega\right\}$$

with the usual norm

$$\|u\|_\infty = \max_{\overline{\Omega}} |u|.$$

It is enough to show that $\mathscr{C}$ and $\mathscr{D}$ are disjoint in $\mathbb{R} \times Y$. We first claim that $\mathscr{D}$ is unbounded in the direction of Y. It suffices to show that $(\lambda_1/f_\infty, 0)$ is a blow-up point of $\mathscr{D}$ in $\mathbb{R} \times Y$. Otherwise, there exists $M > 0$ such that $\|u_n\|_\infty \leqslant M$ for any $(\lambda_n, u_n) \in \mathscr{D}$ with $\lambda_n \to \lambda_1/f_\infty$ as $n \to +\infty$. Applying [**29**, theorem 8.33], we get that $\|u_n\| \leqslant M'$ for some positive constant M' depending on f, N, M, λ_1 and $\partial\Omega$, which contradicts the fact of $\mathscr{D}$ meeting $(\lambda_1/f_\infty, \infty)$.

Suppose, by contradiction, that $\mathscr{C} \cap \mathscr{D} \neq \emptyset$ in $\mathbb{R} \times Y$. Since $\mathscr{D}$ is unbounded in the direction of Y and meets $\mathscr{C}$, there exists $(\lambda, u) \in (\mathscr{C} \cup \mathscr{D})$ such that $\max_{\overline{\Omega}} u = \beta$. Clearly, there exists $0 < m < +\infty$ such that $f(s) \leqslant m(\beta - s)$ for any $s \in [0, \beta]$. Now, let us consider the following problem

$$\begin{cases} -\Delta(\beta - u) + \lambda m\,(\beta - u) = \lambda m\,(\beta - u) - \lambda f(u) & \text{in } \Omega, \\ \beta - u > 0 & \text{on } \partial\Omega. \end{cases}$$

The strong maximum principle of [**29**] implies that $\beta > u$ in Ω. This is a contradiction.

Thus 1^o of theorem 1.6 of [**53**] does not occur. So 2^o of theorem 1.6 of [**53**] occurs. We claim that $\mathscr{D} - \mathscr{M}$ has an unbounded projection on $\mathbb{R}$. Now we show that the case of $\mathscr{D} - \mathscr{M}$ meeting $\lambda_j \times \{\infty\}$ for some $j > 1$ is impossible, where λ_j denotes the jth of $-\Delta$ with 0-Dirichlet boundary condition. Assume on the contrary that $\mathscr{D} - \mathscr{M}$ meets $\lambda_j \times \{\infty\}$ for some $j > 1$. So there exists a neighbourhood $\widetilde{\mathscr{N}} \subset \widetilde{\mathscr{M}}$ of $\lambda_j \times \{\infty\}$ such that u must change sign for any $(\lambda, u) \in (\mathscr{D} - \mathscr{M}) \cap (\widetilde{\mathscr{N}} \setminus (\lambda_j \times \{\infty\}))$, where $\widetilde{\mathscr{M}}$ is a neighbourhood of $\lambda_j \times \{\infty\}$ which satisfies the assumptions of theorem 1.6 of [**53**]. This contradicts the fact of $\mathscr{D} \subset ((\mathbb{R} \times \mathbb{P}) \cup \{(\lambda_1/f_\infty, +\infty)\})$. Now the desired conclusions can be seen. □

See (a) of figure 2 for the global structures of $\mathscr{C}$ and $\mathscr{D}$. Since we are not requiring continuously differentiable property of f, these results of theorem 4.1 improve the corresponding ones of [**2**].

REMARK 4.2. From the argument of theorem 4.1, we can easily see that problem (1.5) has at least two positive solutions for $\lambda \in (\max\{\lambda_1/f_\infty, \lambda_1/f_0\}, +\infty)$ if f_0, $f_\infty \in (0, +\infty)$ with $f_0 = f_\infty$. Moreover, the assumptions of theorem 4.1 implies that there exists a positive constant $\varrho > 0$ such that

$$\frac{f(s)}{s} \leqslant \varrho$$

for any $s > 0$. So it follows from lemma 3.2 that there exists $\tau_9 > 0$ such that problem (1.5) has no positive solution for any $\lambda \in (0, \tau_9)$.

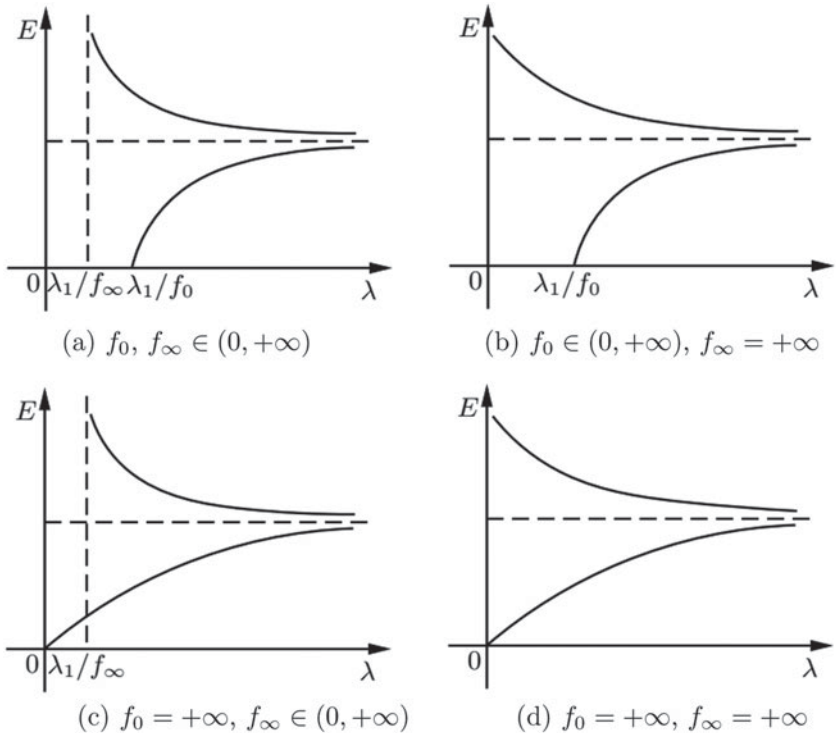

Figure 2. Bifurcation diagrams of Theorems 4.1– 4.4 and corollary 4.5.(a) f_0, $f_\infty \in (0, +\infty)$, (b) $f_0 \in (0, +\infty)$, $f_\infty = +\infty$, (c) $f_0 = +\infty$, $f_\infty \in (0, +\infty)$, (d) $f_0 = +\infty$, $f_\infty = +\infty$.

Next, we consider the case of superlinear growth of f at infinity.

THEOREM 4.3. *Let (G) hold. Assume that $f_0 \in (0, +\infty)$ and $f_\infty = +\infty$. Then*

(i) *if $\lambda \in (0, \lambda_1/f_0]$, problem (1.5) has at least one positive solution;*

(ii) *if $\lambda \in (\lambda_1/f_0, +\infty)$, problem (1.5) has at least two positive solutions.*

Proof. Consider the following problem

$$\begin{cases} -\Delta u = \lambda f_n(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where f_n is defined by

$$f_n(s) = \begin{cases} f(s), & s \in [0, n], \\ \left(2n^2 - f(n)\right)(s - n)\frac{1}{n} + f(n), & s \in (n, 2n), \\ ns, & s \in [2n, +\infty). \end{cases}$$

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Clearly, we can see that $\lim_{n\to+\infty} f_n(s) = f(s)$ and

$$\lim_{s\to+\infty} \frac{f_n(s)}{s} = n.$$

If (λ, u) is any solution of problem (4.3) with $\|u\| \not\equiv 0$, dividing problem (4.3) by $\|u\|^2$ and setting $w = u/\|u\|^2$ yield

$$\begin{cases} -\Delta w = \lambda \frac{f_n(u)}{\|u\|^2} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

Let $\eta_n(s) = f_n(s) - ns$ and

$$\widetilde{\eta}_n(w) = \begin{cases} \|w\|^2 \eta_n\left(\frac{w}{\|w\|^2}\right) & \text{if } w \neq 0, \\ 0 & \text{if } w = 0. \end{cases}$$

Then problem (4.4) is equivalent to

$$\begin{cases} -\Delta w = \lambda\left(nw + \widetilde{\eta}_n(w)\right) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

By a similar argument as in theorem 3.12, we get a sequence unbounded continua $\mathscr{C}_n$ emanating from $(\lambda_1/n, 0)$ such that $\mathscr{C}_n \subset ((\mathbb{R}\times\mathbb{P}) \cup \{(\lambda_1/0, 0)\})$, meets ∞ in $\mathbb{R}\times E$. As theorem 3.19, we obtain that $\mathscr{C} = \limsup_{n\to+\infty} \mathscr{C}_n$ is unbounded and connected such that $(0, 0) \in \mathscr{C}$.

We claim that $\mathscr{C} \cap ((0, +\infty) \times \{0\}) = \emptyset$. Otherwise, there exists a sequence of positive solution $\{(\lambda_n, u_n)\}$ of problem (1.5) such that $(\lambda_n, w_n) \to (\mu, 0)$ as $n \to +\infty$ with some $\mu > 0$. So $\|u_n\| = 1/\|w_n\|^2 \to +\infty$ as $n \to +\infty$. Theorem 1.1 of [**28**] and theorem 8.33 of [**29**] implies that $\|u_n\|$ is uniformly bounded. So we get a contradiction.

Now applying the inversion $w \to w/\|w\|^2 = u$, we obtain a continuum $\mathscr{D}$ of non-trivial solutions of problem (1.5) emanating from $(0, +\infty)$ and satisfying at least one of the following two alternatives:

(i) is unbounded in the direction of λ,

(ii) meets $\{(\lambda, 0) : \lambda \in \mathbb{R}_+\}$.

Similarly as theorem 4.1, we can show that the second alternative is impossible and $\mathscr{D} \subseteq (\{(0, \infty)\} \cup (\mathbb{R}\times\mathbb{P}))$. So we obtain the desired conclusions, see (b) of figure 2. □

Theorem 4.3 can be optimal under the more strong condition, see [**23**]. Now, we consider the case of sublinear growth of f at zero.

THEOREM 4.4. *Assume that* $f_0 = +\infty$ *and* $f_\infty \in (0, +\infty)$. *Then*

(i) *if* $\lambda \in (0, \lambda_1/f_\infty]$, *problem (1.5) has at least one positive solution;*

(ii) *if* $\lambda \in (\lambda_1/f_\infty, +\infty)$, *problem (1.5) has at least two positive solutions.*

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Proof. Consider the following problem

$$\begin{cases} -\Delta u = \lambda f^n(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}$$

where f_n is defined as the proof of Theorem 3.19. Clearly, we can see that $\lim_{n\to+\infty} f^n(s) = f(s)$, $f_0^n = n$ and $f_\infty^n = f_\infty$. The argument of theorem 4.1 implies that there exists one sequence unbounded continua $\mathscr{C}_n$ of solution set of problem (4.6) emanating from $(\lambda_1/n, 0)$. Taking $z^* = (0,0)$, clearly $z^* \in \liminf_{n\to+\infty} \mathscr{C}_n$. Lemma 2.5 of [**16**] implies that $\mathscr{C} = \limsup_{n\to+\infty} \mathscr{C}_n$ is unbounded and connected such that $z^* \in \mathscr{C}$. The rest of the proof is the same as that of theorem 4.1. □

See (c) of figure 2 for the global bifurcation diagram of theorem 4.4. Combing the arguments of theorems 4.3–4.4, we can easily get the following corollary, see (d) of figure 2.

Corollary 4.5. *Let (G) hold. Assume that $f_0 = +\infty$ and $f_\infty = +\infty$. Then problem (1.5) has at least two positive solutions for any $\lambda \in (0, +\infty)$.*

Proof of Theorem 1.2. Let $\beta = l_\kappa(\alpha)$ and

$$f(t) = \frac{1}{\sqrt{1 + 2\kappa h_\kappa^2(t)}} g\left(h_\kappa(t)\right).$$

Then the assumptions of (G4)–(G5) imply that f satisfies $f(\beta) = 0$, $f(s)s > 0$ for $s \in (0, \beta) \cup (\beta, +\infty)$ and there exists a constant $\gamma > 0$ such that

$$\lim_{s\to\beta^-} \frac{f(s)}{\beta - s} = \gamma.$$

From the argument of theorem 1.1, we know that f satisfies the subcritical growth condition (G) and

$$f_0 = g_0, \ f_\infty = \frac{g_\infty}{\kappa}.$$

Using theorems 4.1–4.3, remark 4.2 and corollary 4.5, we can obtain the desired conclusions. □

Proof of Theorem 1.4. (i) If $\kappa \in (g_\infty/g_0, g_\infty/\lambda_1)$, then we obtain

$$\frac{\lambda_1}{g_0} < \frac{\lambda_1\kappa}{g_\infty} \quad \text{and} \quad 1 > \frac{\lambda_1\kappa}{g_\infty}.$$

Theorem 1.2 (i) shows that problem (1.3) has at least two positive solutions. For any $\kappa \in (g_\infty/\lambda_1, +\infty)$, it follows from $\lambda_1 < g_0$ that

$$\frac{\lambda_1}{g_0} < 1 < \frac{\lambda_1\kappa}{g_\infty}.$$

Then theorem 1.2 (i) shows that problem (1.3) has at least one positive solution.

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For any $\kappa < g_\infty/g_0$, one has that

$$\frac{\lambda_1 \kappa}{g_\infty} < \frac{\lambda_1}{g_0} < 1.$$

From theorem 1.2 (i), we get that problem (1.3) has at least two positive solutions.

For $\kappa = g_\infty/g_0$, it is easy to see that

$$\frac{\lambda_1 \kappa}{g_\infty} = \frac{\lambda_1}{g_0} < 1.$$

So the existence of two positive solutions of problem (1.3) can be deduced from theorem 1.2 (ii).

(ii) For any $\kappa \in (0, g_\infty/\lambda_1)$, it follows from $\lambda_1 > g_0$ that

$$\frac{\lambda_1 \kappa}{g_\infty} < 1 < \frac{\lambda_1}{g_0}.$$

Thus, theorem 1.2 (i) follows the desired conclusion.

(iii) For any $\kappa \in (0, g_\infty/\lambda_1)$, we have that

$$\frac{\lambda_1 \kappa}{g_\infty} < 1.$$

Hence theorem 1.2 (iv) shows that problem (1.3) has at least two positive solutions.

For any $\kappa \in [g_\infty/\lambda_1, +\infty)$, we get

$$1 \leqslant \frac{\lambda_1 \kappa}{g_\infty}.$$

Therefore, theorem 1.2 (iv) implies that problem (1.3) has at least one positive solution.

□

5. Proof of Theorem 1.5 and some corollaries

Let $\xi : \mathbb{R}_+ \to \mathbb{R}$ be such that

$$F(s) = F_0 s + \xi(s)$$

with

$$\lim_{s \to 0^+} \frac{\xi(s)}{s} = 0.$$

Let us consider

$$\begin{cases} -\Delta u = (\beta - \lambda F_0)\, u - \lambda \xi(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

as a bifurcation problem from the trivial solution axis. For any fixed $\mu_0 := (\lambda_*, \beta_*)$ such that $\beta_* - \lambda_* F_0 = \lambda_1$, we may apply theorem 2.4 of [**26**] with $\mathscr{O} = \mathbb{R}^2_+ \times E$ and

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$\Gamma = \{(\lambda, \beta_*) : \lambda > 0\}$ and obtain a connected set $\mathscr{C} \subset \mathbb{R}^2 \times E$ of nontrivial solutions of problem (5.1) emanating from $(\mu_0, 0)$ with topological dimension at least 2 at every point. Clearly, the signum condition of $0 < F(s) < \beta/\lambda$ for any $s > 0$ and any given β, $\lambda > 0$ implies $\mathscr{C} \subset ((\mathbb{R}^2 \times \mathbb{P}) \cup \{(\mu_0, 0)\})$. It follows that $\mathscr{C}$ is unbounded in $\mathbb{R}^2 \times E$.

Proof of Theorem 1.5. Let $\mu_\infty = (\lambda^*, \beta_*)$ such that $\beta_* - \lambda^* F_\infty = \lambda_1$. From remark 2.3 of [**26**], we know that there exists an unbounded component $\mathscr{C}_\Gamma$ of the section of $\mathscr{C}$ over Γ. It is sufficient to show that $\mathscr{C}_\Gamma$ joins $(\mu_0, 0)$ to $(\mu_\infty, +\infty)$. Let $(\beta_*, \lambda_n, u_n) \in \mathscr{C}_\Gamma$ where $u_n \not\equiv 0$ satisfies $\sqrt{\beta_*^2 + \lambda_n^2} + \|u_n\| \to +\infty$.

We claim that there exists a constant M such that $\lambda_n \in (0, M]$ for any $n \in \mathbb{N}$. Our assumptions imply that there exists a positive constant $\rho > 0$ such that

$$\frac{F(s)}{s} \geqslant \rho$$

for any $s > 0$. Let φ_1 be a positive eigenfunction associated with λ_1. We multiply the first equation of problem (1.4) by φ_1, and obtain after integrations by parts

$$\lambda_1 \int_\Omega u_n \varphi_1 \,\mathrm{d}x = \int_\Omega \left(\beta_* - \frac{\lambda_n F(u_n)}{u_n}\right) u_n \varphi_1 \,\mathrm{d}x \leqslant (\beta_* - \lambda_n \rho) \int_\Omega u_n \varphi_1 \,\mathrm{d}x.$$

It follows that $\lambda_n \leqslant (\beta_* - \lambda_1)/\rho$. Then by an argument similar to that of theorem 3.3, we obtain

$$-\Delta \overline{u} = \left(\beta_* - \overline{\lambda} F_\infty\right) \overline{u}$$

for some $\overline{u} \in E$, where $\overline{\lambda} = \lim_{n \to +\infty} \lambda_n$. It follows that $\overline{\lambda} = \lambda^*$. Therefore, $\mathscr{C}_\Gamma$ joins $(\mu_0, 0)$ to $(\mu_\infty, +\infty)$. □

Since $F(s) = \Phi(s^2)s$, from theorem 1.5, we can easily derive the following corollary.

Corollary 5.1. *Assume that $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is continuous such that $\Phi(0), \Phi(+\infty) \in (0, +\infty)$ with $\Phi(0) > \Phi(+\infty)$, and $0 < \Phi(t) < \beta/\lambda$ for any $t > 0$ and any given β, $\lambda > 0$. Then for any*

$$(\lambda, \beta) \in \left\{(\lambda, \beta) \in \mathbb{R}^2 : \lambda > 0, \lambda_1 + \lambda\Phi(+\infty) \leqslant \beta \leqslant \lambda_1 + \lambda\Phi(0)\right\},$$

problem (1.4) has at least one positive solution, where $\Phi(+\infty) = \lim_{t \to +\infty} \Phi(t)$.

Now, we consider the problem (1.2). We present the following hypotheses on Φ.

(H1) The function $\Phi : \mathbb{R}_+ \to \mathbb{R}$ is continuous.

(H2) $\Phi(s) < 1$ for $s > 0$.

(H3) There exists $\Phi_\infty \in [-\infty, 0]$ such that

$$\Phi_\infty = \lim_{s \to +\infty} \frac{\Phi(s)}{s}.$$

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(H4) There exists one constant α such that $\Phi(\alpha^2) = 1$, $\Phi(s) < 1$ for $s \in (0, \alpha^2) \cup (\alpha^2, +\infty)$.

(H5) There exists a constant $\delta > 0$ such that

$$\lim_{s \to \alpha^-} \frac{s - \Phi\left(s^2\right) s}{l_\kappa(\alpha) - l_\kappa(s)} = \delta.$$

(H6)

$$\lim_{s \to +\infty} \frac{\Phi(s)}{s^l} = C$$

for some $l \in (1, (N + 2)/(N - 2))$, where C is a positive constant.

Then it is easy to verify that the hypotheses (H1)–(H6) is equivalent to (G1)–(G6) with $g_0 = 1 - \Phi(0)$ and $g_\infty = -\Phi_\infty$. So from theorems 1.1–1.4, we can easily get the following four corollaries.

COROLLARY 5.2. *Assume that (H1)–(H3) and (H6) hold and κ is a positive constant.*

(a) *If $\Phi(0) \in (-\infty, 1)$, $\Phi_\infty \in (-\infty, 0)$ satisfying $\kappa(1 - \Phi(0)) \neq -\Phi_\infty$, then there exist four positive constants μ_1, μ_1', μ_2 and μ_2' with $\mu_1' \leqslant \mu_1$ and $\mu_2' \geqslant \mu_2$ such that problem (1.2) has at least one positive solution for all $\lambda \in (\mu_1, \mu_2)$ and has no positive solution for all $\lambda \in (0, \mu_1') \cup (\mu_2', +\infty)$.*

(b) *If $\Phi(0) \in (-\infty, 1)$ and $\Phi_\infty = 0$, then there exist two positive constants μ_3 and μ_3' with $\mu_3' \leqslant \mu_3$ such that problem (1.2) has at least one positive solution for all $\lambda \in (\mu_3, +\infty)$ and has no positive solution for all $\lambda \in (0, \mu_3')$.*

(c) *If $\Phi(0) \in (-\infty, 1)$ and $\Phi_\infty = -\infty$, then there exist two positive constants μ_4 and μ_4' with $\mu_4' \geqslant \mu_4$ such that problem (1.2) has at least one positive solution for all $\lambda \in (0, \mu_4)$ and has no positive solution for all $\lambda \in (\mu_4', +\infty)$.*

(d) *If $\Phi(0) = 1$ and $\Phi_\infty \in (-\infty, 0)$, then there exist two positive constants μ_5 and μ_5' with $\mu_5' \leqslant \mu_5$ such that problem (1.2) has at least one positive solution for all $\lambda \in (\mu_5, +\infty)$ and has no positive solution for all $\lambda \in (0, \mu_5')$.*

(e) *If $\Phi(0) = 1$ and $\Phi_\infty = 0$, then there exist three positive constants μ_6, μ_6' and μ_7 with $\mu_6' \leqslant \mu_6 \leqslant \mu_7$ such that problem (1.2) has at least two positive solutions for all $\lambda \in (\mu_7, +\infty)$, one positive solution for all $\lambda \in [\mu_6, \mu_7]$ and has no positive solution for all $\lambda \in (0, \mu_6')$.*

(f) *If $\Phi(0) = 1$ (or $-\infty$) and $\Phi_\infty = -\infty$ (or 0), then for any $\lambda \in (0, +\infty)$, problem (1.2) has atleast one positive solution.*

(g) *If $\Phi(0) = -\infty$ and $\Phi_\infty \in (-\infty, 0)$, then there exist two positive constants μ_8 and μ_8' with $\mu_8' \geqslant \mu_8$ such that problem (1.2) has at least one positive solution for all $\lambda \in (0, \mu_8)$ and has no positive solution for all $\lambda \in (\mu_8', +\infty)$.*

(h) *If* $\Phi(0) = -\infty$ *and* $\Phi_\infty = -\infty$, *then there exist tree positive constants* μ_9, μ_{10} *and* μ'_{10} *with* $\mu_9 \leqslant \mu_{10} \leqslant \mu'_{10}$ *such that problem (1.2) has at least two positive solutions for all* $\lambda \in (0, \mu_9)$, *has at least one positive solution for all* $\lambda \in [\mu_9, \mu_{10}]$ *and has no positive solution for all* $\lambda \in (\mu'_{10}, +\infty)$.

COROLLARY 5.3. *Let (H1), (H3)–(H6) hold and* κ *be a positive constant.*

(i) *If* $\Phi(0) \in (-\infty, 1)$, $\Phi_\infty \in (-\infty, 0)$ *with* $\kappa(1 - \Phi(0)) \neq -\Phi_\infty$, *then there exists* $\mu_{11} > 0$ *such that problem (1.2) has at least two positive solutions for all*

$$\lambda \in \left(\max\left\{\frac{\lambda_1\kappa}{-\Phi_\infty}, \frac{\lambda_1}{1-\Phi(0)}\right\}, +\infty\right),$$

has at least one positive solution for all

$$\lambda \in \left(\min\left\{\frac{\lambda_1\kappa}{-\Phi_\infty}, \frac{\lambda_1}{1-\Phi(0)}\right\}, \max\left\{\frac{\lambda_1\kappa}{-\Phi_\infty}, \frac{\lambda_1}{1-\Phi(0)}\right\}\right]$$

and has no positive solution for all $\lambda \in (0, \mu_{11})$.

(ii) *If* $\Phi(0) \in (-\infty, 1)$, $\Phi_\infty \in (-\infty, 0)$ *with* $\kappa(1 - \Phi(0)) = -\Phi_\infty$, *then there exists* $\mu_{12} > 0$ *such that problem (1.2) has at least two positive solutions for all*

$$\lambda \in \left(\frac{\lambda_1}{1-\Phi(0)}, +\infty\right)$$

and has no positive solution for all $\lambda \in (0, \mu_{12})$.

(iii) *If* $\Phi(0) \in (-\infty, 1)$ *and* $\Phi_\infty = -\infty$, *then problem (1.2) has at least two positive solutions for all* $\lambda \in (\lambda_1/(1 - \Phi(0)), +\infty)$, *has at least one positive solution for all* $\lambda \in (0, \lambda_1/(1 - \Phi(0))]$.

(iv) *If* $\Phi(0) = -\infty$ *and* $\Phi_\infty \in (-\infty, 0)$, *then problem (1.2) has at least two positive solutions for all* $\lambda \in ((\lambda_1\kappa)/(-\Phi_\infty), +\infty)$, *has at least one positive solution for all* $\lambda \in (0, (\lambda_1\kappa)/(-\Phi_\infty)]$.

(v) *If* $\Phi(0) = -\infty$ *and* $\Phi_\infty = -\infty$, *then problem (1.2) has at least two positive solutions for all* $\lambda \in (0, +\infty)$.

COROLLARY 5.4. *Assume that* $\lambda = 1$, *(H1)–(H3) and (H6) hold.*

(a) *If* $\Phi(0) \in (-\infty, 1)$, $\Phi_\infty \in (-\infty, 0)$ *satisfying* $1 - \Phi(0) > \lambda_1$, *then problem (1.2) has at least one positive solution for all* $\kappa \in (-\Phi_\infty/\lambda_1, +\infty)$.

(b) *If* $\Phi(0) \in (-\infty, 1)$, $\Phi_\infty \in (-\infty, 0)$ *satisfying* $1 - \Phi(0) < \lambda_1$, *then problem (1.2) has at least one positive solution for all* $\kappa \in (0, -\Phi_\infty/\lambda_1)$.

(c) *If* $\Phi(0) = 1$ *and* $\Phi_\infty \in (-\infty, 0)$, *then problem (1.2) has at least one positive solution for all* $\kappa \in (0, -\Phi_\infty/\lambda_1)$.

(d) *If* $\Phi(0) = -\infty$ *and* $\Phi_\infty \in (-\infty, 0)$, *then problem (1.2) has at least one positive solution for all* $\kappa \in (-\Phi_\infty/\lambda_1, +\infty)$.

Corollary 5.5. *Let (H1), (H3)–(H6) hold and* $\lambda = 1$.

(i) *If* $\Phi(0) \in (-\infty, 1)$, $\Phi_\infty \in (-\infty, 0)$ *with* $1 - \Phi(0) > \lambda_1$, *then problem (1.2) has at least two positive solutions for all* $\kappa \in (0, -\Phi_\infty/\lambda_1)$, *has at least one positive solution for all* $\kappa \in (-\Phi_\infty/\lambda_1, +\infty)$.

(ii) *If* $\Phi(0) \in (-\infty, 1)$, $\Phi_\infty \in (-\infty, 0)$ *with* $1 - \Phi(0) < \lambda_1$, *then problem (1.2) has at least one positive solution for all* $\kappa \in (0, -\Phi_\infty/\lambda_1)$.

(iii) *If* $\Phi(0) = -\infty$ *and* $\Phi_\infty \in (-\infty, 0)$, *then problem (1.2) has at least two positive solutions for all* $\kappa \in (0, -\Phi_\infty/\lambda_1)$, *has at least one positive solution for all* $\kappa \in [-\Phi_\infty/\lambda_1, +\infty)$.

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