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A ROBUSTLY CHAIN TRANSITIVE ATTRACTOR WITH SINGULARITIES OF DIFFERENT INDICES

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Abstract Given a 4-manifold, we build a non-empty C^1-open set of vector fields having a (chain transitive) attractor containing singularities of different indices. Then, we begin the study of the hyperbolic properties of such a robust singular attractor.

Keywords: singular attractor; robustly chain transitive; partially hyperbolic; index

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1. Introduction

1.1. Motivations

The dynamics of flows is much related to the dynamics of diffeomorphisms, and their studies have followed almost parallel paths. From many points of view, the dynamics of vector fields in dimension n looks like diffeomorphisms in dimension $n-1$. However, there is a phenomenon which is really specific to vector fields: the existence of singularities (zeros of the vector field). This specificity is especially important when the singularities are not isolated from the rest of the dynamics, that is, when there are regular recurrent orbits accumulating the singularities.

The first example with this behavior was presented by Lorenz, in [24], exhibiting a simple family of algebraic vector fields in $\mathbb{R}^3$ with experimental evidence that the orbits

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tend to some compact set (called the Lorenz attractor) containing regular periodic orbits and a singularity. In [1, 19, 20], the authors give a geometric construction of C^1-open sets of vector fields on 3-manifolds, having a topological transitive attractor containing periodic orbits and one singularity.

Let us present one specific difficulty brought by the robust coexistence of singularities and periodic orbits: a hyperbolic periodic point of a diffeomorphism on a compact manifold persists by C^1-small perturbation of the dynamics: it has a well-defined continuation for the nearby diffeomorphisms. This periodic point may disappear under a large perturbation; however, before disappearing, it must lose its hyperbolicity. That is, the birth and death of periodic orbits are through non-hyperbolic periodic orbits. A diffeomorphism f is a *star diffeomorphism* if every periodic point of every diffeomorphism in a C^1-neighborhood of f is hyperbolic. As an important step of the structural stability conjecture, [2, 21, 23, 25, 26] show that every star diffeomorphism is Axiom A and satisfies the no cycle condition. The equivalent statement for flows is wrong: every periodic orbit of any vector field in the C^1-open set of 'geometric Lorenz attractors' is hyperbolic. So the geometric Lorenz attractors are star vector fields, but are not Axiom A flows. The reason is that periodic orbits may die remaining hyperbolic until the end, by going to the singularity: the periodic orbits are transformed in a homoclinic orbit of the singularity (in fact [17] shows that non-singular star flows satisfy Axiom A). The difficulties brought by the coexistence of singularities and periodic (or recurrent) orbits explain why the stability conjecture for flows [22] was solved nine years after the stability conjecture for diffeomorphisms [26].

The coexistence of singularities and regular recurrent orbits are mostly understood in dimension 3, by a long sequence of papers by Morales, Pacifico, and/or Pujals (see in particular [29, 30]). They defined the notion of singular hyperbolicity, which requires some compatibility between the hyperbolicity of the singularity and the hyperbolicity of the regular orbits. This allows them to define the notion of singular Axiom A flow, where the global dynamics splits in the disjoint union of transitive invariant compact sets, each of them being either hyperbolic or singular hyperbolic; furthermore, each of the singular hyperbolic sets is either an attractor (and the singularities contained in that set have index 2) or a repeller (and the singularities have index 1). In [28], Morales and Pacifico prove that there is a residual set $\mathcal{R} \subset \mathcal{X}^1(M)$ such that, for any $X \in \mathcal{R}$,

- either X has infinitely many attractors or repellers (in that case, [3] shows that X may be approached by homoclinic tangencies);
- or X is singular Axiom A without cycles; moreover, every non-trivial transitive set with singularities is either an attractor or a repeller.

In higher dimensions, the problem of the robust coexistence of singularities and periodic orbits is very far from being understood. There are easy examples obtained by multiplying three-dimensional examples by a strong contraction. This kind of example produces singular attractors whose singularities have only one expanding Lyapunov exponent, and it has been an open question for a long time if 'Lorenz-like' attractors with more than one expanding Lyapunov exponent could exist. [13] built the first examples of robustly transitive singular attractors having singularities with an arbitrary number of positive Lyapunov exponents. However, in the attractors of [13], the singularities have the same index, and the periodic orbits are all hyperbolic and with the same index. It was clear that these examples were far from covering all the possibilities. Building new examples is important for building a global view of the possibilities. Recently, [4] built the first examples (in dimension 5) of robustly transitive attractors containing periodic orbits with different indices; however, all the singularities in the attractor have the same index.

In this paper, we first give an example (Theorem A) of a robust attractor for flows on a 4-manifold, containing two hyperbolic singularities of different indices. Then Theorem B shows that, for every robust attractor with these properties, small perturbations create homoclinic tangencies and heterodimensional cycles associated to periodic orbits in the attractor. Finally, Theorem C proves that, if a robust singular attractor contains an index 2 singularity, then it has a partially hyperbolic splitting with a one-dimensional strong stable direction.

Our examples and those in [4, 13] open the door for the understanding of the dynamics of robust singular attractors, in dimension larger than 3. We are far from having a global overview of all the possibilities: it remains to understand the relationship between the index of the singularities and the index of the periodic orbits, the compatibility between the hyperbolicity of the singularity and the hyperbolicity of the regular orbits, the global structure (dominated splittings of the flow or of the Linear Poincaré flow) carried by such attractors, and so on; all these questions would need examples showing what is possible and what is not.

In that spirit, in [11], we built a C^1-open set $\mathcal{O}$ of vector fields on a 4-manifold such that every generic $X \in \mathcal{O}$ has no topological transitive (nor chain recurrent) attractors, but has a unique transitive *quasi-attractor* (i.e., intersection of (non-transitive) attractors) containing a singularity and regular periodic orbits.

1.2. Precise statement of our main results

Let X denote a vector field on a closed manifold M, and let $\phi_t^X : M \to M$ denote the time t map of its flow $\phi^X = \{\phi_t^X\}_{t\in\mathbb{R}}$.

The *index* $\mathrm{Ind}(\sigma)$ of a hyperbolic singular point $\sigma \in \mathrm{Sing}(X)$ is the dimension of its stable manifold. The *index* $\mathrm{Ind}(\gamma)$ of a regular hyperbolic periodic orbit γ is the dimension of its strong stable manifold.

A ϕ^X-invariant compact set Λ is *transitive* if there is a point $x \in \Lambda$ such that its positive orbit $\{\phi_t^X(x), t \geqslant 0\}$ is dense in Λ. The transitivity is a notion of indecomposability of the invariant set Λ. We will use here a weaker notion of indecomposability called *chain transitivity*, based on the notion of pseudo-orbits introduced by Conley.

For every $\varepsilon > 0$, a sequence $\{x_0, x_1, \ldots, x_n\}$ is called an *ε-pseudo-orbit*, if there are $\{t_i\}_{i=0}^{n-1}$ verifying $t_i > 1$ for any $0 \leqslant i \leqslant n-1$ such that $d(\phi_{t_i}^X(x_i), x_{i+1}) < \varepsilon$. Λ is called *chain transitive* if, for any $x, y \in \Lambda$, and for any $\varepsilon > 0$, there is an ε-pseudo-orbit $\{x = x_0, x_1, \ldots, x_n = y\}$ with $x_i \in \Lambda$ for any $0 \leqslant i \leqslant n$.

A *topological attractor* of a vector field X is an invariant compact set Λ admitting an open neighborhood U such that

(1) U is *an attracting region of* X: the boundary ∂U is a codimension 1 submanifold of M transverse to X and such that X is entering in U;

(2) Λ is *the maximal invariant set in* U; that is,

$$\Lambda = \bigcap_{t\in\mathbb{R}} \phi_t^X(U).$$

Notice that the maximal invariant set in any attracting region is a topological attractor. One usually requires an attractor to verify some indecomposability condition such as transitivity, or chain transitivity. Here, we will use both notions, so we will specify *transitive attractor* or *chain transitive attractor*.

Theorem A. *Given any closed manifold M with $\dim M \geqslant 4$, there is a C^1-open set $\mathcal{U}^1 \subset \mathcal{X}^1(M)$ and there is an open set $U \subset M$ such that, for any $X \in \mathcal{U}^1$, one has the following properties.*

- *U is an attracting region of X; we denote by Λ_X the maximal invariant set in U.*
- *Λ_X is a chain transitive attractor.*
- *Λ_X contains two singularities $\sigma_1, \sigma_2 \in \Lambda_X$ satisfying $\mathrm{Ind}\sigma_1 = \dim M - 1$ and $\mathrm{Ind}\sigma_2 = \dim M - 2$.*

Furthermore, for any $r \geqslant 1$, let $\mathcal{U}^r = \mathcal{U}^1 \cap \mathcal{X}^r(M)$ denote the C^r-open set of C^r-vector fields in $\mathcal{U}^1$. There is a C^r-residual subset $\mathcal{R}^r$ of $\mathcal{U}^r$ such that Λ_X is transitive for every $X \in \mathcal{R}^r$; moreover, the residual set $\mathcal{R}^r$ is the complement in $\mathcal{U}^r$ of a codimension 1 submanifold of $\mathcal{X}^r(M)$.

Remark 1.1. Here, the codimension 1 submanifold is an immersed submanifold; it is not an embedded submanifold.

We do not know if the chain transitive attractor Λ_X is in fact transitive for every X in $\mathcal{U}$. More generally, we ask the following question.

Problem. Does there exist a robustly chain transitive attractor which is not robustly transitive?

1.2.1. Idea of our construction. Our example is a perturbation of a simple vector field. We start with a vector field Y, on a 3-manifold M_0, having a robustly transitive singular attractor with a unique singular point p_0. Then we consider the 4-manifold $M = M_0 \times S^1$ endowed with the vector field $Z = Y \times \mathbf{0}|_{S^1}$ (that is, the vector field tangent to the factors $M_0 \times \{s\}$ which induces Y on each of these factors). Now, Theorem A consists in performing a small perturbation of the vector field Z in order to get a robust chain transitive attractor. The perturbation turns the circle $\{p_0\} \times S^1$ in a normally hyperbolic circle containing two singular points of different indices.

For proving the chain transitivity of the attractor, we will use that the initial three-dimensional attractor may have a very large expansion in the unstable direction. For this reason, we will not choose the usual Lorenz attractor[4] as our vector field Y. We will choose for Y a geometric model built in [31] which is a very simple and beautiful trick for allowing an arbitrarily large expansion in the unstable direction.

Remark 1.2. The vector field Y, the starting point of our construction, can be realized in a 3-ball B^3. Thus the vector field Z can be realized in $B^3 \times S^1$, which can be embedded in a 4-ball B^4. In other words, the vector field stated by Theorem A in dimension 4 can be realized in a ball B^4 (see Remark 3.1). By multiplying our construction in B^4 by a transversal strong contraction, one gets a C^1 vector field in a ball B^n (for any $n \geqslant 4$) with an attractor contained in a normally hyperbolic (contracting) 4-ball. This explains why Theorem A holds on any manifold with $\dim M \geqslant 4$.

1.3. Hyperbolic properties

The notion of singular hyperbolicity, introduced in dimension 3 by Morales, Pacifico, and Pujals, can be adapted easily in higher dimensions. The singular hyperbolicity implies that all the periodic orbits are hyperbolic and have the same index. Furthermore, this property is robust. Hence, the singular hyperbolicity implies the star condition. Conversely, [16, 18, 27] prove that, *under the star condition, every robustly transitive attractor Λ is singular hyperbolic.* This holds in particular for the examples of robust singular attractors obtained by multiplying a Lorenz-like attractor in dimension 3 by a transverse strong contraction; it also holds for geometric Lorenz attractors with singularities having arbitrarily large expanding directions built in [13].

However, in dimensions larger than 3, robust singular attractors do not need to satisfy the star condition or the singular hyperbolicity: our examples and the examples built in dimensions larger than 5 by [4] do not satisfy the star condition: they present periodic orbits with different indices, which allows us to create heterodimensional cycles, and therefore non-hyperbolic periodic orbits. This is a general phenomenon.

We say that X has a *homoclinic tangency* if X has a periodic point x and $W^s(\mathrm{Orb}(x))$ intersects $W^u(\mathrm{Orb}(x))$ non-transversely at some points.

Theorem B. *If Λ is a robustly chain transitive attractor of X which contains singularities of two different indices, then X can be accumulated in the C^1-topology by vector fields with a homoclinic tangency.*

Remark 1.3. An argument in [18] proves that every robustly transitive (and not chain transitive!) set (not necessarily attractor) which contains singularities of different indices can be C^1-perturbed in order to create a homoclinic tangency.

[4] Recall that the geometric model of a Lorenz attractor uses a cross section which is a rectangle; this rectangle is cut into two subrectangles by the local stable manifold of the singular point, and each of the subrectangles has a first return in the cross section, which is expanded in the unstable direction; this construction forbid the expansion rate of the first return map to be uniformly larger than 2.

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The homoclinic tangencies allow one to produce non-hyperbolic periodic orbits inside the attractor. Hence, robust chain transitive attractors with singularities of different indices are never singular hyperbolic. So it is natural to ask what kind of hyperbolicity satisfies robust singular (chain) transitive attractors?

We need some definitions. We denote by Φ^X the tangent flow (that is the derivative of ϕ^X). Given a compact invariant set Λ of $X \in \mathcal{X}^1(M)$, we say that Λ has a *T-dominated splitting*, for some constant $T > 0$, if there is a Φ^X-invariant continuous splitting $T_\Lambda M = E_1 \oplus E_2$ such that, for any $x \in \Lambda$, we have

$$\|\Phi_T^X|_{E_1(x)}\| \|\Phi_{-T}^X|_{E_2(\phi_T^X(x))}\| \leqslant 1/2.$$

In this case, we say that E_1 is *dominated* by E_2. An invariant bundle E_1 on Λ is called *contracting* if there are constants $C > 0$ and $\lambda < 0$ such that, for any $t > 0$ and $x \in \Lambda$, we have $\|\Phi_t^X|_{E_1(x)}\| \leqslant Ce^{\lambda t}$.

For diffeomorphisms, [7, 15, 25] prove that every robustly transitive set Λ has a dominated splitting; moreover, considering the finest dominated splitting on Λ, the extremal bundles are volume contracting and volume expanding.

For vector fields, the situation is more difficult.

- On the one hand, the argument in [7] proves that every robustly transitive set has some dominated splitting on the normal bundle with respect to the linear Poincaré flow. However, the normal bundle is defined only out of the singularities.
- On the other hand, there are robustly transitive vector fields X without Φ^X-invariant dominated splitting (such examples can be obtained by considering the suspension flow of the robustly transitive diffeomorphisms built in [12]). However, robustly transitive vector fields have no singularities (see [14, 16, 35]).

In fact, the known examples of robust singular attractors are using both structures (dominated splitting for the flow and for the Poincaré flow), but there is no formal proof of the existence of a dominated splitting.

Problem. Do there exist robustly singular (chain) transitive attractors without Φ^X-invariant dominated splitting?

As a partial answer, we show that every robustly chain transitive attractor having an index 2 singularity admits a partially hyperbolic splitting with a strong stable direction.

Theorem C. *There is a dense open set $\mathcal{O} \subset \mathcal{X}^1(M)$ such that, for any $X \in \mathcal{O}$, if Λ is a robustly chain transitive attractor of X containing a singularity of index 2, then Λ has a partially hyperbolic splitting; i.e., there is a dominated splitting $T_\Lambda M = E^{ss} \oplus F$ such that E^{ss} is uniformly contracted. Furthermore,* $\dim E^{ss} = 1$.

2. Introduction to the Morales–Pujals example

2.1. Informal description

Let M_0 be a compact three-dimensional C^∞ Riemannian manifold without boundary. In [31], Morales and Pujals describe a robustly transitive singular attractor Λ_0 of

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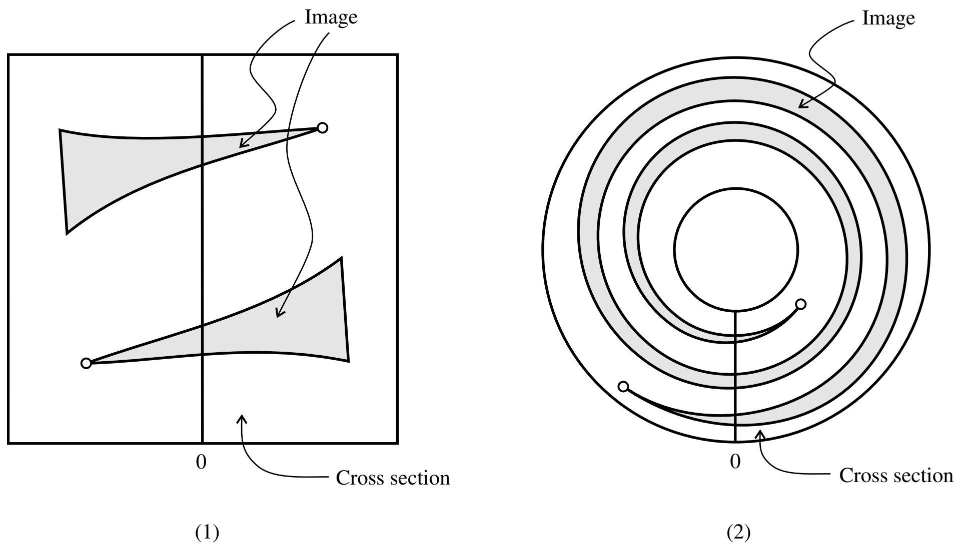

Figure 1. The first return map.

a vector field $Y \in \mathcal{X}^1(M_0)$ which is not equivalent to the classical geometric Lorenz attractor [19]. Our interest in that example is that its first return map on a given global cross section has an arbitrarily large expansion in the unstable direction.

More precisely, the usual geometric Lorenz attractor admits a cross section which is a square $[-1, 1] \times [-1, 1]$ crossing the local stable manifold of the singularity along $[-1, 1] \times \{0\}$, and the first return maps $[-1, 1] \times [-1, 1] \setminus [-1, 1] \times \{0\} \to [-1, 1] \times [-1, 1]$ behave as in Figure 1(1).

In the Morales–Pujals example, there is a cross section which is an annulus $[-1, 1] \times S^1$ crossing the local stable manifold of the singularity along $[-1, 1] \times \{0\}$, and the first return maps $[-1, 1] \times S^1 \setminus [-1, 1] \times \{0\} \to [-1, 1] \times S^1$ behave as in Figure 1(2), allowing an arbitrary expansion.

2.2. Construction of a Morales–Pujals attractor

The Morales–Pujals example was built in [31] as a strange singular attractor appearing through a simple bifurcation from a Morse–Smale system. Here, we are interested in the attractor itself, but not in how it appears. The aim of this section is to give a simple construction of this attractor, focusing on the fact that it is contained in a 3-ball, and that its basin contains a domain U_0 bounded by a surface transverse to the flow (indeed, that is a general feature of attractors, according to Conley theory).

Let us start with a usual Cherry flow X_c on the torus $\mathbb{T}^2 = S^1 \times S^1$, that is, a flow having two singularities, one repelling and one saddle, and transverse to the circle $S^1 \times \{0\}$. By the definition of Cherry flow, every point of $S^1 \times \{0\}$, except one point s_0 (belonging to the local stable manifold of the saddle), admits a (positive) first return on $S^1 \times \{0\}$. The image of this first return map is an open interval $I \times \{0\}$ in the circle $S^1 \times \{0\}$,

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whose extremities are points in both unstable separatrices of the saddle. One can choose this Cherry flow X_c such that

- it is trivial (i.e., $\frac{\partial}{\partial s_2}$, where s_2 is the second coordinate of the torus) on the annulus $S^1 \times [-1/2, 0]$;
- the expanding eigenvalue of the saddle point is stronger than the stable one (that is, the sum of the eigenvalues is positive).

We embed the 2-torus in the 3-ball (for instance as a revolution torus of $\mathbb{R}^3$). And we extend the Cherry flow as a vector field Y_0 in a tubular neighborhood of the torus just by adding a normally contracting component to the flow. More precisely, let $[-1, 1] \times \mathbb{T}^2$ denote this tubular neighborhood of the torus. Then, the expression, in the product coordinates of the vector field at a point (w, s), is

$$Y_0(w, s) = (0, X_c(s)) - \alpha \cdot w \frac{\partial}{\partial w},$$

where $\alpha > 0$ is chosen large enough such that the torus is normally hyperbolic for Y_0. Notice that the tubular neighborhood $[-1, 1] \times \mathbb{T}^2$ of the torus is now an attracting region for Y_0. We complete Y_0 in the whole ball in any way.

By the construction, the local strong stable leaves are the segments $[-1, 1] \times \{s\}$. Remove from $[-1, 1] \times \mathbb{T}^2$ a solid cylinder $[-1, 1] \times D$, where $D \subset \mathbb{T}^2$ is a disc contained in the basin of the repeller of the Cherry flow such that ∂D is transversal to X_c and disjoint from $S^1 \times \{0\}$. Now $[-1, 1] \times (\mathbb{T}^2 \setminus D)$ is an attracting region. Its boundary is homeomorphic to a surface of genus 2. One can smooth this surface in order to get a smooth surface transverse to the flow, and bounding a handlebody U_0 which is an attracting region (and contained in a 3-ball).

Consider now the annulus $\sigma = [-1, 1] \times S^1 \times \{0\}$. The first return map on σ is well defined out of $[-1, 1] \times \{s_0\}$. Its image is a region consisting of strong stable segments through the points of the segment $\{0\} \times I \times \{0\}$ (where I is the image of the first return map of the Cherry flow) whose length tends to 0 at the extremities of I, so that this region has two cusps at the extremities of I.

We now modify this flow on $[-1, 1] \times S^1 \times [-1/2, 0]$, in order to modify the return map. For that, we consider the return map from σ to $[-1, 1] \times S^1 \times \{-1/2\}$. The image is also a region $D_{-1/2}$ bounded by two segments tangent at the extremities of the interval $\{0\} \times I \times \{-1/2\}$, forming a cusp at the extremities of this interval.

We will replace Y_0 by a vector field Y which coincides with Y_0 out of the interior of $[-1, 1] \times S^1 \times [-1/2, 0]$, and which is transverse to the annuli $[-1, 1] \times S^1 \times \{u\}$. Notice that, up to smooth orbital equivalence, such a vector field is determined by the holonomy map $h : [-1, 1] \times S^1 \times \{-1/2\} \to \sigma = [-1, 1] \times S^1 \times \{0\}$, and h can be chosen as an arbitrary diffeomorphisms of these annuli which is isotopic to the identity, and equal to the identity in a neighborhood of the boundaries. As we are simply concerned with the return map of Y on σ, we only need to define h on the region $D_{-1/2}$. We require that h preserves and contracts the restriction to $D_{-1/2}$ of the strong stable foliation, and that h induces an arbitrarily large expansion of the transverse direction to the stable direction. (A more precise construction will be given in the next subsection.)

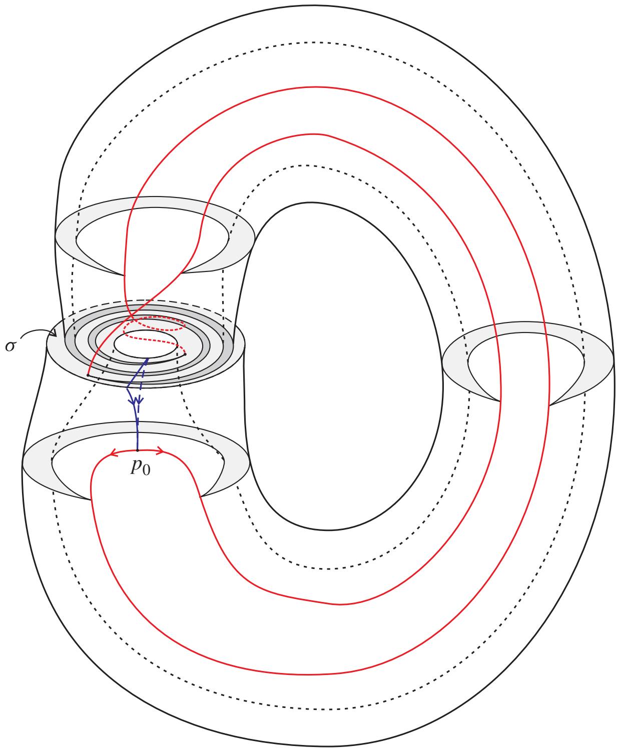

Figure 2. The Morales–Pujals example.

As a consequence, the first return map of Y on σ will be partially hyperbolic with an arbitrary large expansion (as in Figure 2). Now, the fact that the vector field Y admits on U_0 a unique robustly transitive attractor can be proven exactly in the same way as for the classical Lorenz attractor, the transitivity here being easier to get because the expansion is arbitrarily large. (We will give a short idea of the proof at the end of this section.)

2.3. Technical description

Now, we outline the topological construction, the hyperbolic properties of the flow, and the first return map of the Morales–Pujals example [31]. For any $a > 0$ and $\lambda^u > 2$, there exists a vector field Y on M_0 having a transitive attractor Λ_0 with the following properties.

Topological conditions.

$Y.1.$ Λ_0 is a transitive attractor: there is a neighborhood U_0 of Λ_0 whose boundary is a surface transverse to the vector field Y, the vector field Y enters in U_0, and Λ_0 is the maximal invariant set $\bigcap_{t\geqslant 0} \phi_t^Y(U_0)$.

$Y.2.$ Λ_0 has only one singularity p_0, which is an index 2 saddle point.

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$Y.3.$ Λ_0 admits a cross section which is an annulus $\sigma := [-1, 1] \times S^1$. Furthermore, every orbit entirely contained in $U_0 \setminus \{p_0\}$ crosses the interior of the section σ.

We choose the Riemannian metric on M_0 such that the cross section σ is orthogonal to Y.

$Y.4.$ The local stable manifold $W^s_{loc}(p_0)$ cuts σ along a segment $\ell := [-1, 1] \times \{0\}$.

$Y.5.$ The first return map on σ is well defined on $\sigma \setminus \ell$, and we denote it by $f : \sigma \setminus \ell \to \sigma$. Moreover, the image of f is contained in the interior of σ.

$Y.6.$ The lateral limits $\lim_{s\to 0+} f([-1, 1]\times\{s\})$ and $\lim_{s\to 0-} f([-1, 1]\times\{s\})$ exist, and are different points denoted respectively q_0^+ and q_0^-. Furthermore, q_0^+ and q_0^- are each the first intersection point of one unstable separatrix of p_0 (that is, a connected component of $W^u(p_0) \setminus \{p_0\}$) with σ. Finally, q_0^+ and q_0^- are not in the local stable manifold ℓ of the singularity p_0.

Hyperbolic properties of the flow.

$Y.7.$ **Partial hyperbolicity**: Λ_0 has a singular hyperbolic splitting, i.e., there is a dominated splitting $T_{\Lambda_0} M_0 = E^s \oplus E^{cu}$ for the tangent flow Φ^Y, such that E^s is contracting and E^{cu} is volume expanding (see item ($Y.10$(b)) for the definition of *volume expanding*).

$Y.8.$ **The strong stable foliation on** U_0: This implies that the stable bundle E^s admits a unique continuous invariant extension (already denoted by E^s) on U_0. Furthermore, there exists a unique invariant foliation $\mathcal{F}^{ss}$ tangent to E^s in U_0; the foliation $\mathcal{F}^{ss}$ is called the *strong stable foliation*.

$Y.9.$ We assume that $\mathcal{F}^{ss}$ is tangent[5] to σ. Furthermore, the leaves of the restriction of $\mathcal{F}^{ss}$ to σ are the segments $[-1, 1] \times \{s\}$.

$Y.10.$ **Center-unstable cone field**: There is no uniqueness of the extensions of E^{cu} on U_0. By an abuse of notation, we denote by E^{cu} a (non-invariant) continuous extension on U_0 of the bundle E^{cu} (defined on Λ_0) such that, for every $x \in U_0$, the plane $E^{cu}(x)$ is transverse to the bundle $E^s(x)$ and contains $\mathbb{R}Y(x)$.

For $c > 0$ and $x \in U_0$, we denote

$$\mathcal{C}^{cu}_c(x) = \{v \in T_x M_0 : v = v^s + v^{cu}, v^s \in E^s, v^{cu} \in E^{cu}, \text{ and } \|v^s\| \leqslant c\|v^{cu}\|\}.$$

(a) (**invariance of the center-unstable cone field**) The cone field of size a (where $a > 0$ is the constant fixed at the beginning of the section) is positively invariant, and there is $0 < \mu_0 < 1$ such that, for any $t \geqslant 1$,

$$\Phi^Y_t(\mathcal{C}^{cu}_a(x)) \subset \mathcal{C}^{cu}_{\mu_0 a}(\phi^Y_t(x));$$

(b) (**volume expansivity**) moreover, every plane P in the cone field $\mathcal{C}^{cu}_a(x)$ ($x \in U_0$) is volume expanding; more precisely, there is $\lambda^{cu} > 2$ such that, for every $t \geqslant 1$

[5] This hypothesis is not required by Morales and Pujals in [31]; however, it is not difficult to build a Morales–Pujals example having this extra property.

and every plane $P \subset \mathcal{C}^{cu}_a(x)$, $x \in U_0$, the determinant of the restriction to P of the derivative Φ^Y_t satisfies

$$\det(\Phi^Y_t|_P) > \lambda^{cu} > 2.$$

Hyperbolic properties of the first return map.

Y.11. **Stable foliation of the first return map**: By assumption, on σ, the leaves of $\mathcal{F}^{ss}$ are the segments $[-1, 1] \times \{s\}$ for $s \in S^1$; we denote by $\mathcal{F}^{ss}_\sigma$ the restriction of $\mathcal{F}^{ss}$ to σ. The foliation $\mathcal{F}^{ss}_\sigma$ is invariant by the first return map f and is uniformly contracted: there is $\lambda^s > 1$ such that, for every curve $\omega \subset \mathcal{F}^{ss}_\sigma(x) \cap f(\sigma \setminus \ell)$, we have

$$\text{Length}(f^{-1}(\omega)) > \lambda^s \text{Length}(\omega).$$

Y.12. Given any regular periodic orbit $\gamma \subset \Lambda_0$, one has that $W^u(\gamma) \cap \sigma$ intersects every leaf of $\mathcal{F}^{ss}_\sigma$; in particular, it intersects ℓ.

Y.13. **Center-unstable cone field for the first return map**: For $c > 0$ and $x \in \sigma$, we denote

$$\begin{aligned} C^{cu}_c(x) &= \{v \in T_x\sigma : v = v^s + v^{cu}, v^s \in T_x\mathcal{F}^{ss}_\sigma, v^{cu} \in E^{cu} \cap T_x\sigma, \text{ and } \|v^s\| \leqslant c\|v^{cu}\|\} \\ &= \mathcal{C}^{cu}_c(x) \cap T_x\sigma \qquad (\text{because } T_x\mathcal{F}^{ss}_\sigma = E^s(x)). \end{aligned}$$

Then $C^{cu}_c(x)$ is a cone tangent to σ and transverse to the strong stable leaf through x.

We assume that there is $0 < b < a$ such that

(a) the center unstable cone $C^{cu}_b(x)$ of size b is positively invariant and uniformly expanded; more precisely,
- for every $x \in \sigma \setminus \ell$, one has $Df(C^{cu}_b(x)) \subset C^{cu}_{\mu_0 b}(f(x))$, where μ_0 is the constant defined in item (Y.10);
- for every $v \in C^{cu}_b(x)$, we have $\|Df(v)\| > \lambda^u\|v\|$, where $\lambda^u > 2$ is the constant fixed at the beginning of § 2.3;

(b) as $0 < b < a$, for every $x \in \sigma$, one has $C^{cu}_b(x) \subset \mathcal{C}^{cu}_b(x) \subset \mathcal{C}^{cu}_a(x)$; as we choose σ orthogonal to Y and tangent to E^s, this implies that, for every $x \in \sigma$, the sum $C^{cu}_b(x) + \mathbb{R}Y(x) := \{u + v, u \in C^{cu}_b(x), v \in \mathbb{R}Y(x)\}$ is contained in $\mathcal{C}^{cu}_b(x)$ (hence in $\mathcal{C}^{cu}_a(x)$);

(c) the boundary of σ is tangent to the cone field C^{cu}_b; by the previous item, this implies that, at every point $x \in \partial\sigma$, the plane $T_x\partial\sigma + \mathbb{R}Y(x)$ is contained in $\mathcal{C}^{cu}_b(x)$.

In [31], Morales and Pujals built vector fields satisfying all the conditions above and proved that Λ_0 is a robustly transitive attractor. We now give a brief summary of their argument.

Idea of the proof of the Morales–Pujals example. They prove that every vector field Y' close enough to Y admits a well-defined return map on σ outside a segment ℓ', the intersection of the local stable manifold of the singular point with σ. (ℓ' is some kind of continuation of ℓ.) Furthermore, the first return map preserves the cone field $\mathcal{C}^{cu}_a$ and expands the vectors by a factor larger than λ^u. As λ^u is strictly larger than 2, for any

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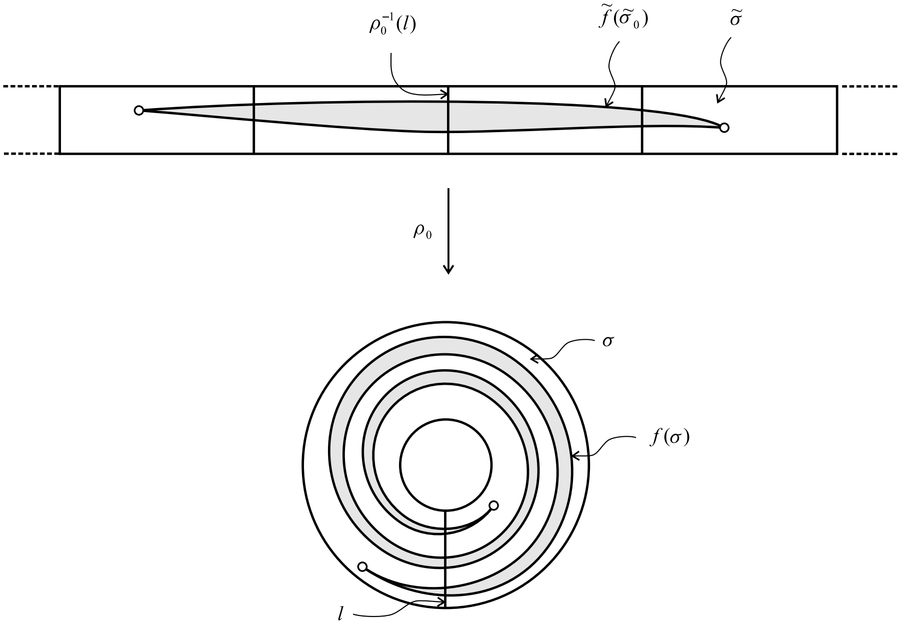

Figure 3. The first return map.

(non-trivial) segment ω in σ tangent to C_b^{cu}, there is $n > 0$ such that $f'^n(\omega)$ cuts ℓ' twice; that is, it makes an entire turn around the annulus σ. This implies that $f'^n(\omega)$ cuts every strong stable leaf on σ. As a consequence, any two periodic orbits in the maximal invariant set $\Lambda_0' = \bigcap_{t\geqslant 0} \phi_t^{Y'}(U_0)$ are homoclinically related, and their unstable manifolds are dense in Λ_0'. The same argument proves that the stable manifold of the singularity and of every periodic orbit is dense in U_0. They deduce that Λ_0' is a transitive attractor containing the singularity; furthermore, Λ_0' is the homoclinic class of a periodic orbit: in particular, the periodic orbits are dense in Λ_0'. □

The proof of Theorem A follows the same framework as the argument above. In that argument, the fact that a segment tangent to the cone field C_b^{cu} and cutting ℓ' twice needs to cut every strong stable leaf will be easier to generalize in higher dimensions if we consider a cyclic cover of the annulus σ. For this reason, we point out the following property, satisfied by the Morales–Pujals example (see Figure 3).

Y.14. **Lift of the return map f on a cyclic cover of σ**: Let $\rho_0 : \widetilde{\sigma} := [-1, 1] \times \mathbb{R} \to \sigma$ be the universal cover of σ. Let $\widetilde{\sigma}_0$ be a connected component of $\rho_0^{-1}(\sigma \setminus \ell)$, and let $\tilde{f} : \widetilde{\sigma}_0 \to [-1, 1] \times \mathbb{R}$ be a lift of f.

Then the boundary $\partial\tilde{f}(\widetilde{\sigma}_0)$ meets every segment $[-1, 1] \times \{s\}$, $s \in \mathbb{R}$, in at most two points.

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Remark 2.1. Recall that the Morales–Pujals example can be realized as a vector field on $\mathbb{R}^3$ or on a compact 3-ball; as a consequence, every compact 3-manifold carries an open set of vector fields having a Morales–Pujals attractor.

3. The details of the construction

3.1. The vector field Z, the product of the Morales–Pujals example by S^1

Let M_0 be a closed 3-manifold endowed with a smooth vector field Y having a robustly transitive attractor which is a Morales–Pujals attractor as described in § 2.

We consider the 4-manifold $M = M_0 \times S^1$ endowed with the orthogonal product Riemann metric of the metric on M_0 by the canonical metric on $S^1 = \mathbb{R}/\mathbb{Z}$. A point $x \in M_0 \times S^1$ is of the form $x = (x^1, x^2)$, where $x^1 \in M_0$ and $x^2 \in S^1$. We denote by Z the vector field defined on the 4-manifold $M_0 \times S^1$ as $Z = Y \times \mathbf{0}|_{S^1}$, where $\mathbf{0}|_{S^1}$ is the zero vector field on S^1. The vector field Z is tangent to each of the factors $M_0 \times \{x^2\}$, for $x^2 \in S^1$, and its restriction to $M_0 \times \{x^2\}$ is the vector field Y.

Remark 3.1. As noticed in Remark 2.1, a Morales–Pujals attractor can be realized on the 3-ball B^3. The product $B^3 \times S^1$ can be embedded in any compact 4-manifold. For this reason, the vector field Z defined above can be carried on any compact 4-manifold.

Let us recall some of the main properties that Z inherits directly from Y.

Topological properties.

Z.1. Recall that U_0 is an attracting region for Y. Then the open subset $U := U_0 \times S^1$ is an attracting region for Z, and the maximal invariant set of Z in U is $\Lambda = \Lambda_0 \times S^1$. It is a (non-robustly) chain transitive attractor for Z.

Z.2. The product $\Sigma := \sigma \times S^1 = [-1, 1] \times S^1 \times S^1$ is a cross section of the attractor, and every orbit entirely contained in U and not contained in $\{p_0\} \times S^1$ cuts the interior of Σ.

Z.3. The circle $S := \{p_0\} \times S^1$ is invariant and its local stable manifold $W^s(S)$ cuts Σ along an annulus $L := [-1, 1] \times \{0\} \times S^1 = \ell \times S^1$.

Z.4. The first return map F of Z on the cross section Σ is well defined (and is a diffeomorphism) on $\Sigma \setminus L$, and its image is contained in the interior of Σ.

Z.5. The image of the first return map F admits a compactification by the two circles $Q^+ := \{q_0^+\} \times S^1$ and $Q^- := \{q_0^-\} \times S^1$ which are connected components of $W^u(S) \cap \Sigma$.

Hyperbolic properties of the S^1-fibration. One of the main difficulty of our example comes from the fact that the one-dimensional lamination on Λ whose leaves are the factors $\{x^1\} \times S^1$ is not normally hyperbolic; and, in fact, it is easy to build perturbations of Z breaking this lamination.

However one has the following.

Z.6. The circle $S = \{p_0\} \times S^1$ is a normally hyperbolic circle.

Z.7. Given any regular periodic orbit $\gamma \subset \Lambda_0$ of Y, the torus $\Gamma := \gamma \times S^1$ is a normally hyperbolic torus.

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Hyperbolic properties of the flow.

$Z.8$. Z admits a partially hyperbolic splitting on Λ:

(a) the (one-dimensional) strong stable direction $E^{ss,Z}$ is well defined on the whole attracting region U, and $E^{ss,Z}(x^1, x^2)$, for $(x^1, x^2) \in U = U_0 \times S^1$, is just the strong stable direction E^s at x^1;

(b) the center-unstable direction $E^{cu,Z}$ at $(x^1, x^2) \in \Lambda$ is the product of the center-unstable direction E^{cu} at x^1 by the tangent lines to the factor $\{x^1\} \times S^1$;

in the previous section, we have chosen a (non-Φ^Y-invariant) extension of E^{cu} on U_0 such that $Y(x^1) \in E^{cu}(x^1)$, for every $x^1 \in U_0$; we denote by $E^{cu,Z}$ the corresponding extension on U of the center-unstable direction: $E^{cu,Z}(x^1, x^2)$ is the product of $E^{cu}(x^1)$ by the tangent lines to the factor $\{x^1\} \times S^1$.

$Z.9$. The splitting $E^{ss,Z} \oplus E^{cu,Z}$ on U allows us to define a cone field:

$$\mathcal{C}_a^{cu,Z} = \{v \in TM \times S^1 : v = v^s + v^{cu}, v^s \in E^{ss,Z}, v^{cu} \in E^{cu,Z}, \text{ and } \|v^s\| \leqslant a\|v^{cu}\|\}.$$

Recall that

- each factor $M_0 \times \{x^2\}$ is invariant by Φ^Z and Φ^Z induces Φ^Y on each factor;
- the cone field $\mathcal{C}_a^{cu}$ is invariant by Φ_t^Y $(t \geqslant 1)$;
- the vectors tangent to $E^{ss,Z}$ are uniformly contracted by Φ_t^Z $(t \geqslant 1)$, and Φ_t^Z preserves the norm of the vectors tangent to the S^1 factor.

One deduces the following lemma.

Lemma 3.2. *The cone field* $\mathcal{C}_a^{cu,Z}$ *is invariant by* Φ_t^Z $(t \geqslant 1)$; *moreover, there is* $\mu \in (0, 1)$ *such that, for any* $t \geqslant 1$,

$$\Phi_t^Z(\mathcal{C}_a^{cu,Z}(x)) \subset \mathcal{C}_{\mu a}^{cu,Z}(\phi_t^Z(x)).$$

Finally, by item ($Y.10$(b)), one gets that every three-dimensional subspace B in the cone field $\mathcal{C}_a^{cu,Z}(x)$ $(x \in U)$ is volume expanding. More precisely, we have the following.

Lemma 3.3. *There is* $T > 0$ *such that, for every* $t \geqslant T$, *every* $x \in U$, *and every three-dimensional subspace* $B \subset \mathcal{C}_a^{cu,Z}(x)$, *one has*

$$\det(\Phi_t^Z|_B) > \lambda^{cu} > 2.$$

Proof. Simply notice that $\Phi_t^Z(B)$ converges uniformly to $E^{cu,Z}(\phi_t^Z(x))$ as $t \to +\infty$. Then the claim follows from the uniform volume expansion on $E^{cu,Z}$. □

Hyperbolic properties of the first return map.

$Z.10$. **Stable foliation of the first return map**: We denote by $\mathcal{F}_\Sigma^{ss}$ the one-dimensional foliation on $\Sigma = \sigma \times S^1$ which is tangent to each factor $\sigma \times \{x^2\}$ and coincides with $\mathcal{F}_\sigma^{ss,Y}$ on this factor. This foliation is invariant by the first return map F and is uniformly contracted by F by the factor $(\lambda^s)^{-1} < 1$.

$Z.11$. Notice that, for any regular periodic orbit $\gamma \subset \Lambda_0$ of Y, $W^u(\Gamma)$, the unstable manifold of the normally hyperbolic torus $\Gamma = \gamma \times S^1$, is $W^u(\gamma) \times S^1$. Thus $W^u(\Gamma) \cap \Sigma$ intersects every leaf of $\mathcal{F}_\Sigma^{ss}$ and L.

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$Z.12.$ **Center-unstable cone field for the first return map**: For $x \in \Sigma$, we denote

$$\begin{aligned} C_b^{cu,Z}(x) &= \{v \in T_x\Sigma : v = v^s + v^{cu},\ v^s \in T_x\mathcal{F}_\Sigma^{ss},\ v^{cu} \in E^{cu,Z} \cap T_x\Sigma, \text{ and } \|v^s\| \leqslant b\|v^{cu}\|\} \\ &= \mathcal{C}_b^{cu,Z}(x) \cap T_x\Sigma. \end{aligned}$$

Then $C_b^{cu,Z}(x)$ is a cone tangent to Σ and transverse to the strong stable leaf through x. Moreover:

(a) notice that $E^{cu,Z} \cap T_x\Sigma$ is the product by the S^1 direction of the distribution $E^{cu} \cap T\sigma$ defined on the three-dimensional model; now, as in item $(Z.9)$, we deduce the DF-invariance of the cone field $C_b^{cu,Z}$ from the Df-invariance of the cone field C_b^{cu}; more precisely, there is $\mu \in (0, 1)$ such that, for every $x \in \Sigma \setminus L$, one has

$$DF(C_b^{cu,Z}(x)) \subset C_{\mu b}^{cu,Z}(F(x));$$

(b) for every $x \in \Sigma$, the sum $C_b^{cu,Z}(x) + \mathbb{R}Z(x)$ is contained in $\mathcal{C}_b^{cu,Z}(x)$ (hence in $\mathcal{C}_a^{cu,Z}(x)$);

(c) the boundary $\partial\Sigma$ is tangent to the cone field $C_b^{cu,Z}$; by the previous item, this implies that, at every point $x \in \partial\Sigma$, the 3-space $T_x\partial\Sigma + \mathbb{R}Z(x)$ is contained in $\mathcal{C}_b^{cu,Z}(x)$.

$Z.13.$ **Volume expansion in the center-unstable cone field**: Recall that the vector field Y has been defined for an arbitrary $a > 0$ and $\lambda^u > 2$, and uses a constant $b \in (0, a)$.

Let $P \subset C_b^{cu,Z}(x)$ be a plane contained in the center-unstable cone field at a point x. If a has been chosen small enough, then b is also very small. Then, the determinant $\det(DF|_P)$ is almost equal to the determinant $\det(DF|_Q)$, where Q is a plane generated by a vector in the cone field C_b^{cu} and the vector tangent to the S^1 factor; as the S^1 direction is preserved by F, and the action of DF on this direction is by isometry, $\det(DF|_Q)$ is larger than λ^u (by definition of the vector field Y). One deduces that for λ^u large enough and a small enough there is $\lambda > 2$ such that, for any $x \in \Sigma$ and any plane $P \subset C_b^{cu,Z}(x)$, one has

$$\det(D_xF|_P) > \lambda > 2.$$

3.2. Vector fields in a neighborhood of Z

Most of the properties of the vector field Z persist in a small neighborhood of Z, sometimes needing to be a little bit changed. More precisely, given any hyperbolic periodic orbits $\gamma, \gamma' \subset \Lambda_0$ of the vector field Y, there is a neighborhood $\mathcal{U}_1 \subset \mathcal{X}^1(M_0 \times S^1)$ of Z such that, for any $X \in \mathcal{U}_1$, one has the following properties.

Topological properties.

$X.1.$ The open subset $U = U_0 \times S^1$ is an attracting region for X. We denote by Λ_X the maximal invariant set of X in U, that is, $\Lambda_X = \bigcap_{t \geqslant 0} \phi_t^X(U)$. It is a (possibly non-chain transitive) attractor for X.

$X.2.$ The circle $S = \{p_0\} \times S^1$ is normally hyperbolic for Z; hence it has a unique continuation S_X for X. The circle S_X is ϕ^X-invariant, normally hyperbolic, and X has no singularities in $U \setminus S_X$.

$X.3$. The torus $\gamma \times S^1 (\gamma' \times S^1)$ has a unique continuation $\Gamma_X (\Gamma'_X)$ which is a ϕ^X-invariant, normally hyperbolic torus.

$X.4$. The product $\Sigma = \sigma \times S^1 = [-1, 1] \times S^1 \times S^1$ is a cross section of X, and every orbit of X entirely contained in U and not contained in S_X cuts the interior of Σ.

$X.5$. The local stable manifold $W^s(S_X)$ cuts Σ along an annulus L^X which varies continuously with X, and whose boundary is contained in the boundary of Σ. In particular, if the neighborhood $\mathcal{U}_1$ is small enough, then L^X is C^1-close to $L = [-1, 1] \times \{0\} \times S^1$.

$X.6$. The local stable manifold $W^s(\Gamma_X)$ cuts Σ along an annulus I^X which varies continuously with X, and whose boundary is contained in the boundary of Σ.

$X.7$. The first return map F_X of X on the cross section Σ is well defined (and is a diffeomorphism) on $\Sigma \setminus L^X$, and its image is contained in the interior of Σ.

$X.8$. The image of the first return map F_X admits a compactification by the two circles $Q^{+,X}$ and $Q^{-,X}$ which are the connected components of $W^u(S_X) \cap \Sigma$ obtained as the unique continuation of Q^+ and Q^- (these intersections are transversal, and hence vary continuously with X). For $\mathcal{U}_1$ small enough, $Q^{+,X}$ and $Q^{-,X}$ are C^1-close to Q^+ and Q^-, respectively.

Hyperbolic properties of the flow.

$X.9$. X admits a partially hyperbolic splitting on Λ_X.

(a) The strong stable direction $E^{ss,X}$ is well defined on the whole attracting region U and varies continuously with X; the center-unstable direction $E^{cu,X}$ is well defined on Λ_X;

(b) the cone field $\mathcal{C}_a^{cu,Z}$ defined on U is transverse to $E^{ss,X}$, and is invariant by Φ_t^X for $t \geqslant 1$; furthermore, up to increase a little bit the constant $\mu < 1$, one has for $t \geqslant 1$,

$$\Phi_t^X(\mathcal{C}_a^{cu,Z}(x)) \subset \mathcal{C}_{\mu a}^{cu,Z}(\phi_t^X(x));$$

(c) moreover, up to shrink a little bit $\lambda^{cu} > 2$, every three-dimensional subspace B in the cone field $\mathcal{C}_a^{cu,Z}(x)$ $(x \in U)$ is volume expanding by Φ_t^X; that is, for $t \geqslant T$, the determinant $\det(\Phi_t^X|_B)$ satisfies

$$\det(\Phi_t^X|_B) > \lambda^{cu}.$$

Hyperbolic properties of the first return map.

$X.10$. **Stable foliation of the first return map**: Notice that the flow ϕ^X admits a strong stable foliation $\mathcal{F}^{ss,X}(x)$. The foliation $\mathcal{F}_\Sigma^{ss,X}$ on $\Sigma = \sigma \times S^1$ whose leaves are the projection along the orbits of X of $\mathcal{F}^{ss,X}(x)$ is invariant by F_X and uniformly contracted by F_X by the factor $(\lambda^s)^{-1} < 1$ (up to shrink a little bit $\lambda^s > 1$).

$X.11$. Notice that the unstable manifold $W^u(\Gamma_X)$ varies continuously with X on the compact parts. One deduces that, for X close enough to Z, one has that $W^u(\Gamma_X) \cap \Sigma$ intersects every leaf of $\mathcal{F}_\Sigma^{ss,X}$ and L^X.

So we choose $\mathcal{U}_1$ such that these properties hold on $\mathcal{U}_1$.

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X.12. **Center-unstable cone field for the first return map**: Then $C_b^{cu,Z}(x)$ is a cone tangent to Σ and transverse to the strong stable leaf of $\mathcal{F}^{ss,X}$ through x. Moreover, one has the following proposition.

Proposition 3.4. *For any $0<\mu<\mu'<1$ and any $2<\lambda'<\lambda$, there is a C^1-neighborhood $\mathcal{U}_3\subset\mathcal{U}_1$ of Z such that, for every $X\in\mathcal{U}_3$ and every $x\in\Sigma\setminus L^X$, one has*

$$DF_X(C_b^{cu,Z}(x))\subset C_{\mu' b}^{cu,Z}(F_X(x)).$$

Furthermore, for every plane P contained in $C_b^{cu,Z}(x)$, one has

$$\det(D_xF_X|_P)>\lambda'>2.$$

Proof. For X close to Z, the Poincaré return time associated to F_X is uniformly bounded out of a small neighborhood of L. Hence, up to increase a little bit $\mu<1$, for X close to Z and x out of this small neighborhood of L, one has

$$DF_X(C_b^{cu,Z}(x))\subset C_{\mu b}^{cu,Z}(F_X(x)).$$

Thus, we only need to consider the points in a small neighborhood of L.

Consider a' such that $0<a'<\mu b$. By the definition of the cone fields $\mathcal{C}_c^{cu,Z}(x)$ and $C_c^{cu,Z}(x)$, for any point $x\in\Sigma$, we have

$$\mathcal{C}_{a'}^{cu,Z}(x)\cap T_x\Sigma\subset C_{\mu b}^{cu,Z}(x).$$

Recall that $C_b^{cu,Z}(x)\oplus\mathbb{R}Z(x)\subset\mathcal{C}_b^{cu,Z}(x)$, and $0<b<a$. As a consequence, we get the following.

Lemma 3.5. *There exists a C^1-neighborhood of Z such that, for any X in this neighborhood and any $x\in\Sigma$, we have*

$$C_b^{cu,Z}(x)\oplus\mathbb{R}X(x)\subset\mathcal{C}_a^{cu,Z}(x).$$

Lemma 3.6. *There are $t_0>0$ and a C^1-neighborhood of Z such that, for any $t>t_0$ and any X in that neighborhood, one has*

$$\Phi_t^X(\mathcal{C}_a^{cu,Z}(x))\subset\mathcal{C}_{a'}^{cu,Z}(\phi_t^X(x)).$$

Proof. Fix some $a''<a'$. Using the partial hyperbolicity of Z, one gets $t_0>0$ such that, for any $t>t_0$, one has $\Phi_t^Z(\mathcal{C}_a^{cu,Z})\subset\mathcal{C}_{a''}^{cu,Z}$. Now, by continuity and compactness, one gets $\Phi_t^X(\mathcal{C}_a^{cu,Z})\subset\mathcal{C}_{a'}^{cu,Z}$, for XC^1-close to Z. □

We now choose a C^1-neighborhood $\widetilde{\mathcal{U}}_3$ of Z and a neighborhood U_δ of L in Σ such that

- $\widetilde{\mathcal{U}}_3$ is contained in the neighborhoods of Z stated in Lemmas 3.5 and 3.6;
- for any $x\in\Sigma\setminus U_\delta$ and any $X\in\widetilde{\mathcal{U}}_3$, one has

$$DF_X(C_b^{cu,Z}(x))\subset C_{\mu' b}^{cu,Z}(F_X(x));$$

- for any $x\in U_\delta$ and any $X\in\widetilde{\mathcal{U}}_3$, the Poincaré return time $\tau^X(x)$ of x is greater than t_0.

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Now, the main step of the proof is the following lemma.

Lemma 3.7. *For any $X \in \widetilde{\mathcal{U}}_3$ and any $x \in \Sigma \setminus L^X$, one has*

$$DF_X(C_b^{cu,Z}(x)) \subset C_{\mu' b}^{cu,Z}(F_X(x)).$$

Proof. Suppose, to the contrary, that there exist $x \in U_\delta$ and $v \in C_b^{cu,Z}(x)$ such that

$$DF_X(v) \notin C_{\mu b}^{cu,Z}(F_X(x)).$$

Hence $DF_X(v) \notin \mathcal{C}_{a'}^{cu,Z}(F_X(x))$.

Notice that $\Phi^X_{\tau^X(x)}(v) - DF_X(v) \in \mathbb{R}X(F_X(x))$; thus there is $v' \in \mathbb{R}X(x)$ such that

$$\Phi^X_{\tau^X(x)}(v - v') = DF_X(v) \notin \mathcal{C}_{a'}^{cu,Z}(F_X(x)).$$

However, $v \in C_b^{cu,Z}(x)$ and $C_b^{cu,Z}(x) + \mathbb{R}X(x) \subset \mathcal{C}_a^{cu,Z}(x)$, because $X \in \widetilde{\mathcal{U}}_3$. As a consequence, one has $v - v' \in \mathcal{C}_a^{cu,Z}(x)$. As $X \in \widetilde{\mathcal{U}}_3$ and $\tau^X(x) > t_0$, this implies that

$$\Phi^X_{\tau^X(x)}(v - v') \in \mathcal{C}_{a'}^{cu,Z}(F_X(x)).$$

This contradiction proves that $C_b^{cu,Z}$ is positively invariant. □

To end the proof of the proposition, it remains to prove the area expansiveness on planes in the cone field $C_b^{cu,Z}$.

As before, by replacing λ by λ' (with $2 < \lambda' < \lambda$), for x out of a small neighborhood of L and X C^1 close to Z, every plane in $C_b^{cu,Z}(x)$ is area expanded by the first return map F_X by a factor larger than λ'.

For any plane P contained in $C_b^{cu,Z}(x)$ and $X \in \widetilde{\mathcal{U}}_3$, we have

$$P \oplus \mathbb{R}X(x) \subset \mathcal{C}_a^{cu,Z}(x).$$

Thus Φ_t^X is volume expanding on $P \oplus \mathbb{R}X(x)$; that is, for $t \geqslant T$, one has

$$\det(\Phi_t^X|_{P\oplus\mathbb{R}X(x)}) > \lambda^{cu}.$$

For a point x close to L and X close to Z, the Poincaré return time $\tau^X(x)$ is arbitrary large. Hence, the expanding rate

$$\det(\Phi^X_{\tau^X(x)}|_{P\oplus\mathbb{R}X(x)}) > (\lambda^{cu})^{\left[\frac{\tau^X(x)}{T}\right]}$$

is arbitrary large.

On the other hand, $\mathbb{R}X(x)$ is invariant by Φ_t^X, and the angle between $\mathbb{R}X(x)$ and $T\Sigma$ is uniformly away from zero. This implies that $\det(D_xF_X|_P) > \lambda'$ as $\tau^X(x)$ is large enough. □

3.3. Performing a perturbation of Z

Recall that $\mathcal{U}_1$ is the C^1-neighborhood of Z built in § 3.2.

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Lemma 3.8. *There exists a non-empty open subset $\mathcal{U}_2 \subset \mathcal{U}_1$ containing Z in its closure (i.e., $Z \in \overline{\mathcal{U}_2}$) such that, for any $X \in \mathcal{U}_2$, we have*

- *the flow restricted on S_X, denoted by $\phi_t^X|_{S_X}$, is Morse–Smale: the chain recurrent set of $\phi_t^X|_{S_X}$ is the set of two singularities p_1^X and p_2^X. The points p_1^X and p_2^X are singular saddles of X, whose indices are $\mathrm{Ind}(p_1^X) = 3$ and $\mathrm{Ind}(p_2^X) = 2$;*
- *the flow restricted on Γ_X, denoted by $\phi_t^X|_{\Gamma_X}$, is Morse–Smale: the chain recurrent set of $\phi_t^X|_{\Gamma_X}$ is the set of two periodic orbits γ_1^X and γ_2^X, whose indices (in the ambient manifold M) are $\mathrm{Ind}(\gamma_1^X) = 2$ and $\mathrm{Ind}(\gamma_2^X) = 1$ (the index of a hyperbolic set is the dimension of its contracting bundle);*
- $\mathrm{cl}(W^u(\Gamma_X)) = \mathrm{cl}(W^u(\gamma_2^X)) = \mathrm{cl}(W^u(\gamma_1^X))$, *and* $\mathrm{cl}(W^s(\Gamma_X)) = \mathrm{cl}(W^s(\gamma_1^X)) = \mathrm{cl}(W^s(\gamma_2^X))$.

Remark 3.9. Note that $Z \notin \mathcal{U}_2$.

Lemma 3.8 is a direct consequence of the following result (be aware that the notation in this proposition is not coherent with the notation in other parts of the paper).

Proposition 3.10. *Let $f : V_0 \to M$ be a diffeomorphism onto its image defined on an open subset V_0 of a manifold M. Assume that $K \subset V_0$ is a f-invariant transitive compact subset which is a uniformly hyperbolic non-trivial basic set for f (non-trivial means that it is not reduced to a periodic orbit); let $V \subset V_0$ be an isolating neighborhood of K. Let $y \in K$ denote a periodic point of f and $\pi(y)$ its period.*

Let $F_0 := (f, id_{S^1}) : V_0 \times S^1 \to M \times S^1$. For every neighborhood $\mathcal{W}_0$ of F_0 in $\mathrm{Diff}^1(V_0 \times S^1, M \times S^1)$ for the classical compact open C^1-topology, there is a diffeomorphism $F_1 \in \mathcal{W}_0$ and there is a neighborhood $\mathcal{W}_1 \subset \mathcal{W}_0$ of F_1 with the following properties.

- *The diffeomorphism F_1 is of the form $F_1(x, s) = (f(x), \varphi_x(s))$, where $\{\varphi_x\}_{x \in V_0}$ is a continuous family of circle diffeomorphisms such that $\varphi_x = id_{S^1}$ for $x \notin V$;*
- *the normally hyperbolic (periodic) circle*[6] *$C = \{y\} \times S^1$ contains precisely 2 periodic points $y_1 = (y, 0)$ and $y_2 = (y, \frac{1}{2})$ for F_1 which are hyperbolic;*
- *the circle C admits a well-defined continuation C_F which is an $F^{\pi(y)}$-invariant normally hyperbolic circle, for every $F \in \mathcal{W}_1$; furthermore, C_F contains exactly two periodic points $y_{1,F}$ and $y_{2,F}$ which are hyperbolic and vary continuously with $F \in \mathcal{W}_1$;*
- *for every $F \in \mathcal{W}_1$, one has*

$$\mathrm{cl}(W^u(C_F)) = \mathrm{cl}(W^u(y_{1,F})) = \mathrm{cl}(W^u(y_{2,F})),$$

and

$$\mathrm{cl}(W^s(C_F)) = \mathrm{cl}(W^s(y_{1,F})) = \mathrm{cl}(W^s(y_{2,F})).$$

In order to apply Proposition 3.10 to Lemma 3.8, we will consider $M = \mathrm{int}(\sigma)$ and $M \times S^1 = \mathrm{int}(\Sigma)$.

[6] In order to be completely rigorous, the orbit of C is a normally hyperbolic invariant manifold.

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Proof of Lemma 3.8. The first return map F of the vector field Z on $\Sigma = \sigma \times S^1$ is the product (f, id_{S^1}), where $f : \sigma \setminus \ell \to \mathrm{int}(\sigma)$ is the first return map on σ of the vector field Y. We consider y and y' being intersection points of γ and γ' with σ. By item Y.12 on the vector field Y, the periodic orbits γ and γ' are homoclinically related. As a consequence, there is a hyperbolic non-trivial basic set[7] $K \subset \sigma \setminus (\ell \cup \partial\sigma)$ of f which contains y and y'. We fix an open neighborhood V_0 of K which is relatively compact in $\sigma \setminus (\ell \cup \partial\sigma)$. Hence the first return map induces a diffeomorphism denoted by $F_0 := (f, id_{S^1}) : V_0 \times S^1 \to \mathrm{int}(\Sigma)$. We choose an isolating neighborhood $V \subset V_0$ of K. We consider a diffeomorphism F_1, arbitrarily C^1-close to F_0, given by Proposition 3.10. Notice that F_1 coincides with F_0 out of $V \times S^1$. We write $F_1 = G \circ F_0$, where $G = F_1 \circ F_0^{-1}$ is a diffeomorphisms of $\Sigma \setminus L$ which is the identity map out of $V \times S^1$; notice that G extends by continuity as being the identity map on L.

This allows us to realize F_1 as the first return map of a vector field X_1 which is C^1-close to Z, leaves invariant the S^1-fibration, and coincides with Z out of $\phi^Z_{[-\frac{1}{2},0]}(\Sigma)$; for that, we choose X_1 in $\phi^Z_{[-\frac{1}{2},0]}(\Sigma)$ in such a way that, for every $x \in \Sigma$, one has

$$\phi^{X_1}_{\frac{1}{2}}(\phi^Z_{-\frac{1}{2}}(x)) = G(x).$$

In this construction, if G is C^1-close to the identity map, then we can choose the vector field X_1 to be C^1-close to Z; more precisely, given a neighborhood $\mathcal{U}$ of Z, Proposition 3.10 ensures that we can choose F_1 arbitrarily C^1-close to F_0 so that the vector field X_1 can be chosen in $\mathcal{U}$. As stated by Proposition 3.10, F_1 has exactly two hyperbolic points, y_1 and y_2, on $\{y\} \times S^1$

Let $\mathcal{W}_1$ be the C^1-neighborhood of F_1 in $\mathrm{Diff}(V_0 \times S^1, \mathrm{int}(\Sigma))$ stated by Proposition 3.10. There is a C^1 neighborhood $\mathcal{U}'$ of X_1 such that the first return map F_X of every vector field $X \in \mathcal{U}'$ belongs to $\mathcal{W}_1$. Every vector field $X \in \mathcal{W}_1$ has a well-defined normally hyperbolic torus Γ_X, a continuation of $\gamma \times S^1$, and Γ_X contains exactly two periodic orbits γ_1^X and γ_2^X with different indices, and one has

$$\mathrm{cl}(W^u(\Gamma_X)) = \mathrm{cl}(W^u(\gamma_2^X)) = \mathrm{cl}(W^u(\gamma_1^X)). \quad \text{and} \quad \mathrm{cl}(W^s(\Gamma_X)) = \mathrm{cl}(W^s(\gamma_1^X)) = \mathrm{cl}(W^s(\gamma_2^X)).$$

We have proved that the vector fields in $\mathcal{U}'$ satisfy items (2) and (3) of Lemma 3.8. To end the proof of Lemma 3.8, it remains to build a smaller open set in which item (1) is also satisfied.

For that, we fix a neighborhood O of the circle $\{p_0\} \times S^1$ which is disjoint from the Z-orbit segments joining the points in $\overline{V_0 \times S^1}$ to their image by the first return map F_0. Notice that O is also disjoint from the X_1-orbit segments joining the points in $\overline{V_0 \times S^1}$ to their image by the first return map F_1. Now we perform a C^1 small perturbation X_2 of X_1 such that $X_2 \in \mathcal{U}'$, X_2 preserves the S^1-fibration, and $\{p_0\} \times S^1$ contains exactly

[7] Indeed, as γ and γ' are hyperbolic periodic orbits, there is a hyperbolic basic set of Y containing γ and γ'; the intersection of this basic set and the cross section σ is the stated basic set K of the first return map f.

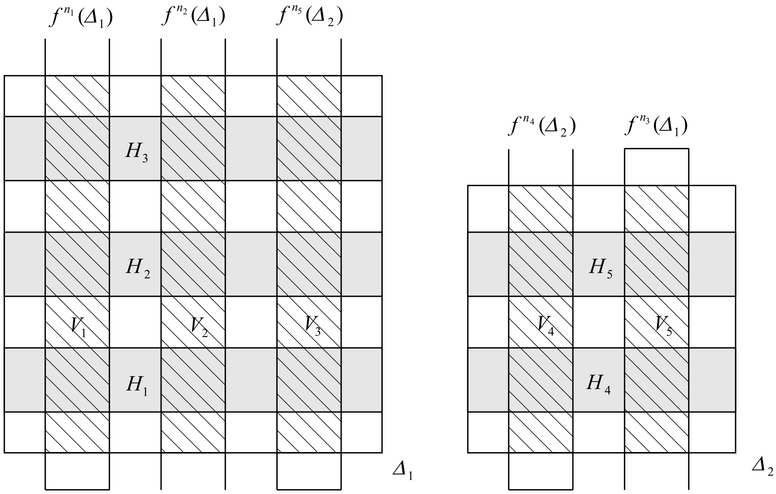

Figure 4. A Markov partition for a basic set $K^1 \subset K$.

two hyperbolic singularities, p_1 and p_2. Now, every vector field X in a sufficiently small C^1-neighborhood $\mathcal{U}_2$ of X_2 satisfies all the stated properties. □

3.4. Proof of Proposition 3.10

3.4.1. Sketch of the proof. Proposition 3.10 is essentially a particular case of [6, Theorem B] which performs a perturbation of a product diffeomorphism (f, id_N), where id_N is the identity map of an arbitrary compact manifold N (for us $N = S^1$) and f admits a hyperbolic attractor Λ; this perturbation turns $\Lambda \times N$ in a robustly transitive attractor.

The proof of Proposition 3.10 follows very closely the one of [6, Theorem B], in particular, by the use of [6, Theorem 3.5].

- We first choose a hyperbolic basic set $K^1 \subset K$ of f, having a Markov partition by disjoint compact cubes with a prescribed incidence matrix (see Figure 4).
- This allows us to perform an explicit smooth 2-parameter family $F_{\alpha,\beta}$ of perturbations of $F_0 = (f, id_{S^1})$ with a prescribed effect on $K^1 \times S^1$ (see Figure 5). In particular, there is a periodic fiber C containing exactly 2 hyperbolic periodic points y_1 and y_2.
- For every small $\alpha > 1$, the map $F_{\alpha,\beta}$ has two disjoint hyperbolic basic sets, and [9] proves that one of them is a *cs*-blender containing y_1 and the other is a *cu*-blender containing y_2, according to the terminology of [6].

  The *cs*-blenders are some kind of fat hyperbolic sets displaying an open region (called here the *characteristic region of the blender*) such that any strong unstable disc crossing this characteristic region cuts (in a C^1-robust way) the strong stable manifold of the blender.

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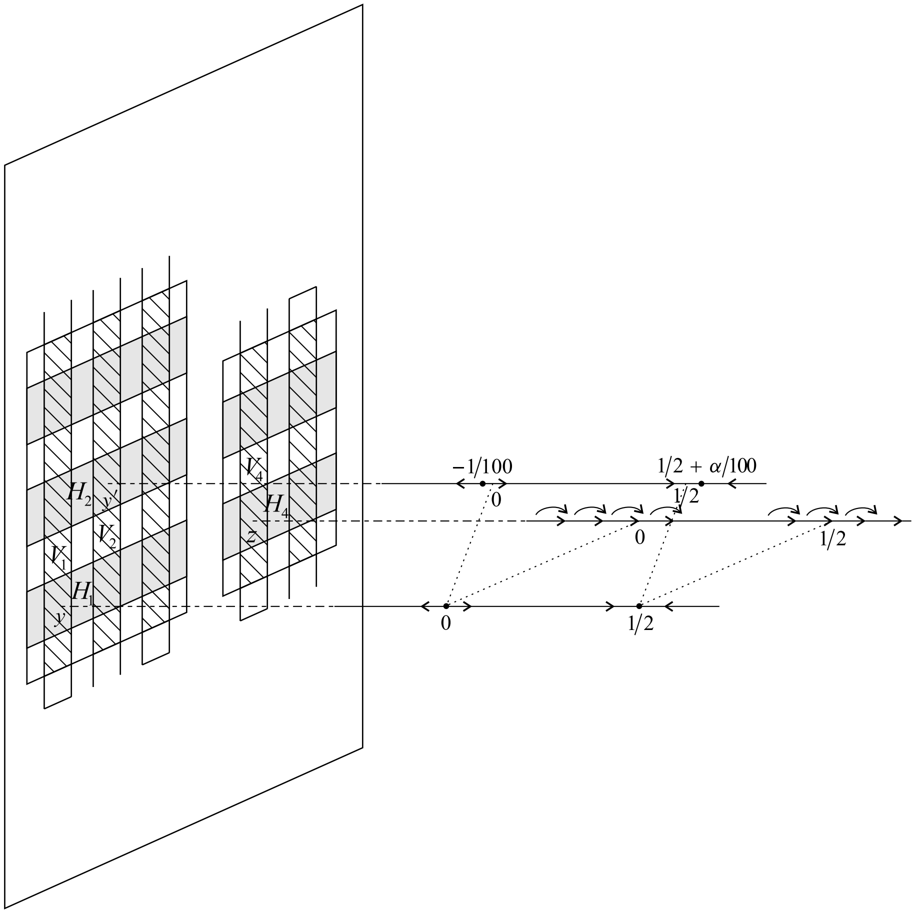

Figure 5. A two-parameter family $F_{\alpha,\beta}$ of perturbations of F_0.

- Then, [6] proves that the stable manifold of every periodic orbit in the *cs*-blenders contains in its closure the stable manifold of every periodic saddle whose unstable manifold cuts the characteristic region of the blender.
- When the parameter β is irrational, we will prove that the unstable (respectively, stable) manifold of any saddle in $K^1 \times S^1$ crosses the characteristic region of the *cs*-blender (respectively, *cu*-blender).
- We will apply this argument to the orbits of y_1 and y_2. We will obtain that, for small $\alpha > 1$ and for $\beta \notin \mathbb{Q}$, the closure of the invariant manifolds of the orbits of y_2 and of y_2 coincide robustly with the closure of the invariant manifold of the fiber C, concluding the proof.

3.4.2. Markov partition and basic set. As K is a non-trivial basic set, there is another periodic point $z \in K$, homoclinically related with y. Let $\pi(z)$ denote its period.

Lemma 3.11. *There exist*

- *a basic set* $K^1 \subset K$,
- *rectangles* Δ_1 *and* Δ_2, *centered at* y *and* z *respectively, which in the local chart at* y *and* z *are the products* $\Delta_i = \Delta_i^s \times \Delta_i^u$, *for* $i = 1, 2$,
- *disjoint horizontal subrectangles* $H_1 = \Delta_1^s \times B_1, H_2 = \Delta_1^s \times B_2, H_3 = \Delta_1^s \times B_3$ *of* Δ_1 *and* $H_4 = \Delta_2^s \times B_4, H_5 = \Delta_2^s \times B_5$ *of* Δ_2,
- *vertical subrectangles* $V_1 = B_1' \times \Delta_1^u, V_2 = B_2' \times \Delta_1^u, V_3 = B_3' \times \Delta_1^u$ *of* Δ_1 *and* $V_4 = B_4' \times \Delta_2^u, V_5 = B_5' \times \Delta_2^u$ *of* Δ_2,
- *and strictly positive integers* $n_1, \ldots, n_5$,

such that

- $n_1 = \pi(y)$ *and* $n_4 = \pi(z)$,
- *the* $f^n(H_j), n \in \{1, \ldots, n_j - 1\}$ *are pairwise disjoint and disjoint from* $\Delta_1 \cup \Delta_2$, *and*
- $f^{n_1}(H_1) = V_1, f^{n_2}(H_2) = V_2, f^{n_3}(H_3) = V_5, f^{n_4}(H_4) = V_4$, *and* $f^{n_5}(H_5) = V_3$.

Idea of the proof. As the points y and z are homoclinically related, they belong to the same basic set K. This basic set K admits a Markov partition $\mathcal{M}_0 = \{\mathcal{R}_i\}_{i \in I}$. By considering a refinement $\mathcal{M}_i = \{f^{-n}(\mathcal{R}_{i_{-n}}) \cap f^{-n+1}(\mathcal{R}_{i_{-n+1}}) \cap \cdots \cap f^n(\mathcal{R}_{i_n})\}$, one gets Markov partitions whose rectangles are arbitrarily small.

We fix a homoclinic orbit p_{yy} of y, a heteroclinic orbit p_{yz} going from y to z, and a heteroclinic orbit p_{zy} going from z to y. Then one gets the stated rectangles as follows.

- Let Δ_1, Δ_2, Δ_{yy}, Δ_{yz}, and Δ_{zy} be the rectangles of $\mathcal{M}_n$ containing y, z, p_{yy}, p_{yz} and p_{zy}, respectively.
- We denote $n_1 = \pi(y)$ and $n_4 = \pi(z)$, and $H_1 = f^{-n_1}(\Delta_1) \cap \Delta_1$ and $H_4 = f^{-n_4}(\Delta_2) \cap \Delta_2$. Now, we put $V_1 = f^{n_1}(\Delta_1) \cap \Delta_1$ and $V_4 = f^{n_4}(\Delta_2) \cap \Delta_2$.
- Consider the smallest number of $T_1 > 0$ and $T_2 > 0$ such that $f^{-T_1}(\Delta_{yy}) \cap \Delta_1 \neq \emptyset$ and $f^{T_2}(\Delta_{yy}) \cap \Delta_1 \neq \emptyset$; then we denote $H_2 = f^{-T_1}(\Delta_{yy}) \cap \Delta_1$ and $V_2 = f^{T_2}(\Delta_{yy}) \cap \Delta_1$. We denote $n_2 = T_1 + T_2$.
- Consider the smallest number of $T_3 > 0$ and $T_4 > 0$ such that $f^{-T_3}(\Delta_{yz}) \cap \Delta_1 \neq \emptyset$ and $f^{T_2}(\Delta_{yz}) \cap \Delta_2 \neq \emptyset$; then we denote $H_3 = f^{-T_3}(\Delta_{yz}) \cap \Delta_1$ and $V_3 = f^{T_3}(\Delta_{yz}) \cap \Delta_2$. We denote $n_3 = T_3 + T_4$.
- Consider the smallest number of $T_5 > 0$ and $T_6 > 0$ such that $f^{-T_5}(\Delta_{zy}) \cap \Delta_2 \neq \emptyset$ and $f^{T_6}(\Delta_{zy}) \cap \Delta_1 \neq \emptyset$; then we denote $H_5 = f^{-T_5}(\Delta_{zy}) \cap \Delta_2$ and $V_5 = f^{T_6}(\Delta_{zy}) \cap \Delta_1$. We denote $n_5 = T_5 + T_6$.

The construction above depends on the Markov partition $\mathcal{M}_n$ we considered; however, we have omitted the dependence on n for simplicity. When the rectangles of $\mathcal{M}_n$ are small enough (i.e., for n large enough), the construction above satisfies all the stated properties.

Finally, $\bigcup_{i=1}^{5}\{H_i, \ldots, f^{n_i-1}H_i\}$ is a Markov partition generating a basic set K^1 contained in K, contained in the interior of $\bigcup_{i=1}^{5}\bigcup_{j=0}^{n_i-1} f^j(H_i)$, and containing y and z. □

We denote by $K^0 \subset K^1$ the basic set which is the maximal invariant set in $\bigcup_{i=1}^{2} \bigcup_{j=0}^{n_i-1} f^j(H_i)$. Notice that the map $g : H_1 \cup H_2 \to \Delta_1$ which is f^{n_i} on H_i is a usual Smale horseshoe. We denote by $y' \in K^0$ the fixed point of f^{n_2} contained in $H_2 \cap V_2$ (hence in Δ_1).

3.4.3. Family of perturbations. Let $\varphi_\alpha : S^1 \to S^1$ be a diffeomorphisms varying smoothly with $\alpha \in [1, 2]$ such that

- φ_1 is the identity map of S^1;
- $\varphi_\alpha(s) = \alpha s$ for $s \in [-\frac{1}{8}, \frac{1}{8}]$;
- $\varphi_\alpha(s) = \frac{1}{2} + \alpha^{-1}(s - \frac{1}{2})$ for $s \in [\frac{1}{2} - \frac{\alpha}{8}, \frac{1}{2} + \frac{\alpha}{8}]$;
- for $\alpha > 1$, the diffeomorphism φ_α has no other fixed points than 0 and $\frac{1}{2}$:

$$\mathrm{Fix}(\varphi_\alpha) = \left\{0, \frac{1}{2}\right\}.$$

Given any $\alpha \in [1, 2]$ and $\beta \in \mathbb{R}$, we choose a diffeomorphism $F_{\alpha,\beta} : V_0 \times S^1 \to M \times S^1$ such that $F_{\alpha,\beta}$ is of the form $F_{\alpha,\beta}(x, s) = (f(x), \varphi_{\alpha,\beta,x}(s))$, where $\{\varphi_{\alpha,\beta,x}\}_{\alpha\in[1,2],\beta\in\mathbb{R},x\in V_0}$ is a continuous family of circle diffeomorphisms such that

- $\varphi_{\alpha,\beta,x} = id_{S^1}$ for $x \notin V$ and for $x \in \bigcup_{i=1}^{5} \bigcup_{j=0}^{n_i-1} f^j(H_i) \setminus (H_1 \cup H_2 \cup H_4)$;
- $\varphi_{\alpha,\beta,x} = \varphi_\alpha$ for $x \in H_1$;
- $\varphi_{\alpha,\beta,x}(s) = s + \beta$ for $x \in H_4$;
- $\varphi_{\alpha,\beta,x}(s) = \varphi_\alpha(s) + \frac{\alpha-1}{100}$ for $x \in H_2$.

Notice that there is $\alpha_0 \in (1, 2]$ and $\beta_0 > 0$ such that, for every $\alpha \in [1, \alpha_0]$ and any $\beta \in [-\beta_0, \beta_0]$, the S^1-bundle over K is normally hyperbolic.

Now, for every $\alpha \in (1, \alpha_0]$ and $\beta \in [-\beta_0, \beta_0]$, the circle $C = \{y\} \times S^1$ is an invariant normally hyperbolic circle of the diffeomorphism $F_{\alpha,\beta}^{\pi(y)}$, and it contains exactly two fixed points (for $F_{\alpha,\beta}^{\pi(y)}$): the point $y_1 = (y, 0)$ (of index equal to the index of y for $f^{\pi(y)}$) and the point $y_2 = (y, \frac{1}{2})$ (of index equal to the index of y for $f^{\pi(y)}$ plus 1).

As a consequence, for the diffeomorphism $F_{\alpha,\beta}$, one has

$$\mathrm{cl}(W^u(C)) = \mathrm{cl}(W^u(y_1)) \supset \mathrm{cl}(W^u(y_2)), \quad \text{and} \quad \mathrm{cl}(W^s(C)) = \mathrm{cl}(W^s(y_2)) \supset \mathrm{cl}(W^s(y_1)).$$

Furthermore, these properties are robust: there is a C^1-neighborhood $\mathcal{W}_3$ of $F_{\alpha,\beta}$ such that the circle C has a normally hyperbolic continuation C_F for $F \in \mathcal{W}_3$, containing the hyperbolic continuations $y_{1,F}$ and $y_{2,F}$, and one has

$$\mathrm{cl}(W^u(C_F)) = \mathrm{cl}(W^u(y_{1,F})) \supset \mathrm{cl}(W^u(y_{2,F})),$$

and

$$\mathrm{cl}(W^s(C_F)) = \mathrm{cl}(W^s(y_{2,F})) \supset \mathrm{cl}(W^s(y_{1,F})).$$

3.4.4. Existence of the blender. Now, Proposition 3.10 is a straightforward consequence of Lemma 3.12 below.

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Lemma 3.12. *For every $\alpha \in (1, \alpha_0]$ and any irrational $\beta \in [-\beta_0, \beta_0] \setminus \mathbb{Q}$, there is a neighborhood $\mathcal{W}_{\alpha,\beta}$ of $F_{\alpha,\beta}$ such that, for every $F \in \mathcal{W}_{\alpha,\beta}$, one has*

$$\mathrm{cl}(W^u(y_{1,F})) \subset \mathrm{cl}(W^u(y_{2,F})),$$

and

$$\mathrm{cl}(W^s(y_{2,F})) \subset \mathrm{cl}(W^s(y_{1,F})).$$

Proof. For any $F_{\alpha,\beta}$, one can define the map $G_\alpha : (H_1 \cup H_2) \times S^1 \to \Delta_1 \times S^1$, which is $F^{n_i}_{\alpha,\beta}$ on $H_i \times S^1$. Notice that G_α coincides with the map $(x, s) \mapsto (f^{n_i}(x), \varphi_{\alpha,\beta,x}(s))$. (Thus G_α does not depend on β, as $\varphi_{\alpha,\beta,x}$ does not depend on β on $H_1 \cup H_2$.) One deduces the following.

- The circle $\{y'\} \times S^1$ is invariant, normally hyperbolic, and contains two fixed points $y'_{1,\alpha} = (y', -\frac{1}{100})$ and $y'_{2,\alpha} = (y', \frac{1}{2} + \frac{\alpha}{100})$.
- The restriction of G_α to the cubes $H_1 \times [-\frac{1}{8}, \frac{1}{8}]$ and $H_2 \times [-\frac{1}{8}, \frac{1}{8}]$ is a hyperbolic basic set; as a consequence, the maximal invariant set of $F_{\alpha,\beta}$ in $\left(\bigcup_{i=1}^{2} \bigcup_{j=0}^{n_i-1} f^j(H_i)\right) \times [-\frac{1}{8}, \frac{1}{8}]$ is a hyperbolic basic set denoted by $K^0_{1,\alpha}$, which contains y_1 and $y'_{1,\alpha}$. Furthermore, the hyperbolic set $K^0_{1,\alpha}$ is also partially hyperbolic, the tangent bundle to the S^1 direction being the center direction, and this direction is the weak unstable direction.
- The restriction of G_α to the cubes $H_1 \times [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}]$ and $H_2 \times [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}]$ is a hyperbolic basic set; as a consequence, the maximal invariant set of $F_{\alpha,\beta}$ in $\left(\bigcup_{i=1}^{2} \bigcup_{j=0}^{n_i-1} f^j(H_i)\right) \times [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}]$ is a hyperbolic basic set denoted by $K^0_{2,\alpha}$, which contains y_2 and $y'_{2,\alpha}$. Furthermore, the hyperbolic set $K^0_{2,\alpha}$ is also partially hyperbolic, the tangent bundle to the S^1 direction being the center direction, and this direction is the weak stable direction.

According to the terminology of [6], it is proved in [9] that the hyperbolic set $K^0_{1,\alpha}$ is a *cs*-blender. Every disc[8] D tangent to a small cone field around the strong unstable direction and crossing $\Delta_1 \times [-\frac{1}{8}, \frac{1}{8}]$ between the local stable manifolds of the points y_1 and $y'_{1,\alpha}$ meets the local stable manifold of $K^0_{1,\alpha}$. Furthermore, this property is robust, that is, it holds for the diffeomorphisms F in a small C^1-neighborhood of $F_{\alpha,\beta}$ for the continuations of $K^0_{1,F}, y_{1,F}, y'_{1,F}$ for F of $K^0_{1,\alpha}, y_1, y'_{1,\alpha}$. As a consequence, [6] proved that, *if the unstable manifold $W^u(y_2)$ contains a disc tangent to the unstable cone field and crossing $\Delta_1 \times [-\frac{1}{8}, \frac{1}{8}]$ between the local stable manifolds of the points y_1 and $y'_{1,\alpha}$*, then $\mathrm{cl}(W^s(y_{2,F})) \subset \mathrm{cl}(W^s(y_{1,F}))$ for every F in a small C^1-neighborhood of $F_{\alpha,\beta}$.

In the same way, the hyperbolic set $K^0_{2,\alpha}$ is a *cu*-blender. Every disc D tangent to a small cone field around the strong stable direction and crossing $\Delta_1 \times [\frac{1}{2} - \frac{1}{8}, \frac{1}{2} + \frac{1}{8}]$ between the local stable manifolds of the points y_2 and $y'_{2,\alpha}$ meets the local unstable manifold of $K^0_{2,\alpha}$. Furthermore, this property is robust, that is, it holds for diffeomorphisms F in a small C^1-neighborhood of $F_{\alpha,\beta}$ for the continuations

[8] A curve is a one-dimensional disc here. So we use the word disc to express a curve sometimes.

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of $K^0_{2,F}$, $y_{2,F}$, and $y'_{2,F}$ for F of $K^0_{2,\alpha}$, y_2, and $y'_{2,\alpha}$. Furthermore, *if the stable manifold* $W^s(y_1)$ *contains a disc tangent to the stable cone field and crossing* $\Delta_1 \times [\frac{1}{2}-\frac{1}{8}, \frac{1}{2}+\frac{1}{8}]$ *between* $W^u_{loc}(y_2)$ *and* $W^u_{loc}(y'_{2,\alpha})$, then $\mathrm{cl}(W^u(y_{1,F})) \subset \mathrm{cl}(W^u(y_{2,F}))$ for every F in a small C^1-neighborhood of $F_{\alpha,\beta}$.

To end the proof, it only remains to prove claim 1 below.

Claim 1. *For every irrational* $\beta \in [-\beta_0, \beta_0] \setminus \mathbb{Q}$, *the unstable manifold* $W^u(y_2)$ *contains a disc tangent to the unstable cone field and crossing* $\Delta_1 \times [-\frac{1}{8}, \frac{1}{8}]$ *between* $W^s_{loc}(y_1)$ *and* $W^s_{loc}(y'_{1,\alpha})$, *and the stable manifold* $W^s(y_1)$ *contains a disc tangent to the stable cone field and crossing* $\Delta_1 \times [\frac{1}{2}-\frac{1}{8}, \frac{1}{2}+\frac{1}{8}]$ *between* $W^u_{loc}(y_2)$ *and* $W^u_{loc}(y'_{2,\alpha})$.

Proof. The circle $\{z\} \times S^1$ is a periodic normally hyperbolic circle of period $\pi(z) = n_4$ on which $F^{n_4}_{\alpha,\beta}$ induces a rotation R_β, hence an irrational rotation. The local dynamics is the product map of f^{n_4} by the rotation R_β. As $f^{n_3}(H_3) = V_5 \subset \Delta_2$, the unstable manifold $W^u(y_2)$ contains an unstable disc crossing $\Delta_2 \times S^1$. Using iteration of $F^{n_4}_{\alpha,\beta}$, one gets that every unstable disc contained in $W^u(\{z\} \times S^1)$ is a limit of unstable discs in $W^u(y_2)$. Now, using the fact that $f^{n_5}(H_5) = V_3$, one gets that $W^u(\{z\} \times S^1)$ contains an unstable disc crossing $\Delta_1 \times [-\frac{1}{8}, \frac{1}{8}]$ between $W^s_{loc}(y_1)$ and $W^s_{loc}(y'_{1,\alpha})$; hence, the same happens for $W^u(y_2)$.

One argues in the same way, using negative iterates of $F_{\alpha,\beta}$ for getting the property on $W^s(y_1)$. □

□

In the next section, we will prove there is a C^1 neighborhood $\mathcal{U}$ and a local generic set $\mathcal{R} \subset \mathcal{U}$ such that, if $X \subset \mathcal{R}$, the attractor Λ_X is transitive.

4. Proof of the transitivity

Lemma 3.8 built an open set $\mathcal{U}_2$ (containing Z in its closure). Vector fields in $\mathcal{U}_2$ have exactly two singular points p^X_1 and p^X_2 contained in the circle S_X, and two periodic orbits in the torus Γ_X.

In this section, we will prove that there exists a C^1 neighborhood $\mathcal{U}_0$ such that, for every vector field $X \in \mathcal{U}_0 \cap \mathcal{U}_2$, Λ_X is a transitive attractor if the following two hypotheses hold.

(H$_1$) $W^u(p^X_1) \cap W^s(p^X_2) = \emptyset$;

(H$_2$) $W^u(p^X_1) \cap W^s(\Gamma_X) = \emptyset$.

In particular, the following is the main theorem of this paper.

Theorem 4.1. *There exists a* C^1 *neighborhood* $\mathcal{U}_0$ *of* Z *such that, for every* $X \in \mathcal{U}_0 \cap \mathcal{U}_2$ *satisfying* (H$_1$) *and* (H$_2$), Λ_X *is a transitive attractor.*

To this end, we will prove of the density of the stable and unstable manifold of Γ_X.

Proposition 4.2. *There exists a* C^1 *neighborhood* $\mathcal{U}_0$ *of* Z *such that, for every* $X \in \mathcal{U}_0 \cap \mathcal{U}_2$ *satisfying* (H$_1$) *and* (H$_2$), $W^s(\Gamma_X)$ *is dense in* U.

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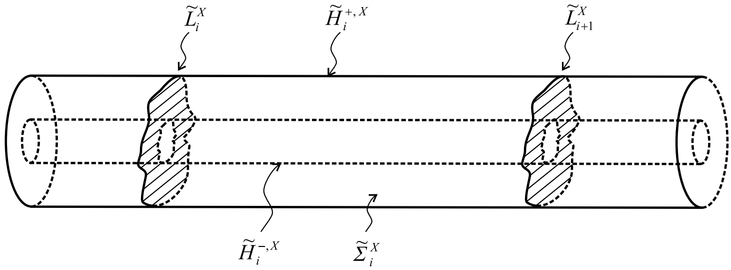

Figure 6. Covering.

Proposition 4.3. *There exists a C^1 neighborhood $\mathcal{U}_0$ of Z such that, for every $X \in \mathcal{U}_0$, $W^u(\Gamma_X)$ is dense in Λ_X.*

We will give the proof of the above two propositions in §§ 3.3 and 3.4. We first introduce the covering of the cross section Σ.

4.1. The covering map

We consider the covering map of Σ,

$$\rho : \widetilde{\Sigma} := [-1, 1] \times \mathbb{R} \times S^1 \to \Sigma.$$

An *essential annulus* of $\widetilde{\Sigma}$ is an embedding of $[-1, 1] \times S^1$ in $\widetilde{\Sigma}$ whose boundary is contained in the boundary $\{-1, 1\} \times \mathbb{R} \times S^1$ of $\widetilde{\Sigma}$ and which is isotopic to $[-1, 1] \times \{0\} \times S^1$ (by an isotopy keeping the boundary on the boundary of $\widetilde{\Sigma}$). Notice that every essential annulus cuts $\widetilde{\Sigma}$ into two components.

For any vector field X in $\mathcal{U}_3$ (where $\mathcal{U}_3$ is the open set built in Proposition 3.4), the preimage $\rho^{-1}(L^X)$ is a sequence of disjoint essential annuli $\widetilde{\Sigma}$, denoted by $\{\widetilde{L}_i^X\}$, where $\{\widetilde{L}_{i+1}^X\}$ is the image of $\{\widetilde{L}_i^X\}$ by the cover automorphism $\mathcal{T} : (r, s, t) \mapsto (r, s + 1, t)$.

For any i, the successive annuli $\widetilde{L}_i^X$ and $\widetilde{L}_{i+1}^X$ split $\widetilde{\Sigma}$ into three (open) connected components; one is bounded, and the other two are unbounded, and we denote by $\widetilde{\Sigma}_i^X$ the bounded component. Notice that ρ induces a diffeomorphism ρ_i from $\widetilde{\Sigma}_i^X$ to $\Sigma \setminus L^X$ which is the definition domain of the first return map.

We denote by $H^{+,X}$ and $H^{-,X}$ the open cylinders

$$H^{\pm,X} := \{\pm 1\} \times S^1 \times S^1 \setminus L^X.$$

We denote $\widetilde{H}_i^{\pm,X} = \rho_i^{-1}(H^{\pm,X}) \subset \widetilde{\Sigma}_i^X$. See Figure 6.

We choose a lift $\widetilde{F}_X : \widetilde{\Sigma} \setminus \bigcup_i \widetilde{L}_i^X \to \widetilde{\Sigma}$ of the first return map F_X, so that

- its restriction $\widetilde{F}_{i,X} : \widetilde{\Sigma}_i^X \to \widetilde{\Sigma}$ is a diffeomorphism onto its image;
- $\widetilde{F}_X$ commutes with the cover automorphism $\mathcal{T}$; that is, $\widetilde{F}_{i+1,X} = \mathcal{T} \circ \widetilde{F}_{i,X} \circ \mathcal{T}^{-1}$.

For each i, the image $\widetilde{F}_X(\widetilde{\Sigma}_i^X)$ is a connected subset of $\widetilde{\Sigma}$. Recall that the circles $Q^{\pm,X}$ are the connected components of the intersection of Σ and $W^u(S_X)$.

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4.2. Geometric properties of the image $\widetilde{F}_X(\widetilde{\Sigma}_i^X)$

The aim of the section is to show that $\widetilde{F}_X(\widetilde{\Sigma}_i^X)$, for X close enough to Z, is bounded by the union of two compact C^1-annuli $\widetilde{A}_i^{+,X}$ and $\widetilde{A}_i^{-,X}$ which intersect exactly on their boundary, and vary continuously with X. The difficulty here is that the first return map F_X does not vary continuously with X (more precisely, it varies continuously far from L^X, but not in the neighborhood of L^X).

Lemma 4.4. *The boundary of $\widetilde{F}_X(\widetilde{\Sigma}_i^X)$ in $\widetilde{\Sigma}$ is the union*

$$\widetilde{F}_X(\widetilde{H}_i^{+,X}) \cup \widetilde{F}_X(\widetilde{H}_i^{-,X}) \cup \widetilde{Q}_i^{+,X} \cup \widetilde{Q}_i^{-,X},$$

where $\widetilde{Q}_i^{\pm,X}$ is a lift of $Q^{\pm,X}$.

Proof. This is a direct consequence of the fact that, for any sequence of regular points which converges to $L^X \subset W^s(p_1^X)$, its image by the first return map must converge to $Q^{+,X}$ or $Q^{-,X}$ (because the positive orbit of a regular point which is close to the stable manifold of S_X follows the unstable manifold of S_X). □

We denote by $\pi : \widetilde{\Sigma} = [-1, 1] \times \mathbb{R} \times S^1 \to \mathbb{R} \times S^1$ the natural projection $(r, s, t) \mapsto (s, t)$.

Notice that the projections $\pi(\widetilde{Q}_i^{+,Z})$ and $\pi(\widetilde{Q}_i^{-,Z})$ are disjoint embedded circles bounding an essential annulus in $\mathbb{R} \times S^1$ (recall that the circles $Q^{+,Z}$ and $Q^{-,Z}$ are $\{q^+\} \times S^1$ and $\{q^-\} \times S^1$, respectively).

As a consequence, one gets the following.

Lemma 4.5. *There is a C^1-neighborhood $\mathcal{U}_4 \subset \mathcal{U}_3$ of Z such that, for every $X \in \mathcal{U}_4$, the projections $\pi(\widetilde{Q}_i^{+,X})$ and $\pi(\widetilde{Q}_i^{-,X})$ are disjoint embedded circles bounding an essential annulus in $\mathbb{R} \times S^1$.*

The cone field $C_b^{cu,Z}$ is transverse to the foliation $\mathcal{F}_\Sigma^{ss,Z}$ whose leaves are the segments $[-1, 1] \times \{(s, t)\}$. This cone field is invariant by the first return maps F_X for every $X \in \mathcal{U}_3$. Furthermore, the boundary of Σ is tangent to this cone field. One deduces that the images $F_X(H^{+,X})$ and $F_X(H^{-,X})$ are open annulus tangent to $C_b^{cu,Z}$, and hence transverse to the foliation $\mathcal{F}_\Sigma^{ss,Z}$. One deduces directly the following lemma.

Lemma 4.6. *For any $X \in \mathcal{U}_3$, the projection π induces a local diffeomorphism from $\widetilde{F}_X(\widetilde{H}_i^{\pm,X})$ to $\mathbb{R} \times S^1$.*

As a consequence, one gets the following corollary.

Corollary 4.7. *There is a neighborhood $\mathcal{U}_5 \subset \mathcal{U}_4$ of Z such that, for any $X \in \mathcal{U}_5$, the projection π induces a homeomorphism from $\widetilde{F}_X(\widetilde{H}_i^{\pm,X})$ to the open annulus bounded by $\pi(\widetilde{Q}_i^{+,X}) \cup \pi(\widetilde{Q}_i^{-,X})$ in $\mathbb{R} \times S^1$.*

Proof. $\widetilde{F}_X(\widetilde{H}_i^{\pm,X})$ is an open C^1-annulus whose boundary is contained in $\widetilde{Q}_i^{+,X} \cup \widetilde{Q}_i^{-,X}$. The projection is a local diffeomorphism. This implies that the boundary of the projection is contained in $\pi(\widetilde{Q}_i^{+,X}) \cup \pi(\widetilde{Q}_i^{-,X})$. Hence the image $\pi(\widetilde{F}_X(\widetilde{H}_i^{\pm,X}))$ is the open annulus bounded by $\pi(\widetilde{Q}_i^{+,X}) \cup \pi(\widetilde{Q}_i^{-,X})$.

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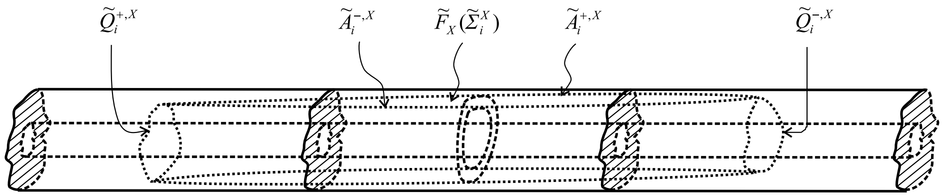

Figure 7. First return map.

One deduces that, for every $X \in \mathcal{U}_4$, the projection induces a cover from $\widetilde{F}_X(\widetilde{H}_i^{\pm,X})$ to the open annulus bounded by $\pi(\widetilde{Q}_i^{+,X}) \cup \pi(\widetilde{Q}_i^{-,X})$.

In order to finish the proof, we need to show that there is one point in the open annulus bounded by $\pi(\widetilde{Q}_i^{+,X}) \cup \pi(\widetilde{Q}_i^{-,X})$ having a unique preimage by π on $\widetilde{F}_X(\widetilde{H}_i^{\pm,X})$.

This property is satisfied by the vector field Z.

Furthermore, $F_X(H^{\pm,X})$ varies continuously on compact subsets, that is, *far from* L^X, and L^X varies continuously with X. Fix a small neighborhood U_L of $L = L^Z$ and a small neighborhood U_Q of $Q^{+,Z} \cup Q^{-,Z}$ such that $F(U_L) \subset U_Q$.

For X C^1-close to Z, one has $F_X(U_L) \subset U_Q$. Now, F_X varies continuously out of U_L. Hence, for X C^1-close to Z, a point in $\mathbb{R} \times S^1 \setminus \pi(\rho^{-1}(U_Q))$ has at most one preimage on $\widetilde{F}_X(\widetilde{H}_i^{\pm,X})$, ending the proof.

Lemma 4.8. *The union* $\widetilde{A}_i^{+,X} = \widetilde{F}_X(\widetilde{H}_i^{+,X}) \cup \widetilde{Q}_i^{+,X} \cup \widetilde{Q}_i^{-,X}$ *is a* C^1 *embedding of the compact cylinder* $[-1, 1] \times S^1$*. The same holds for* $\widetilde{A}_i^{-,X} = \widetilde{F}_X(\widetilde{H}_i^{-,X}) \cup \widetilde{Q}_i^{+,X} \cup \widetilde{Q}_i^{-,X}$.

Proof. First, notice that $\widetilde{A}_i^{+,X} = \widetilde{F}_X(\widetilde{H}_i^{+,X}) \cup \widetilde{Q}_i^{+,X} \cup \widetilde{Q}_i^{-,X}$ is compact. We just need to proof that the tangent plane to $\widetilde{F}_X(\widetilde{H}_i^{+,X})$ at a point x converges to a plane when x tends to a point of $\widetilde{Q}_i^{+,X}$, and that these limit planes form a continuous family.

Notice that $Q^{\pm,X} \subset W^u(S_X)$ is contained in Λ_X so that the center-unstable bundle is well defined on $Q^{\pm,X}$.

The tangent plane $T_x(H^{+,X})$ is contained in the cone field $C_b^{cu,Z}(x)$ (see Z.12(c) of the properties of Z) and the sum $T_x(H^{+,X}) + \mathbb{R}X$ is contained in $\mathcal{C}_a^{cu,Z}(x)$(by Lemma 3.5). Furthermore, the cone field $C_b^{cu,Z}(x)$ is invariant by F_X (Proposition 3.4) so that the tangent plane $T_{F_X(x)}(F_X(H^{+,X})$ is contained in $C_b^{cu,Z}(F_X(x))$.

When x tends to L^X, the return time $\tau^X(x)$ tends to infinity. The partial hyperbolicity of the flow ϕ^X implies that $\Phi^X_{\tau^X(x)}(T_x(H^{+,X}) + \mathbb{R}X)$ tends to $E^{cu,X}(y)$, where y is the limit point of $F_X(x)$. □

Thus $\widetilde{A}_i^{+,X}$ and $\widetilde{A}_i^{-,X}$ are two compact C^1-embeddings of the annulus, whose interiors are disjoint and having the same boundary, which is the union of the two circles $\widetilde{Q}_i^{+,X}$ and $\widetilde{Q}_i^{-,X}$. Furthermore, $\widetilde{A}_i^X := \widetilde{A}_i^{+,X} \cup \widetilde{A}_i^{-,X}$ is the boundary of the image $\widetilde{F}_X(\widetilde{\Sigma}_i^X)$.

Recall that $F_X(\Sigma \setminus L^X) = \rho(\widetilde{F}_X(\widetilde{\Sigma}_i^X))$. Its boundary consists in the union A^X of the annuli $A^{+,X} = \rho(\widetilde{A}_i^{+,X})$ and $A^{-,X} = \rho(\widetilde{A}_i^{-,X})$.

See Figure 7.

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We end the section by proving the following.

Lemma 4.9. *The annuli $\widetilde{A}_i^{+,X}$ and $\widetilde{A}_i^{-,X}$ vary C^1-continuously with $X \in \mathcal{U}_5$.*

Idea of the proof. The first return map varies continuously on compact sets in its definition domain, that is, far from L^X. Furthermore, the boundary of $\widetilde{A}_i^{\pm,X}$ is $\widetilde{Q}_i^{+,X} \cup \widetilde{Q}_i^{-,X}$, which varies continuously with X. It just remains to control the tangent plane to $\widetilde{A}_i^{\pm,X}$ in the neighborhood of its boundary.

For that, we fix a constant T_0 such that the cone field $\Phi_{T_0}^X(\mathcal{C}_a^{cu,Z})$ is arbitrarily thin. Notice that this cone field is strictly invariant by Φ^X, and hence is invariant by $\Phi^{X'}$ for $X' C^1$-close enough to X.

Then we choose a small neighborhood U_{L_X} of L_X on which the time return $\tau^X(x)$ is larger than a very large constant $T_1 \gg T_0$.

Now, for $X' C^1$-close to X and $x \in U_{L_X} \cap H^{+,X'}$, we get that the image by $DF_{X'}$ of the tangent plane $T_x H^{+,X'}$ is contained in the (arbitrarily thin) cone $\Phi_{T_0}^X(\mathcal{C}_a^{cu,Z})(F_{X'}(x)) \cap T_{F_{X'}(x)}\Sigma$ (see the proof of Lemma 4.8).

This means that, in a neighborhood of the boundary of $A^{+,X}$, the annuli $A^{+,X'}$ are tangent to the same arbitrarily thin cone field $\Phi_{T_0}^X(\mathcal{C}_a^{cu,Z}) \cap T\Sigma$, for X' close to X. □

4.3. Quasi-connected transverse section to the strong stable cone

The aim of this section is to define a notion of sections of the strong stable foliation $\mathcal{F}_\Sigma^{ss,X}$ (that is, sets meeting each strong stable leaf in at most one point) having a property of connexity which will be invariant by iteration under the first return map F_X. These will be the *quasi-connected sections* defined at the end of this section.

For $x \in \Sigma$, we denote

$$C_c^{ss,Z}(x) = \{v \in T_x\Sigma : v = v^s + v^{cu}, v^s \in T_x\mathcal{F}_\Sigma^{ss}, v^{cu} \in E^{cu,Z} \cap T_x\Sigma, \text{ and } \|v^{cu}\| \leqslant c\|v^s\|\}.$$

Taking c small enough, $C_c^{ss,Z}$ is a cone field around $\mathcal{F}_\Sigma^{ss,Z}$ transverse to $C_a^{cu,Z}$. Furthermore, $C_c^{ss,Z}$ is negatively invariant by F_X for any X C^1-close to Z; more precisely, there exists a neighborhood $\mathcal{U}_c \subset \mathcal{U}_5$ of Z such that, for any $X \in \mathcal{U}_c$ and any $x \in F_X(\Sigma \setminus L^X)$, we have

$$DF_X^{-1}(C_c^{ss,Z}(x)) \subset C_c^{ss,Z}(F_X^{-1}(x)).$$

Notice that, for $c > 0$ small enough, the length of any segment tangent to $C_c^{ss,Z}$ is uniformly bounded by a constant $K > 0$.

We denote by $\widetilde{C}_c^{ss,Z}$ the lift of $C_c^{ss,Z}$ on $\widetilde{\Sigma}$. Let $\widetilde{\mathcal{F}}^{ss}$ denote the lift on $\widetilde{\Sigma}$ of the strong foliation $\mathcal{F}_\Sigma^{ss,Z}$ of the first return map F of Z.

From item $(Y.14)$, one deduces that, for every $i \in \mathbb{Z}$, every leaf ω of $\widetilde{\mathcal{F}}^{ss}$ cuts each annulus $\widetilde{A}_i^{\pm,Z}$ in at most one point. In particular,

$$\sharp(\omega \cap \widetilde{A}_i^Z) \leqslant 2.$$

One deduces the following lemma.

Lemma 4.10. *There is $c > 0$ such that, for any curve ω tangent to $\widetilde{C}_c^{ss,Z}$ and any $i \in \mathbb{Z}$, one has*

$$\sharp(\omega \cap \widetilde{A}_i^Z) \leqslant 2.$$

Proof. The proof is by contradiction, assuming that there are segment ω_n tangent to $\widetilde{C}_{\frac{1}{n}}^{ss,Z}$ and cutting $\widetilde{A}_i^{\pm,Z}$ at x_n and y_n with $x_n \neq y_n$.

As the lengths of ω_n are uniformly bounded, up to considering a subsequence, one may assume that ω_n converges to a segment ω in a leaf of $\widetilde{\mathcal{F}}^{ss}$, and x_n and y_n tend to points $x \in \omega \cap \widetilde{A}_i^{\pm,Z}$ and $y \in \omega \cap \widetilde{A}_i^{\pm,Z}$, respectively.

As ω cuts $\widetilde{A}_i^{\pm,Z}$ in at most one point, one gets $x = y$; that is, the distance $d(x_n, y_n)$ tends to 0. One deduces that x is a tangency point of $\widetilde{A}_i^{\pm,Z}$ with $\widetilde{\mathcal{F}}^{ss}$, contradicting the fact that $\widetilde{A}_i^{\pm,Z}$ is tangent to $C_b^{cu,Z}$. □

We now fix the constant $c > 0$ small enough to satisfy the properties above, and we denote $C^{ss} = C_c^{ss,Z}$ and $\widetilde{C}^{ss} = \widetilde{C}_c^{ss,Z}$. We denote $\mathcal{U}_6 = \mathcal{U}_c$ for this value of c.

This property of Lemma 4.10 is satisfied for every X C^1-close enough to Z, and this is fundamental for our proof.

Lemma 4.11. *There exists a C^1 neighborhood $\mathcal{U}_7 \subset \mathcal{U}_6$ of Z such that, for any $X \in \mathcal{U}_7$ and any curve ω tangent to $\widetilde{C}^{ss}$, we have*

$$\sharp(\omega \cap \widetilde{A}_i^X) \leqslant 2,$$

for each $i \in \mathbb{Z}$.

Proof. We will prove that, for any curve ω tangent to $\widetilde{C}^{ss}$, $\sharp(\omega \cap \widetilde{A}_i^{+,X}) \leqslant 1$ and $\sharp(\omega \cap \widetilde{A}_i^{-,X}) \leqslant 1$. We only give the proof for $\widetilde{A}_i^{+,X}$. The proof for $\widetilde{A}_i^{-,X}$ is analogous.

Suppose to the contrary that there exist $X_n \to Z$ with curves ω_n tangent to $\widetilde{C}^{ss}$ such that $\sharp(\omega_n \cap \widetilde{A}_i^{+,X_n}) \geqslant 2$. Denote $x_n, y_n \in \omega_n \cap \widetilde{A}_i^{+,X_n}$, $x_n \neq y_n$. Taking converging subsequences if necessary, we may assume that $\omega_n \to \omega$, $x_n \to x$, and $y_n \to y$, as $n \to \infty$, where $x, y \in \omega \cap \widetilde{A}_i^{+,Z}$.

Note that ω is a curve tangent to $\widetilde{C}^{ss}$, so that $\sharp(\omega \cap \widetilde{A}_i^{+,Z}) \leqslant 1$. One deduces that $x = y$. According to Lemma 4.9, the annuli $\widetilde{A}_i^{+,X}$ vary C^1-continuously with X. This implies that ω is tangent to $\widetilde{A}_i^{+,Z}$ at x, contradicting the fact that $\widetilde{A}_i^{\pm,Z}$ is tangent to $C_b^{cu,Z}$. □

A curve tangent to $\widetilde{C}^{ss}$ is *maximal* if it has one extremity on $\{-1\} \times \mathbb{R} \times S^1$ and the other extremity on $\{1\} \times \mathbb{R} \times S^1$. Given any curve ω tangent to $\widetilde{C}^{ss}$, there are curves ω' containing ω and which are maximal. We say that ω' is a maximal extension of ω.

As a corollary of Lemma 4.11, we get the following corollary.

Corollary 4.12. *If ω is a maximal curve tangent to $\widetilde{C}^{ss}$, and if the intersection $\omega \cap \widetilde{F}_X(\widetilde{\Sigma}_i^X)$ is not empty, then it is a compact segment having one extremity on $\widetilde{A}_i^{+,X}$ and another on $\widetilde{A}_i^{-,X}$.*

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From the above lemma, we have the following result.

Lemma 4.13. *For any $X \in \mathcal{U}_7$, let $D \subset \widetilde{\Sigma}_i^X$ be a subset (perhaps not connected) for some i. If, for any curve ω tangent to $\widetilde{C}^{ss}$, we have $\sharp(\omega \cap D) \leqslant 1$, then $\sharp(\omega \cap \widetilde{F}_X(D)) \leqslant 1$ for any curve ω tangent to $\widetilde{C}^{ss}$.*

Proof. Suppose to the contrary that there exists a curve ω tangent to $\widetilde{C}^{ss}$ such that there are $x \neq y$ with $x, y \in \omega \cap \widetilde{F}_X(D)$.

Up to considering a maximal extension of ω, we can assume that ω is maximal. Then $\omega \cap \widetilde{F}_X(\widetilde{\Sigma}_i^X) \neq \emptyset$ is a compact segment ω_0 having one extremity on $\widetilde{A}_i^{+,X}$ and another on $\widetilde{A}_i^{-,X}$. Notice that x and y belong to ω_0.

Notice that $\widetilde{F}_X^{-1}$ is defined on ω_0 and that $\widetilde{F}_X^{-1}(\omega_0)$ is a curve in $\widetilde{\Sigma}_i^X$ tangent to $\widetilde{C}^{ss}$ and meeting D on $\widetilde{F}_X^{-1}(x)$ and $\widetilde{F}_X^{-1}(y)$, leading to a contradiction. □

We now consider vector fields $X \in \mathcal{U}_2 \cap \mathcal{U}_7$, where $\mathcal{U}_2$ is the open set built in Lemma 3.8. Recall that vector fields in $\mathcal{U}_2$ have two singular points, p_1^X and p_2^X, whose indices are 3 and 2, respectively.

Notice that $\dim(W^u(p_1^X)) = 1$, so that $O_X = W^u(p_1^X) \cap \Sigma$ is a countable set. We denote $\widetilde{O}_X = \rho^{-1}(O_X)$ the lift of O_X on $\widetilde{\Sigma}$; it is a countable set.

We call a *quasi-connected $\widetilde{C}^{ss}$-section* any (possibly not connected) two-dimensional open surface $D \subset \widetilde{\Sigma}$ which satisfies the following three properties.

(i) There exists a connected set D' such that

$$D \subset D' \subset D \cup \widetilde{O}_X, \quad \text{and} \quad \sharp(D' \setminus D) < \infty. \tag{4.1}$$

(ii) D is tangent to the cone field $\widetilde{C}_b^{cu,Z}$.

(iii) For any curve ω tangent to $\widetilde{C}^{ss}$, we have $\sharp(\omega \cap D) \leqslant 1$.

We say that the set D' is a *connected extension of* D. Recall that D is a C^1 open surface embedded in a Riemannian manifold. Hence the area of D is well defined. We denote by $\mathrm{Area}(D) \in \mathbb{R} \cup \{\infty\}$ the area of D.

We denote by $\mathcal{D}_X$ the set of quasi-connected $\widetilde{C}^{ss}$-sections.

4.4. Iterations of quasi-connected $\widetilde{C}^{ss}$-sections

The aim of this section is to show that the successive iterates $\widetilde{F}_X^n(D)$ of a quasi-connected $\widetilde{C}^{ss}$-section $D \in \mathcal{D}_X$ either meet the (lift of) the stable manifold of the torus Γ_X, or contain a quasi-connected $\widetilde{C}^{ss}$-section $D_n \in \mathcal{D}_X$ whose area increases exponentially by a factor at least $\frac{\lambda'}{2} > 1$. The difficulty is that, at each iteration, the quasi-connected $\widetilde{C}^{ss}$-section can be cut in infinitely many connected components by the annuli $\widetilde{L}_i^X$ (i.e., the local stable manifold of the singularities in S_X, where the first return map F_X is not defined). We solve this difficulty by showing that our hypothesis (H_1) allows us to recover the connexity of the image, just adding some points in $\widetilde{O}_X$ to the image.

For that, we first analyze the behavior of the first return map F_X in the neighborhood of the local stable manifold of S_X, that is, in the neighborhood of L_X.

4.4.1. The first return map in the neighborhood of L_X. Since $W^u(p_1^X)$ is one dimensional, we denote by $q^{\pm,X} = W^u_{loc}(p_1^X) \cap \Sigma$ the first intersection of Σ and each component of $W^u_{loc}(p_1^X)$. That is, $q^{\pm,X}$ is the point in $Q^{\pm,X}$ which belongs to $W^u(p_1^X)$. For $i \in \mathbb{Z}$ and $\pm \in \{+,-\}$, we denote by $\tilde{q}_i^{\pm,X}$ the lift of $q^{\pm,X}$ on $\widetilde{Q}_i^{\pm,X}$.

Since $\dim W^s(p_2^X) = 2$, we have that $\ell_2^X := W^s_{loc}(p_2^X) \cap \Sigma$ is a one-dimensional segment in L^X, and it is a leaf of the strong stable foliation of F_X. For $i \in \mathbb{Z}$, we denote by $\tilde{\ell}_{2,i}^X$ the component of $\rho^{-1}(\ell_2^X)$ such that $\tilde{\ell}_{2,i}^X \subset \widetilde{L}_i^X$. As ℓ_2^X is a leaf of the strong stable foliation $\mathcal{F}_\Sigma^{ss,X}$, we have that $\tilde{\ell}_{2,i}^X$ is tangent to $\widetilde{C}^{ss}$.

Note that, for any sequence of regular points that converges to $L^X \setminus \ell_2^X \subset W^s(p_1^X)$, its image of the first return map converges to $q^{+,X}$ or $q^{-,X}$.

As a consequence, we get the following lemma.

Lemma 4.14. *For the lift of the first return map, we have*

$$\lim_{x_n \in \widetilde{\Sigma}_{i-1}^X, x_n \to x \in \widetilde{L}_i^X \setminus \tilde{\ell}_{2,i}^X} \widetilde{F}_X(x_n) = \tilde{q}_{i-1}^{-,X} \quad \textit{and} \quad \lim_{x_n \in \widetilde{\Sigma}_{i}^X, x_n \to x \in \widetilde{L}_i^X \setminus \tilde{\ell}_{2,i}^X} \widetilde{F}_X(x_n) = \tilde{q}_{i}^{+,X}.$$

4.4.2. The image by $\widetilde{F}_X$ of a quasi-connected $\widetilde{C}^{ss}$-section. Recall that $\lambda' > 2$ has been fixed in Proposition 3.4.

Lemma 4.15. *For any $X \in \mathcal{U}_7 \cap \mathcal{U}_2$ that satisfies* (H$_1$)*, and any quasi-connected $\widetilde{C}^{ss}$-section $D \in \mathcal{D}_X$, we have one of the following properties.*

(a) *There exists a connected extension D' of D such that $D' \cap \widetilde{L}_i^X = \emptyset$ for any $i \in \mathbb{Z}$. In that case, $\widetilde{F}_X(D) \in \mathcal{D}_X$ and*

$$\mathrm{Area}(\widetilde{F}_X(D)) > \lambda' \mathrm{Area}(D).$$

(b) *There exist a connected extension D' and a unique integer $i \in \mathbb{Z}$ such that $D' \cap \widetilde{L}_i^X \neq \emptyset$. In that case, there are $D_1, D_2 \in \mathcal{D}_X$ such that $D_1 \cup D_2 = \widetilde{F}_X(D \setminus \widetilde{L}_i^X)$, and $\exists j \in \{1, 2\}$ such that*

$$\mathrm{Area}(D_j) > \frac{\lambda'}{2} \mathrm{Area}(D).$$

(c) *For any connected extension D' of D, there is $i \in \mathbb{Z}$ such that $D' \cap \widetilde{L}_i^X \neq \emptyset$ and $D' \cap \widetilde{L}_{i+1}^X \neq \emptyset$.*

Proof. Assuming the hypothesis of item (a): First, we assume that there exists a connected extension D' of D such that $D' \cap \widetilde{L}_i^X = \emptyset$ for any $i \in \mathbb{Z}$. To prove that $\widetilde{F}_X(D) \in \mathcal{D}_X$, we just need to verify that $\widetilde{F}_X(D)$ satisfies properties (i)–(iii) of the definition.

Since D' is connected, we know that $D' \subset \widetilde{\Sigma}_i^X$ for some i. Hence $\widetilde{F}_X(D')$ is connected. Furthermore, as $\widetilde{O}_X$ is invariant under $\widetilde{F}_X$, one gets $\widetilde{F}_X(D) \subset \widetilde{F}_X(D') \subset \widetilde{F}_X(D) \cup \widetilde{O}_X$. Now, Lemma 4.13 implies that every curve ω tangent to $\widetilde{C}^{ss}$ meets $\widetilde{F}_X(D)$ in at most one point.

Finally, Proposition 3.4 ensures the stated area expansion.

Assuming the hypothesis of item (b): Now, we assume that there exist a connected extension D' and a unique integer $i \in \mathbb{Z}$ such that $D' \cap \widetilde{L}_i^X \neq \emptyset$.

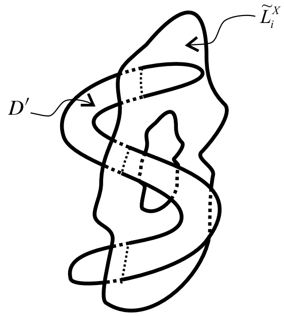

Figure 8. Surface D.

We denote

$$D'_- = D' \cap \widetilde{\Sigma}_{i-1}^X \quad \text{and} \quad D'_+ = D' \cap \widetilde{\Sigma}_i^X.$$
$$D_- = D \cap \widetilde{\Sigma}_{i-1}^X \quad \text{and} \quad D_+ = D \cap \widetilde{\Sigma}_i^X.$$

Thus $D'_- \cup D'_+ \subset D' \subset D'_- \cup D'_+ \cup \widetilde{L}_i^X$.

Lemma 4.13 implies that every curve ω tangent to $\widetilde{C}^{ss}$ meets $\widetilde{F}_X(D_-)$ (respectively, D_+) in at most one point. Moreover, $\widetilde{F}_X(D_-)$ and $\widetilde{F}_X(D_+)$ are open surfaces tangent to $\widetilde{C}_b^{cu,Z}$, and Proposition 3.4 ensures that there is $\pm \in \{+,-\}$ such that

$$\text{Area}(\widetilde{F}_X(D_\pm)) > \frac{\lambda'}{2}\text{Area}(D).$$

Hence we get the conclusion if we show that the following claim holds.

Claim 2. *Each of $\widetilde{F}_X(D_-)$ and $\widetilde{F}_X(D_+)$ admits a connected extension. More precisely, $\widetilde{F}_X(D_+) \cup \{\tilde{q}_i^{+,X}\}$ and $\widetilde{F}_X(D_-) \cup \{\tilde{q}_{i-1}^{-,X}\}$ are connected sets.*

Proof. We write the proof for D_+; the proof for D_- is identical. We assume that D_+ is not empty; otherwise, there is nothing to do.

Note that D'_+ may be not connected: D'_+ may be split into some (maybe infinitely many) connected components.

See Figure 8.

The main step for proving claim 2 is the following claim.

Claim 3. *For any open and closed subset Δ of D'_+, the point $\tilde{q}_i^{+,X}$ is contained in the closure of $\widetilde{F}_X(\Delta)$.*

Proof. According to Lemma 4.14, it is enough to prove that the closure $\overline{\Delta}$ meets $\widetilde{L}_i^X$ in a point out of $\tilde{\ell}_{2,i}^X$. That is,

$$\overline{\Delta} \cap \widetilde{L}_i^X \not\subset \tilde{\ell}_{2,i}^X.$$

Recall that D' is a connected set and that $\widetilde{L}_i^X$ is a compact set, so that D'_+ is an open subset of D'. Hence Δ is an open subset of D'. As D' is connected and $D'_+ \subsetneq D'$, the open

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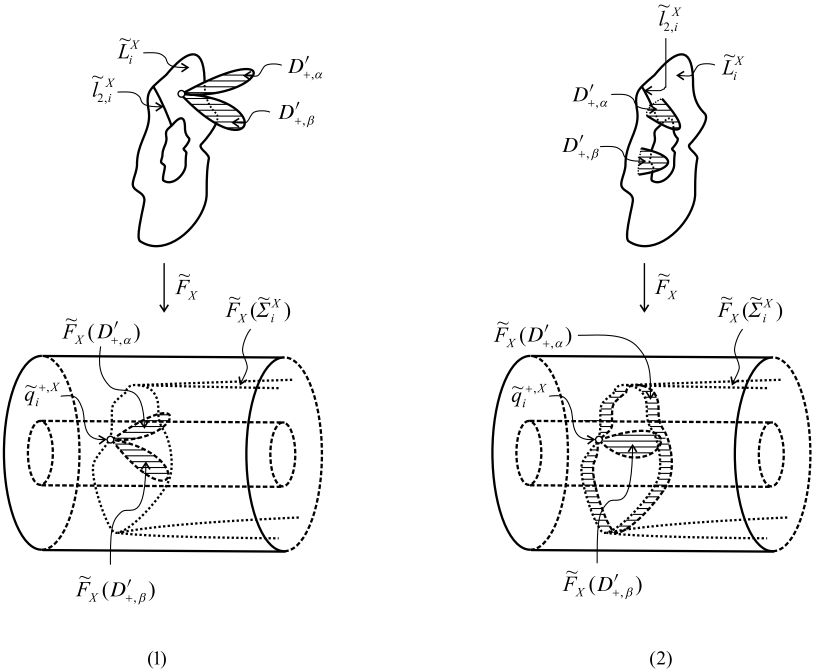

Figure 9. The image of $D'_{\pm,\alpha}$.

subset Δ cannot be closed in D'. Consider a point $x \in (D' \cap \overline{\Delta}) \setminus \Delta$, that is, x belongs to the boundary of Δ in D'. Then x belongs to $\widetilde{L}_i^X \cap D'$. There are two possibilities.

- $x \in \widetilde{O}_X$. In this case, (H_1) asserts that $x \notin \tilde{\ell}_{2,i}^X$, proving the claim. See Figure 9(1).
- $x \in D$. Let D_x be a disc in D centered at x so that $\widetilde{L}_i^X$ cuts D_x into two half discs. Using the fact that the strong stable leaves meet D in at most one point and that Δ contains x in its closure, we deduce that Δ contains one of the half discs. As a consequence, $\Delta \cap \widetilde{L}_i^X$ contains at least one point out of $\tilde{\ell}_{2,i}^X$. See Figure 9(2). □

We can now finish the proof of claim 2. Consider an open and closed subset Δ' of $\widetilde{F}_X(D'_+) \cup \{\tilde{q}_i^{+,X}\}$ and assume that it does not contain $q_i^{+,X}$. Hence it is open and closed in $\widetilde{F}_X(D'_+)$. Hence $\Delta = \widetilde{F}_X^{-1}(\Delta')$ is open and closed in D'_+. Now claim 3 implies that the closure of $\widetilde{F}_X(\Delta)$ contains $\tilde{q}_i^{+,X}$, leading to a contradiction. This proves that $\widetilde{F}_X(D'_+) \cup \{\tilde{q}_i^{+,X}\}$ is connected, ending the proof. □

Assuming that the hypotheses of items (a) and (b) are not satisfied: In this case, any connected extension D' of D meets at least two lifts $\widetilde{L}_i^X$ and $\widetilde{L}_j^X$ with $i < j$. As D' is connected, one deduces that D' meets $\widetilde{L}_{i+1}^X$. □

Remark 4.16. We point that hypothesis (H_1) is necessary in our argument of the above lemma. Otherwise, we cannot get the connectivity of $\widetilde{F}_X(D'_+) \cup \{\tilde{q}_i^{+,X}\}$ or $\widetilde{F}_X(D'_-) \cup \{\tilde{q}_{i-1}^{-,X}\}$. See Figure 10.

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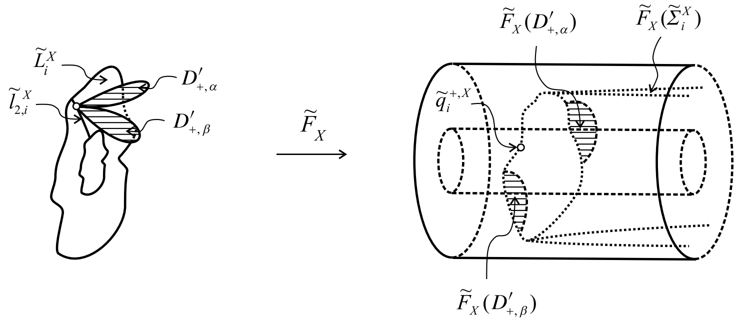

Figure 10. Hypothesis (H_1).

4.4.3. Using hypothesis (H_2). Recall that $\dim W^s(\Gamma_X) = 3$ and that $W^s_{loc}(\Gamma_X)$ cuts Σ along an annulus $I^X = \Sigma \cap W^s_{loc}(\Gamma_X)$ (see the topological properties of the vector fields $X \in \mathcal{U}_1$) which is clearly disjoint from L_X (because L_X is contained in $W^s(S_X)$).

Thus $\rho^{-1}(I^X)$ is a countable sequence of essential annuli in $\widetilde{\Sigma}$ (and disjoint from the annuli $\widetilde{L}_i^X$), denoted by $\widetilde{I}_i^X \in \widetilde{\Sigma}_i$. For each i, $\widetilde{I}_i^X$ splits $\widetilde{\Sigma}_i^X$ into two connected components.

The next lemma shows that, under hypothesis (H_2), if a quasi-connected $\widetilde{C}^{ss}$-section D satisfies item (c) of Lemma 4.15, then D cuts the (lift of the) stable manifold of the torus Γ_X. However, hypothesis (H_2) has no meaning out of $\mathcal{U}_2$: for this reason, the next lemma holds on $\mathcal{U}_7 \cap \mathcal{U}_2$.

Lemma 4.17. *For every $X \in \mathcal{U}_7 \cap \mathcal{U}_2$ satisfying (H_2), and every quasi-connected $\widetilde{C}^{ss}$-section $D \in \mathcal{D}_X$, if $\rho(D) \cap W^s(\Gamma_X) = \emptyset$, then, for any connected extension D', there is at most one integer $i \in \mathbb{Z}$ such that $D' \cap \widetilde{L}_i^X \neq \emptyset$.*

Proof. Suppose to the contrary that there is a connected extension $D' \subset D \cup \widetilde{O}_X$ of D, and that there are two integers $i_1 \neq i_2$ such that $D' \cap \widetilde{L}_{i_1}^X \neq \emptyset$ and $D' \cap \widetilde{L}_{i_2}^X \neq \emptyset$. Thus the connected set D' must intersect some $\widetilde{I}_k^X$ between $\widetilde{L}_{i_1}^X$ and $\widetilde{L}_{i_2}^X$. Moreover, hypothesis (H_2) asserts that $W^u(p_1^X) \cap W^s(\Gamma_X) = \emptyset$, so that $\widetilde{O}_X \cap \widetilde{I}_k^X = \emptyset$. As a consequence, $D \cap \widetilde{I}_k^X \neq \emptyset$, ending the proof of the lemma. □

4.5. Area of the quasi-connected $\widetilde{C}^{ss}$-section in a fundamental domain $\widetilde{\Sigma}_i^X$

The aim of this section is to show that quasi-connected $\widetilde{C}^{ss}$-section D having a large area cannot be contained in the union $\widetilde{\Sigma}_i^X \cup \widetilde{\Sigma}_{i+1}^X$; that is, any connected extension D' of D meets at least two annuli $\widetilde{L}_i^X$ and $\widetilde{L}_{i+1}^X$.

Lemma 4.18. *For any $X \in \mathcal{U}_7$, there is a positive constant $K_X > 0$ such that, for any quasi-connected $\widetilde{C}^{ss}$-section D contained in the union $\widetilde{\Sigma}_i^X \cup \widetilde{\Sigma}_{i+1}^X$, we have*

$$\mathrm{Area}(D) < K_X.$$

Proof. Let $\widetilde{\mathcal{F}}$ denote the foliation $\widetilde{\mathcal{F}}^{ss,Z}$. Recall that $\widetilde{\mathcal{F}}$ is a smooth foliation on $\widetilde{\Sigma} = [-1, 1] \times \mathbb{R} \times S^1$ such that, for any $x = (r, s, t)$, the leaf $\widetilde{\mathcal{F}}(x)$ is the segment $[-1, 1] \times \{(s, t)\}$ and is tangent to the cone field $C_c^{ss,Z}$.

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Denote by $P_{\widetilde{\mathcal{F}}} : \tilde{\Sigma} \to \mathbb{R} \times S^1$ the projection along the leaf of $\widetilde{\mathcal{F}}$. Since $\widetilde{\mathcal{F}}$ is transverse to the cone field $C_b^{cu,Z}$, there exists a constant $C_1 > 0$ such that, for any surface D tangent to $C_b^{cu,Z}$, we have

$$\mathrm{Jac}(P_{\widetilde{\mathcal{F}}}|_D) \geqslant C_1,$$

where $\mathrm{Jac}(P_{\widetilde{\mathcal{F}}}|_D)$ denotes the Jacobian of the restriction to D of the derivative of the projection $P_{\widetilde{\mathcal{F}}}$.

Let $C_{2,X}$ denote the area of $P_{\widetilde{\mathcal{F}}}(\widetilde{\Sigma}_i^X)$; this area is finite because the closure of $\widetilde{\Sigma}_i^X$ is compact[9], and does not depends on $i \in \mathbb{Z}$ because the area and the foliation $\widetilde{\mathcal{F}}$ are invariant by the translation $(r, s, t) \mapsto (r, s+1, t)$. We denote $K_X = \frac{C_{2,X}}{C_1}$.

Let $D \in \mathcal{D}_X$ be a quasi-connected $\widetilde{C}^{ss}$-section contained in $\widetilde{\Sigma}_i^X$. By definition of the quasi-connected $\widetilde{C}^{ss}$-section, each leaf of $\widetilde{\mathcal{F}}$ cuts D in at most one point; that is, the projection $P_{\widetilde{\mathcal{F}}}$ is injective on D. As a consequence, one gets

$$\mathrm{Area}(D) = \int_{P_{\widetilde{\mathcal{F}}}(D)} (\mathrm{Jac}(P_{\widetilde{\mathcal{F}}}|_D))^{-1} dm \leqslant \frac{1}{C_1} \int_{P_{\widetilde{\mathcal{F}}}(D)} dm = \frac{C_{2,X}}{C_1} = K_X,$$

where m is the area measure on $\mathbb{R} \times S^1$. □

As a direct consequence of Lemmas 4.17 and 4.18 we get the following corollary.

Corollary 4.19. *Given $X \in \mathcal{U}_7 \cap \mathcal{U}_2$ satisfying hypothesis* (H$_2$). *Let D be a quasi-connected $\widetilde{C}^{ss}$-section. Then*

$$\mathrm{Area}(D) \geqslant K_X \Longrightarrow \rho(D) \cap W^s(\Gamma_X) \neq \emptyset.$$

Proof. Lemma 4.18 implies that any connected extension of D meets at least two annuli, $\widetilde{L}_i^X$ and $\widetilde{L}_j^X$, for $i \neq j$. As X satisfies (H$_2$), Lemma 4.17 implies that $\rho(D)$ meets $W^s(\Gamma_X)$. □

4.6. Density of the stable manifold $W^s(\Gamma_X)$

Now we finish the proof of the density of $W^s(\Gamma_X)$.

Proof of Proposition 4.2 for $\mathcal{U}_0 = \mathcal{U}_7$. Consider a vector field $X \in \mathcal{U}_7 \cap \mathcal{U}_2$ satisfying hypotheses (H$_1$) and (H$_2$).

Suppose, arguing by contradiction, that there exists a non-empty open set $V \subset U$ such that $V \cap W^s(\Gamma_X) = \emptyset$. This implies that $\phi_t^X(V) \cap W^s(\Gamma_X) = \emptyset$ for any $t \in \mathbb{R}$.

The union of the orbits through the point in V cuts Σ along a (non-empty) open subset $V_\Sigma \subset \Sigma$, disjoint from $W^s(\Gamma_X)$. Thus we can choose a disc $D \in \mathcal{D}_X$ in $\widetilde{\Sigma}$ whose projection $\rho(D)$ is contained in V_Σ.

Since $\rho(D) \cap W^s(\Gamma_X) = \emptyset$, by using Lemma 4.17, for any connected extension D', we have

- either $D' \cap \widetilde{L}_i^X = \emptyset$ for any $i \in \mathbb{Z}$,
- or there exists a unique integer $i \in \mathbb{Z}$ such that $D' \cap \widetilde{L}_i^X \neq \emptyset$.

[9] By shrinking $\mathcal{U}_7$ if necessary, we also could assume that $C_{2,X}$ is uniformly bounded, but we do not need it.

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In any case, Lemma 4.15 implies that $\widetilde{F}_X(D \setminus \bigcup_i \widetilde{L}_i^X)$ contains a quasi-connected $\widetilde{C}^{ss}$-section D_1 with $\mathrm{Area}(D_1) > \frac{\lambda'}{2}\mathrm{Area}(D)$. Furthermore, $\rho(D_1)$ is disjoint from $W^s(\Gamma_X)$.

Repeating the above process, one builds a sequence of quasi-connected $\widetilde{C}^{ss}$-sections $D_n \subset \widetilde{F}_X(D_{n-1} \setminus \bigcup_i \widetilde{L}_i^X)$ with $\mathrm{Area}(D_n) > \frac{\lambda'}{2}\mathrm{Area}(D_{n-1})$, and $\rho(D_n)$ is disjoint from $W^s(\Gamma_X)$.

Hence $\mathrm{Area}(D_n) \to \infty$ as $n \to \infty$. In particular, $\mathrm{Area}(D_n) \geqslant K_X$ for n large enough. Now, Corollary 4.19 implies that $\rho(D_n) \cap W^s(\Gamma_X) \neq \emptyset$. This contradiction ends the proof of the proposition. □

4.7. Density of $W^u(\Gamma_X)$

Recall that, by definition of $\mathcal{U}_6 = \mathcal{U}_c$, the length of the leaves of $\mathcal{F}_\Sigma^{ss,X}$ is uniformly bounded by a constant $K > 0$.

Proof of Proposition 4.3 for $\mathcal{U}_0 = \mathcal{U}_7$. As for the proof of Proposition 4.2, we only need to prove that $W^u(\Gamma_X)$ is dense in $\Lambda_X \cap \Sigma$.

Recall that, as X belongs to $\mathcal{U}_1$, the unstable manifold $W^u(\Gamma_X)$ cuts every leaf of the strong stable foliation $\mathcal{F}_\Sigma^{ss,X}$ (see item $(X.11)$ in § 3.2).

Notice that $\Lambda_X \cap \Sigma$ is contained in $F_X(\Sigma \setminus L_X) \cup Q^{+,X} \cup Q^{-,X}$. In particular, the negative iterates F_X^{-n}, $n \geqslant 0$, of the first return map are defined on $\Lambda_X \setminus (W^u(S_X) \cap \Sigma)$.

Consider first a point $x \in \Lambda_X \setminus (W^u(S_X) \cap \Sigma)$, an open neighborhood U' of x, and a segment of strong stable leaf $\omega \subset \mathcal{F}_\Sigma^{ss,X}(x) \cap U'$ containing x in its interior. For any $n \geqslant 0$, consider the strong stable leaf $L_n = \mathcal{F}_\Sigma^{ss,X}(F_X^{-n}(x))$. Notice that L_n is disjoint from L_X, and that $F_X^n(L_n)$ is a segment of strong stable leaf centered at x and of length less than $(\lambda^s)^{-n}$; in particular, this length tends to 0. Hence, for n large enough, one has $F_X^n(L_n) \subset \omega$. As L_n cuts $W^u(\Gamma_X)$, one deduces that ω cuts $W^u(\Gamma_X)$. In particular, x belongs to the closure of $W^u(\Gamma_X)$.

To conclude the proof, it is now enough to notice that $W^u(S_X)$ is contained in the closure of $W^u(\Gamma_X)$. That is a direct consequence of the λ-lemma (also called the *inclination lemma*) using the transverse intersection of $W^u(\Gamma_X)$ with $W^s(p_i^X)$ for $i = 1, 2$. □

4.8. Proof of Theorem 4.1

We fix $\mathcal{U}_0 = \mathcal{U}_7$.

Proof of Theorem 4.1. Consider a vector field $X \in \mathcal{U}_7 \cap \mathcal{U}_2$ satisfying hypotheses (H_1) and (H_2). We will prove that the homoclinic related points of γ_1^X are dense in Λ_X.

Take any $x \in \Lambda_X$ and any open neighborhood V of x. By Lemma 3.8 and Proposition 4.3, we have $\Lambda_X \subset \mathrm{cl}(W^u(\Gamma_X)) \subset \mathrm{cl}(W^u(\gamma_1^X))$. Thus $V \cap W^u(\gamma_1^X) \neq \emptyset$. And since $\dim W^u(\gamma_1^X) = 3$, there exists a two-dimensional surface $D \subset \phi_{t_0}^X(V \cap W^u(\gamma_1^X)) \cap \Sigma$ for some $t_0 \geqslant 0$. Note that D is tangent to $C_b^{cu,Z}$.

On the other hand, by Lemma 3.8 and Proposition 4.2, we have $U \subset \mathrm{cl}(W^s(\Gamma_X)) \subset \mathrm{cl}(W^s(\gamma_1^X))$. One deduces that $W^s(\gamma_1^X) \cap \Sigma$ contains a dense subset of strong stable leaves (of $\mathcal{F}_\Sigma^{ss,X}$), and hence cuts D transversally.

In other words, we have proved that $\Lambda_X = \mathrm{cl}(W^s(\gamma_1^X) \pitchfork W^u(\gamma_1^X))$; that is, that Λ_X is the homoclinic class of the periodic orbit γ_1^X. Hence Λ_X is transitive. □

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Proof of Theorem A. We denote $\mathcal{U}^1 = \mathcal{U}_7 \cap \mathcal{U}_2$. It is a non empty C^1-open set.

First, notice that hypotheses (H_1) and (H_2) hold on the complement of an immersed codimension 1 submanifold in $\mathcal{U}^1 \cap \mathcal{X}^r(M)$ for the C^r topology, and for every integer $r \geqslant 1$.

Thus, considering $\mathcal{U}^r = \mathcal{U}^1 \cap \mathcal{X}^r(M)$, there exists a residual set $\mathcal{R}^r \subset \mathcal{U}^r$ such that any $X \in \mathcal{R}^r$ satisfies hypotheses (H_1) and (H_2). Then Theorem 4.1 implies that Λ_X is a transitive attractor.

To end the proof of Theorem A, we need to prove that Λ_X is chain recurrent for every $X \in \mathcal{U}^1$.

Suppose to the contrary that there is $X \in \mathcal{U}^1$ such that $\Lambda_X = \cap_{t>0}\phi_t^X(U)$ is not chain transitive. By Conley's theory, there is an open set $U_1 \subset \mathrm{int}(U)$ such that U_1 is an attracting region of X such that Λ_X is not included in U_1.

Thus, there is a compact invariant set $\Gamma \subset U \setminus U_1$. Consider a minimal set in Γ. By using Pugh's C^1 closing lemma [32], there is $X' \in \mathcal{U}^1$, C^1 close to X, such that U_1 is an attracting region of X' and there is a periodic orbit $\gamma \subset U \setminus U_1$. By performing a C^1 generic perturbation, there is $X'' \in \mathcal{R}^1$ such that U_1 is an attracting region of X'', and there is a periodic orbit $\gamma'' \subset U \setminus U_1$. Then $\Lambda_{X''}$ is not transitive, and this contradicts that $X'' \in \mathcal{R}^1$. □

5. Proof of Theorems B and C

In this section, we try to understand robustly chain transitive singular attractors, on compact manifolds, in any number of dimensions. The first step is to understand how the indices of the singularities and of the periodic orbits are related. For our results, the Shilnikov theory of homoclinic bifurcations associated to hyperbolic singular points will be enough, and we recall it in the next section.

5.1. Index of periodic orbits created by homoclinic bifurcation from a singular point

From now on, we assume that M is a compact d-dimensional C^∞ Riemannian manifold without boundary. We recall some results about the Shilnikov bifurcation. Let p be a hyperbolic saddle-type singularity of X. The eigenvalues of $\mathrm{D}X(p)$ are

$$\mathrm{Re}(\lambda_m) \leqslant \cdots \leqslant \mathrm{Re}(\lambda_2) \leqslant \mathrm{Re}(\lambda_1) < 0 < \mathrm{Re}(\alpha_1) \leqslant \mathrm{Re}(\alpha_2) \leqslant \cdots \leqslant \mathrm{Re}(\alpha_n),$$

where $m + n = d$. We denote

$$\mathrm{SV}(p) = \mathrm{Re}(\lambda_1) + \mathrm{Re}(\alpha_1),$$

and we call $\mathrm{SV}(p)$ the *saddle value of* p.

The next statement is a simplified version of results in [34] (see [18] for an analogous use of Shilnikov theory).

Theorem 5.1. *Let p be a singular point of a vector field X on a compact manifold. Assume that there is a homoclinic orbit Γ associated to p.*

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Then there are vector fields Y arbitrarily C^1-close to X having a periodic orbit γ arbitrarily close to Γ and such that

- $\mathrm{Ind}(\gamma) = \mathrm{Ind}(p)$, *if* $\mathrm{SV}(p) \leqslant 0$,
- $\mathrm{Ind}(\gamma) = \mathrm{Ind}(p) - 1$, *if* $\mathrm{SV}(p) \geqslant 0$.

5.2. Generic properties of flows

We also need some results on chain transitive and *robustly chain transitive sets* for C^1-generic vector fields, which we summarize in the following lemma.

An invariant compact set Λ of a vector field X is a *robustly chain transitive set* if there are a C^1-neighborhood $\mathcal{U}$ of X and a neighborhood V of Λ such that

- for every $Y \in \mathcal{U}$, the maximal invariant set $\Lambda_Y = \bigcap_{t\in\mathbb{R}} \phi_t^Y(V)$ is a chain transitive compact set, and
- $\Lambda = \Lambda_X$.

The neighborhood V is called an *isolating neighborhood* of Λ, and Λ_Y is the *continuation of* Λ for Y. Notice that the map $Y \mapsto \Lambda_Y$ is upper semi-continuous.

Lemma 5.2. *There exists a residual set $\mathcal{R}_1 \subset \mathcal{X}^1(M)$ such that, for any $X \in \mathcal{R}_1$ and any chain transitive set Λ of X, the following three properties are satisfied.*

- *X is Kupka–Smale.*
- *If there exists a periodic orbit* $\mathrm{Orb}(x) \subset \Lambda$, *then Λ is contained in the homoclinic class of x. As a consequence, if Λ is robustly chain transitive and V an isolating neighborhood of Λ, then Λ is the homoclinic class of x relative to V (i.e., the closure of the homoclinic orbits of x which are contained in V).*
- *If there exist $X_n \to X$, $x_n \in \mathrm{Per}(X_n)$ with $\mathrm{Ind}(x_n) = I$ and $x_n \to x \in \Lambda$, then there exists $y_n \in \mathrm{Per}(X)$ with $\mathrm{Ind}(y_n) = I$ such that $y_n \to y \in \Lambda$. As a consequence, if Λ is an attractor, then there exists a periodic orbit $\mathrm{Orb}(y_0) \subset \Lambda$ of X with $\mathrm{Ind}(y_0) = I$.*

(The first property is the classical Kupka–Smale theorem. The second and the third properties are straightforward adaptations for flows of the results in [5, Corollaire 1.11] and [38, Lemma 3.5], respectively.)

By using the connecting lemma for pseudo-orbits proved in [5], we have that, for any $X \in \mathcal{R}_1$, any chain transitive attractor Λ of X is a homoclinic class and it varies continuously in the vector field; i.e., for any $X_n \to X$, we have

$$\lim_H \Lambda_{X_n} = \Lambda,$$

where $\lim_H$ is the Hausdorff limit.

5.3. The linear Poincaré flow

Given a vector field X on a Riemannian manifold M, we denote by N^X its normal bundle, defined on $M \setminus \mathrm{Sing}(X)$: the fiber $N^X(x)$ is the orthogonal subspace of $X(x)$ in T_xM.

The flow of X defines a natural flow on N^X, denoted by ψ_t^X, as follows: the linear map $\psi_t^X : N^X(x) \to N^X(\phi_t^X(x))$ is the composition of the differential $\Phi_t^X : N^X(x) \to T_{\phi_t^X(x)}M$ followed by the linear projection $T_{\phi_t^X(x)}M \to N^X(\phi_t^X(x))$ parallel to the direction of the vector $X(\phi_t^X(x))$.

Given an invariant set Λ of X and $T > 0$, we say that Λ has a *T-dominated splitting* with respect to (shortly w.r.t.) the linear Poincaré flow if there is a ψ^X-invariant continuous splitting $N^X|_{\Lambda'} = E_1 \oplus E_2$, where Λ' denotes $\Lambda \setminus \mathrm{Sing}(X)$, such that, for any $x \in \Lambda'$, we have

$$\|\psi_T^X|_{E_1(x)}\| \|\psi_{-T}^X|_{E_2(\phi_T^X(x))}\| \leqslant 1/2.$$

In this case, we say that E_1 is *dominated* by E_2, and we write $E_1 \oplus_< E_2$.

An invariant bundle E_1 on Λ' is called *contracting* if there are constants $C > 0$ and $\lambda < 0$ such that, for any $t > 0$ and $x \in \Lambda'$, we have

$$\|\psi_t^X|_{E_1(x)}\| \leqslant Ce^{\lambda t}.$$

The following remark summarizes some properties of dominated splittings that we will use.

Remark 5.3.
- The dominated splitting is unique (when it exists) if we fix the dimension of the bundles: for every $i \in \{1, \dots, d-2\}$ and $x \in M \setminus \mathrm{Sing}(X)$, there is at most one T-dominated splitting $N^X = E \oplus_< F$ w.r.t. the linear Poincaré flow of X on the orbit of x, with $\dim(E) = i$.
- For every $T > 0$, the existence of a T-dominated splitting is a closed property in the following sense.

  Let x_n be a sequence of regular points of vector fields X_n, and assume that the sequence X_n converges to a vector field X for the C^1-topology, and that x_n converges to a regular point x of X.

  If there is a T-dominated splitting $E_n \oplus_< F_n$ w.r.t. the linear Poincaré flow of X_n on the orbit of x_n and $\dim(E_n) = i$, then there is a T-dominated splitting $E \oplus_< F$ w.r.t. the linear Poincaré flow of X on the orbit of x, with $\dim(E) = i$.

  Furthermore, $E(x)$ and $F(x)$ are the limits of the $E_n(x_n)$ and $F_n(x_n)$, respectively.

5.4. Hyperbolic properties of the linear Poincaré flow, far from tangencies

For the vector fields far away from tangency, we have the following lemma about the uniform dominated splitting on the periodic orbits. One can read [10, 33, 37] for more details.

Denote

$$\mathrm{HT}(M) = \{X \in \mathcal{X}^1(M) : X \text{ has a homoclinic tangency}\}.$$

Lemma 5.4. *Let $X \in \mathcal{X}^1(M) \setminus \overline{\mathrm{HT}(M)}$. Then there exist a C^1 neighborhood $\mathcal{U}$ of X and a constant $T > 0$ such that, for every $Y \in \mathcal{U}$ and $x \in \mathrm{Per}(Y)$, there exists a T-dominated*

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splitting

$$N^Y_{\mathrm{Orb}_Y(x)} = E^Y \oplus_< F^Y$$

w.r.t. the linear Poincaré flow ψ^Y_t*, where* $\dim E^Y = \mathrm{Ind}(x)$.

Thus we have the following lemma about the uniform dominated splitting on robustly chain transitive sets.

Lemma 5.5. *Let* Λ *be a singular robustly chain transitive set of* $X \in \mathcal{X}^1(M) \setminus \overline{\mathrm{HT}(M)}$*, let* V *be an isolating neighborhood of* Λ*, and let* $\mathcal{U}_0$ *be a* C^1*-neighborhood of* X*, where the continuation* Λ_Y *is defined for* $Y \in \mathcal{U}_0$.

Let $p \in \Lambda$ *be a hyperbolic singularity with* $\mathrm{SV}(p) \neq 0$*. Then there exist a* C^1 *neighborhood* $\mathcal{U}$ *of* X *and a constant* $T > 0$ *such that, for every* $Y \in \mathcal{U}$ *and* $x \in \mathrm{Per}(Y)$ *with* $\mathrm{Orb}_Y(x) \subset \Lambda_Y$*, there exists a* T*-dominated splitting*

$$N^Y_{\mathrm{Orb}_Y(x)} = E^Y \oplus_< F^Y$$

w.r.t. the linear Poincaré flow ψ^Y_t*, where the dimension of* E^Y *is* $\mathrm{Ind}(p)$ *when* $\mathrm{SV}(p) < 0$*, or* $\mathrm{Ind}(p) - 1$ *when* $\mathrm{SV}(p) > 0$.

Proof. Let $\mathcal{U} \subset \mathcal{U}_0$ and T be given by Lemma 5.4. By decreasing $\mathcal{U}$ if necessary, we may assume that the index of p and the sign of $\mathrm{SV}(p)$ are fixed in $\mathcal{U}$. Take $Y \in \mathcal{U}$ and $x \in \mathrm{Per}(Y)$ with $\mathrm{Orb}_Y(x) \subset \Lambda_Y$.

Notice that x is the limit of hyperbolic periodic point x_n of vector fields Y_n converging to Y in the C^1-topology. So, by using Remark 5.3, we only need to consider the case where the orbit $\mathrm{Orb}_Y(x)$ is hyperbolic. Moreover, once more using Remark 5.3 and the fact that the hyperbolic periodic orbits vary continuously with the vector field, we also can assume that Y is Kupka–Smale.

By using the connecting lemma for pseudo-orbits (see [5, Théorème 1.2]), we can C^1-approximate Y by a sequence of $X_n \in \mathcal{U}$ exhibiting a homoclinic orbit $\Gamma_n \subset V$ of the singularity p^n (continuation of p). Notice that Γ_n is contained in Λ_{X_n}, because V is an isolating neighborhood of Λ. We denote by x^n the continuation of the periodic point x for X_n.

Let i denote $\mathrm{Ind}(p)$ when $\mathrm{SV}(p) < 0$, and $\mathrm{Ind}(p) - 1$ when $\mathrm{SV}(p) > 0$. Then, by Theorem 5.1, there exists $Y_n \in \mathcal{U}$ arbitrarily C^1 close to X_n which has a hyperbolic periodic orbit $\mathrm{Orb}_{Y_n}(y_n)$ arbitrarily close to Γ_n with index i. Notice that the orbit of y_n is contained in V and $Y_n \in \mathcal{U}$ so that $y_n \in \Lambda_{Y_n}$.

With another arbitrarily C^1-small perturbation, we can get $Z_n \in \mathcal{R}_1 \cap \mathcal{U}$, where $\mathcal{R}_1$ is the residual subset stated in Lemma 5.2. We denote by x^{Z_n} and $y_n^{Z_n}$ the continuations of or Z_n of x^n and y_n, respectively.

Thus, by Lemma 5.2, Λ_{Z_n} is the relative homoclinic class of $y_n^{Z_n}$ in V. This implies that x^{Z_n} is the limit of the periodic points contained in Λ_{Z_n} and whose index is i.

Then, by definition of $\mathcal{U}$, and by using Remark 5.3, there exists a T-dominated splitting

$$N^{Z_n}_{\mathrm{Orb}_{Z_n}(x^{Z_n})} = E^{Z_n} \oplus_{<} F^{Z_n}$$

on $\mathrm{Orb}_{Z_n}(x^{Z_n})$ w.r.t. the linear Poincaré flow $\psi_t^{Z_n}$, where the dimension of E^{Z_n} is i.

As Z_n converges to Y and x^{Z_n} converges to x, we conclude, by Remark 5.3, that the orbits of x for Y admit a T-dominated splitting $E^Y \oplus_{<} F^Y$ with $\dim(E^Y) = i$, concluding the proof. □

5.5. Homoclinic tangencies for robust attractors with different indices of singularities

The aim of this section is the proof of Theorem B. We argue by contradiction, assuming that there exists a non-empty C^1-open set $\mathcal{U}$ of vector fields far from tangencies, that is, $\mathcal{U} \subset \mathcal{X}^1(M) \setminus \overline{\mathrm{HT}(M)}$, and an open set $V \subset M$ such that, for every $X \in \mathcal{U}$, the following hold.

- The maximal invariant set Λ_X of X in V is a chain transitive attractor.;
- Λ_X contains two hyperbolic singular points, p_1^X and p_2^X, whose indices satisfy
$$i_1 := \mathrm{Ind}(p_1^X) > \mathrm{Ind}(p_2^X) =: i_2.$$
- The saddle value of p_1^X does not vanish for $X \in \mathcal{U}$, and its sign does not depend on $X \in \mathcal{U}$.

We denote $i = i_1$ if the saddle value of p_1^X is negative, and $i = i_1 - 1$ if it is positive, for $X \in \mathcal{U}$. Notice that $i_2 \leqslant i$, by hypothesis.

For $X \in \mathcal{U}$, we denote by $\lambda^{uu,X}$ the upper bound of the real part of the eigenvalues of X at p_2^X. We denote by $W^{uu}(p_2^X)$ the strong unstable manifold associated to the sum $E^{uu}(p_2^X)$ of the generalized eigenspaces associated to eigenvalues with real part equal to $\lambda^{uu,X}$.

We denote by $E^s(p_2^X)$ the stable space of p_2^X, that is, the subspace of $T_{p_2^X}M$ corresponding to the eigenvalues with negative real part.

The next step is the following lemma.

Lemma 5.6. *There is $X \in \mathcal{U}$ and there is a homoclinic orbit Γ associated to p_2^X such that*

- *Γ is contained in $W^{uu}(p_2^X)$, and*
- *there are smooth local coordinates $\varphi : O_0 \to \mathbb{R}^d$ in a neighborhood O_0 of p_2^X such that $\varphi_*(X)$ is a linear vector field.*

Proof. First, notice the following claim.

Claim 4. *For every $X \in \mathcal{U}$, the stable manifold $W^s(p_2^X)$ contains a regular orbit in Λ_X. In other words,*

$$W^s(p_2^X) \cap \Lambda_X \neq \{p_2^X\}.$$

Proof. We notice that, as Λ_X is chain recurrent, there are closed ε-pseudo orbits of Λ_X, for $\varepsilon > 0$ arbitrarily small, having a point arbitrarily close to p_2^X. The upper limit for the Hausdorff topology of these pseudo-orbits contains a regular point in $W^s(p_2^X) \cap \Lambda_X$. □

Consider a Kupka–Smale vector field $Y \in \mathcal{U}$. The unstable manifold $W^u(p_2^Y)$ (hence *a fortiori* $W^{uu}(p_2^Y)$) is contained in the attractor Λ_Y.

As Λ_Y is chain transitive, contains points in $W^{uu}(p_2^Y)$ and of $W^s(p_2^Y)$, and Y is Kupka–Smale, the connecting lemma for pseudo-orbits in [5] allows us to get a small perturbation $Z \in \mathcal{U}$ of Y having a homoclinic orbit Γ contained in $W^{uu}(p_2^Z)$. Now, an extra C^1-small perturbation X of Z allows us to linearize the vector field in the neighborhood of the singular point, preserving the homoclinic orbit. □

We fix a compact neighborhood O_1 of p_u^X, contained in the interior of O_0. We denote by $\mathcal{P} \subset \mathbb{R}^d$ the vector subspace which is tangent to the image by φ_* of $E^s(p_2^X) \oplus E^{uu}(p_2^X)$. Notice that $\mathcal{P}$ is invariant by the linear vector field $\varphi_*(X)$. We denote $P = \varphi^{-1}(\mathcal{P}) \cap O_0$: it is a submanifold of M contained in O_1, tangent at p_2^X to $E^s(p_2^X) \oplus E^{uu}(p_2^X)$, and tangent to X.

Lemma 5.7. *With the notation above, there is a sequence of vector fields $X_n \in \mathcal{U}$ satisfying the following properties:*

- *the sequence X_n converges to X in the C^1-topology;*
- *$X_n(x) = X(x)$ for $x \in O_1$;*
- *X_n has a periodic orbit γ_n such that $\gamma_n \cap P$ contains a point x_n such that $\lim_{n\to\infty} x_n = p_2^X$.*

Idea of the proof. The proof consists in performing a small perturbation of the homoclinic orbit Γ, in a neighborhood of a point $p^{uu} \in \Gamma \cap P \cap W^{uu}_{loc}(p_2^X)$, and of a point $p^s \in \Gamma \cap P \cap W^s_{loc}(p_2^X)$ (where $W^{uu}_{loc}(p_2^X)$ and $W^s_{loc}(p_2^X)$ denote the connected components of $W^{uu}(p_2^X) \cap P$ and $W^s(p_2^X) \cap P$ containing p_2^X). The perturbation consists in pushing Γ in a direction tangent to P, turning Γ in a periodic orbit.

More precisely, we denote Γ^s_{loc} and Γ^{uu}_{loc} the connected components of $\Gamma \cap W^s_{loc}(p_2^X)$ and $\Gamma \cap W^{uu}_{loc}(p_2^X)$, and $\Gamma_0 = \Gamma \setminus (\Gamma^s_{loc} \cup \Gamma^{uu}_{loc})$. Then the periodic orbit γ_n of X_n consists in the union of the segment Γ_0 and a segment in P joining the extremities of Γ_0. □

We denote by σ_n the segment of $\gamma_n \cap P \cap O_1$ containing the point x_n. Notice that σ_n is an orbit segment of X_n but also an orbit segment of X, as X and X_n coincide on O_1.

Lemma 5.5 provides a C^1-neighborhood $\mathcal{U}_1 \subset \mathcal{U}$ of X and a constant T such that, for every $Y \in \mathcal{U}_1$ and every periodic orbit $\gamma \subset V$ of Y, there is a T-dominated splitting $N^Y|_\gamma = E^Y \oplus_< F^Y$ w.r.t. the Poincaré flow of Y over γ, such that $\dim(E^Y) = i$.

Hence, for n large enough, there is a T-dominated splitting $N^{X_n}|_{\gamma_n} = E^n \oplus_< F^n$ w.r.t. the Poincaré flow of X_n over γ_n, such that $\dim(E^n) = i$.

The next lemma concludes the proof of Theorem B.

Lemma 5.8. *With the notation above, there is n_0 such that, for any $n \geqslant n_0$, if $N^{X_n}|_{\gamma_n} = E^n \oplus_< F^n$ is a dominated splitting w.r.t. the Poincaré flow of X_n over γ_n, then $\dim(E^n) < i_2 \leqslant i$, where i_2 is the index of p_2^X.*

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Proof. This lemma is proved by the same argument as in [16, Lemma 4.3], which is also very similar to that for [35, Proposition 4.1]. Let us just sketch this argument here.

We argue by contradiction, assuming that there is $j \geqslant i_2$ such that (up to extract a subsequence) all the splittings $N^{X_n}|_{\gamma_n} = E^n \oplus_< F^n$ satisfy $\dim(E^n) = j$.

Notice that over σ_n the splittings are all invariant by the Poincaré flow of X, and are T-dominated for the Poincaré flow of X, because $X_n = X$ on O_1. We denote by ϕ_t, Φ_t, and ψ_t the flow, its derivative, and its Poincaré flow for X (and X_n) in restriction to O_1.

Let y_n and z_n denote the first and last extremity of the orbit segments σ_n. By considering a subsequence one may assume that y_n and z_n converge to points $y \in W^s(p_2^X) \setminus \{p_2^X\}$ and $z \in W^{uu}(p_2^X) \setminus \{p_2^X\}$. Furthermore, there is a sequence $t_n > 0$ tending to infinity such that

- the C^1-distance between the orbit segments $\phi_{[0,2t_n]}(y_n)$ and $\phi_{[0,2t_n]}(y)$ tends to 0, and
- the C^1-distance between the orbit segments $\phi_{[-2t_n,0]}(z_n)$ and $\phi_{[-2t_n,0]}(z)$ tends to 0.

We consider the points $a_n = \phi_{t_n}(y_n)$ and $b_n = \phi_{-t_n}(z_n)$.

We first want to understand $\psi_{t_n}(b_n) : N^X(b_n) \to B^X(z_n)$.

Claim 5. *The projection $D_n^s(b_n)$ of $E^s(p_2^X)$ on $N^X(b_n)$ parallel to $X(b_n)$ is contained in an arbitrarily small cone around $E^n(b_n)$, for n large enough.*

Ingredients of the proof. The orbit segment $\phi_{[0,t_n]}(b_n)$ is almost tangent to the $E^{uu}(p_2^X)$ direction. The derivative $\Phi_{t_n}(b_n)$ leaves invariant the stable direction $E^s(p_2^X)$, and it induces a uniform contraction. Finally, $\dim(E^s(p_2^X)) = i_2 \leqslant \dim(E^n(b_n))$. □

Let $s_n > 0$ such that $\phi_{s_n}(a_n) = b_n$. Consider

$$D_n^s(a_n) = \psi_{-s_n}(D_n^s(b_n)) \quad \text{and} \quad D_n^s(y_n) = \psi_{-s_n-t_n}(D_n^s(b_n)).$$

Because of the dominated splitting $E^n \oplus_< F^n$, negative iteration by ψ preserves a small cone in N^X around E^n. As a consequence, $D_n^s(y_n)$ and $D_n^s(a_n)$ are contained in an arbitrarily small cone around $E^n(y_n)$ and $E^n(a_n)$, respectively.

Claim 6. *$D_n^s(y_n)$ and $D_n^s(a_n)$ are tangent to P.*

Proof. $D_n^s(b_n)$ is tangent to P, the space P is invariant by Ψ_{-s_n}, and $D_n^s(a_n)$ is the projection of $\Psi_{-s_n}(D_n^s(b_n))$ on $N^X(a_n)$ parallel to $X(a_n)$ which is tangent to P. The case of $D_n^s(y_n)$ is identical. □

Now, $\dim D_n^s(y_n) = i_2$, and $X(y_n)$ is almost tangent to $E^s(p_2^X)$. This implies that $D_n^s(y_n)$ contains a unit vector v_n which is almost orthogonal to $E^s(p_2^X)$. This implies that v_n has a large component in the $E^{uu}(p_2^X)$ direction.

One deduces that the rate of expansion on the vector v_n of $\psi_{t_n} : D_n^s(y_n) \to D_n^s(a_n)$ is close to $(\lambda^u u)^{t_n}$, i.e., almost the largest possible. This contradicts the fact that $\psi_t(v_n)$ remains in a small cone around $E^n(\phi_t(y_n))$ for $t \in [0, t_n]$. This contradiction concludes the proof. □

5.6. Existence of a strong stable direction for a robust singular attractor

The aim of this section is the proof of Theorem C.

The proof uses many arguments in other papers, some of them expressed in other context. For this reason, we will not always presents all the details.

Let $\mathcal{R}_1$ be the residual set in $\mathcal{X}^1(M)$ given in Lemma 5.2. Consider a vector field $X \in \mathcal{R}_1$, an attracting region V of X, and assume that the maximal invariant set $\Lambda \subset V$ is a robustly chain transitive attractor of X which contains a singularity p of index 2 (and possibly other singularities).

We fix an open neighborhood $\mathcal{U}$ of X such that V is an attracting region for every $Y \in \mathcal{U}$ and the maximal invariant set of Y in V is a chain transitive attractor. By shrinking $\mathcal{U}$ if necessary, one may assume that the singularities of $Y \in \mathcal{U}$ are all hyperbolic and vary continuously with Y.

5.6.1. No index 1 singularities. First notice that, according to [16, 35], the attractor Λ is not the whole manifold; that is, X is not robustly chain transitive. Hence Λ is not Lyapunov stable for the negative times of the flow.

The first step is to show the following.

Lemma 5.9. *For every $Y \in \mathcal{U}$ and any singularity $p \in \mathrm{Sing}(X)$, one has* $\mathrm{Ind}(p) \geqslant 2$.

Proof. This is a simple consequence of Hayashi's connecting lemma, already used in [30, Theorem E]: if p is a singularity of index 1 then its stable manifold consists in p and two regular orbits. As the attractor is not Lyapunov stable for the negative times, a small perturbation of the vector field make each of these regular orbits go out of V, implying that Λ cannot accumulate on them. That is impossible, as we have shown in the proof of Lemma 5.6. □

5.6.2. Dominated splitting for the Poincaré flow. By using the method in [7], one can prove the following lemma.

Lemma 5.10. *There exists a C^1 neighborhood (already denoted by $\mathcal{U}$) of X in $\mathcal{X}^1(M)$ and $T > 0$ such that, for any $Y \in \mathcal{R}_1 \cap \mathcal{U}$, there exists a T-dominated splitting*

$$N^Y_{\Lambda_Y - \mathrm{Sing}(Y)} = E^Y \oplus_< F^Y \tag{5.2}$$

w.r.t. the linear Poincaré flow ψ^Y_t.

Consider the set $I(Y)$ of integers i such that there is a T-dominated splitting $E^Y \oplus_< F^Y$ w.r.t. ψ^Y with $\dim(E^Y) = i$. This set varies semi-upper continuously with Y: if Y has no T-dominated splitting with $\dim(E^Y) = i$, then (by using Remark 5.3 and the fact that the periodic orbits of $Y \in \mathcal{R}_1 \cap \mathcal{U}$ are dense in Λ_Y) there is a periodic orbit γ of Y which does not admit a dominated splitting $E \oplus_< F$ with $\dim(E) = i$; this property holds for the continuation of γ in a small neighborhood of Y, proving the semi-continuity.

As a consequence, we get the following lemma.

Lemma 5.11. *There are open subsets $\mathcal{U}_i \subset \mathcal{U}$, $i \in \{1, \ldots, d-2\}$ such that*

- *$\bigcup_i \mathcal{U}_i$ is dense in $\mathcal{U}$, and,*

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- *for any* $Y \in \mathcal{U}_i \cap \mathcal{R}_1$, *there exists a* T*-dominated splitting*

$$N^Y_{\Lambda_Y - \mathrm{Sing}(Y)} = E^Y \oplus_< F^Y \tag{5.3}$$

w.r.t. the linear Poincaré flow ψ^Y_t, *with* $\dim(E^Y) = i$.

Furthermore, the following lemma holds.

Lemma 5.12. *For every (not necessarily generic)* $Y \in \mathcal{U}_i$, *every periodic orbit* $\gamma \subset \Lambda_Y$ *admits a* T*-dominated splitting* $E \oplus_< F$ *w.r.t. the linear Poincaré flow over* γ, *with* $\dim(E^Y) = i$.

Proof. Just consider $Y_n \in \mathcal{R}_1 \cap \mathcal{U}_i$ such that Y_n tends to Y and each Y_n has a periodic orbit γ_n which tends to γ. Then the dominated splitting associated to Y_n induces the stated dominated splitting on γ, by Remark 5.3. □

5.6.3. The dimension of E^Y is 1.

Lemma 5.13. *With the notation of the previous section, for every* $i > 1$, *the open set* $\mathcal{U}_i$ *is empty.*

Proof. The proof is very similar to the argument of the proof of Theorem B. Assuming that $\mathcal{U}_i \neq \emptyset$, we consider $Y \in \mathcal{R}_1 \cap \mathcal{U}_i$. Let $p \in \Lambda_Y$ be a singularity of index 2. By a small perturbation of Y ,we create a homoclinic orbit contained in the strong unstable manifold of p. Keeping this homoclinic orbit, a new perturbation linearizes Y in the neighborhood of the singularity (that is exactly Lemma 5.6).

Then small extra perturbations Y (keeping the linearization of Y in the neighborhood of p) create a vector field $Y_n \in \mathcal{U}_i$ having a periodic orbit γ_n which contains a segment in the strong unstable plane passing arbitrarily close to the singularity (see Lemma 5.7).

Finally, Lemma 5.8 implies that the periodic orbits γ_n do not admit a T-dominated splitting $E \oplus_< F$ w.r.t. the linear Poincaré flow over γ_n, with $\dim(E^Y) = i \geqslant \mathrm{Ind}(p)$. This contradicts Lemma 5.12. □

Corollary 5.14. *Every* $Y \in \mathcal{R}_1 \cap \mathcal{U}$ *admits a* T*-dominated splitting*

$$N^Y_{\Lambda_Y - \mathrm{Sing}(Y)} = E^Y \oplus_< F^Y \tag{5.4}$$

w.r.t. the linear Poincaré flow ψ^Y_t, *with* $\dim(E^Y) = 1$.

Proof. Lemma 5.13 implies that Y can be approximated by $Y_n \in \mathcal{U}_1$. However, $Y \in \mathcal{R}_1$ is a continuity point of the map $Z \mapsto \Lambda_Z$. Then Λ_Y inherits the T-dominated splitting of the Y_n. □

Corollary 5.14 means that $\mathcal{U} = \mathcal{U}_1$.

5.6.4. Strong stable direction at the singular points. Given a hyperbolic singularity q of non-zero index of a vector field X, we denote by $\lambda^{ss}(q) < 0$ the infimum of the real part of the eigenvalues of D_qX. We denote by $E^{ss}(q) \subset T_qM$ the sum of the (generalized) eigenspaces associated to the eigenvalues whose real part is λ^{ss}, and we call it the *strong stable space* of q.

We denote by $W^{ss}(q)$ the strong stable manifold of q tangent to $E^{ss}(q)$.

The aim of this section is to prove the following lemma.

Lemma 5.15. *For every $Y \in \mathcal{U}$, and for every singular point $q \in \mathrm{Sing}(X)$, the dimension of $E^{ss}(q)$ is 1.*

Proof. Assume, arguing by contradiction, that there is $Y \in \mathcal{U}$ with a singularity q such that $\dim(E^{ss}(q)) \geqslant 2$. If some of the corresponding eigenvalues are complex, then this property holds in a neighborhood of Y. Otherwise, there is real negative eigenvalue with multiplicity at least 2. A small perturbation of Y creates a complex strong stable eigenvalue. Hence in both cases, one may assume that $Y \in \mathcal{R}_1 \cap \mathcal{U}$ and that q has a complex strong stable eigenvalue.

As Y belongs to $\mathcal{R}_1$, there are periodic orbits passing arbitrarily close to q, and hence close to some point $z \in W^u(q)$. Hence, there are periodic orbits γ_n containing segment σ_n whose C^1 distance to the segments $\phi^Y_{[-n,0]}(z)$ tends to 0. Let z_n be the extremity of σ_n.

As $Y \in \mathcal{R}_1 \cap \mathcal{U} = \mathcal{U}_1$, the attractor Λ_Y admits a dominated splitting $E \oplus_< F$ on $\Lambda_Y \setminus \mathrm{Sing}(Y)$ for the linear Poincaré flow, with $\dim(E) = 1$.

Recall that Y admits a (non-unique) invariant strong stable foliation in a neighborhood of q. We denote by $E^{ss}(z)$ the tangent space of this foliation at z, for z in a neighborhood of q. Furthermore, there is a negatively invariant cone field around the strong stable direction E^{ss}.

As Y is almost tangent to the unstable space on the whole segment σ_n, one gets that the strong stable space $E^{uu}(z)$ makes a bounded angle with $N^Y(z)$ for every $z \in \sigma_n$. Let $D^{uu}(z)$ denote the projection of $E^{uu}(s)$ on $N^Y(z)$ parallel to $Y(z)$. This projection is injective, and its minimal norm (that is the inverse of the norm of its inverse) is larger than an uniform constant. The dimension of $D^{uu}(z)$ is equal to the dimension of $E^{uu}(q)$, and hence is, by assumption, at least 2.

Consider $F(z_n)$. It is a vectorial subspace of $N^Y(z_n)$ of dimension $n-2$. Then

$$\dim F(z_n) \cap D^{ss}(z_n) \geqslant 1.$$

Consider a vector $v \in F(z_n) \cap D^{ss}(z_n)$, and let $u \in E^{ss}(z_n)$ whose projection is v. Now, $\psi^Y_{-t}(v)$ is the projection on $N^Y(\phi_{-t}(z_n))$ of $\Phi^Y_{-t}(u)$, for $t \in [0, n]$. So the rates of expansion of v for ψ^Y_{-t} and of u for Φ^Y_{-t} are in a bounded ratio, on σ_n; as a consequence, this expansion is the largest possible, corresponding to $-\lambda^{ss}$. This contradicts the fact that $v \in F(z_n)$ (the vectors of $F(z_n)$ are dominated by those of $E(z_n)$ for negative iterations). □

Lemma 5.16. *Given any $Y \in \mathcal{U}$ and $q \in \mathrm{Sing}(Y)$, there is no homoclinic orbit in $W^{ss}(q)$; in other words,*

$$W^{ss}(q) \cap W^u(q) = \{q\}.$$

Proof. The proof is by contradiction. Assuming that Y has an homoclinic orbit Γ in $W^{ss}(q)$, we perform a perturbation preserving this homoclinic orbit, and linearizing Y

in the neighborhood of q. Then the proof follows the same argument as the proof of Theorem B and Lemma 5.13, but for negative times of the flow, leading to a contradiction with the fact that $\dim(F^Y) = d - 2 \geqslant \dim(E^u(q))$. □

As a direct corollary, one gets the following.

Corollary 5.17. *For any vector field $Y \in \mathcal{R}_1 \cap \mathcal{U}$ and $q \in \mathrm{Sing}(Y)$, one has*

$$W^{ss}(q) \cap \Lambda_Y = \{q\}.$$

Proof. As Y is Kupka–Smale and Λ_Y is chain recurrent, if $W^{uu}(q) \cap \Lambda_Y$ contains a regular orbit then the connecting lemma for pseudo-orbits allows us to create a homoclinic orbit of q contained in $W^{ss}(q)$, contradicting Lemma 5.16. □

5.6.5. Extension of the bundles E^Y and $F^Y \oplus \mathbb{R}Y$ to the singularity. Consider a vector field $Y \in \mathcal{R}_1 \cap \mathcal{U}$. Changing the Riemannian metric on M if necessary, one may assume that, for every $q \in \mathrm{Sing}(Y)$, the strong stable space $E^{ss}(q)$ is orthogonal to the sum $G(q)$ of all the generalized eigenspaces corresponding to the other eigenvalues. Notice that the splitting $E^{ss}(q) \oplus G(q)$ is dominated.

Corollary 5.17 implies that, for any singular point q of Y, and any small cone around $G(q)$, the vector $Y(z)$ is contained in that cone for z close enough to q. This implies that $E^{ss}(q)$ is almost orthogonal to $Y(z)$ for z close to q. More precisely, $E^{ss}(q)$ is contained in all the possible limits of $N^Y(z)$ when $z \in \Lambda_Y$ tends to q. As a consequence, one gets the following lemma.

Lemma 5.18. *The bundle E^Y defined on $\Lambda_Y \setminus \mathrm{Sing}(Y)$ extends continuously at every singular point $q \in \Lambda_Y$ by*

$$E^Y(q) = E^{ss}(q).$$

Furthermore, the linear Poincaré flow ψ^Y extends continuously on $E^{ss}(q)$ by

$$\psi^Y|_{E^{ss}(q)} = \Phi^Y|_{E^{ss}(q)}.$$

Notice that the bundle $G^Y = \mathbb{R}Y \oplus F^Y$, defined on $\Lambda_Y \setminus \mathrm{Sing}(Y)$, is invariant by the flow Φ^Y. One gets the following.

Lemma 5.19. *The bundle $G^Y = \mathbb{R}Y \oplus F^Y$ defined on $\Lambda_Y \setminus \mathrm{Sing}(Y)$ extends continuously at every singular point $q \in \Lambda_Y$ by*

$$G^Y(q) = G(q).$$

Furthermore, this bundle is Φ^Y-invariant.

5.6.6. The uniform contraction of the bundle E^Y for ψ^Y. The aim of this section is to prove the following lemma.

Lemma 5.20. *For any $Y \in \mathcal{R}_1 \cap \mathcal{U}$, the bundle E^Y is uniformly contracted by the linear Poincaré flow ψ^Y.*

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Proof. The proof is an adaptation of the argument in [25] using his ergodic closing lemma for proving that robustly transitive diffeomorphisms on surfaces are Anosov. The same argument has been already used in [7] for getting the volume hyperbolicity of robustly transitive diffeomorphisms in higher dimensions. The difficulty here is that we deal with singular flows. This adaptation has already been used in [16] in the context of star flows (see also [36]).

We argue by contradiction, assuming that the bundle E^Y is not uniformly contracted by the linear Poincaré flow ψ^Y.

We consider ψ^Y as a continuous linear cocycle over the flow ϕ^Y. For every $z \in \Lambda_Y$, we denote $v(z)$ to be a unit vector directing $E^Y(z)$ and we denote $\alpha(z) = \log(\|\psi_1^Y(v(z)\|)$.

As E^Y is not uniformly contracted, there is a ϕ^Y-invariant probability measure μ supported on Λ_Y and such that

$$\int_{\Lambda_Y} \alpha(z) d\mu(z) \geqslant 0.$$

We consider a disintegration of μ in ergodic measure. As a consequence, we get that there is an ergodic measure ν supported on Λ_Y satisfying

$$\int_{\Lambda_Y} \alpha(z) d\mu(z) \geqslant 0.$$

Notice that ν cannot be supported on a singular point q, because $E^Y(q) = E^{ss}(q)$ is uniformly contracted, and hence it is supported on regular orbits. More precisely, ν-almost every point spends an arbitrarily large fraction of time out of a small neighborhood of the singularities.

Now, Mañé's ergodic closing lemma asserts that ν-almost every point x is *well closable*, i.e., there is Y' arbitrarily C^1-close to Y, such that the orbit γ of x for Y' is periodic; furthermore, $\phi_t^{Y'}(x)$ remains arbitrarily close to $\phi_t^Y(x)$ for $0 < t < \pi(x, Y')$, where $\pi(x, Y')$ is the period of x for Y'.

Notice that, by Lemma 5.12 and the fact that $\mathcal{U} = \mathcal{U}_1$, one has that γ has a splitting $E^{Y'} \oplus_< F^{Y'}$ for $\psi^{Y'}$.

Claim 7. *The orbit γ above can be chosen such that the Lyapunov exponent of $\psi^{Y'}$ along γ in $E^{Y'}$ is arbitrarily close to $\int_{\Lambda_Y} \alpha(z) d\mu(z)$.*

Proof. For ν-almost every x and any t large enough, $|\int_{\Lambda_Y} \alpha(z) d\mu(z) - \frac{1}{t} \log \|D\psi_t^Y|_{E^Y(x)}\||$ is arbitrarily small. Furthermore, x is well closable.

Furthermore, by Remark 5.3, the bundle $E^{Y'}(\phi_t^{Y'}(x))$ is arbitrarily close to $E^Y(\phi_t^Y(x))$, for $t \in [0, \pi(x, Y')]$ such that $E^Y(\phi_t^Y(x))$ is out of a small neighborhood of the singularity. As a consequence, $E^{Y'}(\phi_t^{Y'}(x))$ is arbitrarily close to $E^Y(\phi_t^Y(x))$ on an arbitrarily large fraction of $t \in [0, \pi(x, Y')]$. For these times, the actions of the flow $\psi^{Y'}$ and ψ^Y are also arbitrarily close. The actions are bounded, and abelian (because the dimension of E^Y is 1). Then one may neglect the effect of the few times t where $\phi_t^Y(x)$ is very close to the singularities, leading to the conclusion. $\square$

Hence, for any $\varepsilon > 0$, there are Y' arbitrarily close to Y and a periodic orbit γ of Y' having a Lyapunov exponent for $\psi^{Y'}$ in the $E^{Y'}$ direction larger than $-\varepsilon$. An extra perturbation, using Franks' lemma, creates a vector field Y'' arbitrarily close to Y having a periodic orbit $\gamma \in \Lambda_{Y''}$ whose Lyapunov exponent in $E^{Y''}$ is positive (see [10] for an adaptation of Frank's lemma for flows). However, $E^{Y''}$ is dominated by $F^{Y''}$, so that γ is a source, contradicting the fact that $\Lambda_{Y''}$ is a chain transitive attractor. □

5.6.7. The partial hyperbolicity of the flow Φ^Y. Up to now we have that every $Y \in \mathcal{R}_1 \cap \mathcal{U}$ admits a splitting

$$TM|_{\Lambda_Y} = E^Y \oplus G^Y,$$

such that

- E^Y is one dimensional, continuous, invariant by ψ^Y on $\Lambda_Y \setminus \mathrm{Sing}(Y)$, and uniformly contracted by ψ^Y;
- G^Y is continuous and Φ^Y-invariant, and can be written $G^Y = \mathbb{R}Y \oplus F^Y$ on $\Lambda_Y \setminus \mathrm{Sing}(Y)$;
- the sum $E^Y \oplus F^Y$ is the normal bundle N^Y and induces a T-dominated splitting of ψ^Y.

To end the proof, we need to get a Φ^Y-invariant stable bundle. We will briefly explain the argument; details can be found in [16].

Choose a Riemannian metric on M such that E^Y and F^Y are orthogonal. One writes the action of Φ^Y on T_xM, for $x \in \Lambda_Y$, in an orthogonal basis whose first $d-1$ vectors belong to G^Y and whose last vector belongs to E^Y. The expression of Φ^Y is a block-trigonal matrix with two blocks on the diagonal, the second block being the action of ψ^Y on E^Y:

$$\Phi^Y_t|_{T_xM} = \begin{pmatrix} A_t(x) & C_t(x) \\ 0 & B_t(x) \end{pmatrix}.$$

We first prove that, for negative times, $B_t(x)$ dominates $A_t(x)$. That is, $\lim\ \sup_{t\to+\infty} \frac{\|A_{-t}(x)\|}{\|B_{-t}(x)\|} < 0$, and this limit is uniform in x. That is an easy consequence of the facts that

- far from the singularity, the splitting $E^Y \oplus F^Y$ is L-dominated, and the action on the Y direction is almost an isometry, and,
- on a singularity q, $E^Y = E^{ss}(q)$.

One deduces that the ratio $\frac{\|C_{-t}(x)\|}{\|B_{-t}(x)\|}$ remains uniformly bounded on Λ_Y for large $t > 0$. This means that the iterates by Φ^Y_{-t} of the bundle E^Y keep an angle bounded from zero with G^Y, leading to a Φ^Y-invariant bundle $E^{s,Y}$. This bundle is uniformly contracted and dominated by G^Y.

Hence we get the following lemma.

Lemma 5.21. *For every $Y \in \mathcal{R}_1 \cap \mathcal{U}$, the attractor Λ_Y admits a partially hyperbolic splitting $E^{s,Y} \oplus_< G^Y$, where $E^{s,Y}$ is one dimensional and uniformly contracted.*

Note that the map $Z \mapsto \Lambda_Z$ is upper semi-continuous. Hence the attractor Λ_Z are contained in an arbitrarily small neighborhood of Λ_Y for Z close enough to Y. Hence

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the partially hyperbolic splitting w.r.t. the tangent flow Φ_t^Y on Λ_Y can be continuously extended on Λ_Z for the nearby vector fields Z. Hence we can get the open and dense set $\mathcal{O} \subset \mathcal{U}$ of vector fields Z for which Λ_Z is partially hyperbolic, ending the proof of Theorem C.

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