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# Oscillations of a first order functional differential equation

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Oscillation results are obtained for a first order functional differential equation, by transforming it to an equation for which oscillatory information exists in the literature already.

## 1. Introduction

To underline the importance of the study of functional differential equations of the first order, our attention will be focused at first on some samples of them which are of particular interest. Thus, in [1], information has been gathered concerning the functional differential equation

$$x'(t) = g\big(x(t)\big) - g\big(x(t-L)\big) ,$$

which models population growth and gonorrhea epidemiology. In [12], the functional differential equation

$$s'(t) = -\beta(t)s(t)[2\gamma+s(t-14)-s(t-12)+\gamma] ,$$

which describes how measles spread through a population as a function of time, has been studied. Finally, in [2], [4], [5], the functional differential equation

$$y'(t) = ay(\lambda t) + by(t) ,$$

which arises as the mathematical idealization and simplification of an industrial problem involving wave motion in the overhead supply line to an electrified railway system (*cf.* also [3], [8]), has been discussed in detail.

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The last of the above samples led to the investigation - as far as oscillation results are concerned (see [9], [10]) - of the functional differential equations

$$y'(t) = p(t)y\big(g(t)\big) + q(t)y(t) , \tag{1.1}$$

$$y'(t) = \sum_{i=1}^{n} p_i(t)y\big(g_i(t)\big) + q(t)y(t) . \tag{1.2}$$

Going further, we deal here with the oscillatory behaviour of the functional differential equation

$$y'(t) + \sum_{i=1}^{m} p_i(t)y^{1-n}(t)y^{n}\big(g_i(t)\big) = 0 , \quad n \geq 1 , \tag{1.3}$$

for which we assume that the following conditions hold always:

(C1): $p_i(t), g_i(t) \in C\big[[0,\infty),\mathbb{R}\big]$ , $p_i(t) \geq 0$ , $i = 1, 2, \ldots, m$ ;

(C2): $g_i(t) \leq t$ , $\lim_{t\to\infty} g_i(t) = \infty$ , and $g_i'(t) \geq 0$ , $i = 1, 2, \ldots, m$ .

In the sequel, a solution $y(t)$ of (1.3) is called "oscillatory" if it has arbitrarily large zeros, and "nonoscillatory" if it is eventually of constant sign.

## 2. Main results

In [9], [10], it has been possible to derive "qualitative" information concerning (1.1) and (1.2), by transforming them - through the transformation $y(t)\exp\left[-\int q(t)dt\right] = z(t)$ - to the equations

$$z'(t) + l(t)z\big(g(t)\big) = 0 , \tag{2.1}$$

$$z'(t) + \sum_{i=1}^{n} l_i(t)z\big(g_i(t)\big) = 0 , \tag{2.2}$$

respectively, for which there exist oscillation results in the literature already (*cf.* [11]). This "transformation technique" is going to be used to give oscillation results for (1.3). To this end, set

(2.3) $$np_i(t) = P_i(t) \ , \quad i = 1, 2, \ldots, m \quad \text{and} \quad y^n(t) = z(t) \ ,$$

to obtain (1.3) in the form

(2.4) $$z'(t) + \sum_{i=1}^{m} P_i(t)z\big(g_i(t)\big) = 0 \ ,$$

which, in fact, is similar to (2.2).

Now, taking account of Lemma 2.1 in [*10*], which applies to (2.2) (or to (2.4) here) and observing, from (2.3), that if $z(t)$ oscillates, so does $y(t)$, we establish the following result for (1.3).

THEOREM. *Consider the functional differential equation* (1.3), *subject to the conditions* (C1), (C2). *If, in addition,*

(2.5) $$\limsup_{t\to\infty} \sum_{i=1}^{m} \int_{K(t)}^{t} p_i(s)ds > 1/n \ ,$$

*where* $K(t) = \max g_i(t)$ , $i = 1, 2, \ldots, m$ , *then every solution* $y(t)$ *of* (1.3) *is oscillatory.*

COROLLARY. *Consider the functional differential equation*

(2.6) $$y'(t) + p(t)y^{1-n}(t)y^n\big(g(t)\big) = 0 \ , \quad n \ge 1 \ ,$$

*subject to the following conditions:*

*(i)* $p(t), g(t) \in C\big[[0, \infty), \mathbb{R}\big]$ , $p(t) \ge 0$ ;

*(ii)* $g(t) \le t$ , $\lim_{t\to\infty} g(t) = \infty$ , *and* $g'(t) \ge 0$ .

*If, in addition,*

(2.7) $$\limsup_{t\to\infty} \int_{g(t)}^{t} p(s)ds > 1/n \ ,$$

*then every solution* $y(t)$ *of* (2.6) *oscillates.*

REMARKS. It has been possible so far (see also [*10*]) to obtain different oscillatory information for (1.2) and (1.3) easily, by changing both - through different transformations - to the form (2.2) (or the similar form (2.4)). This would seem more difficult should usual oscillation approaches be used, which would anyway be different for (1.2)

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and (1.3).

Consider now (1.3) for $n = 1$ . Then the established theorem would take the form of Lemma 2.1 in [10]. Next, consider (2.6) for $n = 1$ . Then the established corollary would take the form of Corollary 2.1 in [7].

Finally, add a forcing term to (1.3), that is, consider the functional differential equation

$$y'(t) + \sum_{i=1}^{m} p_i(t)y^{1-n}(t)y^n\left(g_i(t)\right) = r(t) \ , \quad n \geq 1 \ ,$$

and use (2.3) to obtain it in the form

$$z'(t) + \sum_{i=1}^{m} p_i(t)z\left(g_i(t)\right) = q(t)z^k(t) \ , \quad k \geq 1 \ ,$$

which will be called the "Bernoulli type" functional differential equation (*cf.* [6, p. 98]). It has not been possible for the author to find an appropriate "transformation technique" to this functional differential equation, in order to derive oscillation results for it. The question of investigating, in general, the oscillatory behaviour of the "Bernoulli type" functional differential equation - which, to the author's knowledge, has not been considered yet - remains open.

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