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Unfolding plane curves with cusps and nodes

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Given an irreducible surface germ $(X, 0) \subset (\mathbb{C}^3, 0)$ with a one-dimensional singular set Σ, we denote by $\delta_1(X, 0)$ the delta invariant of a transverse slice. We show that $\delta_1(X, 0) \geqslant m_0(\Sigma, 0)$, with equality if and only if $(X, 0)$ admits a corank 1 parametrization $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ whose only singularities outside the origin are transverse double points and semi-cubic cuspidal edges. We then use the local Euler obstruction $\mathrm{Eu}(X, 0)$ in order to characterize those surfaces that have finite codimension with respect to $\mathcal{A}$-equivalence or as a frontal-type singularity.

1. Introduction

Any irreducible complex plane curve singularity $(Y, 0)$ can be parametrized, that is, it can be seen as the image of a finite and generically one-to-one map germ $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$. We can then look at it either as a finitely determined map germ with respect to the $\mathcal{A}$-equivalence or as a frontal-type singularity (using the terminology of Kurbatskiĭ and Zakalyukin [16]) of finite codimension in some sense. This phenomenon becomes explicit when we consider a suitable deformation Y_t, parametrized by a stable map γ_t. In the first case, Y_t is a morsification of Y, since the degenerated singularity splits into a finite number of nodes, that is, transverse double points A_1. In the second case, besides the nodes, we also allow the birth of simple cusps A_2, which are stable singularities in this context. As an example, we see in figure 1 the two deformations of the E_6 singularity, parametrized by $\gamma(v) = (v^3, v^4)$.

The total space of the deformation $(X, 0)$ is an irreducible surface in $(\mathbb{C}^3, 0)$ with one-dimensional singular locus Σ, which has special properties. It can be parametrized as the image of a map germ $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ given by $f(u, v) = (u, \gamma_u(v))$. If γ_u is a morsification, then f is $\mathcal{A}$-finite, that is, it has finite codimension with respect to the $\mathcal{A}$-equivalence. Otherwise, if γ_u is a deformation as a frontal, then f is itself a frontal-type surface of finite codimension as a frontal (see § 3). In figure 2 we show the two surfaces constructed with the two deformations of E_6. On the left-hand side we have the $P_3(c)$ singularity of Mond [12], and on the right-hand side we have the swallowtail.

Another interesting property of $(X, 0)$ is the equality $\delta_1(X, 0) = m_0(\Sigma, 0)$, where $\delta_1(X, 0)$ is the transverse delta invariant (i.e. the delta invariant of a generic plane section) and $m_0(\Sigma, 0)$ is the multiplicity of its singular locus. Since this is the minimal possible value for $\delta_1(X, 0)$, we say that $(X, 0)$ is a δ_1-minimal surface. In fact, we show in theorem 2.1 that, for any irreducible surface $(X, 0)$ with non-isolated

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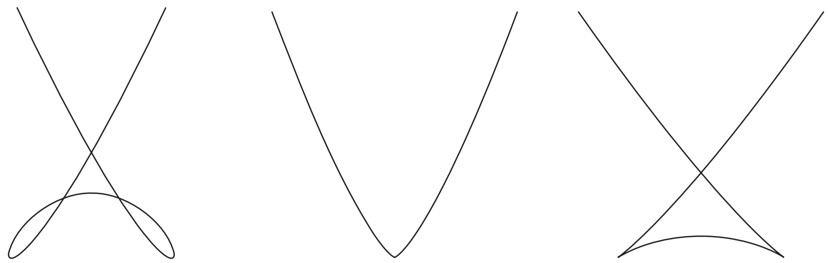

Figure 1. Two deformations of the E_6 singularity.

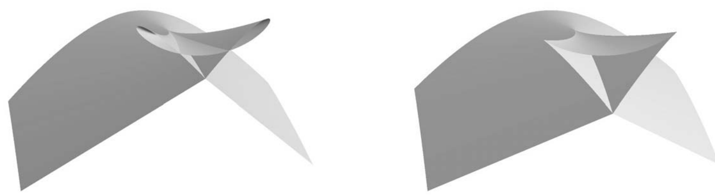

Figure 2. The total spaces of the deformations.

singularity, we have $\delta_1(X,0) \geqslant m_0(\Sigma,0)$, with equality if and only if $(X,0)$ admits a corank 1 parametrization $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$, and such that the only singularities outside the origin are transverse double points or semi-cubic cuspidal edges.

In the last part of the paper, we use the local Euler obstruction $\mathrm{Eu}(X,0)$ in order to characterize those surfaces among the δ_1-minimal ones that are stable unfoldings of plane curves or frontals. We show that if $(X,0)$ is δ_1-minimal, then

$$1 \leqslant \mathrm{Eu}(X,0) \leqslant m_0(X,0).$$

Moreover, we deduce the following (see corollary 4.3).

(1) $(X,0)$ is the image of a corank 1 $\mathcal{A}$-finite map germ if and only if it is δ_1-minimal and $\mathrm{Eu}(X,0) = 1$.

(2) $(X,0)$ is the image of a corank 1 frontal of finite codimension if and only if it is δ_1-minimal and $\mathrm{Eu}(X,0) = m_0(X,0)$.

Note that Jorge-Pérez and Saia proved in [6] that if $(X,0)$ is the image of a corank 1 $\mathcal{A}$-finite map germ, then $\mathrm{Eu}(X,0) = 1$. The results presented here are also related to those of [10], where we consider the classification and the invariants of corank 1 $\mathcal{A}$-finite map germs $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ by looking at the transverse slice.

2. δ_1-minimal surfaces

Let $(X,0) \subset (\mathbb{C}^3,0)$ be a singular surface. Given $0 \in H \subset \mathbb{C}^3$ a generic plane we consider the plane curve $Y = X \cap H$ and we call it a *transverse slice* of X. The delta invariant of Y at 0 is an invariant of $(X,0)$ that is independent of the choice of H. We define $\delta_1(X,0) := \delta(Y,0)$ and call it the *transverse delta invariant*.

Given an analytic set germ $(V,0) \subset (\mathbb{C}^n,0)$, we denote by $m_0(V,0)$ its multiplicity. We recall that this can be computed by means of a generic linear projection

$\ell\colon \mathbb{C}^n \to \mathbb{C}^d$, where $d = \dim(V,0)$. Then, $m_0(V,0) = \#V \cap H_t$, where $H_t = \ell^{-1}(t)$ and $t \in \mathbb{C}^d$ is a generic value.

THEOREM 2.1. *Let $(X,0) \subset (\mathbb{C}^3,0)$ be an irreducible surface with singular locus $(\Sigma,0)$ of dimension* 1*; then*

$$\delta_1(X,0) \geqslant m_0(\Sigma,0).$$

Moreover, the equality holds if and only if $(X,0)$ admits a corank 1 *parametrization $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ such that the only singularities outside the origin are transverse double points and semi-cubic cuspidal edges.*

Proof. We consider a linear projection $\ell\colon \mathbb{C}^3 \to \mathbb{C}$ such that $H = \ell^{-1}(0)$ is a generic plane and $Y = X \cap H$ is a transverse slice of X. Moreover, for each $t \in \mathbb{C}$ we can take $H_t = \ell^{-1}(t)$ in such a way that $Y_t = X \cap H_t$ defines a flat deformation of $(Y,0)$.

Since $(X,0)$ is irreducible, it has a normalization $n\colon (\tilde{X},0) \to (X,0)$, where $(\tilde{X},0)$ is a normal surface and n is finite and generically one-to-one. By taking the composition $\tilde{p} = p \circ n\colon (\tilde{X},0) \to (\mathbb{C},0)$ we also have a flat deformation of $\tilde{Y} = n^{-1}(Y)$.

We now use a result of Lejeune *et al.* [8] (see also [3, § 4.1.14]): for any $t \neq 0$ small enough,

$$\delta(Y,0) = \delta(\tilde{Y},0) + \sum_{p \in S(Y_t)} \delta(Y_t,p), \tag{2.1}$$

where $S(Y_t)$ denotes the singular set of Y_t. Obviously, $S(Y_t) = Y_t \cap \Sigma = H_t \cap \Sigma$ and, for each $p \in S(Y_t)$, $\delta(Y_t,p) \geqslant 1$. Therefore,

$$\delta(Y,0) \geqslant \#H_t \cap \Sigma = m_0(\Sigma,0).$$

We prove the equality in the case when $(X,0)$ admits a corank 1 parametrization $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ and the only singularities of $(X,0)$ outside the origin are transverse double points and semi-cubic cuspidal edges. In fact, after making a linear coordinate change in $\mathbb{C}^3$, and after reparametrization, we can assume that f is given in the form

$$f(u,v) = (u, p(u,v), q(u,v))$$

for some function germs p, q and such that generic plane is $x = 0$ (here, we denote by (x,y,z) the coordinates in $\mathbb{C}^3$). Then, $\tilde{Y}$ is the curve $u = 0$, which is smooth, and thus $\delta(\tilde{Y},0) = 0$.

On the other hand, for each $t \neq 0$, the deformation Y_t is given by $x = t$. The only singularities of Y_t are cusps and nodes, both having delta invariant equal to 1. By (2.1), $\delta(Y,0) = m_0(\Sigma,0)$.

We now prove the converse. If $\delta(Y,0) = m_0(\Sigma,0)$, we deduce from (2.1) that $\delta(\tilde{Y},0) = 0$ and $\delta(Y_t,p) = 1$ for each $t \neq 0$ and $p \in S(Y_t)$. In other words, $\tilde{Y}$ is smooth at 0, and the only singularities of Y_t are cusps and nodes when $t \neq 0$.

Since $\delta(\tilde{Y},0) = 0$, we have from (2.1) that Y_t is a delta-constant family of curves in the sense of Teissier. By [3, § 7.1.3], Y_t admits a normalization in family. This means that there exists a normalization of the form $\varphi\colon (\tilde{Y} \times \mathbb{C},0) \to (X,0)$. But the uniqueness of the normalization implies that $\tilde{X}$ is smooth at 0 and we can assume

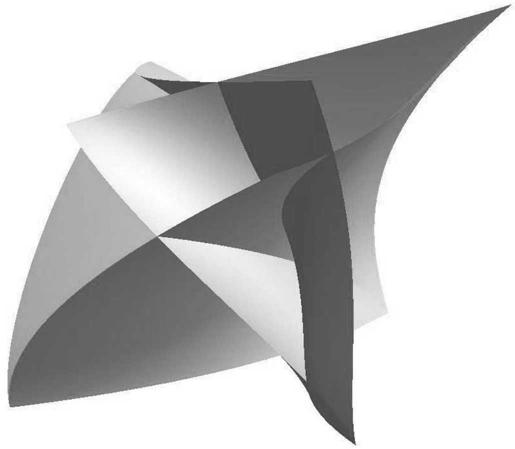

Figure 3. A double-fold corank 2 singularity.

that $\tilde{X} = \mathbb{C}^2$. Thus, $(X, 0)$ is the image of $f = i \circ n\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, where i denotes the inclusion map.

Because $\tilde{Y}$ is smooth at 0, f must have corank 1. Moreover, the only singularities of f outside the origin will be semi-cubic cuspidal edges and transverse double points (having as transverse slice cusps and nodes, respectively). □

Definition 2.2. We say that a surface $(X, 0) \subset (\mathbb{C}^3, 0)$ is δ_1-*minimal* if it is irreducible with one-dimensional singular locus Σ and $\delta_1(X, 0) = m_0(\Sigma, 0)$.

It follows from theorem 2.1 that $(X, 0)$ is δ_1-minimal if and only if it admits a corank 1 parametrization $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ such that the only singularities outside the origin are semi-cubic cuspidal edges and transverse double points. We see in the following example that the corank 1 condition is necessary.

Example 2.3. Let $(X, 0)$ be the surface parametrized by the double-fold map germ $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ given by $f(u, v) = (u^2, v^2, u^5 + v^5 + 2u^3v^3)$ (see [9]). Then, $(X, 0)$ is irreducible, its singular set Σ has dimension 1, and all the singularities outside the origin are semi-cubic cuspidal edges and transverse double points (see figure 3). But, since f has corank 2, we expect to get $\delta_1(X, 0) > m_0(\Sigma, 0)$.

In fact, according to [9], Σ is the curve in $(\mathbb{C}^3, 0)$ given by the zeros of the 3×3 minors of the matrix

$$\begin{pmatrix} -z & x^2 & y^2 & 2xy \\ x^3 & -z & 2x^2y & y^2 \\ y^3 & 2xy^2 & -z & x^2 \\ 2x^2y^2 & y^3 & x^3 & -z \end{pmatrix}.$$

Using the computer algebra system Singular [4], we compute $m_0(\Sigma, 0) = 13$. On the other hand, $(X, 0)$ is given by the determinant of the above matrix:

$$x^{10} - 8x^8y^3 + 16x^6y^6 - 2x^5y^5 - 2x^5z^2 - 16x^4y^4z \\ - 8x^3y^8 - 8x^3y^3z^2 + y^{10} - 2y^5z^2 + z^4 = 0.$$

In order to compute the transverse slice, we just substitute $z = ax + by$ for some generic coefficients $a, b \in \mathbb{C}$. Again with the aid of Singular, we get $\delta_1(X, 0) = 14$.

We can associate two invariants with each δ_1-minimal surface $(X, 0)$. Let $\ell\colon \mathbb{C}^3 \to \mathbb{C}$ be a generic linear projection and set $H_t = \ell^{-1}(t)$ and $Y_t = X \cap H_t$. Since $(X, 0)$ is δ_1-minimal, the only singularities of Y_t for $t \neq 0$ small enough are cusps and nodes.

Definition 2.4. We define the *numbers* of *transverse cusps* and *transverse nodes* of $(X, 0)$, respectively, as the following:

- κ is the number of cusps (A_2) of Y_t,
- ν is the number of nodes (A_1) of Y_t.

The numbers κ, ν are well defined and do not depend on the choice of the generic linear projection ℓ nor the parameter t. In fact, we can split the singular locus into $\Sigma = \Sigma_1 \cup \Sigma_2$, where Σ_1 contains the transverse double points and Σ_2 contains the points of semi-cubic cuspidal edge type. Then, $\kappa = m_0(\Sigma_2, 0)$ and $\nu = m_0(\Sigma_1, 0)$. Moreover, we also deduce from the additivity of the multiplicity that

$$\kappa + \nu = m_0(\Sigma_2, 0) + m_0(\Sigma_1, 0) = m_0(\Sigma, 0) = \delta_1(X, 0).$$

Another consequence of theorem 2.1 is that a surface $(X, 0)$ is δ_1-minimal if and only if it is the image of an unfolding of a plane curve with only cusps and nodes. If $(X, 0)$ admits a corank 1 parametrization $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, then, after making a linear coordinate change in $\mathbb{C}^3$, and after reparametrization, we can assume that f is given in the form

$$f(u, v) = (u, \gamma_u(v)),$$

where $\gamma_u(v)$ is the parametrization of the plane curve $Y_u = X \cap \{x = u\}$.

Proposition 2.5. *Let $(X, 0)$ be a δ_1-minimal surface, parametrized by $f(u, v) = (u, \gamma_u(v))$, where $x = 0$ is a generic plane. The following statements are equivalent:*

(1) $\kappa = 0$,

(2) *f is $\mathcal{A}$-finite,*

(3) *for each $t \neq 0$, γ_t is $\mathcal{A}$-stable.*

Proof. The equivalence between (1) and (3) follows from the fact that the only $\mathcal{A}$-stable singularities of plane curves are nodes. The equivalence between (1) and (2) is a consequence of the Mather–Gaffney determinacy criterion: the map germ $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ is $\mathcal{A}$-finite if and only if there is a proper representative $f\colon U \to V$ such that $f^{-1}(0) = \{0\}$ and the restriction to $U \setminus \{0\}$ is $\mathcal{A}$-stable. But since the cross-caps and the transverse triple points are isolated, by shrinking U if necessary, this is equivalent to that f has only transverse double points on $U \setminus \{0\}$. □

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Example 2.6. Let $(X, 0)$ be an irreducible surface with one-dimensional singular set whose transverse slice has type E_6. We parametrize the curve by $\gamma(v) = (v^3, v^4)$ and take the miniversal deformation

$$\Gamma(v; a, b, c) = (v^3 + av, v^4 + bv^2 + cv).$$

Then, after a linear coordinate change, $(X, 0)$ admits a parametrization of the form

$$f(u, v) = (u, v^3 + a(u)v, v^4 + b(u)v^2 + c(u)v)$$

for some $a, b, c \in \mathbb{C}\{u\}$, with $a(0) = b(0) = c(0) = 0$.

The discriminant of the deformation Δ is the set of points $(a, b, c) \in \mathbb{C}^3$ such that the curve $\gamma_{a,b,c}(v) = (v^3 + av, v^4 + bv^2 + cv)$ is not $\mathcal{A}$-stable. According to [10], Δ has the equation $P_1P_2P_3 = 0$, where

$$\begin{aligned} P_1 &= 16a^3 - 48a^2b + 36ab^2 + 27c^2, \\ P_2 &= 32a^3 - 48a^2b + 24ab^2 - 4b^3 + 27c^2, \\ P_3 &= a - b. \end{aligned}$$

The three factors P_1, P_2, P_3 correspond to the strata of singular points, self-tangencies and triple points, respectively.

If we also define $P_i = P_i(a(u), b(u), c(u))$, we have the following three types of δ_1-minimal surfaces.

(1) $(X, 0)$ is δ_1-minimal with $\kappa = 0$ and $\nu = 3$ if and only if $P_1P_2P_3 \neq 0$.

(2) $(X, 0)$ is δ_1-minimal with $\kappa = 1$ and $\nu = 2$ if and only if $P_1 = 0$ but $(c, 2a - 3b) \neq (0, 0)$ and $P_2P_3 \neq 0$.

(3) $(X, 0)$ is δ_1-minimal with $\kappa = 2$ and $\nu = 1$ if and only if $(c, 2a - 3b) = (0, 0)$ but $P_2P_3 \neq 0$.

The condition $(c, 2a - 3b) \neq (0, 0)$ to distinguish between (2) and (3) follows from the analysis of the number of solutions of the system

$$\gamma'_{a,b,c}(v) = (3v^2 + a, 4v^3 + 2bv + c) = 0.$$

There are two solutions if and only if $3v^2+a$ divides $4v^3+2bv+c$, which is equivalent to $c = 0$, $2a - 3b = 0$.

3. Frontals

In this section, we consider frontal-type singularities. This concept was introduced by Zakalyukin and Kurbatskiĭ in [16] and it is the generalization of a front. Roughly speaking, a frontal is the projection of a Legendrian submanifold with singularities. We also refer the reader to Ishikawa's paper [5] for basic definitions and notation about Legendre singularities. Although we only consider the complex case here, many of the results are also valid in the real case.

Let $PT^*\mathbb{C}^{n+1}$ be the projectivized cotangent bundle of $\mathbb{C}^{n+1}$ with the canonical contact structure defined by the contact form α, and denote the projection by $\pi\colon PT^*\mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$. By definition, a holomorphic map germ $\mathcal{L}\colon (\mathbb{C}^n, 0) \to$

$PT^*\mathbb{C}^{n+1}$ is said to be *integral* if $\mathcal{L}^*\alpha \equiv 0$. This means that $\mathcal{L} = (f, [\nu])$, where $f\colon (\mathbb{C}^n, 0) \to \mathbb{C}^{n+1}$ is a holomorphic map germ and $\nu\colon (\mathbb{C}^n, 0) \to T^*\mathbb{C}^{n+1}$ is a holomorphic, everywhere non-zero 1-form along f such that $\nu(\mathrm{d}f \circ \xi) = 0$, for any $\xi \in V_n$, the space of all germs of vector fields in $(\mathbb{C}^n, 0)$. If ν is given in coordinates by $\nu = \sum_{j=1}^{n+1} \nu_j \,\mathrm{d}x_j$, this is also equivalent to

$$\sum_{j=1}^{n+1} \nu_j \frac{\partial f_j}{\partial u_i} = 0 \quad \forall i = 1, \dots, n$$

for all u in a neighbourhood of the origin in $\mathbb{C}^n$.

Definition 3.1. We say that a map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a *frontal* map germ if there exists an integral map germ $\mathcal{L}\colon (\mathbb{C}^n, 0) \to PT^*\mathbb{C}^{n+1}$ such that $\pi \circ \mathcal{L} = f$. If, in addition, $\mathcal{L}$ is an embedding, then we say that f is a *front*.

When $\mathcal{L}$ is an integral embedding, then its image in $PT^*\mathbb{C}^{n+1}$ is called a Legendrian submanifold. If it is not an embedding, then it is usual to call the image a Legendrian submanifold with singularities. A hypersurface singularity $(X, 0)$ in $(\mathbb{C}^{n+1}, 0)$ is called a *frontal* (respectively, *front*) if there exists a frontal (respectively, front) map germ $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ whose image is $(X, 0)$.

Remark 3.2. If the map germ f is itself an embedding, then it is always a frontal and the class $[\nu]$ is determined uniquely by the components of the cross product

$$\frac{\partial f}{\partial u_1} \wedge \cdots \wedge \frac{\partial f}{\partial u_n}.$$

If f is not an embedding, but it is generically immersive (for instance, when it is finite and generically one-to-one), then the class $[\nu]$ is also uniquely determined, if it exists.

Example 3.3. We consider some examples.

(1) Any irreducible plane curve singularity is always a frontal. Assume that $(Y, 0)$ is parametrized in $(\mathbb{C}^2, 0)$ by $\gamma(v) = (p(v), q(v))$, where

$$p(v) = a_n v^n + a_{n+1} v^{n+1} + \cdots,$$
$$q(v) = b_m v^m + b_{m+1} v^{m+1} + \cdots,$$

with $a_n, b_m \neq 0$ and $n \leqslant m$. We then take the 1-form:

$$\nu = \frac{1}{v^{n-1}}(-q'(v)\,\mathrm{d}x + p'(v)\,\mathrm{d}y).$$

Note that $(Y, 0)$ is a front if and only if $m = n + 1$.

(2) The double-fold surface $(X, 0)$ of example 2.3 is a corank 2 frontal surface in $(\mathbb{C}^3, 0)$. In fact, since

$$\frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v} = uv(-2u(5u^2 + 6v^3), -2v(6u^3 + 5v^2), 4),$$

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we may take

$$\nu = -2u(5u^2 + 6v^3)\,\mathrm{d}x - 2v(6u^3 + 5v^2)\,\mathrm{d}y + 4\,\mathrm{d}z.$$

(3) Not every parametrized surface $(X, 0) \subset (\mathbb{C}^3, 0)$ is a frontal. For instance, given the cross-cap $f(u, v) = (u, v^2, uv)$ we have

$$\frac{\partial f}{\partial u} \wedge \frac{\partial f}{\partial v} = (-2v^2, -u, 2v).$$

Assume that there exists an everywhere non-zero 1-form ν such that $\mathcal{L} = (f, [\nu])$. We could then write

$$-2v^2 = \alpha\nu_1, \qquad -u = \alpha\nu_2, \qquad 2v = \alpha\nu_3$$

for some function α. Since α divides u and v, α should be a unit. But then

$$\nu = \frac{1}{\alpha}(-2v^2\,\mathrm{d}x - u\,\mathrm{d}y + 2v\,\mathrm{d}z),$$

and $\nu(0) = 0$, in contradiction with the hypothesis.

In general, we have the following criterion for corank 1 hypersurfaces.

Proposition 3.4. *Consider a hypersurface $(X, 0) \subset (\mathbb{C}^{n+1}, 0)$ parametrized by a corank 1 map germ $f(u, v) = (u, p(u, v), q(u, v))$, with $u \in \mathbb{C}^{n-1}$, $v \in \mathbb{C}$. Then, $(X, 0)$ is a frontal if and only if either $\partial p/\partial v$ divides $\partial q/\partial v$ or $\partial q/\partial v$ divides $\partial p/\partial v$.*

Proof. We have that

$$\frac{\partial f}{\partial u_1} \wedge \cdots \wedge \frac{\partial f}{\partial u_{n-1}} \wedge \frac{\partial f}{\partial v} = \left(\Delta_1, \dots, \Delta_{n-1}, -\frac{\partial q}{\partial v}, \frac{\partial p}{\partial v}\right),$$

where

$$\Delta_i = \frac{\partial q}{\partial v}\frac{\partial p}{\partial u_i} - \frac{\partial q}{\partial u_i}\frac{\partial p}{\partial v}.$$

Assume, for instance, that $\partial q/\partial v = \lambda(\partial p/\partial v)$ for some function λ. Then, $\Delta_i = \mu_i(\partial p/\partial v)$, with $\mu_i = \lambda(\partial p/\partial u_i) - \partial q/\partial u_i$, and thus we can take

$$\nu = \mu_1\,\mathrm{d}x_1 + \cdots + \mu_{n-1}\,\mathrm{d}x_{n-1} - \lambda\,\mathrm{d}x_n + \mathrm{d}x_{n+1}.$$

Conversely, suppose that there exists a non-zero 1-form ν such that $\mathcal{L} = (f, [\nu])$ is integral. There then exists a function α such that

$$\Delta_i = \alpha\nu_i, \quad i = 1, \dots, n-1, \qquad -\frac{\partial q}{\partial v} = \alpha\nu_n, \qquad \frac{\partial p}{\partial v} = \alpha\nu_{n+1},$$

and hence

$$\alpha\nu_i = -\alpha\left(\nu_n\frac{\partial p}{\partial u_i} + \nu_{n+1}\frac{\partial q}{\partial u_i}\right), \quad i = 1, \dots, n-1.$$

If $\alpha = 0$, we have that $\partial p/\partial v = \partial p/\partial v = 0$ and the result is obvious. Otherwise, if $\alpha \neq 0$, we have that

$$\nu_i = -\nu_n\frac{\partial p}{\partial u_i} - \nu_{n+1}\frac{\partial q}{\partial u_i}, \quad i = 1, \dots, n-1.$$

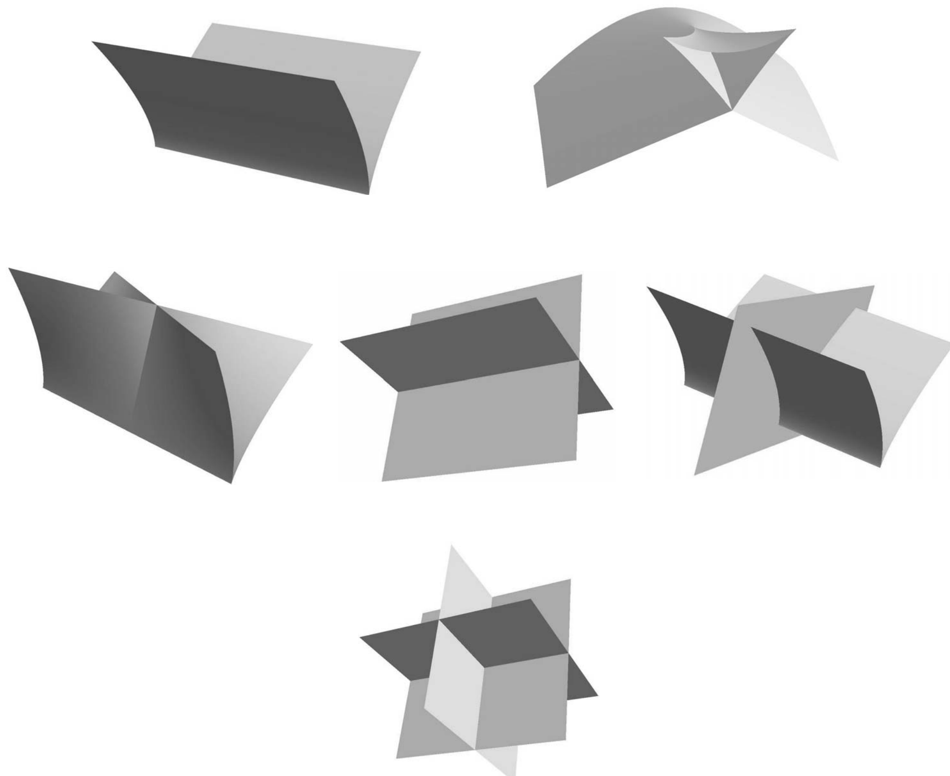

Figure 4. Stable frontal surfaces.

Since $\nu(0) \neq 0$, necessarily either $\nu_n(0) \neq 0$ or $\nu_{n+1}(0) \neq 0$, so either $\partial p/\partial v \mid \partial q/\partial v$ or $\partial q/\partial v \mid \partial p/\partial v$. □

EXAMPLE 3.5. We apply this criterion to see some examples of frontal surfaces.

(1) The swallowtail $(X, 0)$ is a frontal surface (see the right-hand side of figure 2). In fact, it is parametrized by $f(u, v) = (u, v^3 + uv, v^4 + \frac{2}{3}uv^2)$, and we have that $\partial p/\partial v = 3v^2 + u$ and $\partial q/\partial v = \frac{4}{3}v(3v^2 + u)$.

(2) The folded Whitney umbrella is the surface $(X, 0)$ in $(\mathbb{C}^3, 0)$ parametrized by $f(u, v) = (u, v^2, uv^3 + v^5)$ (see figure 4). This is also a frontal, since $\partial p/\partial v = 2v$ and $\partial q/\partial v = v(3uv + 5v^3)$.

We now define the codimension of a frontal as the codimension of the Legendrian singularity whose projection is the frontal, with respect to Legendre equivalence. We define $W = PT^*\mathbb{C}^{n+1}$ for simplicity and let $\mathcal{L}\colon (\mathbb{C}^n, 0) \to (W, w_0)$ be the integral map germ given by $\mathcal{L} = (f, [\nu])$. We recall the following notation from [5].

(1) $VI_{\mathcal{L}}$ is the space of all integral infinitesimal deformations of $\mathcal{L}$, that is, germs of vector fields along $\mathcal{L}$ that preserve the contact structure.

(2) VL_{W,w_0} is the space of all germs of Legendre vector fields in (W, w_0).

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Definition 3.6. We define the $\mathcal{F}_e$-*codimension* of f as

$$\mathcal{F}_e - \operatorname{codim}(f) = \dim_{\mathbb{C}} \frac{VI_{\mathcal{L}}}{\{d\mathcal{L} \circ \xi + \tilde{\eta} \circ \mathcal{L} \colon \xi \in V_n,\ \tilde{\eta} \in VL_{W,w_0}\}}.$$

If the $\mathcal{F}_e$-codimension is finite, we say that f is $\mathcal{F}$-*finite*, and if the $\mathcal{F}_e$-codimension is 0, we say that f is $\mathcal{F}$-*stable*.

According to [5], the space $VI_{\mathcal{L}}$ can be interpreted as the space of all infinitesimal integral deformations of $\mathcal{L}$, and the subspace

$$\{d\mathcal{L} \circ \xi + \tilde{\eta} \circ \mathcal{L} \colon \xi \in V_n,\ \tilde{\eta} \in VL_{W,w_0}\}$$

should be understood as the extended tangent space to the orbit of $\mathcal{L}$ under the action of Legendre equivalences. It follows from the definition that f is $\mathcal{F}$-stable if and only if $\mathcal{L}$ is infinitesimally Legendre stable in the sense of [5]. By [5, 4.1], any corank 1 $\mathcal{F}$-stable frontal is the projection of an open Whitney umbrella.

All the above definitions are also valid if instead of germs we consider multigerms $f\colon (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, y)$, where $S \subset \mathbb{C}^n$ is any finite set and $y \in \mathbb{C}^{n+1}$. We use this remark to classify the $\mathcal{F}$-stable singularities of curves and surfaces. Note that all the $\mathcal{F}$-stable singularities of frontal surfaces except folded Whitney umbrellas are generic fronts, and their classification is well known (see, for instance, [1]).

Proposition 3.7.

(1) *The $\mathcal{F}$-stable singularities of a frontal curve are cusps and nodes.*

(2) *The $\mathcal{F}$-stable singularities of a frontal surface are either semi-cubic cuspidal edges, swallowtails, folded Whitney umbrellas or their transverse self-intersections (see figure 4).*

The following property is an adapted version of the Mather–Gaffney finite determinacy criterion for frontals (see [15]).

Proposition 3.8. *A frontal $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is $\mathcal{F}$-finite if and only if there exists a proper and finite-to-one representative $f\colon U \to V$ such that $f^{-1}(0) = \{0\}$ and the multigerm at any point $y \in V \setminus \{0\}$ is $\mathcal{F}$-stable.*

By shrinking the neighbourhoods U, V if necessary, all the isolated $\mathcal{F}$-stable singularities can be avoided. We then have the following direct consequence of propositions 3.7 and 3.8.

Corollary 3.9.

(1) *A frontal curve is $\mathcal{F}$-finite if and only if it has an isolated singularity.*

(2) *A frontal surface of corank 1 is $\mathcal{F}$-finite if and only if the only singularities outside the origin are transverse double points and semi-cubic cuspidal edges.*

Recall that if $(X, 0)$ is δ_1-minimal, then $0 \leqslant \kappa \leqslant m_0(X, 0) - 1$, where κ is the number of cusps. We then have the following property, which is, in some sense, dual to proposition 2.5.

Proposition 3.10. *Let $(X,0)$ be a δ_1-minimal surface parametrized by $f(u,v) = (u, \gamma_u(v))$, where $x = 0$ is a generic plane. The following statements are equivalent:*

(1) $\kappa = m_0(X,0) - 1$,

(2) f *is an $\mathcal{F}$-finite frontal,*

(3) f *is a frontal unfolding of γ_0 and, for each $t \neq 0$, γ_t is $\mathcal{F}$-stable.*

Proof. Since $(X,0)$ is δ_1-minimal, the only singularities outside the origin are transverse double points and semi-cubic cuspidal edges. Moreover, for each t, the transverse slice Y_t is parametrized by $\gamma_t(v) = (p(t,v), q(t,v))$ and it has only cusps and nodes if $t \neq 0$. By proposition 3.7 and corollary 3.9, in order to show the equivalence between the three statements, we need only show that $\kappa = m_0(X,0) - 1$ if and only if f is a frontal.

Given $h \in \mathcal{O}_2$, we denote by $o_v(h)$ the order of h in v, that is, the order of $h(0,v) \in \mathcal{O}_1$. Assume that $o_v(p) = m$ and $o_v(q) = k$ with $m \leqslant k$. Then, because of the genericity assumption, we have that $m_0(X,0) = m$.

For a fixed small enough $t \neq 0$, κ is equal to the number of solutions of $p_v(t,v) = q_v(t,v) = 0$ in v. If $h = \gcd(p_v, q_v)$, then κ is less than or equal to the number of solutions of $h(t,v) = 0$ in v. In particular,

$$\kappa \leqslant o_v(h) \leqslant o_v(p_v) = m - 1 = m_0(X,0) - 1.$$

Thus, we have the following equivalences:

$$\kappa = m_0(X,0) - 1 \iff o_v(h) = o_v(p_v) \iff p_v \mid q_v \iff f \text{ is a frontal.}$$

□

4. Local Euler obstruction

The local Euler obstruction was first introduced by McPherson [11] as an ingredient in the construction of characteristic classes of singular algebraic varieties. Here, we prefer to use the approach of Lê and Teissier [7] in terms of polar multiplicities. Given an analytic set germ $(V,0) \subset (\mathbb{C}^n,0)$ of dimension d, its local Euler obstruction is computed as an alternate sum

$$\mathrm{Eu}(V,0) = \sum_{i=0}^{d-1} (-1)^i m_i(V,0),$$

where $m_i(V,0)$ denotes the ith-polar multiplicity (see [7] for definitions and details). In particular, for a surface $(X,0)$,

$$\mathrm{Eu}(X,0) = m_0(X,0) - m_1(X,0),$$

and hence $\mathrm{Eu}(X,0) \leqslant m_0(X,0)$.

In the next theorem, we compute the local Euler obstruction of a δ_1-minimal surface in terms of the number of transverse cusps κ. To do this, we first characterize the number ν of transverse nodes in terms of the number of vanishing cycles of the transverse slice Y_t.

Lemma 4.1. *Let $(X,0)$ be a δ_1-minimal surface. Then, for each $t \neq 0$ small enough, the Euler characteristic of Y_t is*

$$\chi(Y_t) = 1 - \nu.$$

Proof. We define $\delta = \delta_1(X,0) = \delta(Y,0)$. Since $(X,0)$ is δ_1-minimal, we have seen in the proof of theorem 2.1 that $(Y,0)$ is irreducible, and hence its Milnor number is $\mu(Y,0) = 2\delta$ (by Milnor's formula).

On the other hand, $\chi(Y_t)$ is related to the Milnor number by the following formula [3]:

$$\mu(Y,0) - \sum_{p \in S(Y_t)} \mu(Y_t,p) = \dim_{\mathbb{C}} H^1(Y_t;\mathbb{C}) = 1 - \chi(Y_t).$$

For each $t \neq 0$ small enough, the only singularities of Y_t are simple cusps, with Milnor number 2, and nodes, with Milnor number 1. Hence, we obtain

$$\mu(Y,0) - \sum_{p \in S(Y_t)} \mu(Y_t,p) = 2\delta - (2\kappa + \nu) = \nu.$$

□

Theorem 4.2. *Let $(X,0)$ be a δ_1-minimal surface. Then,*

$$\mathrm{Eu}(X,0) = 1 + \kappa.$$

In particular, $1 \leqslant \mathrm{Eu}(X,0) \leqslant m_0(X,0)$.

Proof. We use a formula of Brasselet *et al.* [2] that is valid whenever $(X,0)$ is equidimensional and has one-dimensional singular locus Σ. We take $t \neq 0$ small enough, and assume that $Y_t \cap \Sigma = \{x_1,\dots,x_m\}$. Then,

$$\mathrm{Eu}(X,0) = \chi(Y_t) - m + \sum_{i=1}^{m} \mathrm{Eu}(X,x_i).$$

Note that $Y_t \cap \Sigma$ is the singular locus of Y_t, and, since each singular point has delta invariant 1, we have that $m = \delta_1(X,0) = \kappa + \nu$. By lemma 4.1, $\chi(Y_t) = 1 - \nu$. For each $i = 1,\dots,m$, $\mathrm{Eu}(X,x_i) = 2$ if X is either a semi-cubic cuspidal edge or a transverse double point at x_i. Thus,

$$\mathrm{Eu}(X,0) = 1 - \nu - (\kappa + \nu) + 2\kappa + 2\nu = 1 + \kappa.$$

□

As a consequence, we arrive at the following result, which characterizes those surfaces that are stable unfoldings of plane curves or frontals.

Corollary 4.3. *Let $(X,0) \subset (\mathbb{C}^3,0)$ be an irreducible surface with singular locus of dimension 1. The following then hold.*

(1) *$(X,0)$ is the image of a corank 1 $\mathcal{A}$-finite germ if and only if it is δ_1-minimal and* $\mathrm{Eu}(X,0) = 1$.

(2) $(X,0)$ *is the image of a corank* 1 $\mathcal{F}$*-finite front if and only if it is* δ_1*-minimal and* $\mathrm{Eu}(X,0) = m_0(X,0)$.

Proof. It follows directly from theorem 2.1, propositions 2.5, 3.10 and theorem 4.2. □

We finish with a result where we consider irreducible surfaces with one-dimensional locus in any ambient space and without any finiteness assumption. Given a space curve $(Y,0) \subset (\mathbb{C}^N,0)$, the *first polar multiplicity* was introduced by the author and Tomazella in [14] as

$$m_1(Y,0) := \mu(\ell|_{(Y,0)}),$$

where $\ell\colon \mathbb{C}^N \to \mathbb{C}$ is a generic linear form and $\mu(\ell|_{(Y,0)})$ is the Milnor number in the sense of Mond and van Straten [13]. It is then shown that

$$m_1(Y,0) = \mu(Y,0) + m_0(Y,0) - 1, \tag{4.1}$$

where $\mu(Y,0)$ is now the Milnor number of a space curve as defined by Buchweitz and Greuel [3].

PROPOSITION 4.4. *Let* $(X,0) \subset (\mathbb{C}^{N+1},0)$ *be an equidimensional surface with one-dimensional singular set* Σ*. Then, for* $t \neq 0$,

$$m_1(X,0) = m_1(Y,0) - \sum_{x\in S(Y_t)} m_1(Y_t,x),$$

where Y_t *is the transverse slice of* $(X,0)$.

Proof. This is again a consequence of the Brasselet–Lê–Seade formula together with (4.1):

$$\begin{aligned} m_1(X,0) &= m_0(X,0) - \mathrm{Eu}(X,0) \\ &= m_0(X,0) - \chi(Y_t) + \sum_{x\in S(Y_t)} (\mathrm{Eu}(X,x) - 1) \\ &= m_0(Y_0,0) - 1 + (1-\chi(Y_t)) + \sum_{x\in S(Y_t)} (m_0(Y_t,x) - 1) \\ &= m_0(Y_0,0) - 1 + \mu(Y_0,0) - \sum_{x\in S(Y_t)} (\mu(Y_t,x) - m_0(Y_t,x) + 1) \\ &= m_1(Y_0,0) + \sum_{x\in S(Y_t)} m_1(Y_t,x). \end{aligned}$$

□

COROLLARY 4.5. *With the above hypothesis, the following statements are equivalent:*

(1) $m_1(X,0) = 0$,

(2) $(X,0)$ *defines a* m_1*-constant deformation of* $(Y,0)$.

Moreover, if $N = 2$ and $(X, 0)$ admits a parametrization, then any of the two above statements is also equivalent to the following one:

(3) $(X, 0)$ *is a frontal.*

Proof. The equivalence between the first two statements follows directly from proposition 4.4. According to Lê and Teissier [7], the condition $m_1(X, 0) = 0$ is also equivalent to the fact that $(X, 0)$ has a finite number of limiting tangent planes at the origin. But, in the particular case that $(X, 0)$ admits a parametrization $f\colon (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, this condition is equivalent to the fact that $(X, 0)$ is a frontal. □

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