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Automata theory based on unsharp quantum logic[†]

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By studying two unsharp quantum structures, namely extended lattice ordered effect algebras and lattice ordered QMV algebras, we obtain some characteristic theorems of MV algebras. We go on to discuss automata theory based on these two unsharp quantum structures. In particular, we prove that an extended lattice ordered effect algebra (or a lattice ordered QMV algebra) is an MV algebra if and only if a certain kind of distributive law holds for the sum operation. We introduce the notions of (quantum) finite automata based on these two unsharp quantum structures, and discuss closure properties of languages and the subset construction of automata. We show that the universal validity of some important properties (such as sum, concatenation and subset constructions) depend heavily on the above distributive law. These generalise results about automata theory based on sharp quantum logic.

1. Introduction

Based on the Hilbert space formalisation of quantum mechanics, Birkhoff and von Neumann proposed the concept of quantum logic in 1936, where projectors on a Hilbert space are regarded as quantum events of the logic. In quantum theory, quantum events reflect the projector valued (PV) measure of an observable. Since the set $\mathscr{P}(\mathscr{H})$ of all projection operators of a separable Hilbert space is an orthomodular lattice, orthomodular lattices have been the main model used in the study of quantum logic (Husimi 1937; Kaplansky 1955; Mackey 1963; Kalmbach 1983). However, the set of projection operators is not the set of maximal possible events produced by the statistical rules of quantum theory, so the PV measure is generalised to the positive operator valued (POV) measure. $\mathscr{E}(\mathscr{H})$ denotes the set of all positive operators of Hilbert space, and its elements are called effects (Ludwig 1983). Any event in $\mathscr{P}(\mathscr{H})$ always satisfies the non-contradiction principle, and such an event is called a sharp event. The quantum logic corresponding to $\mathscr{P}(\mathscr{H})$ is then called sharp quantum logic. Since quantum events reflected by $\mathscr{E}(\mathscr{H})$ do not satisfy the non-contradiction principle, they are called unsharp events, and the quantum logic corresponding to $\mathscr{E}(\mathscr{H})$ is called unsharp quantum logic (Chiara *et al.* 2004). Recently, many algebraic structures have been proposed to reflect quantum effects. In 1994, Foulis

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introduced effect algebras equivalent to difference posets (Kôpka and Chovanec 1994) and unsharp orthoalgebras (Giuntini and Greuling 1989). Effect algebras can be regarded as one of the main models for unsharp quantum logic. As special kinds of effect algebras, multiple valued (MV) algebras play an analogous role to that of Boolean algebras in sharp quantum logic (Chang 1958; Dvurečenskig and Pulmannová 2000). In addition, quantum MV (QMV) algebras are another important kind of unsharp quantum structure (Giuntini 1996), which are not only a non-lattice theoretic generalisation of MV algebras, but also a non-idempotent generalisation of orthomodular lattices.

Finite automata are among the simplest abstract mathematical models of computing machines, and automata theory is an essential part of computation theory. In order to set up a theory of computation based on quantum logic, automata theories based on orthomodular lattices have been established (Qiu 2003, 2004; Ying 2000a, 2000b, 2005; Qiu and Ying 2004). With this approach, the authors revealed an essential difference between classical computation theory and computation theory based on quantum logic. They found that many important properties of automata depend heavily on the distributivity of the underlying logic. Since unsharp quantum logic embodies the general laws of quantum theory, it is necessary to establish automata theory based on unsharp quantum structures.

In this paper, we mainly consider two algebraic models of unsharp quantum logic: extended lattice ordered effect algebras and lattice ordered QMV algebras. We find that extended lattice ordered effect algebras (or lattice ordered QMV algebras) are MV algebras if and only if they satisfy a certain kind of distributive law relating to the sum operation, which is the main operation on unsharp quantum structures. Interestingly, when setting up automata theory based on these unsharp quantum logics, we find that some important properties (such as the sum, concatenation and subset construction of automata) depend heavily on this kind of distributivity of truth-valued lattices. We conclude that distributivity of the underlying lattice is essential for building automata theory based on either orthomodular lattice or more general unsharp quantum structures. This generalises the results of automata theory based on sharp quantum logic.

2. Preliminaries

A partial binary operation on a non-empty set P is a map $\oplus : D(\oplus) \longrightarrow P$ with domain $D(\oplus) \subseteq P \times P$. If $D(\oplus) = P \times P$, then $\oplus$ is a total binary operation or simply a binary operation. If $\boxplus$ is a binary operation that extends a partial binary operation $\oplus$, we call $\boxplus$ a total extension of $\oplus$.

Definition 2.1 (Foulis and Bennett 1994). An effect algebra is a system $\varphi = (E, 0, 1, \oplus)$, where $0, 1$ are distinct elements of E, and $\oplus$ is a partial binary operation on E that satisfies the following conditions:

(E1) If $(a, b) \in D(\oplus)$, then $(b, a) \in D(\oplus)$ and $b \oplus a = a \oplus b$.
(E2) If $(a, b), (a \oplus b, c) \in D(\oplus)$, then $(b, c), (a, b \oplus c) \in D(\oplus)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
(E3) For every $a \in E$, there exists a unique $a' \in E$ such that $(a, a') \in D(\oplus)$ and $a \oplus a' = 1$.
(E4) If $(a, 1) \in D(\oplus)$, then $a = 0$.

Remark 2.1. Let $\varphi = (E, 0, 1, \oplus)$ be an effect algebra.

(i) Define $a \leqslant b$ if and only if there exists an element $c \in E$ such that $a \oplus c = b$. Then the relation $\leqslant$ is a partial order relation such that $0 \leqslant a \leqslant 1$ for all $a \in E$. If $(E; \leqslant)$ is a lattice, then E is called a lattice ordered effect algebra.
(ii) If b is the unique element of E such that $a \oplus b = 1$, then b is called the orthosupplement of a and denoted a'.
(iii) $a \oplus b$ is defined if and only if $a \leqslant b'$.
(iv) $0' = 1, a'' = a$ for all $a \in E$ and $a \leqslant b$ implies $b' \leqslant a'$.
(v) Given $a \leqslant b, c \leqslant d \in E$, the existence of $b \oplus d$ implies the existence of $a \oplus c$, in which case $a \oplus c \leqslant b \oplus d$.
(vi) For all $a, b, c, d \in E$, if $a \oplus b = a \oplus c$, then $b = c$.

Example 2.1 (Foulis and Bennett 1994). Let H be a complex Hilbert space and $\mathscr{E}(H)$ be the set of self-adjoint linear operators on H whose inner product $\langle , \rangle$ satisfies $\forall \phi \in H, 0 \leqslant \langle A\phi, \phi \rangle \leqslant \|\phi\|^2$. It is easy to see that $\mathscr{E}(H)$ is a poset with respect to the partial ordering $A_1 \leqslant A_2$ if and only if $\forall \phi \in H, \langle A_1\phi, \phi \rangle \leqslant \langle A_2\phi, \phi \rangle$. Define $0 = 0, 1 = I, A' = I - A$ and, for $A, B \in \mathscr{E}(H)$, $A \oplus B = A + B$, if $A + B$ is defined in $\mathscr{E}(H)$. Then $(\mathscr{E}(H), 0, I, \oplus)$ is an effect algebra.

Definition 2.2 (Gudder 1995). A supplement algebra (S-algebra for short) is an algebraic structure $\mathfrak{M} = (M, \boxplus, ', 0, 1)$ consisting of a set M with two constant elements $0, 1$, a unary operation $'$ and a binary operation $\boxplus$ on M satisfying the following axioms:

(S1) $a \boxplus b = b \boxplus a$.
(S2) $a \boxplus (b \boxplus c) = (a \boxplus b) \boxplus c$.
(S3) $a \boxplus a' = 1$.
(S4) $a \boxplus 0 = a$.
(S5) $a'' = a$.
(S6) $a \boxplus 1 = 1$.

A multiple-valued (MV) algebra (Chang 1958) is an S-algebra satisfying:

(MV) $(a' \boxplus b)' \boxplus b = (a \boxplus b')' \boxplus a$.

For an S-algebra, we define the following three binary operations:

$$a \odot b = (a' \boxplus b')'$$
$$a \sqcap b = (a \odot b') \boxplus b$$
$$a \sqcup b = (a \odot b') \boxplus b.$$

A quantum MV (QMV) algebra (Giuntini 1996) is an S-algebra satisfying:

(QMV1) $a \sqcup (b \sqcap a) = a$.
(QMV2) $(a \sqcap b) \sqcap c = (a \sqcap b) \sqcap (b \sqcap c)$.
(QMV3) $a \boxplus [b \sqcap (a \boxplus c)'] = (a \boxplus b) \sqcap (a \boxplus (a \boxplus c)']$.
(QMV4) $a \boxplus (a' \sqcap b) = a \boxplus b$.
(QMV5) $(a' \boxplus b) \sqcup (b' \boxplus a) = 1$.

It is easy to see that under the operations $\sqcap$ and $\sqcup$, a QMV algebra cannot be a lattice (Giuntini 1996).

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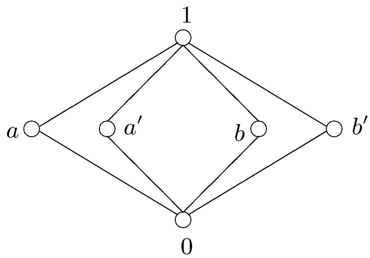

Fig. 1. $\mathfrak{M}_6$ in Example 2.3.

Example 2.2. Let $(Q, \boxplus, ', 0, 1)$ be a QMV algebra. Define $a \oplus b = a \boxplus b$ if and only if $a \boxplus b < 1$ or $a = b'$. Then $(Q, \oplus, 0, 1)$ is an effect algebra.

It is easy to see from the definition that every MV algebra is a QMV algebra. However, the converse is not true.

Example 2.3. $\mathfrak{M}_6 = \{a, a', b, b', 0, 1\}$, which is determined by a particular modular sublattice of the spin $\frac{1}{2}$ (Svozil 1998), is a QMV algebra but not an MV algebra. The operation $\boxplus$ is taken as the sup of the lattice and $'$ as the orthocomplement.

Example 2.4. Let $\mathscr{E}(H)$ be the set of effects on H. Define $0 = 0, 1 = I, A' = I - A$ and, for $A, B \in \mathscr{E}(H)$,

$$A \boxplus B = \begin{cases} A \oplus B, & \text{if } A \oplus B \text{ is defined} \\ I, & \text{otherwise.} \end{cases}$$

Then $\varphi = (\mathscr{E}(H), 0, 1, \boxplus, ')$ is a QMV algebra but not an MV-algebra. Again, φ is not an effect algebra.

If a, b are elements of a QMV-algebra, we write $a \leqslant b$ if $a = a \sqcap b$.

A QMV-algebra M is quasi-linear if $a \nleqslant b$ implies $a \sqcap b = b$ (Giuntini 1996).

A QMV-algebra (respectively, an MV-algebra) M is linear if $\forall a, b \in M$, either $a \leqslant b$ or $b \leqslant a$.

Example 2.5. Let $\varphi = (E, \oplus, 0, 1)$ be an effect algebra. The operation $\oplus$ could be extended to a total operation $\boxplus : E \times E \longrightarrow E$ by defining

$$a \boxplus b = \begin{cases} a \oplus b, & \text{if } (a \oplus b) \text{ is defined} \\ 1, & \text{otherwise.} \end{cases}$$

We use $\bar{\varphi} = (E, 0, 1, \boxplus)$ to denote the resulting structure and call it an extended effect algebra. From Gudder (1995), we can see that an extended effect algebra $\bar{\varphi}$ preserves the order of effect algebra and is equivalent to a quasilinear QMV algebra.

The following distributive laws hold for lattice ordered effect algebras.

Proposition 2.1 (Dvurečenskig and Pulmannová 2000). Let $\varphi = (E, \oplus, 0, 1)$ be a lattice ordered effect algebra. If $a \oplus b$ and $a \oplus c$ exist, then $(a \oplus b) \wedge (a \oplus c) = a \oplus (b \wedge c)$ for any $a, b, c \in E$.

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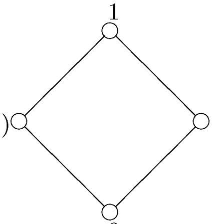

Fig. 2. $\mathfrak{M}_4$ in Example 2.6

However, distributive laws for the operation $\boxplus$ may not hold for quasilinear QMV algebras, as shown by the following example.

Example 2.6. Consider $\mathfrak{M} = \{0, 1, a, b\}$ with the operations $\oplus$ and $'$ defined as $a = a' \neq b = b'$. Then $(M, \oplus, 0, 1)$ is a lattice ordered effect algebra. However, its extension is just $\mathfrak{M}_4$, where $a \boxplus b = 1, 1 \boxplus x = 1$ for any $x \in \mathfrak{M}$. Furthermore, it is also the smallest QMV algebra that is not an MV algebra (Giuntini 1996).

Obviously, $a \boxplus (a' \wedge b) = a \boxplus 0 = a$, but $(a \boxplus a') \wedge (a \boxplus b) = 1$. Thus the distributive law $\boxplus$ over $\wedge$ does not hold.

But the distributive law is true for MV algebras.

Proposition 2.2 (Dvurečenskig and Pulmannová 2000). Let $\varphi = (M, \boxplus, 0, 1)$ be an MV algebra. Then for all $a, b, c \in M$, we have $a \boxplus (b \wedge c) = (a \boxplus b) \wedge (a \boxplus c)$.

3. Characterising MV algebras

From Example 2.6, we know that there are lattice ordered QMV (quasilinear QMV) algebras that are not MV algebras. However, when they satisfy the distributive law, they become MV algebras.

In this section, we give a characterisation of MV (linear MV) algebras using the distributive law.

Theorem 3.1. Let $\varphi = (Q, \boxplus, 0, 1)$ be a lattice ordered quasilinear QMV algebra. The following conditions are equivalent:

(i) φ is a linear MV algebra.
(ii) For all $u, v, w \in Q$, $(u \boxplus v) \wedge (u \boxplus w) = u \boxplus (v \wedge w)$.

Proof. '(i) implies (ii)' follows from Proposition 2.2.

We now show that '(ii) implies (i)'. Since a quasilinear QMV algebra is an MV algebra if and only if it is linear, we only need to prove that any two elements are comparable in Q. To show a contradiction, assume that there are a and b that are incomparable.

(1) First we prove there exists no $x \in Q$ such that $0 < x < a$. Otherwise, if such an x existed, there would be $x < b$. Indeed, let $u = a', v = x, w = b$, so $v < u'$. Thus $v \oplus u$ exists. Since a and b are incomparable, u' and w are incomparable. So $w \not\leq u'$, that is, $u \boxplus w = 1$. Then $(u \boxplus v) \wedge (u \boxplus w) = (u \oplus v) \wedge 1 = u \oplus v$, by the distributive law, and the equality is equal to $u \oplus v \wedge w$. Furthermore, we have $v = v \wedge w$ by Remark 2.1.

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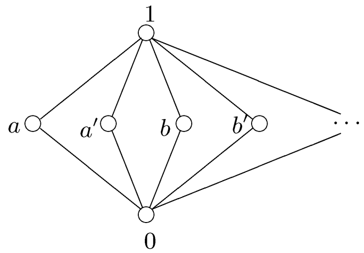

Fig. 3. in Theorem 3.1

Obviously, $v \neq w$. Thus $v < w$, that is, $x < b$. Since Q is a lattice, $0 < x \leqslant a \wedge b$. There are c and d such that $a = (a \wedge b) \oplus c$ and $b = (a \wedge b) \oplus d$ because $(Q, \oplus, 0, 1)$ is also an effect algebra. Clearly, c and d are incomparable. Otherwise, if we suppose $c \leqslant d$, then $a = (a \wedge b) \oplus c \leqslant (a \wedge b) \oplus d = b$ by monotony of the operation $\oplus$ in effect algebras, which contradicts our assumption of the incomparability of a and b. Similarly, from $c < a$, we have $c < b$ by the incomparability of a and b. Thus there exists one $e \in Q$ such that $b = c \oplus e$. Again $e < a$ since a, b are incomparable. That is, $e \leqslant a \wedge b$. But $a = (a \wedge b) \oplus c$ and $b = c \oplus e$ lead to $b \leqslant a$, which contradicts our assumption of the incomparability of a and b. So there exists no x such that $0 < x < a$ or $0 < x < b$.

(2) Similarly, we prove there exists no y such that $a < y < 1$. If such a y existed, then we would have $0 < y' < a'$. Clearly, a' and b' are incomparable. By the same reasoning as in (1), we have $0 < y' < b'$. This contradicts (1). So there exists no y such that $a < y < 1$ or $b < y < 1$.

From the discussion in (1) and (2), we conclude that all elements are incomparable with each other except for 0 and 1.

Since Q is a lattice ordered quasilinear QMV algebra, for any $a \in Q, 0 < a < 1$, we have $a \boxplus a \neq a$. Indeed, if $a \boxplus a = a$, then $a \nleqslant a'$, otherwise, $a \oplus a = a$, which means $a = 0$. But $a \sqcap a' = (a \boxplus a) \odot a' = a \odot a' = (a' \boxplus a)' = 0 \neq a'$, which is in contradiction with the definition of quasilinear QMV algebras. Hence, for any $0 < a < 1$, we have $a \boxplus a = 1$. As for the other elements, if $0 < a, b < 1, a \neq b$, then $a \boxplus b \geqslant a$ and $a \boxplus b \geqslant b$. So $a \boxplus b \geqslant a \vee b = 1$, that is, $a \boxplus b = 1$. For complement operation $'$, there are only two choices for any $a \in Q$ if $a \neq 0, 1$: either $a' = a$ or $a' \neq a$. From the discussion, φ is just given by the three cases shown in Figures 3, 4 and 5. In the following, we show that these quasilinear QMV algebras do not satisfy the distributive law. For the case shown in Figure 3, considering elements $a, a', b \in Q$, we have $(a \boxplus a') \wedge (a \boxplus b) = 1 \wedge 1 = 1$, but $a \boxplus (a' \wedge b) = a \boxplus 0 = a$, which destroys the distributive law. Similarly, the distributive law does not hold for the other two cases (Figure 4 and 5) either. So there are no incomparable elements in Q, which shows that φ must be linear, that is, φ is a linear MV algebra. □

Theorem 3.2. Let $\varphi = (Q, \boxplus, 0, 1)$ be a lattice ordered QMV algebras. The following conditions are equivalent:

(i) φ is an MV algebra.

(ii) For all $u, v, w \in Q$, $(u \boxplus v) \wedge (u \boxplus w) = u \boxplus (v \wedge w)$.

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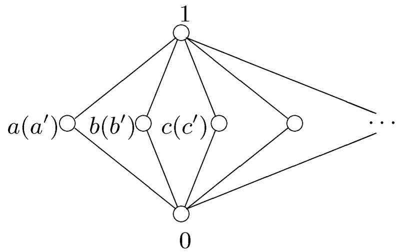

Fig. 4. in Theorem 3.1

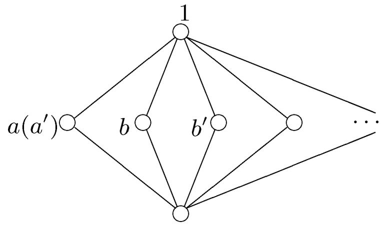

Fig. 5. in Theorem 3.1

Proof. '(i) implies (ii)' follows from Proposition 2.2.

We now prove that '(ii) implies (i)'. For any $a, b \in Q$, assume $a' \boxplus b = 1$. Let $u = a', v = b, w = a$. Then $(a' \boxplus b) \wedge (a' \boxplus a) = a' \boxplus (b \wedge a)$, namely, $a' \boxplus b = a' \boxplus (b \wedge a)$. By $a' \boxplus b = 1$, then $a' \boxplus (b \wedge a) = a' \boxplus a$. Since $b \wedge a \leqslant a, a \leqslant a$, we have $b \wedge a = a$ by Giuntini (1996, Theorem 2.5). That is, $a \leqslant b$. So φ is an MV algebra from Giuntini (1996, Theorem 2.14). □

Remark 3.1. Since an MV algebra is linear if and only if it is quasilinear, Theorem 3.2 gives us an alternative way to prove Theorem 3.1.

Remark 3.2. From Theorems 3.1 and 3.2, we see that the distributive law $\boxplus$ over $\wedge$ plays an important role in transforming a QMV algebra into an MV algebra. What about the distributive law for $\boxplus$ over $\sqcap$? The $\sqcap$ operation is another prime operation in QMV algebra in addition to $\boxplus$. However, there exists a QMV algebra with all $u, v, w \in E$, $(u \boxplus v) \sqcap (u \boxplus w) = u \boxplus (v \sqcap w)$ that is not an MV algebra. For example, $\mathfrak{M}_4$ is such a QMV algebra.

4. $\mathscr{E}$-valued automata

As we know that MV algebras play an important role in the development of unsharp quantum logic where Lukasiewicz disjunction, denoted $\oplus$, and conjunction, denoted $\odot$, are the main operations in MV algebras. Using these two operations along with $\vee$ and $\wedge$ in lattices, Di Nola and Gerla (Di Nola and Gerla 2004; Gerla 2003; Gerla 2004) proposed the semiring reduction of MV algebras. The authors gave the definition of automata on MV algebras from a semiring perspective and found languages of automata on MV algebras that retain some of the regularity of formal power series. Given the relation between MV algebras and effect algebras, we naturally ask: from the point of view of unsharp quantum logic, how can we set up automata theory based on these quantum structures and how can it be characterised?

In this section, we extend the truth lattice to lattice ordered effect algebras to ensure that some relevant definitions are well defined, and we give the definition of automata based on extended lattice ordered effect algebras. Similarly, we can obtain automata theory based on lattice ordered QMV algebras without changing anything.

We first recall some notions from classical automata theory. An automaton is a quintuple $\mathscr{R} = \langle Q, \Sigma, I, T, E\rangle$ in which:

(i) Q is a finite non-empty states set.
(ii) Σ is a finite alphabet whose elements are called labels.
(iii) $I \subseteq Q$ is the initial states set.
(iv) $T \subseteq Q$ is the terminal states set.
(v) $E \subseteq Q \times \Sigma \times E$, and each $(p, \sigma, q) \in E$ is called a transition in $\mathscr{R}$ and means that input σ makes state p become q.

Obviously, conditions (iii), (iv) and (v) in the above definition can be treated as the following propositions with 'yes/no' as their truth values:

(a) For any $q \in Q$, is q an initial state?
(b) For any $p \in Q$, is p a terminal state?
(c) Does σ make state p become q?

Hence, it is easy to see that classical automata theory is indeed based on boolean logic.

In a similar way, we let quantum logic denote the truth value of the propositions, and can set up automata theory based on quantum logic. In the following, $\mathscr{E}$ denotes an extended lattice ordered effect algebra (a lattice ordered quasilinear QMV algebras). If we use $\mathscr{E}$ now to denote a lattice ordered QMV algebra, we can obtain automata theory based on lattice ordered QMV algebras without changing anything.

Let Σ^*, Σ^+ be the sets of strings over Σ with $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$ and $\Sigma^+ = \bigcup_{n=1}^{\infty} \Sigma^n$, and let $\epsilon = \Sigma^0$ denote the empty string.

Definition 4.1 ($\mathscr{E}$-valued non-deterministic finite automaton). An $\mathscr{E}$-valued non-deterministic finite automaton is a quintuple $M = (Q, \Sigma, I, T, \delta)$ in which:

(i) Q is a finite non-empty state set.
(ii) Σ is a finite alphabet.
(iii) $I : Q \to \mathscr{E}$ is the initial state function.
(iv) $T : Q \to \mathscr{E}$ is the terminal state function.
(v) $\delta : Q \times \Sigma \cup \{\epsilon\} \times Q \to \mathscr{E}$ is the transition function, where $\delta(p, \epsilon, q) = \begin{cases} 0, & p = q \\ 1, & p \neq q. \end{cases}$
As in the classical case, $\delta(p, \sigma, q)$ indicates the truth value of the proposition that input σ causes state p to become q.

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Definition 4.2 (n-path). An n-path between p and q in M is a finite sequence of states of the form $\pi = (p_0 = p, p_1, p_2, \cdots, p_n = q)$. In a given $\mathscr{E}$-automaton M, the set of all paths $\pi = (p_0 = p, p_1, p_2, \cdots, p_n = q)$ of length n between p and q will be denoted by $P_M^n(p, q)$.

The n-path π is assigned with the function $||\pi|| : \Sigma^n \to \mathscr{E}$, such that

$$||\pi||(\sigma_1 \cdots \sigma_n) = \boxplus_{i=0,1,\cdots,n-1} \delta(p_i, \sigma_{i+1}, p_{i+1}).$$

Then a word $s = \sigma_1 \sigma_2 \cdots \sigma_n \in \Sigma^+$ is accepted with degree

$$|M|(s) = \wedge_{p,q \in Q} \wedge_{\pi \in P_M^n(p,q)} I(p) \boxplus ||\pi||(s) \boxplus T(q).$$

Now we give the definitions of a general $\mathscr{E}$-valued language and an $\mathscr{E}$-valued recognisable language.

Definition 4.3. An $\mathscr{E}$-valued language L on Σ is a map $L : \Sigma^* \to \mathscr{E}$.

An $\mathscr{E}$-valued language L on Σ is called a recognisable language if there exists an $\mathscr{E}$-valued automaton M such that $L = |M|$. In detail, for any word $s = \sigma_1 \sigma_2 \cdots \sigma_n \in \Sigma^+$,

$$L(s) = |M|(s) = \wedge_{p,q \in Q} \wedge_{\pi \in P_M^n(p,q)} I(p) \boxplus ||\pi||(s) \boxplus T(q).$$

Let $L(\mathscr{E})$ denote the class of $\mathscr{E}$-valued recognisable languages of Σ^*. Obviously, $L(\mathscr{E})$ is a subset of $(\mathscr{E})^{\Sigma^*}$.

Definition 4.4. Let $f, g \in (\mathscr{E})^{\Sigma^*}$ be $\mathscr{E}$-valued subsets.

(i) The intersection of two $\mathscr{E}$-valued languages f and g, denoted $f \wedge g$, is defined by $(f \wedge g)(s) = f(s) \wedge g(s)$ for any $s \in \Sigma^*$.

(ii) The sum of $\mathscr{E}$-valued languages f and g, denoted $f \boxplus g$, is defined by $(f \boxplus g)(s) = f(s) \boxplus g(s)$ for any $s \in \Sigma^*$.

(iii) Denote $s^R = \sigma_n \cdots \sigma_1$ for any $s = \sigma_1 \cdots \sigma_n \in \Sigma^n (n \geqslant 1)$, and $\epsilon^R = \epsilon$. The reversal of an $\mathscr{E}$-valued language L is defined by $f^R(s) = f(s^R)$.

(iv) The concatenation of two $\mathscr{E}$-valued languages f and g, denoted $f \cdot g$, is defined by $(f \cdot g)(s) = \bigwedge_{s_1 s_2 = s} [f(s_1) \boxplus g(s_2)]$ for any $s \in \Sigma^*$.

5. Closure properties of an $\mathscr{E}$-valued language

In this section, we discuss the closure properties of an $\mathscr{E}$-valued language.

Theorem 5.1. $L(\mathscr{E})$ is closed under the intersection operation.

Proof. Suppose $M_1 = (Q_1, \Sigma, I_1, T_1, \delta_1)$ and $M_2 = (Q_2, \Sigma, I_2, T_2, \delta_2)$ are two $\mathscr{E}$-valued automata with $Q_1 \cap Q_2 = \phi$. The languages they recognise are L_1 and L_2, respectively.

Construct an $\mathscr{E}$-valued automaton $M_1 \wedge M_2 = (Q_1 \cup Q_2, \Sigma, I^{M_1 \wedge M_2}, T^{M_1 \wedge M_2}, \delta^{M_1 \wedge M_2})$ as follows:

$$I^{M_1 \wedge M_2} : Q_1 \cup Q_2 \longrightarrow \mathscr{E}, \ p \longmapsto \begin{cases} I_1(p), & p \in Q_1 \\ I_2(p), & p \in Q_2 \end{cases}$$

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$$T^{M_1 \wedge M_2} \; : \; Q_1 \cup Q_2 \longrightarrow \mathscr{E}, \; p \longmapsto \begin{cases} T_1(p), & p \in Q_1 \\ T_2(p), & p \in Q_2 \end{cases}$$

$$\delta^{M_1 \wedge M_2} \; : \; Q_1 \cup Q_2 \times \Sigma \cup \{\epsilon\} \times Q_1 \cup Q_2 \longrightarrow \mathscr{E},$$

$$(p, \sigma, q) \longmapsto \begin{cases} \delta_1(p, \sigma, q), & p, q \in Q_1 \\ \delta_2(p, \sigma, q), & p, q \in Q_2 \\ 1, & \text{otherwise.} \end{cases}$$

Let $P^n_{M_1 \wedge M_2}$ denote the set of paths $\pi = (p_0, p_1, \cdots, p_n)$ with $p_i \in Q_1 \cup Q_2$ for every $i = 0, 1, 2, \cdots, n$. For every $s = \sigma_1 \cdots \sigma_n \in \Sigma^*$ and $\pi \in P^n_{M_1 \wedge M_2}$, we have

$$\begin{aligned} ||\pi||^{M_1 \wedge M_2}(s) &= \delta^{M_1 \wedge M_2}(p_0, \sigma_1, p_1) \boxplus \cdots \boxplus \delta^{M_1 \wedge M_2}(p_{n-1}, \sigma_n, p_n) \\ &= \begin{cases} ||\pi||^{M_1}(s), & \text{if } \pi \in P^n_{M_1} \\ ||\pi||^{M_2}(s), & \text{if } \pi \in P^n_{M_2} \\ 1, & \text{otherwise.} \end{cases} \end{aligned}$$

Hence, for any $s = \sigma_1 \sigma_2 \cdots \sigma_n \in \Sigma^+$,

$$\begin{aligned} |M_1 \wedge M_2|(s) &= \wedge_{p,q \in Q^{M_1 \wedge M_2}} \wedge_{\pi \in P^n_{M_1 \wedge M_2}(p,q)} (I^{M_1 \wedge M_2}(p) \boxplus ||\pi||(s) \boxplus T^{M_1 \wedge M_2}(q)) \\ &= \wedge_{p,q \in Q_1 \cup Q_2} \wedge_{\pi \in P^n_{M_1}(p,q)} \wedge_{\pi \in P^n_{M_2}(p,q)} (I^{M_1 \wedge M_2}(p) \\ &\qquad \boxplus ||\pi||(s) \boxplus T^{M_1 \wedge M_2}(q)) \\ &= \left[\wedge_{p,q \in Q_1} \wedge_{\pi \in P^n_{M_1}(p,q)} (I_1(p) \boxplus ||\pi||(s) \boxplus T_1(q)) \right] \\ &\quad \wedge \left[\wedge_{p,q \in Q_2} \wedge_{\pi \in P^n_{M_2}(p,q)} (I_2(p) \boxplus ||\pi||(s) \boxplus T_2(q)) \right] \\ &= L_1(s) \wedge L_2(s) \end{aligned}$$

and

$$\begin{aligned} |M_1 \wedge M_2|(\epsilon) &= \bigwedge_{p \in Q_1 \cup Q_2} (I^{M_1 \wedge M_2}(p) \boxplus T^{M_1 \wedge M_2}(p)) \\ &= \left(\bigwedge_{p \in Q_1} I_1(p) \boxplus T_1(p) \right) \wedge \left(\bigwedge_{p \in Q_2} I_2(p) \boxplus T_2(p) \right) \\ &= |M_1|(\epsilon) \wedge |M_2|(\epsilon). \end{aligned}$$

So we have proved that $M_1 \wedge M_2$ is the $\mathscr{E}$-valued automaton corresponding to $L_1 \wedge L_2$. □

Suppose $M_1 = (Q_1, \Sigma, I_1, T_1, \delta_1)$ and $M_2 = (Q_2, \Sigma, I_2, T_2, \delta_2)$ are $\mathscr{E}$-valued automata with $Q_1 \cap Q_2 = \phi$. Construct an $\mathscr{E}$-valued automaton

$$M_1 \boxplus M_2 = (Q_1 \times Q_2, \Sigma, I^{M_1 \boxplus M_2}, T^{M_1 \boxplus M_2}, \delta^{M_1 \boxplus M_2})$$

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where

$$\begin{aligned} I^{M_1 \boxplus M_2} &: (p,q) \in Q_1 \times Q_2 \longmapsto I_1(p) \boxplus I_2(q) \\ T^{M_1 \boxplus M_2} &: (p,q) \in Q_1 \times Q_2 \longmapsto T_1(p) \boxplus T_2(q) \\ \delta^{M_1 \boxplus M_2} &: ((p_0,q_0),\sigma,(p_1,q_1)) \in (Q_1 \times Q_2) \times \Sigma \cup \{\epsilon\} \times (Q_1 \times Q_2) \\ &\longmapsto \delta_1(p_0,\sigma,p_1) \boxplus \delta_2(q_0,\sigma,q_1) \end{aligned}$$

Theorem 5.2. Let $M_1 = (Q_1, \Sigma, I_1, T_1, \delta_1)$ and $M_2 = (Q_2, \Sigma, I_2, T_2, \delta_2)$ be two $\mathscr{E}$-valued automata, and L_1, L_2 be two $\mathscr{E}$-valued languages corresponding to M_1, M_2, respectively. If one of the Q_1, Q_2 contains at least two states, then $|M_1 \boxplus M_2| = L_1 \boxplus L_2$ if and only if $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$ for any $a, b, c \in E$.

Proof.

— *If part:*
Suppose $s = \sigma_1 \cdots \sigma_n \in \Sigma^n (n \geqslant 1)$ and for any $a, b, c \in E$, that $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$. Then

$$\begin{aligned} (L_1 \boxplus L_2)(s) &= |M_1|(s) \boxplus |M_2|(s) \\ &= \wedge_{p_0,\cdots,p_n \in Q_1} (I_1(p_0) \boxplus \delta_1(p_0,\sigma_1,p_1) \boxplus \cdots \boxplus T_1(p_n)) \\ &\quad \boxplus \wedge_{q_0,\cdots,q_n \in Q_2} (I_2(q_0) \boxplus \delta_2(q_0,\sigma_1,q_1) \boxplus \cdots \boxplus T_2(q_n)) \\ &= \wedge_{p_i \in Q_1} (I_1(p_0) \boxplus \cdots \boxplus T_1(p_n) \boxplus (\wedge_{q_i \in Q_2} (I_2(q_0) \boxplus \cdots \boxplus T_2(q_n)))) \\ &= \wedge_{p_i \in Q_1} (\wedge_{q_i \in Q_2} (I_1(p_0) \boxplus \cdots \boxplus T_1(p_n) \boxplus I_2(q_0) \boxplus \cdots \boxplus T_2(q_n))) \\ &= \wedge_{p_i \in Q_1, q_i \in Q_2} (I_1(p_0) \boxplus I_2(q_0) \boxplus \delta_1(p_0,\sigma_1,p_1) \boxplus \delta_2(q_0,\sigma_1,q_1) \boxplus \cdots \\ &\qquad \boxplus T_1(p_n) \boxplus T_2(q_n)) \\ &= \wedge_{p_i \in Q_1, q_i \in Q_2} (I^{M_1 \boxplus M_2}(p_0,q_0) \boxplus \delta^{M_1 \boxplus M_2}((p_0,q_0),\sigma_1,(p_1,q_1)) \boxplus \cdots \\ &\qquad \boxplus T^{M_1 \boxplus M_2}(p_n,q_n)) \\ &= |M_1 \boxplus M_2|(s) \end{aligned}$$

It is easy to see that $|M_1|(\epsilon) \boxplus |M_2|(\epsilon) = |M_1 \boxplus M_2|(\epsilon)$.

— *Only if part:*
If the distributive law $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$ does not hold, there exist $a, b, c \in E$ such that $(a \boxplus b) \wedge (a \boxplus c) \neq a \boxplus (b \wedge c)$. Let

$$\begin{aligned} M_1 &= (Q_1, \Sigma, I_1, \delta_1, T_1) \\ M_2 &= (Q_2, \Sigma, I_2, \delta_2, T_2) \end{aligned}$$

be two automata, where

$$\begin{aligned} \Sigma &= \{\sigma\} \\ Q_1 &= \{p_0, p_1\} \\ Q_2 &= \{q_0, q_1\} \\ I_1(p_0) &= a \\ I_1(p_1) &= 0 \end{aligned}$$

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$$
\begin{aligned}
\delta_1(p_0,\sigma,q) &= 0 \quad \text{for any } q \in Q_1 \\
\delta_1(p,\sigma,q) &= 1 \quad \text{for any } p \neq p_0 \\
T_1(p) &= 0 \quad \text{for any } p \in Q_1 \\
I_2(q_0) &= 0 \\
I_2(q_1) &= 0 \\
\delta_2(q_0,\sigma,q_1) &= b \\
\delta_2(q_1,\sigma,q_0) &= c \\
\delta_2(\cdot,\sigma,\cdot) &= 1 \quad \text{for any other cases} \\
T_2(p) &= 0 \quad \text{for any } p \in Q_2.
\end{aligned}
$$

Then $|M_1|(\sigma) = a$, $|M_2|(\sigma) = b \wedge c$, and $(|M_1| \boxplus |M_2|)(\sigma) = a \boxplus (b \wedge c)$. By construction,

$$M_1 \boxplus M_2 = (Q_1 \cup Q_2, \Sigma, I^{M_1 \boxplus M_2}, T^{M_1 \boxplus M_2}, \delta^{M_1 \boxplus M_2}).$$

It is easy to see that

$$
\begin{aligned}
(|M_1 \boxplus M_2|)(\sigma) &= \wedge_{p_i \in Q_1, q_i \in Q_2}(I^{M_1 \boxplus M_2}(p_0,q_0) \boxplus \\
&\qquad \delta^{M_1 \boxplus M_2}((p_0,q_0),\sigma_1,(p_1,q_1)) \boxplus \cdots \boxplus T^{M_1 \boxplus M_2}(p_n,q_n)) \\
&= (a \boxplus b) \wedge (a \boxplus c) \neq a \boxplus (b \wedge c) \\
&= (|M_1| \boxplus |M_2|)(\sigma).
\end{aligned}
$$

□

From the above result and Theorem 3.1, we obtain the following result.

Corollary 5.1. If $\mathscr{E}$ is an MV algebra, then $L(\mathscr{E})$ is closed under the sum operation.

Theorem 5.3. $L(\mathscr{E})$ is closed under the reversal operation.

Proof. Assume that $L \in L(\mathscr{E})$ and $M = (Q, \Sigma, I, T, \delta)$ is the automaton corresponding to L. Construct an $\mathscr{E}$-valued automaton $M^R = (Q, \Sigma, I^R, T^R, \delta^R)$ as follows:

$$
\begin{aligned}
I^R(p) &= T(p) \\
T^R(p) &= I(p) \\
\delta^R(p,\sigma,q) &= \delta(q,\sigma,p).
\end{aligned}
$$

Thus, for any $s = \sigma_1 \cdots \sigma_n \in \Sigma^n (n \geqslant 1)$,

$$
\begin{aligned}
|M^R|(s) &= \wedge_{p_0,\cdots,p_n \in Q}(I^R(p_0) \boxplus \delta^R(p_0,\sigma_1,p_1) \boxplus \cdots \boxplus \delta^R(p_{n-1},\sigma_n,p_n) \boxplus T^R(p_n)) \\
&= \wedge_{p_0,\cdots,p_n \in Q}(T(p_0) \boxplus \delta(p_1,\sigma_1,p_0) \boxplus \cdots \boxplus \delta(p_n,\sigma_n,p_{n-1}) \boxplus I(p_n)) \\
&= \wedge_{p_n,\cdots,p_0 \in Q}(I(p_n) \boxplus \delta(p_n,\sigma_n,p_{n-1}) \boxplus \cdots \boxplus \delta(p_1,\sigma_1,p_0) \boxplus T(p_0)) \\
&= |M|(s^R).
\end{aligned}
$$

It is easy to see that $(|M_1| \boxplus |M_2|)(\epsilon) = |M_1 \boxplus M_2|(\epsilon)$. That is $|M^R|(s) = L(s^R) = L^R(s)$. Hence $L^R \in L(\mathscr{E})$. □

Suppose $M_1 = (Q_1, \Sigma, I_1, T_1, \delta_1)$ and $M_2 = (Q_2, \Sigma, I_2, T_2, \delta_2)$ are $\mathscr{E}$-valued automata with $Q_1 \cap Q_2 = \phi$.

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Construct an $\mathscr{E}$-valued automaton $M_1 \cdot M_2 = (Q, \Sigma, I^{M_1 \cdot M_2}, T^{M_1 \cdot M_2}, \delta^{M_1 \cdot M_2})$ with

$$
\begin{aligned}
Q &= Q_1 \cup Q_2 \\
I^{M_1 \cdot M_2}(p) &= \begin{cases} I_1(p), & p \in Q_1 \\ 1, & p \in Q_2 \end{cases} \\
T^{M_1 \cdot M_2}(p) &= \begin{cases} T_1(p) \boxplus |M_2|(\epsilon), & p \in Q_1 \\ T_2(p), & p \in Q_2 \end{cases}
\end{aligned}
$$

and $\delta^{M_1 \cdot M_2} : Q \times \Sigma \cup \{\epsilon\} \times Q \longrightarrow \mathscr{E}$ defined as follows:

(i) When $\sigma \in \Sigma$,

$$
\delta^{M_1 \cdot M_2}(p, \sigma, q) = \begin{cases} \delta_1(p, \sigma, q), & p, q \in Q_1 \\ \delta_2(p, \sigma, q), & p, q \in Q_2 \\ a, & p \in Q_1, q \in Q_2 \\ 1, & p \in Q_2, q \in Q_1 \end{cases}
$$

where $a = \bigwedge_{p' \in Q_1} [\delta_1(p, \sigma, p') \boxplus T_1(p') \boxplus I_2(q)] \wedge \bigwedge_{p'' \in Q_2} [T_1(p) \boxplus I_2(p'') \boxplus \delta_2(p'', \sigma, q)]$.

(ii) When $\sigma = \epsilon$,

$$
\delta^{M_1 \cdot M_2}(p, \epsilon, q) = \begin{cases} 0, q = p \\ 1, q \neq p. \end{cases}
$$

Theorem 5.4. Let $M_1 = (Q_1, \Sigma, I_1, T_1, \delta_1), M_2 = (Q_2, \Sigma, I_2, T_2, \delta_2)$ be an $\mathscr{E}$-valued automata, and L_1, L_2 be two $\mathscr{E}$-valued languages corresponding to M_1, M_2, respectively. Then $L_1 \cdot L_2$ is the $\mathscr{E}$-valued language of $M_1 \cdot M_2$ if and only if $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$ for any $a, b, c \in E$.

Proof.

— *If part:*
We use M to denote $M_1 \cdot M_2$. Assume $n \geqslant 1$, $s = u_1 u_2 \cdots u_n \in \Sigma^+$, $\pi = (p_0 = p, p_1, p_2, \cdots, p_n = q) \in P_M^n(p, q)$. If $p_0 \in Q_1$ and $p_n \in Q_2$, there is one k $(0 \leqslant k \leqslant n-1)$ such that $p_k \in Q_1, p_{k+1} \in Q_2$.
If $q \in Q_2$, then:

$$
\begin{aligned}
|M_1 \cdot M_2|(s) &= \bigwedge_{p,q \in Q} \bigwedge_{\pi \in P_M^n(p,q)} \left(I^{M_1 \cdot M_2}(p) \boxplus ||\pi||(s) \boxplus T^{M_1 \cdot M_2}(q) \right) \\
&= \bigwedge_{p \in Q_1, q \in Q_2} \Bigg[I_1(p) \\
&\qquad \boxplus \bigwedge_{0 \leqslant k \leqslant n-1} \bigwedge_{p_1, \cdots, p_k \in Q_1} \bigwedge_{p_{k+1}, \cdots, p_{n-1} \in Q_2} \left(\sum_{i=1}^{k} \delta_1(p_{i-1}, u_i, p_i) \right. \\
&\qquad\qquad \left. \boxplus \delta^{M_1 \cdot M_2}(p_k, u_{k+1}, p_{k+1}) \boxplus \sum_{i=k+2}^{n} \delta_2(p_{i-1}, u_i, p_i) \right) \\
&\qquad \boxplus T_2(q) \Bigg]
\end{aligned}
$$

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$$
\begin{aligned}
= \bigwedge_{p\in Q_1, q\in Q_2} \Bigg[& I_1(p) \\
& \boxplus \bigwedge_{0\leqslant k\leqslant n-1} \bigwedge_{p_1,\cdots,p_k\in Q_1} \bigwedge_{p_{k+1},\cdots,p_{n-1}\in Q_2} \left(\sum_{i=1}^{k} \delta_1(p_{i-1}, u_i, p_i) \right. \\
& \qquad \boxplus \left[\bigwedge_{p'\in Q_1} (\delta_1(p_k, u_{k+1}, p') \boxplus T_1(p') \boxplus I_2(p_{k+1})) \right. \\
& \qquad\qquad \left. \wedge \bigwedge_{p''\in Q_2} (T_1(p_k) \boxplus I_2(p'') \boxplus \ \delta_2(p'', u_{k+1}, p_{k+1}) \right] \\
& \qquad \left. \boxplus \sum_{i=k+2}^{n} \delta_2(p_{i-1}, u_i, p_i) \right) \\
& \boxplus \ T_2(q) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \bigwedge_{0\leqslant k\leqslant n-1} \Bigg[& \bigwedge_{p_0,\cdots,p_k,p'\in Q_1} \bigwedge_{p_{k+1},\cdots,p_n\in Q_2} \Bigg[I_1(p_0) \boxplus \sum_{i=1}^{k} \delta_1(p_{i-1}, u_i, p_i) \\
& \boxplus \delta_1(p_k, u_{k+1}, p') \\
& \boxplus T_1(p') \boxplus I_2(p_{k+1}) \boxplus \sum_{i=k+2}^{n} \delta_2(p_{i-1}, u_i, p_i) \boxplus T_2(p_n) \Bigg] \\
& \wedge \bigwedge_{p_0,\cdots,p_k\in Q_1} \bigwedge_{p_{k+1},\cdots,p_n,p''\in Q_2} \Bigg[I_1(p_0) \\
& \boxplus \sum_{i=1}^{k} \delta_1(p_{i-1}, u_i, p_i) \boxplus T_1(p_k) \boxplus I_2(p'') \\
& \boxplus \delta_2(p'', u_{k+1}, p_{k+1}) \boxplus \sum_{i=k+2}^{n} \delta_2(p_{i-1}, u_i, p_i) \boxplus T_2(p_n) \Bigg] \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
= \bigwedge_{0\leqslant k\leqslant n-1} & (|M_1|(u_1 \cdots u_{k+1}) \boxplus |M_2|(u_{k+2} \cdots u_n)) \\
& \wedge (|M_1|(u_1 \cdots u_k) \boxplus |M_2|(u_{k+1} \cdots u_n))
\end{aligned}
$$

$$
= \bigwedge_{v_1 v_2 = s, v_1, v_2 \in \Sigma^*} |M_1|(v_1) \boxplus |M_2|(v_2).
$$

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If $q \in Q_1$, we have

$$
\begin{aligned}
|M_1 \cdot M_2|(s) &= \bigwedge_{p,q\in Q} \bigwedge_{\pi\in P^n_M(p,q)} \left(I^{M_1\cdot M_2}(p) \boxplus ||\pi||(s) \boxplus T^{M_1\cdot M_2}(q)\right) \\
&= \bigwedge_{p,q\in Q_1} \bigwedge_{\pi\in P^n_{M_1}(p,q)} I_1(p) \boxplus ||\pi||(s) \boxplus T_1(q) \boxplus |M_2|(\epsilon) \\
&= |M_1|(s) \boxplus |M_2|(\epsilon).
\end{aligned}
$$

Thus

$$
|M_1 \cdot M_2|(s) = \bigwedge_{v_1 v_2 = s, v_1, v_2 \in \Sigma^*} (|M_1|(v_1) \boxplus |M_2|(v_2))
$$

for $n \geqslant 1$.

If $s = \epsilon$, we have

$$
\begin{aligned}
|M_1 \cdot M_2|(\epsilon) &= \bigwedge_{p\in Q} (I^{M_1\cdot M_2}(p) \boxplus T^{M_1\cdot M_2}(p)) \\
&= \bigwedge_{p\in Q_1} (I_1(p) \boxplus T^{M_1\cdot M_2}(p)) \\
&= \bigwedge_{p\in Q_1} (I_1(p) \boxplus T_1(p) \boxplus |M_2|(\epsilon)) \\
&= \bigwedge_{p\in Q_1} (I_1(p) \boxplus T_1(p)) \boxplus |M_2|(\epsilon) \\
&= |M_1|(\epsilon) \boxplus |M_2|(\epsilon).
\end{aligned}
$$

— *Only if part:*

If the distributive law is not true, there exist $a, b, c \in \mathscr{E}$ such that $(a \boxplus b) \wedge (a \boxplus c) \neq a \boxplus (b \wedge c)$. Let $M_1 = (Q_1, \Sigma, \delta_1, I_1, T_1)$ be an automaton where

$$
\begin{aligned}
Q_1 &= \{p_0, p_1\} \\
\Sigma &= \{\sigma\} \\
I_1(p_0) &= T_1(p_1) = 0 \\
I_1(p_1) &= T_1(p_0) = 1 \\
\delta_1(p_0, \sigma, p_1) &= a \\
\delta_1 &= 1 \text{ for other arguments.}
\end{aligned}
$$

And $M_2 = (Q_2, \Sigma, \delta_2, I_2, T_2)$ where

$$
\begin{aligned}
Q_2 &= \{q_0, q_1, q_2\} \\
I_2(q_0) &= T_2(q_1) = T_2(q_2) = 0 \\
I_2(q_1) &= I_2(q_2) = T_2(q_0) = 1 \\
\delta_2(q_0, \sigma, q_1) &= b \\
\delta_2(q_0, \sigma, q_2) &= c \\
\delta_2 &= 1 \text{ for other arguments.}
\end{aligned}
$$

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Let $s = \sigma\sigma$. Then

$$\begin{aligned}|M_1 \cdot M_2|(s) &= \bigwedge_{r_i \in Q} I^{M_1 \cdot M_2}(r_0) \boxplus \delta^{M_1 \cdot M_2}(r_0, \sigma, r_1) \\ &\qquad \boxplus \delta^{M_1 \cdot M_2}(r_1, \sigma, r_2) \boxplus T^{M_1 \cdot M_2}(r_2) \\ &= \bigwedge_{r_1, r_2 \in Q} I_1(p_0) \boxplus \delta^{M_1 \cdot M_2}(p_0, \sigma, r_1) \\ &\qquad \boxplus \delta^{M_1 \cdot M_2}(r_1, \sigma, r_2) \boxplus T^{M_1 \cdot M_2}(r_2) \\ &= (I_1(p_0) \boxplus \delta_1(p_0, \sigma, p_1) \boxplus [1 \wedge \delta_2(q_0, \sigma, q_1)] \boxplus T_2(q_1)) \\ &\quad \wedge (I_1(p_0) \boxplus \delta_1(p_0, \sigma, p_1) \boxplus [1 \wedge \delta_2(q_0, \sigma, q_2)] \boxplus T_2(q_2)) \\ &\quad \wedge (I_1(p_0) \boxplus [\delta_1(p_0, \sigma, p_1) \wedge 1] \boxplus \delta_2(q_0, \sigma, q_1) \boxplus T_2(q_1)) \\ &\quad \wedge (I_1(p_0) \boxplus [\delta_1(p_0, \sigma, p_1) \wedge 1] \boxplus \delta_2(q_0, \sigma, q_2) \boxplus T_2(q_2)) \\ &= (a \boxplus b) \wedge (a \boxplus c).\end{aligned}$$

In addition,

$$\begin{aligned}(L_1 \cdot L_2)(s) &= (|M_1|(\epsilon) \boxplus |M_2|(\sigma\sigma)) \wedge (|M_1|(\sigma) \boxplus |M_2|(\sigma)) \wedge (|M_1|(\sigma\sigma) \boxplus |M_2|(\epsilon)) \\ &= a \boxplus (b \wedge c).\end{aligned}$$

Then, by $a \boxplus (b \wedge c) \neq (a \boxplus b) \wedge (a \boxplus c)$, we have $|M|(s) \neq (L_1 \cdot L_2)(s)$. □

From Theorems 5.4 and 3.1, we have the following results.

Corollary 5.2. If $\mathscr{E}$ is an MV algebra, $L(\mathscr{E})$ is closed under the concatenation operation.

Corollary 5.3. If $\mathscr{E}$ is an MV algebra, $L(\mathscr{E})$ is closed under the Kleene closure.

Furthermore, letting $x \odot y = (x' \boxplus y')'$, we can define an automaton on extended lattice ordered effect algebras as follows.

An $\mathscr{E}$-valued nondetermined finite automaton is a quintuple $M = (Q, \Sigma, I, T, \delta)$ in which:

(i) Q is a finite non-empty state set.
(ii) Σ is a finite non-empty set of input symbols.
(iii) $I : Q \to \mathscr{E}$ is the initial state function.
(iv) $T : Q \to \mathscr{E}$ is the terminal state function.
(v) $\delta : Q \times \Sigma \cup \{\epsilon\} \times Q \to \mathscr{E}$ is the transition function, where $\delta(p, \epsilon, q) = \begin{cases} 1, & p = q \\ 0, & p \neq q. \end{cases}$

Let Σ^*, Σ^+ be the sets of strings over Σ with $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n$ and $\Sigma^+ = \bigcup_{n=1}^{\infty} \Sigma^n$, and let $\epsilon = \Sigma^0$ denote the empty word.

The n-path π is assigned with the label $||\pi|| \in \mathscr{E}^{\Sigma^n} (n \geqslant 1)$ such that

$$||\pi||(\sigma_1 \cdots \sigma_n) = \odot_{i=0,1,\cdots,n-1} \delta(p_i, \sigma_{i+1}, p_{i+1}).$$

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In a given $\mathscr{E}$-valued automaton M, the set of all paths $\pi = (p_0 = p, p_1, p_2, \cdots, p_n = q)$ of length n between p and q will be denoted by $P^n_M(p,q)$. Then a word $s = \sigma_1\sigma_2\cdots\sigma_n \in \Sigma^+$ is accepted with $\mathscr{E}$-value

$$|M|(s) = \vee_{p,q\in Q} \vee_{\pi \in P^n_M(p,q)} I(p) \odot ||\pi||(s) \odot T(q).$$

Similarly, we can define language of the automata and discuss the corresponding properties of the language.

Remark 5.1. Let $\mathscr{E}$ denote an orthomodular lattice. Then $\odot$ becomes $\wedge$ in orthomodular lattices. Thus, we can obtain the automata theory based on orthomodular lattices (Ying 2000a, 2000b, 2005; Qiu 2003, 2004).

6. Subset construction of $\mathscr{E}$-valued automata

In this section, we give the subset construction of $\mathscr{E}$-valued automata.

Definition 6.1. Given an $\mathscr{E}$-valued automaton $M = (Q, \Sigma, I, T, \delta)$, if

(i) there is a unique q_0 in Q with $I(q_0) \neq 1$,
(ii) there exists at most one q for any pair $(p, \sigma) \in Q \times \Sigma$ such that $\delta(p, \sigma, q) \neq 1$,

then we call M a determined $\mathscr{E}$-valued automaton.

Definition 6.2 (Subset construction of $\mathscr{E}$-valued automata). If $\mathscr{E}$ is finite, the subset construction of an $\mathscr{E}$-valued automaton $M = (Q, \Sigma, I, T, \delta)$, denoted $\mathscr{E}^M$, is defined as $\mathscr{E}^M = (\mathscr{E}^Q, \Sigma, \bar{I}, \bar{T}, \bar{\delta})$, where:

(i) $\mathscr{E}^Q : Q \to \mathscr{E}$, that is, the set of all $\mathscr{E}$-valued subsets of Q.

(ii) $\bar{I} : \mathscr{E}^Q \to \mathscr{E}, \bar{I}(X) = \begin{cases} 0, & X = I \\ 1, & \text{otherwise.} \end{cases}$

(iii) $\bar{T} : \mathscr{E}^Q \to \mathscr{E}. \bar{T}(X) = \wedge_{p\in Q}(X(p) \boxplus T(p))$.

(iv) Defining $Y_{X,\sigma}(q) = \wedge_{p\in Q}(X(p) \boxplus \delta(p, \sigma, q))$ for any $\sigma \in \Sigma, X \in \mathscr{E}^Q$, we have $\bar{\delta}$: $\mathscr{E}^Q \times \Sigma \cup \{\epsilon\} \times \mathscr{E}^Q \longrightarrow \mathscr{E}$, is defined as

(a) When $\sigma \in \Sigma$,

$$\bar{\delta}(X, \sigma, Y) = \begin{cases} 0, & Y = Y_{X,\sigma} \\ 1, & \text{otherwise.} \end{cases}$$

(b) When $\sigma = \epsilon$,

$$\bar{\delta}(X, \epsilon, Y) = \begin{cases} 0, & X = Y \\ 1, & \text{otherwise.} \end{cases}$$

Since $\mathscr{E}$ is finite, $\mathscr{E}^Q$ is finite too. Thus, it is easy to see that $\mathscr{E}^M$ is an $\mathscr{E}$-valued determined automaton.

In the following, we will show that only the distributive law can warrant that an $\mathscr{E}$ automaton M and its subset construction have the same ability to recognise languages.

Theorem 6.1. If $\mathscr{E}$ is finite and M is an $\mathscr{E}$-valued automaton, then $|M| = |\mathscr{E}^M|$ if and only if $(a \boxplus b) \wedge (a \boxplus c) = a \boxplus (b \wedge c)$ for any $a, b, c \in \mathscr{E}$.

Proof.

— *If part:*
For any $s = \sigma_1\sigma_2\cdots\sigma_n \in \Sigma^+$,

$$\begin{aligned}
|\mathscr{E}^M|(s) &= \wedge_{X,Y\in\mathscr{E}^Q} \wedge_{\pi\in P^n_{\mathscr{E}^M}(X,Y)} (\bar{I}(X) \boxplus ||\pi||(s) \boxplus \bar{T}(Y)) \\
&= \wedge_{X_0,\cdots,X_n\in\mathscr{E}^Q} (\bar{I}(X_0) \boxplus \bar{\delta}(X_0,\sigma_1,X_1) \boxplus \cdots \\
&\qquad \boxplus \bar{\delta}(X_{n-1},\sigma_n,X_n) \boxplus \bar{T}(X_n)) \\
&= \bar{I}(I) \boxplus \bar{\delta}(I,\sigma_1,Y_{I,\sigma_1}) \boxplus \cdots \boxplus \bar{T}(X_n) \\
&= \bar{T}(X_n) \\
&= \wedge_{p_n\in Q}(X_n(p_n) \boxplus T(p_n)) \\
&= \wedge_{p_n\in Q} \left(\wedge_{p_{n-1}\in Q}(X_{n-1}(p_{n-1}) \boxplus \delta(p_{n-1},\sigma_n,p_n)\right) \boxplus T(p_n)) \\
&= \wedge_{p_n\in Q} \left(\left(\wedge_{p_{n-1}\in Q} \left(\cdots \left(\wedge_{p_0\in Q} I(p_0) \boxplus \delta(p_0,\sigma_1,p_1)\right) \cdots\right)\right.\right. \\
&\qquad \boxplus \delta(p_{n-1},\sigma_n,p_n)) \\
&\qquad \boxplus T(p_n)) \\
&= \wedge_{p_n\in Q} \wedge_{p_{n-1}\in Q} \cdots \wedge_{p_0\in Q} (I(p_0) \boxplus \delta(p_0,\sigma_1,p_1) \cdots \\
&\qquad \boxplus \delta(p_{n-1},\sigma_n,p_n) \boxplus T(p_n)) \\
&= \wedge_{p_i\in Q}(I(p_0) \boxplus \delta(p_0,\sigma_1,p_1) \cdots \boxplus \delta(p_{n-1},\sigma_n,p_n) \boxplus T(p_n)) \\
&= |M|(s),
\end{aligned}$$

and $|\mathscr{E}^M|(\epsilon) = \bar{I}(I) \boxplus \bar{T}(I) = \bigwedge_{p\in Q}(I(p) \boxplus T(p)) = |M|(\epsilon)$.

— *Only if part:*
If the distributive law does not hold, then there exist $a, b, c \in E$ such that $(a \boxplus b) \wedge (a \boxplus c) \neq a \boxplus (b \wedge c)$. Let $M = (Q, \{\sigma\}, I, T, \delta)$ be an automaton where

$$\begin{aligned}
Q &= \{p, q\} \\
I(p) &= 0 \\
I(q) &= 0 \\
T(p) &= a \\
T(q) &= 1 \\
\delta(p,\sigma,p) &= b \\
\delta(q,\sigma,p) &= c \\
\delta &= 1 \quad \text{for other cases.}
\end{aligned}$$

Thus

$$\begin{aligned}
|\mathscr{E}^M|(\sigma) &= \wedge_{p_1\in Q} \left(\wedge_{p_0\in Q} I(p_0) \boxplus \delta(p_0,\sigma,p_1)\right) \boxplus T(p_1) \\
&= \wedge_{p_1\in Q} \left(\wedge_{p_0\in Q} \delta(p_0,\sigma,p_1)\right) \boxplus T(p_1) \\
&= \wedge_{p_1\in Q} (\delta(p,\sigma,p) \wedge \delta(q,\sigma,p)) \boxplus T(p_1) \\
&= (\delta(p,\sigma,p) \wedge \delta(q,\sigma,p)) \boxplus T(p) \\
&= (b \wedge c) \boxplus a.
\end{aligned}$$

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But

$$\begin{aligned} |M|(\sigma) &= \wedge_{p_0,p_1\in Q}(I(p_0) \boxplus \delta(p_0,\sigma,p_1) \boxplus T(p_1)) \\ &= (b \boxplus a) \wedge (c \boxplus a) \\ &\neq |\mathscr{E}^M|(\sigma). \end{aligned}$$

□

Corollary 6.1. If $\mathscr{E}$ is a finite MV algebra, an $\mathscr{E}$-valued non-determined finite automaton has an equal ability to recognise languages as its subset construction.

Remark 6.1. When $\mathscr{E}$ is a lattice ordered QMV algebra, it is easy to see that the results still hold.

7. Conclusion

In this paper, we have considered two algebraic models of unsharp quantum logic: extended lattice ordered effect algebras (lattice ordered quasilinear QMV algebras) and lattice ordered QMV algebras. They are the main models for unsharp quantum logic. For unsharp quantum structures, the sum (or partial sum) operation is the main operation. By studying properties of the sum operation on these two quantum structures, we find that if they satisfy a certain kind of distributive law, they become MV algebras. Interestingly, when we set up automata theory based on these quantum structures, although some constructions are valid without distributivity, we find it is essential for the more interesting ones (such as sum, concatenation and the subset construction of automata). From these results, we can conclude that one can only do automata theory in the presence of a distributivity law, and this holds even when we go from orthomodular lattices to the general unsharp quantum structures.

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