

## Dispersionless integrable hierarchies and $GL(2, \mathbb{R})$ geometry

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### Abstract

Paraconformal or  $GL(2, \mathbb{R})$  geometry on an  $n$ -dimensional manifold  $M$  is defined by a field of rational normal curves of degree  $n - 1$  in the projectivised cotangent bundle  $\mathbb{P}T^*M$ . Such geometry is known to arise on solution spaces of ODEs with vanishing Wünschmann (Dobrov–Wilczynski) invariants. In this paper we discuss yet another natural source of  $GL(2, \mathbb{R})$  structures, namely dispersionless integrable hierarchies of PDEs such as the dispersionless Kadomtsev–Petviashvili (dKP) hierarchy. In the latter context,  $GL(2, \mathbb{R})$  structures coincide with the characteristic variety (principal symbol) of the hierarchy.

Dispersionless hierarchies provide explicit examples of particularly interesting classes of involutive  $GL(2, \mathbb{R})$  structures studied in the literature. Thus, we obtain torsion-free  $GL(2, \mathbb{R})$  structures of Bryant [5] that appeared in the context of exotic holonomy in dimension four, as well as totally geodesic  $GL(2, \mathbb{R})$  structures of Krynski [33]. The latter possess a compatible affine connection (with torsion) and a two-parameter family of totally geodesic  $\alpha$ -manifolds (coming from the dispersionless Lax equations), which makes them a natural generalisation of the Einstein–Weyl geometry.

Our main result states that involutive  $GL(2, \mathbb{R})$  structures are governed by a dispersionless integrable system whose general local solution depends on  $2n - 4$  arbitrary functions of 3 variables. This establishes integrability of the system of Wünschmann conditions.

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### 1. Introduction

#### 1.1. $GL(2, \mathbb{R})$ geometry

On an  $n$ -dimensional manifold  $M$ , a  $GL(2, \mathbb{R})$  geometry (also known as paraconformal geometry [12], a rational normal structure [7], or a special case of the cone structure [21])

is defined by a field of rational normal curves of degree  $n - 1$  in the projectivised cotangent bundle  $\mathbb{P}T^*M$ . Equivalently, it can be viewed as a field of 1-forms  $\omega(\lambda)$  polynomial of degree  $n - 1$  in  $\lambda$ ,

$$\omega(\lambda) = \omega_0 + \lambda\omega_1 + \cdots + \lambda^{n-1}\omega_{n-1}, \quad (1.1)$$

where  $\omega_i$  is a basis of 1-forms (a coframe) on  $M$ . The parameter  $\lambda$  and the 1-form  $\omega(\lambda)$  are defined up to transformations  $\lambda \rightarrow (a\lambda + b)/(c\lambda + d)$ ,  $\omega(\lambda) \rightarrow r(c\lambda + d)^{n-1}\omega(\lambda)$ , where  $a, b, c, d, r$  are arbitrary smooth functions on  $M$  such that  $ad - bc \neq 0$ ,  $r \neq 0$ . Without any loss of generality we can assume  $ad - bc = 1$ .

Conventionally, a  $GL(2, \mathbb{R})$  geometry is defined by a field of rational normal curves in the projectivised *tangent* bundle  $\mathbb{P}TM$ . Our choice of the cotangent bundle is motivated by the fact that characteristic varieties of PDEs, which will be our main source of  $GL(2, \mathbb{R})$  structures, are subvarieties of  $\mathbb{P}T^*M$ . In any case, both pictures are projectively dual: the equation  $\omega(\lambda) = 0$  defines a one-parameter family of hyperplanes that osculate a dual rational normal curve  $\tilde{\omega}(\lambda) \subset \mathbb{P}TM$ . Below we discuss some of the most natural occurrences of  $GL(2, \mathbb{R})$  structures.

**Poisson geometry:** given a generic pair of compatible Poisson bivectors  $\eta_1, \eta_2$  of Kronecker type on an odd-dimensional manifold  $N^{2k+1}$ , there is a canonical  $GL(2, \mathbb{R})$  structure on the base  $M^{k+1}$  (leaf space) of the corresponding action foliation (see [47]). As shown by Gelfand and Zakharevich such structures, also known as Veronese webs, arise in the theory of bi-Hamiltonian integrable systems [20].

**Exotic holonomy:** it was observed by Bryant in [5] that, in four dimensions, there exist torsion-free affine connections whose holonomy group is the irreducible representation of  $GL(2, \mathbb{R})$ . Such connections give rise to canonically defined parallel  $GL(2, \mathbb{R})$  structures. Historically, this was the first example of an ‘exotic’ holonomy not appearing on the Berger list [3], we refer to [6, 39] for the development of the holonomy problem.

**Submanifolds in Grassmannians:** let  $M$  be a submanifold of the Grassmannian  $Gr(k, n)$ . The flat Segre structure of  $Gr(k, n)$  induces on  $M$  a generalised conformal structure. Particular instances of this construction result in a  $GL(2, \mathbb{R})$  geometry on  $M$ .

Thus, let  $M^4$  be a fourfold in the Grassmannian  $Gr(3, 5)$ . The flat Segre structure of  $Gr(3, 5)$  induces a field of twisted cubics on  $\mathbb{P}TM^4$ , that is, a  $GL(2, \mathbb{R})$  structure on  $M^4$ . These structures were investigated in [11] in the context of integrability in Grassmann geometries.

Similarly, let  $\Lambda(3)$  be the Grassmannian of 3-dimensional Lagrangian subspaces of a 6-dimensional symplectic space. Given a hypersurface  $M^5 \subset \Lambda(3)$ , the flat Veronese structure of  $\Lambda(3)$  induces a  $GL(2, \mathbb{R})$  structure on  $M^5$ . Such structures were discussed in [17, 42] in the context of integrability of dispersionless Hirota type equations.

**Algebraic geometry:** given a compact complex surface  $X$  and a rational curve  $C \subset X$  with the normal bundle  $\nu \simeq \mathcal{O}(n)$ , the results of Kodaira [35] show that there is a complex-analytic  $(n + 1)$ -dimensional moduli space  $M$  consisting of deformations of  $C$ , which carries a canonical  $GL(2, \mathbb{R})$  structure. This was studied in detail by Hitchin [26] for  $n = 2$  (using É. Cartan’s work on Einstein-Weyl geometry) and by Bryant [5] for  $n = 3$ . The case of general  $n$  was discussed by Dunajski, Tod [12] and Krynski [33]. The construction generalises to the case when  $X$  is a holomorphic contact 3-fold and  $C \subset X$  is a contact rational curve with the normal bundle  $\nu \simeq \mathcal{O}(n - 1) \oplus \mathcal{O}(n - 1)$  [5, 12, 7].

**Ordinary differential equations:** for every scalar (higher order) ODE with vanishing Wünschmann (Dobrov-Wilczynski) invariants, the space  $M$  of its solutions is canonically endowed with a  $GL(2, \mathbb{R})$  structure. ODEs of this type have been thoroughly investigated in the literature, see e.g. [10, 12, 14, 22, 33, 40] and references therein.

**Dispersionless integrable hierarchies:** given a dispersionless integrable hierarchy of PDEs, it will be demonstrated in this paper that the corresponding characteristic variety (zero locus of the principal symbol) determines canonically a  $GL(2, \mathbb{R})$  structure on every solution. In a somewhat different language examples of this type appeared in [13, 33], although the observation that these structures coincide with the characteristic variety is apparently new. We will show that the  $GL(2, \mathbb{R})$  structures appearing on solutions to integrable hierarchies are not arbitrary, and must satisfy an important property of *involutivity*.

1.2. *Involutive  $GL(2, \mathbb{R})$  structures and dispersionless hierarchies*

For every  $x \in M$ , the equation  $\omega(\lambda) = 0$  defines a 1-parameter family of hyperplanes in  $T_x M$  parametrised by  $\lambda$ ; these are known as  $\alpha$ -hyperplanes. A codimension one submanifold of  $M$  is said to be an  $\alpha$ -manifold if its tangent spaces are  $\alpha$ -hyperplanes [33].

*Definition.* A  $GL(2, \mathbb{R})$  structure is said to be *involutive* [7] or  $\alpha$ -integrable [33] if every  $\alpha$ -hyperplane is tangential to some  $\alpha$ -manifold.

We will relate different approaches to involutivity in Section 3.4. One can show that  $\alpha$ -manifolds of an involutive  $GL(2, \mathbb{R})$  structure depend on 1 arbitrary function of 1 variable (Section 3.3). These manifolds are governed by the so-called ‘eikonal’ system (see the review [43] for a general discussion). The existence of  $\alpha$ -manifolds suggests that involutive  $GL(2, \mathbb{R})$  structures are amenable to twistor-theoretic methods, cf. [21].

In particular,  $GL(2, \mathbb{R})$  structures that arise on solution spaces of ODEs with vanishing Wünschmann invariants are involutive. It was shown in [33] that conversely, every involutive ( $\alpha$ -integrable)  $GL(2, \mathbb{R})$  structure can be obtained from an ODE of this type. Four-dimensional involutive  $GL(2, \mathbb{R})$  structures were extensively studied in [5] in the context of exotic holonomy. These investigations were developed further in [10, 12, 14, 22, 40].

Our main observation is that involutive  $GL(2, \mathbb{R})$  structures are induced, as characteristic varieties, on solutions to dispersionless integrable hierarchies of PDEs. Moreover,  $\alpha$ -manifolds can be obtained as projections of integral manifolds of the associated dispersionless Lax equations.

The following example is based on [14, 33, 47]. Equations of the Veronese web hierarchy have the form

$$(c_i - c_j)u_k u_{ij} + (c_j - c_k)u_i u_{jk} + (c_k - c_i)u_j u_{ik} = 0, \tag{1.2}$$

one equation for every triple  $(i, j, k)$  of distinct indices. Here  $u$  is a function on the  $n$ -dimensional manifold  $M$  with local coordinates  $x^1, \dots, x^n$ , coefficients  $c_i$  are pairwise distinct constants, and  $u_i = u_{x^i}$  denote partial derivatives.

The term ‘hierarchy’ refers to the fact that the overdetermined system (1.2) is in involution for every  $n$  so that any two equations can be viewed as Lie–Bäcklund symmetries of each other: if we take two different triples and unite the indices, then the system of equations of type (1.2) corresponding to all sub-triples of the union is compatible.

The characteristic variety of system (1.2) is defined by a system of quadrics,

$$(c_i - c_j)u_k p_i p_j + (c_j - c_k)u_i p_j p_k + (c_k - c_i)u_j p_i p_k = 0,$$

which specify a rational normal curve in  $\mathbb{P}T^*M$  parametrised as  $p_i = u_i/(\lambda - c_i)$  (the ideal of a rational normal curve is generated by quadrics, see e.g. [25]). Explicitly,

$$\omega(\lambda) = p_i dx^i = \sum \frac{u_i}{\lambda - c_i} dx^i; \tag{1.3}$$

note that expression (1.3) takes form (1.1) on clearing the denominators (since only the conformal class of  $\omega(\lambda)$  is essential we will not make a distinction in what follows). This supplies  $M$  with a  $GL(2, \mathbb{R})$  geometry which depends on the solution  $u$  (otherwise said:  $GL(2, \mathbb{R})$  geometry on the solution  $u$  considered as a submanifold  $\text{graph}(u) \subset M \times \mathbb{R}$ ).

System (1.2) is equivalent to the commutativity conditions of the following vector fields ( $\lambda = \text{const}$ ),

$$\partial_{x^j} - \frac{\lambda - c_1}{\lambda - c_j} \frac{u_j}{u_1} \partial_{x^1}, \quad 1 < j \leq n.$$

Such  $\lambda$ -dependent vector fields are said to define a dispersionless Lax representation for system (1.2). Note that these vector fields are annihilated by  $\omega(\lambda)$ . Their integral manifolds supply  $M$  with a two-parameter family of  $\alpha$ -manifolds. Thus,  $GL(2, \mathbb{R})$  structure (1.3) is involutive. Equivalently, the commutativity of these vector fields can be interpreted as the involutivity of the corresponding corank 2 vector distribution on the  $(n + 1)$ -dimensional manifold  $\hat{M}$  with coordinates  $x^1, \dots, x^n, \lambda$ , known as the correspondence space. The (complexified) space of integral manifolds of this distribution plays important role in the twistor-theoretic approach to the Veronese web hierarchy.

In Section 2 we provide further examples of involutive  $GL(2, \mathbb{R})$  structures supported on solutions to other well-known dispersionless integrable hierarchies. As it was rightly pointed out by the referee of this paper, the involutivity of  $GL(2, \mathbb{R})$  structures defined by characteristic varieties of integrable hierarchies is a manifestation of a general phenomenon known as the ‘integrability of characteristics’ [19, 23, 43]. This is a feature of any compatible system, hence the integrability of a dispersionless equation is not characterized by the involutivity ( $\alpha$ -integrability) of the corresponding  $GL(2, \mathbb{R})$  structures. The integrability is recovered by the canonical connections that we discuss next.

### 1.3. Affine connections associated with involutive $GL(2, \mathbb{R})$ structures

There are several types of canonical connections defined on the tangent bundle of a manifold  $M$  that can be naturally associated with a  $GL(2, \mathbb{R})$  structure on  $M$ . Recall that an affine connection  $\nabla$  is said to be compatible with a  $GL(2, \mathbb{R})$  structure (paraconformal or  $GL(2, \mathbb{R})$  connection), if for every  $v \in TM$

$$\nabla_v \omega(\lambda) \in \text{span}\langle \omega(\lambda), \omega'(\lambda) \rangle, \tag{1.4}$$

where prime denotes differentiation by  $\lambda$ , see [33]. Condition (1.4) means that the parallel transport defined by  $\nabla$  preserves rational normal cones of the  $GL(2, \mathbb{R})$  structure. Equivalently, identifying quadratic equations from the ideal of the rational normal curve

$\omega(\lambda)$  with symmetric bivectors  $g_s$  on  $M$  and denoting  $g = \text{span}\langle g_s \rangle$ , we can represent (1.4) as  $\nabla_v g = g \forall v \neq 0$ .

Condition (1.4) alone does not specify  $\nabla$  uniquely: for this, additional constraints should be imposed. In what follows we discuss four types of canonical connections associated with involutive  $GL(2, \mathbb{R})$  structures, of which the first two are based on the previous works and do not exist universally, while the other two are new and exist for all dispersionless integrable hierarchies studied so far (let us stress that there exist no general theory or complete description of such hierarchies). We use the convention  $\nabla_j \partial_k = \Gamma_{jk}^i \partial_i$ .

*Torsion-free  $GL(2, \mathbb{R})$  connection in 4D*

Torsion-free  $GL(2, \mathbb{R})$  connections can only exist in four dimensions. Indeed, based on the Berger criteria, it was shown in [5] that there exist no non-trivial torsion-free  $GL(2, \mathbb{R})$  connections in higher dimensions. On the contrary, in four dimensions, involutivity of a  $GL(2, \mathbb{R})$  structure is equivalent to the existence of a torsion-free  $GL(2, \mathbb{R})$  connection.

Since  $GL(2, \mathbb{R})$  structures coming from principal symbols of dispersionless integrable hierarchies are automatically involutive (due to the existence of a Lax representation), we obtain an abundance of explicit examples of torsion-free  $GL(2, \mathbb{R})$  connections in four dimensions parametrised by solutions to well-known integrable PDEs, see Section 2.

For the Veronese web hierarchy, the Christoffel symbols of the torsion-free  $GL(2, \mathbb{R})$  connection associated with four-dimensional  $GL(2, \mathbb{R})$  structure (1.3) are computed to be equal to

$$\Gamma_{ii}^i = \frac{u_{ii}}{u_i} - \frac{1}{9} \sum_{j \neq i} \frac{(c_{ik}c_{jl} + c_{il}c_{jk})^2 u_{ij}}{c_{ik}c_{il}c_{jk}c_{jl} u_j}, \quad \Gamma_{ii}^j = \frac{1}{9} \frac{c_{jk}c_{jl} u_i}{c_{ij}c_{lk} u_j} \left( \frac{u_{ik}}{u_k} - \frac{u_{il}}{u_l} \right),$$

$$\Gamma_{ij}^i = \frac{1}{3} \frac{u_{ij}}{u_i} - \frac{1}{9} \left( 1 + \frac{c_{ik}c_{jl}}{c_{ij}c_{kl}} \right) \frac{u_{jl}}{u_l} - \frac{1}{9} \left( 1 + \frac{c_{il}c_{jk}}{c_{ij}c_{lk}} \right) \frac{u_{jk}}{u_k}, \quad \Gamma_{ik}^j = \frac{1}{9} \frac{c_{lj} u_k}{c_{lk} u_j} \left( \frac{u_{ik}}{u_k} - \frac{u_{il}}{u_l} \right),$$

here  $c_{ij} = c_i - c_j$ , and  $i, j, k, l$  are pairwise distinct indices taking values  $1, \dots, 4$ .

*Totally geodesic  $GL(2, \mathbb{R})$  connections*

A particularly interesting subclass of involutive  $GL(2, \mathbb{R})$  structures was introduced by Krynski in [33]: such structures possess a  $GL(2, \mathbb{R})$  connection (with torsion) and a two-parameter family of totally geodesic  $\alpha$ -manifolds. We will refer to such structures/connections as *totally geodesic  $GL(2, \mathbb{R})$  structures/connections*, respectively.

The requirement that  $\nabla$  is a totally geodesic  $GL(2, \mathbb{R})$  connection specifies it up to transformation  $\Gamma_{jk}^i \rightarrow \Gamma_{jk}^i + \phi_j \delta_k^i$  for a covector  $\phi$ . This freedom can be eliminated by requiring that the torsion  $T_\nabla$  is trace-free,  $T_{ik}^i = 0$ . In what follows this will be included into the totally geodesic condition. For  $GL(2, \mathbb{R})$  structures (1.3) coming from the Veronese web hierarchy, the condition  $\text{tr}(T_\nabla(\cdot, X)) = 0$  is equivalent to the constraint  $T_\nabla(\tilde{\omega}(\lambda), \tilde{\omega}'(\lambda)) \in \text{span}\langle \tilde{\omega}(\lambda) \rangle$  used in [33].

Examples of totally geodesic  $GL(2, \mathbb{R})$  structures include the following:

- (i) *four-dimensional  $GL(2, \mathbb{R})$  structures* arising, as characteristic varieties, on solutions to various integrable hierarchies (see Appendix B). We emphasise that, in

general, this is a merely 4-dimensional phenomenon. For instance, 5-dimensional  $GL(2, \mathbb{R})$  structures associated with the dKP hierarchy do not possess totally geodesic  $GL(2, \mathbb{R})$  connections;

- (ii) *multi-dimensional*  $GL(2, \mathbb{R})$  structures arising, as characteristic varieties, on solutions to *linearly degenerate* integrable hierarchies (those having no  $\partial_\lambda$  in the Lax fields, such as the Veronese web hierarchy and the ‘universal’ hierarchy). The two-parameter family of totally geodesic  $\alpha$ -manifolds is the projection of integral manifolds of the Lax distribution.

It was shown in [33] that totally geodesic  $GL(2, \mathbb{R})$  connections  $\nabla$  satisfy the following multi-dimensional *generalized Einstein-Weyl property*. Namely, the symmetrised Ricci tensor of such  $\nabla$  belongs to the span  $\tilde{g}$  of symmetric bivectors defining the dual rational normal curve  $\tilde{\omega}(\lambda)$ :  $Ric_{\nabla}^{sym} \in \tilde{g}$ . Note that in 3D this is precisely the classical Einstein-Weyl condition.

*Normal  $GL(2, \mathbb{R})$  connections*

We call a  $GL(2, \mathbb{R})$  connection  $\nabla$  *normal* if its torsion satisfies the following properties:

- (i)  $T_{\nabla}$  is trace-free:  $\text{tr}(T_{\nabla}(\cdot, X)) = 0 \quad \forall X$ ;
- (ii)  $T_{\nabla}$  preserves  $\alpha$ -hyperplanes as a (2,1)-map:  $X, Y \in \omega(\lambda)^{\perp} \Rightarrow T_{\nabla}(X, Y) \in \omega(\lambda)^{\perp}$ .

Every totally geodesic  $GL(2, \mathbb{R})$  connection is necessarily normal, although the converse is not true in general. It turns out that for all hierarchies we investigated, the normal  $GL(2, \mathbb{R})$  connection exists, and is unique (we point out that there are no totally geodesic connections associated with higher-dimensional  $GL(2, \mathbb{R})$  structures coming from the dKP and the Adler-Shabat hierarchies, starting from dimension 5). The importance of normal  $GL(2, \mathbb{R})$  connections lies in the fact that every such  $\nabla$  satisfies the generalized Einstein-Weyl property.

The totally geodesic (and thus normal)  $GL(2, \mathbb{R})$  connection associated with  $GL(2, \mathbb{R})$  structure (1.3) of the Veronese web hierarchy is given by the formula

$$\nabla_j \partial_k = \left( \frac{u_{jk}}{u_k} + \phi_j \right) \partial_k, \quad \text{or} \quad \Gamma_{jk}^i = \left( \frac{u_{jk}}{u_k} + \phi_j \right) \delta_k^i;$$

here the covector  $\phi_j$  is still arbitrary [33]. It can be fixed uniquely by requiring the torsion to be trace-free:

$$\phi_j = -\frac{1}{n-1} \sum_{k \neq j} \frac{u_{jk}}{u_k}.$$

*A canonical projective connection*

There exists yet another class of affine connections associated with involutive  $GL(2, \mathbb{R})$  structures, namely, torsion-free connections possessing a two-parameter family of totally geodesic  $\alpha$ -manifolds; note that they do not preserve the  $GL(2, \mathbb{R})$  structure in general.

For  $GL(2, \mathbb{R})$  structures defined by the characteristic varieties of dispersionless hierarchies, the two-parameter family of totally geodesic  $\alpha$ -manifolds come from projections of integral manifolds of the corresponding dispersionless Lax equations.

The requirement that  $\nabla$  is a torsion-free connection with a two-parameter family of totally geodesic  $\alpha$ -manifolds specifies it uniquely up to projective equivalence,  $\Gamma_{jk}^i \rightarrow$



$\Gamma_{jk}^i + \phi_j \delta_k^i + \phi_k \delta_j^i$ , for a 1-form  $\phi$ . Thus, we obtain a canonically defined totally geodesic projective connection.

For the involutive  $GL(2, \mathbb{R})$  structure (1.3) of the Veronese web hierarchy, an affine representative of this projective connection is computed to be equal to

$$\nabla_j \partial_k = \frac{u_{jk}}{2} \left( \frac{\partial_j}{u_j} + \frac{\partial_k}{u_k} \right).$$

On every solution, geodesics of this projective connection (considered as unparametrised curves) can be obtained by intersecting  $n - 2$  generic totally geodesic  $\alpha$ -manifolds.

#### 1.4. Summary of the main results

In Section 2 we provide further explicit examples of involutive  $GL(2, \mathbb{R})$  structures given by characteristic varieties of various dispersionless integrable hierarchies, namely the dKP hierarchy, the ‘universal’ hierarchy of Martinez–Alonso and Shabat, and the consistent Adler–Shabat triples. In each case we calculate Christoffel’s symbols of the canonical connections discussed in Section 1.3 (these results are relegated to Appendix B).

Section 3 contains the main results of the paper. In Theorem 1 we demonstrate that the general involutive  $GL(2, \mathbb{R})$  structure can be brought to the normal form

$$\omega(\lambda) = \sum_{i=1}^n \frac{u_i}{\lambda - \frac{u_i}{v_i}} dx^i, \tag{1.5}$$

which can be reduced to (1.1) by clearing denominators. Here the functions  $u$  and  $v$  satisfy a system of second-order PDEs, 2 equations for each quadruple of indices  $1 \leq i < j < k < l \leq n$ :

$$\mathfrak{S}_{(jkl)} (a_i - a_j)(a_k - a_l) \left( \frac{2u_{ij} - (a_i + a_j)v_{ij}}{u_i u_j} + \frac{2u_{kl} - (a_k + a_l)v_{kl}}{u_k u_l} \right) = 0, \tag{1.6}$$

$$\mathfrak{S}_{(jkl)} (b_i - b_j)(b_k - b_l) \left( \frac{2v_{ij} - (b_i + b_j)u_{ij}}{v_i v_j} + \frac{2v_{kl} - (b_k + b_l)u_{kl}}{v_k v_l} \right) = 0, \tag{1.7}$$

where  $a_i = u_i/v_i$ ,  $b_i = v_i/u_i$ , and  $\mathfrak{S}$  denotes cyclic summation over the indicated indices.

In Theorem 2 we prove that overdetermined system (1.6), (1.7) is in involution, and its characteristic variety is the tangential variety of the rational normal curve  $\omega(\lambda)$  given by (1.5). Since the degree of the tangential variety equals  $2n - 4$ , we conclude that general involutive  $GL(2, \mathbb{R})$  structures depend (modulo diffeomorphisms) on  $2n - 4$  arbitrary functions of 3 variables. For  $n = 4$  this reproduces the count in [5]; we also refer to [34] for an alternative PDE system governing involutive  $GL(2, \mathbb{R})$  structures for  $n = 4$ . For general  $n$ , the functional freedom of  $2n - 4$  arbitrary functions of 3 variables was announced by Robert Bryant in a series of talks in the early 2000s [7] (we thank him for sending us the slides), but no proofs have appeared. Our proof is based on the formal theory of PDEs developed in recent years.

Finally, in Theorem 3 we show that equations (1.6), (1.7) governing general involutive  $GL(2, \mathbb{R})$  structures constitute a dispersionless integrable hierarchy with Lax representation in parameter-dependent vector fields.

It was shown in [33] that involutive  $GL(2, \mathbb{R})$  structures are in one-to-one correspondence with ODEs having vanishing Wünschmann invariants. Thus, integrability of system (1.6), (1.7) implies integrability of the Wünschmann conditions.

Our considerations are local. All results on the functional freedom in the general solution referring to the Cartan–Kähler theorem hold in the analytic or formal categories.

### 2. Examples of involutive $GL(2, \mathbb{R})$ structures

In this section we give further examples of involutive  $GL(2, \mathbb{R})$  structures arising on solutions of various dispersionless integrable hierarchies. Our main observation is that  $GL(2, \mathbb{R})$  structures discussed in a similar context by Dunajski and Krynski in [13, 33] are nothing but characteristic varieties of the corresponding PDEs. This makes the construction entirely explicit and intrinsic.

We mainly focus on  $GL(2, \mathbb{R})$  geometry in four dimensions, defined by the first three equations of the corresponding hierarchies. Higher-dimensional generalisations are then obtained by adding higher flows (with higher time variables). Christoffel’s symbols of the canonical connections associated with these examples are presented in Appendix B.

#### 2.1. $GL(2, \mathbb{R})$ structures via dKP hierarchy

The first three equations of the dKP hierarchy have the form

$$\begin{aligned} u_{xt} - u_{yy} - u_x u_{xx} &= 0, \\ u_{xz} - u_{yt} - u_x u_{xy} - u_y u_{xx} &= 0, \\ u_{yz} - u_{tt} + u_x^2 u_{xx} - u_y u_{xy} &= 0. \end{aligned} \tag{2.1}$$

Here  $u$  is a function on the 4-dimensional manifold  $M$  with local coordinates  $x, y, t, z$ . The characteristic variety of this system is the intersection of three quadrics,

$$\begin{aligned} p_x p_t - p_y^2 - u_x p_x^2 &= 0, \\ p_x p_z - p_y p_t - u_x p_x p_y - u_y p_x^2 &= 0, \\ p_y p_z - p_t^2 + u_x^2 p_x^2 - u_y p_x p_y &= 0, \end{aligned} \tag{2.2}$$

which specify a rational normal curve (twisted cubic) in  $\mathbb{P}T^*M$  parametrised as

$$p_x = 1, \quad p_y = \lambda, \quad p_t = \lambda^2 + u_x, \quad p_z = \lambda^3 + 2u_x \lambda + u_y,$$

so that

$$\omega(\lambda) = dx + \lambda dy + (\lambda^2 + u_x) dt + (\lambda^3 + 2u_x \lambda + u_y) dz.$$

This supplies  $M$  with a  $GL(2, \mathbb{R})$  geometry which depends on the solution  $u$ . The occurrence of a rational normal curve in the theory of dKP hierarchy was also noted in [28] in the context of coisotropic deformations of algebraic curves. Equations (2.1) are equivalent to the commutativity conditions of the following vector fields,

$$\begin{aligned} \partial_y - \lambda \partial_x + u_{xx} \partial_\lambda, \\ \partial_t - (\lambda^2 + u_x) \partial_x + (\lambda u_{xx} + u_{xy}) \partial_\lambda, \\ \partial_z - (\lambda^3 + 2u_x \lambda + u_y) \partial_x + (\lambda^2 u_{xx} + \lambda u_{xy} + u_{xt} + u_x u_{xx}) \partial_\lambda, \end{aligned} \tag{2.3}$$

which constitute a dispersionless Lax representation. These vector fields live in the extended 5-dimensional space  $\hat{M}$  with coordinates  $x, y, t, z, \lambda$ ; note the explicit presence of  $\partial_\lambda$ .



Projecting integral manifolds of these vector fields from  $\hat{M}$  to  $M$  we obtain a two-parameter family of  $\alpha$ -manifolds of the corresponding  $GL(2, \mathbb{R})$  structure, thus establishing its involutivity.

Higher-dimensional generalisation of this construction can be obtained by taking higher flows of the dKP hierarchy,

$$u_{i,j+1} - u_{j,i+1} + \sum_{k=1}^i u_{i-k} u_{jk} - \sum_{k=1}^j u_{j-k} u_{ik} = 0, \quad 1 \leq i < j,$$

see e.g. [27]. For  $(i, j) = (1, 2), (1, 3)$  and  $(2, 3)$  this reproduces equations (2.1). Here we use the notation  $u = u(x^1, x^2, x^3, x^4, \dots)$  where  $x^1 = x, x^2 = y, x^3 = t, x^4 = z$ , etc, and subscripts of  $u$  denote partial derivatives. The corresponding characteristic variety is the intersection of quadrics,

$$p_i p_{j+1} - p_j p_{i+1} + \sum_{k=1}^i u_{i-k} p_j p_k - \sum_{k=1}^j u_{j-k} p_i p_k = 0.$$

It defines a rational normal curve; setting  $p_1 = 1$  we can parametrise it recurrently as

$$p_{i+1} = \lambda p_i + \sum_{k=1}^{i-1} u_{i-k} p_k, \quad i \geq 1.$$

Explicitly, this gives

$$p_1 = 1, \quad p_2 = \lambda, \quad p_3 = \lambda^2 + u_1, \quad p_4 = \lambda^3 + 2u_1 \lambda + u_2, \quad p_5 = \lambda^4 + 3u_1 \lambda^2 + 2u_2 \lambda + u_3 + u_1^2,$$

etc. The dispersionless Lax representation of the dKP hierarchy is given by a family of involutive parameter-dependent vector fields

$$X_i = \partial_{x^{i+1}} - \lambda \partial_{x^i} - \sum_{k=1}^{i-1} u_{i-k} \partial_{x^k} + u_{1i} \partial_\lambda, \quad i \geq 1.$$

### 2.2. $GL(2, \mathbb{R})$ structures via the universal hierarchy

The first three equations of the universal hierarchy of Martínez–Alonso and Shabat [38] have the form

$$\begin{aligned} u_{xt} - u_{yy} + u_y u_{xx} - u_x u_{xy} &= 0, \\ u_{xz} - u_{yt} + u_t u_{xx} - u_x u_{xt} &= 0, \\ u_{yz} - u_{tt} + u_t u_{xy} - u_y u_{xt} &= 0. \end{aligned} \tag{2.4}$$

Here  $u$  is a function on the 4-dimensional manifold  $M$  with local coordinates  $x, y, t, z$ . The characteristic variety of this system is the intersection of three quadrics,

$$\begin{aligned} p_x p_t - p_y^2 + u_y p_x^2 - u_x p_x p_y &= 0, \\ p_x p_z - p_y p_t + u_t p_x^2 - u_x p_x p_t &= 0, \\ p_y p_z - p_t^2 + u_t p_x p_y - u_y p_x p_t &= 0, \end{aligned}$$

which specify a rational normal curve in  $\mathbb{P}T^*M$  parametrised as

$$p_x = 1, \quad p_y = \lambda - u_x, \quad p_t = \lambda^2 - u_x \lambda - u_y, \quad p_z = \lambda^3 - u_x \lambda^2 - u_y \lambda - u_t,$$

so that

$$\omega(\lambda) = dx + (\lambda - u_x)dy + (\lambda^2 - u_x\lambda - u_y)dt + (\lambda^3 - u_x\lambda^2 - u_y\lambda - u_t)dz.$$

Equations (2.4) are equivalent to the commutativity conditions of the following vector fields,

$$\begin{aligned} &\partial_y - (\lambda - u_x)\partial_x, \\ &\partial_t - (\lambda^2 - u_x\lambda - u_y)\partial_x, \\ &\partial_z - (\lambda^3 - u_x\lambda^2 - u_y\lambda - u_t)\partial_x, \end{aligned}$$

which constitute a dispersionless Lax representation. Note the absence of  $\partial_\lambda$ , which indicates a close similarity with the Veronese web hierarchy. Integral manifolds of these vector fields provide a two-parameter family of  $\alpha$ -manifolds of the corresponding  $GL(2, \mathbb{R})$  structure.

This has a straightforward higher-dimensional generalisation: the equations are

$$u_{i,j+1} - u_{i+1,j} + u_j u_{1,i} - u_i u_{1,j} = 0, \quad 0 < i < j < n;$$

the  $GL(2, \mathbb{R})$  structure is given by

$$\omega(\lambda) = \sum_{i=1}^n (\lambda^{i-1} - u_1\lambda^{i-2} - \dots - u_{i-1}) dx^i;$$

the Lax representation is

$$X_i = \partial_{x^i} - (\lambda^{i-1} - u_1\lambda^{i-2} - \dots - u_{i-1})\partial_{x^1}, \quad 1 < i \leq n.$$

Considered altogether, these equations form an integrable hierarchy.

### 2.3. $GL(2, \mathbb{R})$ structures via Adler–Shabat triples

Further examples of  $GL(2, \mathbb{R})$  structures arise as characteristic varieties on solutions to triples of consistent 3D second-order PDEs discussed by Adler and Shabat in [1],

$$\begin{aligned} u_{23} &= f(u_1, u_2, u_3, u_{12}, u_{13}), \\ u_{24} &= g(u_1, u_2, u_4, u_{12}, u_{14}), \\ u_{34} &= h(u_1, u_3, u_4, u_{13}, u_{14}), \end{aligned} \tag{2.5}$$

where  $u$  is a function on the 4-dimensional manifold  $M$  with local coordinates  $x^1, \dots, x^4$ . Note that system (1.2) belongs to class (2.5). As yet another example of this type let us consider the system

$$u_{23} = \frac{u_{12} - u_{13}}{u_2 - u_3}, \quad u_{24} = \frac{u_{12} - u_{14}}{u_2 - u_4}, \quad u_{34} = \frac{u_{13} - u_{14}}{u_3 - u_4}. \tag{2.6}$$

Its characteristic variety is defined by a system of quadrics,

$$p_2 p_3 = \frac{p_1 p_2 - p_1 p_3}{u_2 - u_3}, \quad p_2 p_4 = \frac{p_1 p_2 - p_1 p_4}{u_2 - u_4}, \quad p_3 p_4 = \frac{p_1 p_3 - p_1 p_4}{u_3 - u_4},$$

which specify a rational normal curve in  $\mathbb{P}T^*M$  parametrised as  $p_1 = 1, p_i = 1/(\lambda - u_i)$ , so that

$$\omega(\lambda) = dx^1 + \frac{1}{\lambda - u_2} dx^2 + \frac{1}{\lambda - u_3} dx^3 + \frac{1}{\lambda - u_4} dx^4.$$

System (2.6) is equivalent to the conditions of commutativity of the following vector fields,

$$\partial_{x^2} + \frac{1}{u_2 - \lambda} \partial_{x^1} + \frac{u_{12}}{u_2 - \lambda} \partial_\lambda, \quad \partial_{x^3} + \frac{1}{u_3 - \lambda} \partial_{x^1} + \frac{u_{13}}{u_3 - \lambda} \partial_\lambda, \quad \partial_{x^4} + \frac{1}{u_4 - \lambda} \partial_{x^1} + \frac{u_{14}}{u_4 - \lambda} \partial_\lambda,$$

note the explicit presence of  $\partial_\lambda$ . Projecting their integral manifolds from the extended space  $\hat{M}$  to  $M$  we obtain a two-parameter family of  $\alpha$ -manifolds of the corresponding  $GL(2, \mathbb{R})$  structure.

This has a straightforward higher-dimensional generalization: the equations are

$$(u_i - u_j)u_{ij} = u_{1i} - u_{1j}, \quad 1 < i < j \leq n;$$

the  $GL(2, \mathbb{R})$  structure is given by

$$\omega(\lambda) = dx^1 + \sum_{i=2}^n \frac{1}{\lambda - u_i} dx^i;$$

the Lax representation is

$$X_i = \partial_{x^i} - \frac{1}{\lambda - u_i} \partial_{x^1} - \frac{u_{1i}}{\lambda - u_i} \partial_\lambda, \quad 1 < i \leq n.$$

Considered altogether, these equations form an integrable hierarchy.

### 3. General involutive $GL(2, \mathbb{R})$ structures

In this section we demonstrate that general involutive  $GL(2, \mathbb{R})$  structures are governed by a dispersionless integrable hierarchy and derive the corresponding Lax system describing  $\alpha$ -manifolds.

#### 3.1. Parametrisation of involutive $GL(2, \mathbb{R})$ structures

We begin by encoding all involutive structures in a simple ansatz.

**THEOREM 1.** *Every involutive  $GL(2, \mathbb{R})$  structure can be locally represented by formula (1.5), which upon clearing the denominators takes the form*

$$\omega(\lambda) = \sum_{i=1}^n \left[ \prod_{j \neq i} \left( \lambda - \frac{u_j}{v_j} \right) \right] u_i dx^i. \tag{3.1}$$

Here  $u$  and  $v$  are functions of  $(x^1, \dots, x^n)$  and subscripts denote partial derivatives:  $u_i = u_{x^i}$ ,  $v_i = v_{x^i}$ . The functions  $u$  and  $v$  must satisfy a system of PDEs (1.6), (1.7) coming from the integrability condition  $d\omega(\lambda) \wedge \omega(\lambda) = 0$ .

*Proof.* Let (1.1) be an involutive  $GL(2, \mathbb{R})$  structure on  $n$ -dimensional manifold  $M$ . It is easy to see that the space of  $\alpha$ -manifolds is at least 2-dimensional (in fact, it is parametrised by 1 arbitrary function of 1 variable, see Section 3.3). Choosing a 1-parameter family of  $\alpha$ -manifolds we obtain a (local) foliation of  $M$ . This foliation consists of integral manifolds of an integrable distribution  $\omega(a) = 0$  obtained by substituting  $\lambda$  with some function  $a$  on  $M$ . We can thus set  $\omega(a) = f dx$  for some functions  $f$  and  $x$ . Let us now choose  $n$  different 1-parameter families of  $\alpha$ -manifolds that correspond to the choice of  $n$  functions  $a_i$  such that  $\omega(a_i) = f_i dx^i$  (no summation). We will use  $x^i$  as a local coordinate system on  $M^n$ . Note that although one can always set, say,  $f_1 = 1$  by using conformal freedom in  $\omega$ , it is not

always possible to eliminate all  $f_i$  simultaneously. Taking into account that  $\omega$  is polynomial (of degree  $n - 1$ ) in  $\lambda$ , the above conditions fix  $\omega$  uniquely:

$$\omega(\lambda) = \sum_{i=1}^n \left[ \prod_{j \neq i} \frac{\lambda - a_j}{a_i - a_j} \right] f_i dx^i.$$

Let us choose two extra 1-parameter families of  $\alpha$ -manifolds such that  $\omega(a_{n+1}) = f_{n+1} du$  and  $\omega(a_{n+2}) = f_{n+2} dv$  (here  $u, v$  are precisely the functions that will appear later in formula (1.5)). Explicitly, this gives

$$f_i \prod_{j \neq i} \frac{a_{n+1} - a_j}{a_i - a_j} = f_{n+1} u_i, \quad f_i \prod_{j \neq i} \frac{a_{n+2} - a_j}{a_i - a_j} = f_{n+2} v_i. \tag{3.2}$$

The first of these relations allows one to rewrite  $\omega$  as

$$\omega(\lambda) = f_{n+1} \sum_{i=1}^n \left[ \prod_{j \neq i} \frac{\lambda - a_j}{a_{n+1} - a_j} \right] u_i dx^i. \tag{3.3}$$

Taking the ratio of relations (3.2) we obtain

$$\prod_{j \neq i} \frac{a_{n+1} - a_j}{a_{n+2} - a_j} = \frac{f_{n+1} u_i}{f_{n+2} v_i},$$

which is equivalent to

$$\frac{a_{n+2} - a_i}{a_{n+1} - a_i} = s \frac{u_i}{v_i}, \quad s = \frac{f_{n+1}}{f_{n+2}} \prod_{k=1}^n \frac{a_{n+2} - a_k}{a_{n+1} - a_k}.$$

Solving the last relation for  $a_i$  and substituting the result into (3.3) yields

$$\omega(\lambda) = f_{n+1} \sum_{i=1}^n \left[ \prod_{j \neq i} \frac{\lambda - a_{n+2} - s(\lambda - a_{n+1}) \frac{u_j}{v_j}}{a_{n+1} - a_{n+2}} \right] u_i dx^i.$$

Using the linear-fractional freedom in  $\lambda$  (sending  $a_{n+1}$  and  $a_{n+2}$  to  $\infty$  and 0, respectively), as well as the conformal freedom in  $\omega$ , we can reduce the last expression to form (3.1).

Calculating the integrability condition  $d\omega(\lambda) \wedge \omega(\lambda) = 0$  (it is more convenient to use (1.5) for this purpose) and collecting coefficients at  $dx^i \wedge dx^j \wedge dx^k$  we obtain

$$\frac{\lambda - a_i}{u_i} \left( \frac{1}{\lambda - a_k} - \frac{1}{\lambda - a_j} \right) \lambda_i + \frac{\lambda - a_j}{u_j} \left( \frac{1}{\lambda - a_i} - \frac{1}{\lambda - a_k} \right) \lambda_j + \frac{\lambda - a_k}{u_k} \left( \frac{1}{\lambda - a_j} - \frac{1}{\lambda - a_i} \right) \lambda_k + S_{ijk} = 0. \tag{3.4}$$

Here  $\lambda_i = \lambda_{,x^i}$  ( $\lambda$  is viewed as a function of  $x$ ), and

$$S_{ijk} = u_{ij} \frac{a_j - a_i}{u_i u_j} \left( \frac{\lambda}{\lambda - a_i} + \frac{\lambda}{\lambda - a_j} \right) + u_{ik} \frac{a_i - a_k}{u_i u_k} \left( \frac{\lambda}{\lambda - a_i} + \frac{\lambda}{\lambda - a_k} \right) + u_{jk} \frac{a_k - a_j}{u_j u_k} \left( \frac{\lambda}{\lambda - a_j} + \frac{\lambda}{\lambda - a_k} \right) - v_{ij} \frac{a_j - a_i}{u_i u_j} \left( \frac{\lambda a_i}{\lambda - a_i} + \frac{\lambda a_j}{\lambda - a_j} \right) - v_{ik} \frac{a_i - a_k}{u_i u_k} \left( \frac{\lambda a_i}{\lambda - a_i} + \frac{\lambda a_k}{\lambda - a_k} \right) - v_{jk} \frac{a_k - a_j}{u_j u_k} \left( \frac{\lambda a_j}{\lambda - a_j} + \frac{\lambda a_k}{\lambda - a_k} \right).$$

System (1.6), (1.7) governing general involutive  $GL(2, \mathbb{R})$  structures results on elimination of the derivatives of  $\lambda$  from equations (3.4). This can be done as follows. Let us denote  $T_{ijk}$  the left-hand side of (3.4). Taking 4 distinct indices  $i \neq j \neq k \neq l$  one can verify that there are only two non-trivial linear combinations, namely

$$T_{ikj} + T_{ijl} + T_{ilk} + T_{jkl}$$

and

$$\frac{1}{\lambda - a_l} T_{ikj} + \frac{1}{\lambda - a_k} T_{ijl} + \frac{1}{\lambda - a_j} T_{ilk} + \frac{1}{\lambda - a_i} T_{jkl},$$

that do not contain derivatives of  $\lambda$ . The first linear combination is equal to zero identically, while the second combination vanishes (identically in  $\lambda$ ) if and only if relations (1.6) and (1.7) are satisfied, namely the following expression must vanish:

$$E_{ijkl} = \mathfrak{S}_{(jkl)} (a_i - a_j)(a_k - a_l) \left( \frac{2u_{ij} - (a_i + a_j)v_{ij}}{u_i u_j} + \frac{2u_{kl} - (a_k + a_l)v_{kl}}{u_k u_l} \right),$$

as well as similar expressions obtained by interchanging  $u$  and  $v$ ,

$$F_{ijkl} = \mathfrak{S}_{(jkl)} (b_i - b_j)(b_k - b_l) \left( \frac{2v_{ij} - (b_i + b_j)u_{ij}}{v_i v_j} + \frac{2v_{kl} - (b_k + b_l)u_{kl}}{v_k v_l} \right),$$

recall that  $a_i = u_i/v_i$ ,  $b_i = v_i/u_i$ .

Although system (1.6), (1.7) formally consists of  $2\binom{n}{4}$  equations, only  $2\binom{n-2}{2}$  of them are linearly independent. Indeed, we can restrict to equations  $E_{12kl} = 0$  and  $F_{12kl} = 0$  for  $3 \leq k < l \leq n$  since all other equations are their linear combinations: denoting  $\alpha_{ij} = a_i - a_j$  we have

$$\alpha_{12} E_{ijkl} = \alpha_{kl} E_{12ij} + \alpha_{jl} E_{12ki} + \alpha_{jk} E_{12il} + \alpha_{il} E_{12jk} + \alpha_{ik} E_{12lj} + \alpha_{ij} E_{12kl} \tag{3.5}$$

for all indices distinct (note that  $\alpha_{ij} \neq 0$  for  $i \neq j$ ), and similarly for  $F_{ijkl}$ .

For  $n = 4$  system (1.6), (1.7) is determined: it consists of 2 second-order PDEs for 2 functions  $u$  and  $v$  of 4 independent variables, so its general solution is parametrised by 4 arbitrary functions of 3 variables. This gives an explicit confirmation of the result of [5] that modulo diffeomorphisms general involutive  $GL(2, \mathbb{R})$  structures in four dimensions depend on 4 functions of 3 variables. The case of general  $n$  is more complicated because system (1.6), (1.7) becomes overdetermined.

**THEOREM 2.** *For every value of  $n$ , the following holds:*

- (a) *the characteristic variety of system (1.6), (1.7) is the tangential variety of rational normal curve (1.5); it has degree  $2n - 4$ . Rational normal curve (1.5) can be recovered as the singular locus of the characteristic variety;*
- (b) *system (1.6), (1.7) is in involution;*
- (c) *the general solution of system (1.6), (1.7) depends on  $2n - 4$  functions of 3 variables (in the analytic or formal categories).*

*Proof.* (a) Let us parametrise rational normal curve (1.5) as

$$\lambda \longrightarrow [p_1 : \dots : p_n] \in \mathbb{P}T^*M, \quad p_i = \frac{u_i}{\lambda - a_i}, \quad a_i = \frac{u_i}{v_i}, \tag{3.6}$$

so that its tangential variety is given by

$$(\lambda, \mu) \longrightarrow [p_1 : \dots : p_n] \in \mathbb{P}T^*M, \quad p_i = \frac{u_i}{\lambda - a_i} + \frac{u_i \mu}{(\lambda - a_i)^2}. \tag{3.7}$$

Let  $E = E[u, v]$  and  $F = F[u, v]$  be non-linear differential operators on the left-hand sides of (1.6) and (1.7). The symbol of the system  $\mathcal{E} = \{E = 0, F = 0\}$  is given by the matrix

$$\ell_{\mathcal{E}}(p) = \begin{bmatrix} \ell_E^u(p) & \ell_E^v(p) \\ \ell_F^u(p) & \ell_F^v(p) \end{bmatrix}, \tag{3.8}$$

where  $\ell_E^u(p) = \sum_{a \leq b} (\partial E / \partial u_{ab}) p_a p_b$  is the symbol of  $u$ -linearisation of  $E$ , etc. As noted after Theorem 1,  $E = (E_{ijkl})$  has  $\binom{n-2}{2}$  independent components, and similarly for  $F = (F_{ijkl})$ , so that the matrix  $\ell_{\mathcal{E}}$  is of the size  $2\binom{n-2}{2} \times 2$ . The characteristic variety is defined by the formula

$$\text{Char}(\mathcal{E}) = \{[p] \in \mathbb{P}T^*M : \text{rank}(\ell_{\mathcal{E}}(p)) < 2\}.$$

From (1.6) we have

$$\ell_{E_{ijkl}}^u(p) = 2 \sum_{(jkl)} (a_i - a_j)(a_k - a_l) \left( \frac{p_i p_j}{u_i u_j} + \frac{p_k p_l}{u_k u_l} \right).$$

This expression vanishes if we substitute  $p$  from (3.6). Similarly, all other components  $\ell_{E_{ijkl}}^v(p), \ell_{F_{ijkl}}^u(p), \ell_{F_{ijkl}}^v(p)$  of the symbolic matrix vanish, and we conclude that  $\ell_{\mathcal{E}}(p) = 0$  modulo (3.6).

For the tangential variety (3.7), the entries of  $\ell_{\mathcal{E}}(p)$  do not vanish identically, however, a straightforward computation shows that independently of  $(ijkl)$  we get

$$\lambda \ell_{E_{ijkl}}^u(p) + \ell_{E_{ijkl}}^v(p) = 0 \quad \text{and} \quad \lambda \ell_{F_{ijkl}}^u(p) + \ell_{F_{ijkl}}^v(p) = 0,$$

and these identities characterise (3.7). Thus, all columns of  $\ell_{\mathcal{E}}(p)$  are proportional whenever  $p$  satisfies (3.7), and  $\text{rank}(\ell_{\mathcal{E}}(p)) = 1$  unless  $p$  belongs to the rational normal curve (in which case we have  $\text{rank}(\ell_{\mathcal{E}}(p)) = 0$ ). Finally, for a rational normal curve of degree  $n - 1$ , the degree of its tangential variety equals  $2n - 4$ . This variety is known to be generated by quartics [25].

(b) System  $\mathcal{E} = \{E_{12kl} = 0, F_{12kl} = 0\}$  given by (1.6), (1.7) is involutive iff its compatibility conditions are identically satisfied modulo  $\mathcal{E}$ . A long computation, which we present in Appendix A, shows that these conditions are numerated by 5-tuples of distinct indices  $(12ijk)$  where  $2 < i < j < k \leq n$ . More precisely, the compatibility conditions corresponding to any such 5-tuple are first-order differential operators applied to  $E_{12ij}, E_{12jk}, E_{12ki}$  and  $F_{12ij}, F_{12jk}, F_{12ki}$ , and involving only differentiations by variables  $x^1, x^2, x^i, x^j, x^k$ . There are four compatibility conditions for each 5-tuple  $(12ijk)$ .

Thus it suffices to check compatibility for  $n = 5$  to conclude it for general  $n$ . For  $n = 5$  the resolution from Appendix A becomes a short exact sequence  $\mathcal{R}^2 \xrightarrow{\ell_{\mathcal{E}}} \mathcal{R}^6 \xrightarrow{C_{\mathcal{E}}} \mathcal{R}^4$ , where  $\mathcal{R} = \mathbb{R}[p_1, \dots, p_n]$  is the algebra of homogeneous polynomials on  $T^*M$  and  $C_{\mathcal{E}}$  is the compatibility operator. From this we read off the 4 compatibility conditions. A direct verification (using symbolic computations in Maple) shows that they are satisfied. This implies the involutivity.

(c) By a classical result going back to Cartan the general local solution of an involutive PDE system  $\mathcal{E}$  depends on  $d$  arbitrary functions of  $m$  variables where the numbers  $d$  (formal rank) and  $m$  (formal dimension) can be read off the Cartan characters characterising involutivity. The result is formal, but it also holds in the analytic category due to the Cartan–Kähler theorem. Serre reformulated this criterion in homological terms, relating the numbers  $d, m$



to the Hilbert function of the symbolic module. Since the characteristic variety is the support of this module, these numbers can be read off the geometry of this variety and the sheaf  $\ker(\ell_{\mathcal{E}})$  over  $\text{Char}(\mathcal{E})$ .

We refer to [8, chapter 5] and [29] for a modern exposition of these results. In the case when  $\text{Char}(\mathcal{E})$  is irreducible the number  $m$  is the affine dimension of this variety, while  $d$  is its degree multiplied by the rank of the sheaf  $\ker(\ell_{\mathcal{E}}(p))$  at generic point  $p \in \text{Char}(\mathcal{E})$ . Since system (1.6), (1.7) is in involution and its characteristic variety has affine dimension  $m = 3$ , degree  $d = 2n - 4$  and the kernel sheaf of dimension  $2 - \text{rank}(\ell_{\mathcal{E}}(p)) = 1$  at any point  $p \in \text{Char}(\mathcal{E})$  that belongs to (3.7) with  $\mu \neq 0$ , the general solution depends on  $2n - 4$  arbitrary functions of 3 variables.

*Remark 1.* The system  $\mathcal{E}$  can be represented in a simple parametric form ( $1 \leq i < j \leq n$ )

$$\frac{2u_{ij} - (a_i + a_j)v_{ij}}{u_i u_j} = r_i + r_j + \sum_{k=3}^{n-1} l_k \frac{a_i^k - a_j^k}{a_i - a_j}, \quad \frac{2v_{ij} - (b_i + b_j)u_{ij}}{v_i v_j} = s_i + s_j + \sum_{k=3}^{n-1} m_k \frac{b_i^k - b_j^k}{b_i - b_j},$$

This system has  $n(n - 1)$  equations and  $4n - 6$  parameters  $r_1, \dots, r_n, s_1, \dots, s_n, l_3, \dots, l_{n-1}, m_3, \dots, m_{n-1}$ . Elimination of these parameters yields  $(n - 2)(n - 3)$  equations (1.6), (1.7).

### 3.2. Integrability of involutive $GL(2, \mathbb{R})$ structures

**THEOREM 3.** *For every  $n$ , system (1.6), (1.7) is integrable via a dispersionless Lax representation in parameter-dependent vector fields. Letting  $n \rightarrow \infty$  we obtain the corresponding dispersionless integrable hierarchy.*

*Proof.* Let us associate with equations (3.4) the following family of  $\lambda$ -dependent vector fields,

$$V_{ijk} = \frac{\lambda - a_i}{u_i} \left( \frac{1}{\lambda - a_k} - \frac{1}{\lambda - a_j} \right) \partial_{x^i} + \frac{\lambda - a_j}{u_j} \left( \frac{1}{\lambda - a_i} - \frac{1}{\lambda - a_k} \right) \partial_{x^j} + \frac{\lambda - a_k}{u_k} \left( \frac{1}{\lambda - a_j} - \frac{1}{\lambda - a_i} \right) \partial_{x^k} - S_{ijk} \partial_{\lambda},$$

which live in the extended space  $\hat{M}$  with coordinates  $x^1, \dots, x^n, \lambda$ . These vector fields generate a distribution  $V = \text{span}\langle V_{ijk} \rangle$  in  $T\hat{M}$  of dimension  $n - 2$ . Indeed, the identities noted in the proof of Theorem 1 for  $T_{ijk}$  hold for  $V_{ijk}$ , so these latter vector fields are expressed as linear combinations of  $V_{12l}$  for  $3 \leq l \leq n$ . This, in particular, implies that modulo (1.6), (1.7) there are only  $n - 2$  linearly independent relations (3.4).

The geometry behind system (3.4) and the distribution  $V$  is as follows. Consider a hypersurface  $H$  in  $\hat{M}$  defined explicitly as  $\lambda = \lambda(x^1, \dots, x^n)$ . Then the distribution  $V$  is tangential to  $H$  if and only if the function  $\lambda(x^1, \dots, x^n)$  solves system (3.4). Thus system (3.4) is compatible if and only if the associated distribution  $V$  is involutive. In this case the general solution of system (3.4) depends on 1 arbitrary function of 2 variables: there exists a 3-parametric family of integral manifolds of  $V$ , and a generic hypersurface  $H \subset \hat{M}$  with  $V|_H \subset TH$  is formed by a 2-parametric subfamily of integral manifolds of  $V$ , whence the functional freedom.

Direct calculation based on the Frobenius theorem shows that by virtue of equations (1.6), (1.7) the distribution  $V$  is involutive. Thus,  $\lambda$ -dependent vector fields  $V_{ijk}$  constitute a dispersionless Lax representation for system (1.6), (1.7). Projecting integral manifolds of  $V$  from  $\hat{M}$  to  $M$  we obtain a 3-parameter family of codimension 2 submanifolds of  $M$ . Tangent spaces to these submanifolds are  $(n - 2)$ -dimensional osculating spaces of the dual curve  $\tilde{\omega}(\lambda)$ . Indeed, the distribution  $V$  is annihilated by the (pulled-back) 1-forms  $\omega(\lambda)$  and  $\omega'(\lambda)$ .

Equations (1.6), (1.7) for  $u$  and  $v$  are organised in pairs, each pair involving 4 independent variables indexed from 1 to  $n$ . As  $n$  grows, the collection of PDEs is nested and compatible. Ultimately when  $n \rightarrow \infty$  we obtain the corresponding dispersionless hierarchy.

In the context of the general heavenly hierarchy, similar Lax equations appeared recently in [4]. A modification of the inverse scattering transform for Lax equations in parameter-dependent vector fields was developed in [37].

*Remark 2.* System (1.6), (1.7) governing general involutive  $GL(2, \mathbb{R})$  structures can be viewed as a generalisation of the Veronese web hierarchy. Indeed, the Veronese web hierarchy results upon setting  $v_i = 1/c_i q u_i$ , where  $c_i$  are constants and  $q$  is some function. Then the reparametrisation  $\lambda \rightarrow \lambda/q$  identifies  $GL(2, \mathbb{R})$  structure (1.5) with (1.3) (up to unessential conformal factor  $q$ ), so that system (1.6), (1.7) reduces to equations (1.2) of the Veronese web hierarchy. Note that reductions of the general system (1.6), (1.7) to other examples of Sect. 2 (say, the dKP hierarchy) are far more complicated, requiring highly transcendental nonlocal changes of the independent variables  $x^i$  and the dependent variables  $u, v$ . Indeed, although the coordinate planes  $x^i = const$  constitute  $\alpha$ -manifolds for  $GL(2, \mathbb{R})$  structure (1.5), this is not the case for the dKP hierarchy.

Another class of (translationally non-invariant) integrable deformations of the Veronese web hierarchy was considered recently in [32]: the corresponding Lax equations do not however contain  $\partial_\lambda$ , and are specifically 3-dimensional.

*Remark 3.* For  $n = 4$  there exists a unique torsion-free  $GL(2, \mathbb{R})$  connection associated with  $GL(2, \mathbb{R})$  structure (1.5). It can be parametrised as

$$\Gamma^i_{jk} = \frac{u_i^2 u_j u_k}{(a_i - a_j)(a_i - a_k)} \psi_i, \quad \Gamma^i_{jj} = \frac{u_i^2 u_j^2}{(a_i - a_j)^2} \psi_i, \quad \Gamma^i_{ij} = \Gamma^i_{ji} = \frac{u_i u_j}{a_i - a_j} \phi_i, \quad \Gamma^i_{ii} = \rho_i,$$

where  $i, j, k \in \{1, \dots, 4\}$  are pairwise distinct indices, and the quantities  $\psi_i, \phi_i, \rho_i$  are yet to be determined from the following linear system with extra parameters  $s_j, \tilde{s}_j$  to be eliminated.

$$\begin{aligned} \frac{u_{ij}}{u_i u_j} - \sum_k \Gamma^k_{ij} \frac{u_k}{u_i u_j} &= s_2 a_i a_j + s_1 (a_i + a_j) + s_0, \\ \frac{v_{ij}}{v_i v_j} - \sum_k \Gamma^k_{ij} \frac{v_k}{v_i v_j} &= \tilde{s}_2 b_i b_j + \tilde{s}_1 (b_i + b_j) + \tilde{s}_0. \end{aligned}$$

This system contains 20 linear equations for the 18 unknowns  $\psi_i, \phi_i, \rho_i, s_j, \tilde{s}_j$ . These equations are consistent modulo (1.6), (1.7), and lead to a unique torsion-free  $GL(2, \mathbb{R})$  connection.

3.3. Counting  $\alpha$ -manifolds

The dispersionless Lax representation provides a two-parametric family of  $\alpha$ -manifolds. The totality of all  $\alpha$ -manifolds is bigger.

PROPOSITION 1. For an involutive  $GL(2, \mathbb{R})$  structure, its local  $\alpha$ -manifolds are parametrised by 1 function of 1 variable.

*Proof.* Let us invoke a relation with ordinary differential equations having all Wünschmann invariants zero, see [33] for details (recall that all involutive structures arise on solution spaces of such ODEs). An ODE  $\mathcal{E}$  of order  $n$  is given by a submanifold  $x_n = F(t, x_0, x_1, \dots, x_{n-1})$  in the jet-space  $J^n = \mathbb{R}^{n+2}(t, x_0, \dots, x_n)$ , and  $\mathcal{E}$  is diffeomorphic (via the jet-projection) to the jet-space  $J^{n-1}$ . The solution space  $M^n$  is identified with the space of integral curves of the field  $X_F = \partial_t + x_1 \partial_0 + \dots + x_{n-1} \partial_{n-2} + F \partial_{n-1}$ , where  $\partial_i = \partial_{x_i}$  and  $F = F(t, x_0, x_1, \dots, x_{n-1})$ .

Denote by  $\pi : J^{n-1} \rightarrow M = J^{n-1}/X_F$  the projection (since the construction is local, this quotient exists, and is non-singular), and let  $\mathcal{D}_{n-1} = \langle \partial_1, \dots, \partial_{n-1} \rangle$  be the vertical distribution in  $J^{n-1}$  with respect to the projection of  $J^{n-1}$  to  $J^0 = \mathbb{R}^2(t, x_0)$ . The family of hyperplanes  $\pi_* \mathcal{D}_{n-1} \subset TM$  parametrised by the coordinate  $\lambda = t$  along integral curves of  $X_F$  coincides with  $\alpha$ -hyperplanes of a  $GL(2, \mathbb{R})$  structure on  $M$  provided the Wünschmann invariants vanish.

Thus  $\alpha$ -manifolds are projections of integral manifolds of (maximal possible) dimension  $n - 1$  for the (non-holonomic) distribution

$$\mathcal{D}_n = \pi_*^{-1} \pi_*(\mathcal{D}_{n-1}) = \langle X_F, \partial_1, \dots, \partial_{n-1} \rangle = \langle \partial_t + x_1 \partial_0, \partial_1, \dots, \partial_{n-1} \rangle.$$

This distribution has rank  $n$  and possesses a sub-distribution of Cauchy characteristics of rank  $n - 2$  given by  $Ch(\mathcal{D}_n) = \langle \partial_2, \dots, \partial_{n-1} \rangle$ . Consequently, integral manifolds of  $\mathcal{D}_n$  are foliated by the Cauchy characteristics, and therefore coincide with vertical lifts of Legendrian curves of the standard contact structure on the quotient  $J^1 = J^{n-1}/Ch(\mathcal{D}_n)$ .

Note that generic Legendrian curves in  $J^1 = \mathbb{R}^3(t, x_0, x_1)$  are uniquely determined by their projection to the plane  $J^0 = \mathbb{R}^2(t, x_0)$ ; the curves whose projections degenerate to a point correspond to the standard two-parameter family of  $\alpha$ -manifolds. Since curves in the plane are parametrised by 1 function of 1 variable, the claim follows.

Remark 4. By a theorem of Sophus Lie a system of PDEs with the general solution depending on 1 function of 1 variable is solvable via ODEs [31, 36]. Thus  $\alpha$ -manifolds of any involutive  $GL(2, \mathbb{R})$  structure can be found as solutions to a system of ODEs.

The eikonal system  $\Sigma_{\mathcal{E}}$  is defined by the characteristic variety of a system  $\mathcal{E}$  over  $M$  (on a background solution  $u$ ) as follows: a hypersurface  $N = \{S = 0\} \subset M$  is a solution of  $\Sigma_{\mathcal{E}}$  if  $\text{Ann}(T_x N) \in \text{Char}_u(\mathcal{E})(x)$ . For example, if  $\mathcal{E}$  is the dKP equation we get from the first equation of (2.2) that  $\Sigma_{\mathcal{E}}$  is  $S_x S_t - S_y^2 - u_x S_x^2 = 0$ .

In the case of  $GL(2, \mathbb{R})$  structure over  $n$ -dimensional manifold  $M$  the eikonal system is locally given by  $(n - 2)$  PDEs of the first order. These equations are compatible and hence have solutions given by 1 function of 2 variables. In fact,  $\{S = \text{const}\}$  gives a foliation of  $M$  by  $\alpha$ -surfaces, i.e., a curve in the space of  $\alpha$ -surfaces. This directly corresponds to the count of Proposition 1.

3.4. *Equivalent definitions of involutivity*

PROPOSITION 3.1. *For a  $GL(2, \mathbb{R})$  structure, the definitions of involutivity in the sense of Bryant [7] and  $\alpha$ -integrability in the sense of Krynski [33] are equivalent.*

*Proof.* Consider a manifold  $M^n$ , the associated contact manifold  $\mathbb{P}T^*M$  of dimension  $2n - 1$  with the contact distribution  $\mathcal{C}_M$ , and a submanifold  $\mathcal{Z} \subset \mathbb{P}T^*M$  of dimension  $n + 1$  that corresponds to a  $GL(2, \mathbb{R})$  structure on  $M$ . We have  $\dim(T\mathcal{Z} \cap \mathcal{C}_M) = n$ . Since the projection  $\pi : \mathbb{P}T^*M \rightarrow M$  is surjective on  $\mathcal{Z}$ , the intersection  $\mathcal{Z}_x = \mathcal{Z} \cap \pi^{-1}(x) \subset \mathbb{P}T_x^*M$  is a curve (rational normal curve) for each  $x \in M$ . For  $p \in \mathcal{Z}_x$  we have  $d_p\pi(T\mathcal{Z} \cap \mathcal{C}_M) = p^\perp \subset T_xM$ ,  $p^\perp \simeq T\mathcal{Z} \cap \mathcal{C}_M/T_p\mathcal{Z}_x$ .

Denote the contact form by  $\omega$ . Then  $\mathcal{Z}$  is involutive in the sense of [7] if  $\omega \wedge (d\omega)|_{\mathcal{Z}} = 0 \Leftrightarrow (d\omega|_{T\mathcal{Z} \cap \mathcal{C}_M})^2 = 0$  (and  $d\omega|_{T\mathcal{Z} \cap \mathcal{C}_M} \neq 0$  for dimensional reasons), whence for the subbundle  $\Pi_M = \text{Ker}(d\omega|_{T\mathcal{Z} \cap \mathcal{C}_M})$  we have  $\text{rank } \Pi_M = n - 2$ . The local quotient  $(S_M, D_M) = (\mathcal{Z}, T\mathcal{Z} \cap \mathcal{C}_M)/\Pi_M$  is a 3-dimensional contact manifold. Denoting the projection by  $\rho : \mathcal{Z} \rightarrow S_M$ , the corresponding  $\alpha$ -manifolds can be represented in the form  $\rho^{-1}(L)$  where  $L \subset S_M$  is a Legendrian curve with respect to  $D_M$  (compare with the proof of Proposition 1 from Section 3.3).

Conversely, if for every  $x \in M$ ,  $p \in \mathcal{Z}_x$  there exists an  $\alpha$ -manifold tangent to  $p^\perp \subset T_xM$ , then the restriction of the canonical conformally symplectic form  $[d\omega]$  to  $T\mathcal{Z} \cap \mathcal{C}_M$  has rank 2, so that  $\alpha$ -integrability implies involutivity in the sense of [7].

4. *Concluding remarks*

We conclude with two general comments.

(1) It was demonstrated that involutive  $GL(2, \mathbb{R})$  structures in 4D or, equivalently, torsion-free affine connections with the irreducible  $GL(2, \mathbb{R})$  holonomy, are governed by a dispersionless integrable system. It would be interesting to understand which special holonomies lead to nonlinear PDEs that are either explicitly solvable/linearisable, or belong to the class of integrable systems.

(2) Interesting generalisations of involutive  $GL(2, \mathbb{R})$  structures arise in the context of integrable hierarchies whose characteristic varieties are *elliptic curves*. For instance, the first two equations of the dispersionless Pfaff-Toda hierarchy [44] have the form (see [2])

$$e^{F_{xx}} F_{xt} = e^{F_{yy}} F_{yz},$$

$$F_{zt} = 2e^{F_{xx} + F_{yy}} \sinh(2F_{xy}).$$

Here  $F$  is a function on the 4-dimensional manifold  $M$  with coordinates  $x, y, t, z$ . The characteristic variety of this system is a complete intersection of two quadrics in  $\mathbb{P}^3$ :

$$e^{F_{xx}} p_x p_t + e^{F_{xx}} F_{xt} p_x^2 = e^{F_{yy}} p_y p_z + e^{F_{yy}} F_{yz} p_y^2,$$

$$p_z p_t = e^{F_{xx} + F_{yy}} (e^{2F_{xy}} (p_x + p_y)^2 - e^{-2F_{xy}} (p_x - p_y)^2).$$

This specifies a field of elliptic curves in the projectivised cotangent bundle  $\mathbb{P}T^*M$ , recall that the genus  $g$  of a nonsingular complete intersection of two nonsingular surfaces of degrees  $d, e$  in  $\mathbb{P}^3$  equals  $g = de(d + e - 4)/2 + 1$ , see e.g. [24, chapter 2, exercise 8.4 (g)]. For  $d = e = 2$  this gives  $g = 1$ . The geometry of such structures is yet unclear, primarily due

to the lack of a naturally adapted connection (analogous to  $GL(2, \mathbb{R})$  connection) compatible with the above family of elliptic curves in the spirit of (1.4) (indeed, any such connection would automatically preserve all scalar differential invariants of the curves, however, their  $j$ -invariants are non-constant).

Appendix A. Compatibility conditions via free resolutions

In this section we explain how techniques from commutative algebra can be used to effectively compute compatibility conditions of overdetermined systems of PDEs. Then we apply this to our overdetermined system (1.6)-(1.7) encoding involutive  $GL(2, \mathbb{R})$  structures.

We refer to [29, 30] for details on this approach to involutivity of overdetermined systems of PDEs, and for a background on jet machinery in the formal theory of differential equations.

A.1. Projective resolutions and linear differential operators

Let us first consider the case of a linear system  $\mathcal{E}$  of PDEs given by a  $k$ -th order differential operator  $\Delta : \Gamma(\pi) \rightarrow \Gamma(\nu)$  on sections of vector bundles  $\pi, \nu$  over  $M$ . Such an operator corresponds to a morphism of vector bundles  $\psi_k^\Delta : J^k \pi \rightarrow \nu$  with jet-prolongations  $\psi_{k+i}^\Delta : J^{k+i} \pi \rightarrow J^i \nu$ . Then  $\mathcal{E}_{k+i} = \text{Ker}(\psi_{k+i}^\Delta)$  for  $i \geq 0$  and  $\mathcal{E}_l = J^l \pi$  for  $0 \leq l < k$ . The bundle  $\mathcal{E}_\infty$  is the projective limit of  $\mathcal{E}_k$  with respect to projections  $\pi_{i+1,i} : \mathcal{E}_{i+1} \rightarrow \mathcal{E}_i$ .

The dual bundle  $\mathcal{E}^* = \{\mathcal{E}_k^*\}$  allows to characterise involutivity as follows: the system  $\mathcal{E}$  is compatible (involutive) iff  $\Gamma(\mathcal{E}_k^*)$  are projective  $C^\infty(M)$ -modules and  $\pi_{i+1,i}^*$  are injective. Compatibility complex is related to projective resolution of the module  $\Gamma(\mathcal{E}_\infty^*)$ , but it is more convenient to construct this at the symbolic level.

At a point  $x \in M$  the symbol sequence of  $\mathcal{E}$  is  $g_k = \text{Ker}(d\pi_{k,k-1} : T_x \mathcal{E}_k \rightarrow T_x \mathcal{E}_{k-1}) \subset S^k T_x^* M \otimes \pi$ . The dual (over  $\mathbb{R}$ ) sequence determines the module  $\mathcal{M}_\mathcal{E} = \bigoplus g_k^*$  over the algebra  $\mathcal{R} = ST = \bigoplus_{i=0}^\infty S^i T_x M$  of homogeneous polynomials on  $T_x^* M$ , called the symbolic module of  $\mathcal{E}$ .

Since localisation of a projective module is free, we can construct a minimal free resolution of this module, where  $\sigma_\Delta$  is the symbol of  $\Delta$ , the dual of which defines relations among the generators of  $\mathcal{M}_\mathcal{E}$ , and  $\psi^*$  is the first syzygy (we use  $*$  for further convenience):

$$\dots \longrightarrow \mathcal{R} \otimes \varpi^* \xrightarrow{\psi^*} \mathcal{R} \otimes \nu^* \xrightarrow{\sigma_\Delta^*} \mathcal{R} \otimes \pi^* \longrightarrow \mathcal{M}_\mathcal{E} \longrightarrow 0. \tag{A.1}$$

Applying to this the functor  $* = \text{Hom}_\mathbb{R}(\cdot, \mathbb{R})$  we get the following exact sequence

$$0 \longrightarrow g \longrightarrow ST^* \otimes \pi \xrightarrow{\sigma_\Delta} ST^* \otimes \nu \xrightarrow{\psi} ST^* \otimes \varpi \longrightarrow \dots$$

from which we obtain the compatibility condition for  $\mathcal{E} = \{\Delta = 0\}$  as follows. Let  $\Psi \in \text{Diff}(\nu, \varpi)$  be a differential operator with the symbol  $\psi$  at  $x$ . Then the compatibility is  $\Psi \circ \Delta|_\mathcal{E} = 0$ .

More specifically, if the operator  $\Delta$  has order  $k$  and  $\Psi$  has order  $m$  (we consider the simplest case when we have only one order), then  $\Psi \circ \Delta$  has order  $\leq k + m - 1$  and it should be in the differential ideal of  $\mathcal{E}$ , so that  $\Psi \circ \Delta = \Xi \circ \Delta$  for a differential operator  $\Xi$  of order  $< m$ . Modification  $\Psi \mapsto \Psi' = \Psi - \Xi$  does not change the symbol and we get what is called the differential syzygy:

$$\Psi' \circ \Delta = 0. \tag{A.2}$$

This is how algebraic syzygy determines compatibility conditions in the linear case.

For nonlinear equations, apply the linearisation operator on a solution instead of  $\Delta$ . Its symbol again leads to a syzygy, from which we deduce compatibility operators; in this case however  $\Psi$  is an operator in total derivatives. Differential syzygy (A.2), considered as a differential corollary of  $\mathcal{E}$ , yields the complete compatibility condition for this system.

A.2. Application to involutive  $GL(2, \mathbb{R})$  structures

Let us indicate how to construct the syzygy  $\psi$  corresponding to system (1.6)-(1.7). In this case the order of the system is  $k = 2$  and the order of the syzygy will be  $m = 1$ .

Recall that at fixed point  $x \in M$  we denote

$$\mathcal{R} = \mathbb{R}[p_1, \dots, p_n] = ST_x M = \bigoplus_{k=0}^{\infty} S^k T_x^* M$$

the algebra of homogeneous polynomials on  $T_x^* M$ . Denote also  $\mathcal{R}^q = \mathcal{R} \otimes_{\mathbb{R}} \mathbb{R}^q$ .

The symbol  $\ell_{\mathcal{E}}$  of the nonlinear vector-operator defining  $\mathcal{E}$  is given by matrix (3.8), and in new coordinates  $\xi_i = \frac{p_i}{u_i}$  on  $T_x^* M$  it has components

$$\begin{aligned} \ell_{E_{ijkl}}^u(\xi) &= 2 \mathfrak{S}_{(jkl)}(a_i - a_j)(a_k - a_l)(\xi_i \xi_j + \xi_k \xi_l), \\ \ell_{E_{ijkl}}^v(\xi) &= - \mathfrak{S}_{(jkl)}(a_i - a_j)(a_k - a_l)((a_i + a_j)\xi_i \xi_j + (a_k + a_l)\xi_k \xi_l), \\ \ell_{F_{ijkl}}^u(\xi) &= - \mathfrak{S}_{(jkl)} \frac{(a_i - a_j)(a_k - a_l)}{a_i a_j a_k a_l} ((a_i + a_j)\xi_i \xi_j + (a_k + a_l)\xi_k \xi_l), \\ \ell_{F_{ijkl}}^v(\xi) &= 2 \mathfrak{S}_{(jkl)}(a_i - a_j)(a_k - a_l) \left( \frac{\xi_i \xi_j}{a_k a_l} + \frac{\xi_k \xi_l}{a_i a_j} \right), \end{aligned}$$

in the basis  $e_u, e_v$  of  $\mathcal{R}^2$  and basis  $e_{E_{ijkl}}, e_{F_{ijkl}}$  of  $\mathcal{R}^{2\binom{n-2}{2}}$ , where due to relations (3.5) we restrict to indices  $i = 1, j = 2, 2 < k < l \leq n$ . This means that the homomorphism  $\ell_{\mathcal{E}}$  maps  $f(\xi)e_u$  to  $f(\xi) \sum_{k < l} (\ell_{E_{12kl}}^u(\xi)e_{E_{12kl}} + \ell_{F_{12kl}}^u(\xi)e_{F_{12kl}})$  and similarly for  $h(\xi)e_v$ .

Now we resolve  $\ell_{\mathcal{E}}$  by a homomorphism  $\mathcal{C} = \mathcal{C}_{\mathcal{E}}$ . For  $w = \sum_{i < j} (w_{E_{12ij}}e_{E_{12ij}} + w_{F_{12ij}}e_{F_{12ij}})$  the image  $\mathcal{C}(\xi)(w)$  has the following components ( $2 < i < j < k \leq n$ ):

$$\begin{aligned} \mathcal{C}_{ijk}^I &= \mathfrak{S}_{(ijk)}((a_2 - a_k)\xi_1 + (a_k - a_1)\xi_2 + (a_1 - a_2)\xi_k)w_{E_{12ij}}, \\ \mathcal{C}_{ijk}^{II} &= \mathfrak{S}_{(ijk)} \left[ ((a_1 - a_2)(a_2 - a_k)a_1\xi_1 + (a_2 - a_1)(a_1 - a_k)a_2\xi_2 \right. \\ &\quad \left. + ((a_2 - a_k)^2 a_1 + (a_1 - a_k)^2 a_2)\xi_k \right] w_{E_{12ij}} + 2a_1 a_2 a_i a_j (a_1 - a_k)(a_2 - a_k)\xi_k w_{F_{12ij}}, \\ \mathcal{C}_{ijk}^{III} &= \mathfrak{S}_{(ijk)} \left[ 2(a_1 - a_k)(a_2 - a_k)\xi_k w_{E_{12ij}} + ((a_1 - a_2)(a_2 - a_k)a_1\xi_1 \right. \\ &\quad \left. + (a_2 - a_1)(a_1 - a_k)a_2\xi_2 + ((a_2 - a_k)^2 a_1 + (a_1 - a_k)^2 a_2)\xi_k \right] a_i a_j w_{F_{12ij}}, \\ \mathcal{C}_{ijk}^{IV} &= \mathfrak{S}_{(ijk)}((a_2 - a_k)a_1^2\xi_1 + (a_k - a_1)a_2^2\xi_2 + (a_1 - a_2)a_k^2\xi_k) a_i a_j w_{F_{12ij}}. \end{aligned}$$

One verifies that with these homomorphisms the following sequence is exact:

$$\mathcal{R}^2 \xrightarrow{\ell_{\mathcal{E}}} \mathcal{R}^{2\binom{n-2}{2}} \xrightarrow{\mathcal{C}_{\mathcal{E}}} \mathcal{R}^{4\binom{n-2}{3}}. \tag{A.3}$$

In other words,  $\mathcal{C}_{\mathcal{E}}$  is the first syzygy for the module  $\mathcal{M}_{\mathcal{E}}^* = \text{Ker}(\ell_{\mathcal{E}}) = \text{Hom}_{\mathcal{R}}(\mathcal{M}_{\mathcal{E}}, \mathcal{R})$ . Therefore, the differential syzygies (A.2) for  $\mathcal{E}$  given by system (1.6)-(1.7) are enumerated



by 5 different indices  $(12ijk)$ ,  $2 < i < j < k \leq n$ . Consequently to verify compatibility conditions for each of these 5-tuples one can work in the corresponding 5-dimensional space, and this justifies the key argument used in the proof of part (b) of Theorem 2.

**A.3. Constructing the minimal free resolution**

The higher syzygies resolve the dual  $\mathcal{M}_{\mathcal{E}}^*$  of the symbolic module as follows. Let us recall a construction from the commutative algebra adapted to our situation. For a homomorphism  $\varphi : \mathcal{R}^{n-2} \rightarrow \mathcal{R}^2$  the following sequence is known as the Eagon-Northcott complex [15, appendix A2] (all tensor products are over  $\mathcal{R}$ , and  $\star$  is the dualisation over  $\mathcal{R}$ )

$$\dots \longrightarrow S^3\mathcal{R}^{\star 2} \otimes \Lambda^5\mathcal{R}^{n-2} \xrightarrow{\partial} S^2\mathcal{R}^{\star 2} \otimes \Lambda^4\mathcal{R}^{n-2} \xrightarrow{\partial} \mathcal{R}^{\star 2} \otimes \Lambda^3\mathcal{R}^{n-2} \xrightarrow{\partial} \Lambda^2\mathcal{R}^{n-2} \xrightarrow{\epsilon} \mathcal{R}.$$

Here  $\epsilon = \Lambda^2\varphi$  and the differential  $\partial$  is the following composition, in which  $\delta$  is the Spencer differential and  $\ast$  is the Hodge dual via a volume form:

$$\begin{aligned} S^{d+1}\mathcal{R}^{\star 2} \otimes \Lambda^{d+3}\mathcal{R}^{n-2} &\xrightarrow{\delta \otimes \ast} S^d\mathcal{R}^{\star 2} \otimes \mathcal{R}^{\star 2} \otimes \Lambda^{n-d-5}\mathcal{R}^{\star(n-2)} \\ &\xrightarrow{1 \otimes \varphi^* \otimes 1} S^d\mathcal{R}^{\star 2} \otimes \mathcal{R}^{\star(n-2)} \otimes \Lambda^{n-d-5}\mathcal{R}^{\star(n-2)} \\ &\xrightarrow{1 \otimes \wedge} S^d\mathcal{R}^{\star 2} \otimes \Lambda^{n-d-4}\mathcal{R}^{\star(n-2)} \xrightarrow{1 \otimes \ast} S^d\mathcal{R}^{\star 2} \otimes \Lambda^{d+2}\mathcal{R}^{n-2}. \end{aligned}$$

The Eagon–Northcott complex is exact when the Fitting ideal  $I(\varphi)$ , obtained by taking all  $2 \times 2$  determinants of the matrix of  $\varphi$ , contains a regular sequence of length  $(n - 3)$ .

For the system  $\mathcal{E}$  that we study the map  $\ell_{\mathcal{E}}$  can be split. Indeed, one easily checks that  $\ell_{\mathcal{E}}(e_u)$  and  $\ell_{\mathcal{E}}(e_v)$  in sequence (A.3) generate two complementary submodules  $\Lambda^2\mathcal{R}^{n-2} \subset \mathcal{R}^{2\binom{n-2}{2}}$ . Therefore this splitting generates two copies of the  $\star$ -dual Eagon-Northcott complex (in particular  $\mathcal{C}_{\mathcal{E}}^*$  is the doubling of the first differential  $\partial$ ) implying the following resolution of the  $\star$ -dual symbolic module, i.e. dualisation of (A.1) over  $\mathcal{R}$  for  $\mathcal{E}$  given by system (1.6)-(1.7) is

$$\begin{aligned} 0 \longrightarrow \mathcal{M}_{\mathcal{E}}^* \longrightarrow \mathcal{R}^2 \xrightarrow{\ell_{\mathcal{E}}} \mathcal{R}^2 \otimes \Lambda^2\mathcal{R}^{n-2} \\ \xrightarrow{\mathcal{C}_{\mathcal{E}}} \mathcal{R}^{\star 2} \otimes \mathcal{R}^2 \otimes \Lambda^3\mathcal{R}^{n-2} \xrightarrow{\partial^*} S^2\mathcal{R}^{\star 2} \otimes \mathcal{R}^2 \otimes \Lambda^4\mathcal{R}^{n-2} \longrightarrow \dots \end{aligned}$$

The Fitting condition mentioned above follows from the fact that the zero set of  $I(\ell_{\mathcal{E}})$  is the tangential variety to the rational normal curve (see the initial steps of the proof of Theorem 2) that has codimension  $n - 3$ .

The claim that only 5-tuples of distinct indices enter the compatibility conditions can be visualised in terms of the above complex as follows: the factor  $\Lambda^3\mathcal{R}^{n-2}$  in the space of compatibility conditions refers to triples of indices  $(ijk)$  that yield the equations  $E_{12ij}, E_{12jk}, E_{12ki}$  and  $F_{12ij}, F_{12jk}, F_{12ki}$ . For each such 5-tuple  $(12ijk)$  the number of compatibility conditions is four, which equals the rank of the factor  $\mathcal{R}^{\star 2} \otimes \mathcal{R}^2$ .

It is not surprising that the Eagon-Northcott complex arises in our study since it is used to show that the ideal of a rational normal curve is Cohen-Macaulay [15, A2.19].

*Appendix B. Canonical connections for integrable PDEs*

In this section we calculate Christoffel’s symbols of the canonical connections associated with 4D examples of Section 2. This provides explicit formulae for involutive

$GL(2, \mathbb{R})$  structures and the associated connections parametrised by solutions of dispersionless integrable systems (exact solutions can be constructed by the method of hydrodynamic reductions [16]).

In all cases, totally geodesic  $\alpha$ -manifolds are projections of integral manifolds of commuting vector fields from the dispersionless Lax representation. These computations together with the corresponding higher-dimensional counterparts (5D etc, not included here) were performed in Maple’s DIFFERENTIALGEOMETRY package, [arXiv:1607.01966v2](https://arxiv.org/abs/1607.01966v2).

Note that in the 4D cases considered here the normal connections coincide with totally geodesic ones; in higher dimensions the totally geodesic connection does not exist for dKP and Adler-Shabat hierarchies (the normal connection exists), while for the universal hierarchy it exists and coincides with the normal connection.

**B.1. Connections associated with the dKP hierarchy**

We use the notation  $(x^1, x^2, x^3, x^4) = (x, y, t, z)$ , note that  $\Gamma^i_{jk} \neq \Gamma^i_{kj}$  in general. Not listed Christoffel symbols are zero (unless the connection is torsion-free, in which case  $\Gamma^i_{jk} = \Gamma^i_{kj}$ ).

**Torsion-free  $GL(2, \mathbb{R})$  connection** is given by

$$\begin{aligned} \Gamma^1_{13} &= u_{11}, \Gamma^1_{14} = u_{12}, \Gamma^2_{14} = 2u_{11}, \Gamma^1_{22} = \frac{4}{9}u_{11}, \Gamma^1_{23} = u_{12}, \Gamma^2_{23} = \frac{8}{9}u_{11}, \\ \Gamma^1_{24} &= \frac{13}{9}u_{22} - \frac{4}{9}u_{13}, \Gamma^2_{24} = 2u_{12}, \Gamma^3_{24} = \frac{4}{3}u_{11}, \Gamma^1_{33} = \frac{11}{9}u_{13} - \frac{2}{9}u_{22}, \\ \Gamma^2_{33} &= 2u_{12}, \Gamma^3_{33} = \frac{7}{9}u_{11}, \Gamma^1_{34} = 2u_{23} - u_{14} + \frac{4}{3}u_2u_{11}, \Gamma^2_{34} = \frac{22}{9}u_{13} - \frac{4}{9}u_{22}, \\ \Gamma^3_{34} &= 3u_{12}, \Gamma^4_{34} = \frac{2}{3}u_{11}, \Gamma^1_{44} = u_{33} + \frac{32}{9}u_1u_{22} - \frac{41}{9}u_1u_{13} - 2u_2u_{12}, \\ \Gamma^2_{44} &= 4u_{23} - 2u_{14} + 6u_2u_{11}, \Gamma^3_{44} = \frac{16}{3}u_{13} - \frac{7}{3}u_{22}, \Gamma^4_{44} = 4u_{12}. \end{aligned}$$

**Normal (totally geodesic)  $GL(2, \mathbb{R})$  connection with trace-free torsion** is given by

$$\begin{aligned} \Gamma^1_{13} &= u_{11}, \Gamma^1_{14} = u_{12}, \Gamma^2_{14} = 2u_{11}, \Gamma^1_{22} = u_{11}, \Gamma^1_{23} = u_{12}, \Gamma^2_{23} = 2u_{11}, \\ \Gamma^1_{24} &= 2u_{22} - u_{13}, \Gamma^2_{24} = 2u_{12}, \Gamma^3_{24} = 3u_{11}, \Gamma^1_{31} = -\frac{2}{3}u_{11}, \Gamma^1_{32} = u_{12}, \\ \Gamma^2_{32} &= \frac{1}{3}u_{11}, \Gamma^1_{33} = 2u_{22} - u_{13}, \Gamma^2_{33} = 2u_{12}, \Gamma^3_{33} = \frac{4}{3}u_{11}, \Gamma^2_{34} = 4u_{22} - 2u_{13}, \\ \Gamma^1_{34} &= 2u_{23} - u_{14} - 2u_2u_{11}, \Gamma^3_{34} = 3u_{12}, \Gamma^4_{34} = \frac{7}{3}u_{11}, \Gamma^1_{41} = u_{12}, \Gamma^2_{41} = -3u_{11}, \\ \Gamma^1_{42} &= 4u_{13} - 3u_{22}, \Gamma^2_{42} = 2u_{12}, \Gamma^3_{42} = -2u_{11}, \Gamma^1_{43} = 2u_{23} - u_{14} + 3u_2u_{11}, \\ \Gamma^2_{43} &= 3u_{13} - u_{22}, \Gamma^3_{43} = 3u_{12}, \Gamma^4_{43} = -u_{11}, \Gamma^1_{44} = 3u_{33} - u_1u_{13} - 2u_{24}, \\ \Gamma^2_{44} &= 4u_{23} - 2u_{14} + u_2u_{11}, \Gamma^3_{44} = 2u_{13} + u_{22}, \Gamma^4_{44} = 4u_{12}. \end{aligned}$$

**Totally geodesic projective connection** is given by

$$\begin{aligned} \Gamma^1_{13} &= -\frac{1}{2}u_{11}, \Gamma^1_{14} = -u_{12}, \Gamma^2_{14} = -\frac{1}{2}u_{11}, \Gamma^1_{22} = u_{11}, \Gamma^1_{23} = u_{12}, \Gamma^2_{23} = \frac{1}{2}u_{11}, \\ \Gamma^1_{24} &= \frac{3}{2}u_1u_{11} + u_{22}, \Gamma^3_{24} = \frac{1}{2}u_{11}, \Gamma^1_{33} = u_{22} - u_1u_{11}, \Gamma^2_{33} = 2u_{12}, \\ \Gamma^1_{34} &= u_{14} - 2u_1u_{12} - \frac{3}{2}u_2u_{11}, \Gamma^2_{34} = \frac{1}{2}u_1u_{11} + 2u_{22}, \Gamma^3_{34} = u_{12}, \\ \Gamma^1_{44} &= 2u_{33} - u_1u_{22} - u_2u_{12} - u_{24}, \Gamma^2_{44} = 2u_{14} - 4u_1u_{12} - 3u_2u_{11}, \Gamma^3_{44} = 3u_{13} - u_1u_{11}. \end{aligned}$$

**B.2. Connections associated with the universal hierarchy**

We again use the notation  $(x^1, x^2, x^3, x^4) = (x, y, t, z)$ , note that  $\Gamma^i_{jk} \neq \Gamma^i_{kj}$  in general. Not listed Christoffel symbols are zero (unless the connection is torsion-free, in which case  $\Gamma^i_{jk} = \Gamma^i_{kj}$ ).

**Torsion-free  $GL(2, \mathbb{R})$  connection** is given by

$$\begin{aligned} \Gamma_{11}^2 &= -\frac{2}{3}u_{11}, \quad \Gamma_{11}^3 = -\frac{2}{3}u_1u_{11} - u_{12}, \quad \Gamma_{21}^3 = -\frac{1}{3}u_{11}, \\ \Gamma_{11}^4 &= -\frac{1}{3}(2u_1^2 + u_2)u_{11} - u_1u_{12} - u_{13}, \quad \Gamma_{21}^4 = -\frac{2}{3}u_1u_{11} - u_{12}, \\ \Gamma_{12}^2 &= -\frac{8}{9}u_{12}, \quad \Gamma_{22}^2 = -\frac{4}{9}u_{11}, \quad \Gamma_{12}^3 = \frac{1}{9}u_1u_{12} - \frac{5}{9}u_2u_{11} - u_{13}, \\ \Gamma_{22}^3 &= -\frac{2}{9}u_1u_{11} - \frac{7}{9}u_{12}, \quad \Gamma_{32}^3 = -\frac{2}{9}u_{11}, \\ \Gamma_{12}^4 &= \frac{1}{9}(u_1^2 - 7u_2)u_{12} - \frac{1}{9}(7u_1u_2 + 3u_3)u_{11} - u_{14}, \\ \Gamma_{22}^4 &= \frac{1}{9}u_1u_{12} - \frac{5}{9}u_2u_{11} - u_{13}, \quad \Gamma_{32}^4 = -\frac{4}{9}u_1u_{11} - \frac{2}{3}u_{12}, \\ \Gamma_{13}^3 &= \frac{1}{9}(u_1^2 + 4u_2)u_{12} - \frac{10}{9}u_3u_{11} - u_{14}, \quad \Gamma_{23}^3 = -\frac{1}{9}(u_1^2 + 4u_2)u_{11} - u_{13}, \\ \Gamma_{33}^3 &= -\frac{2}{9}u_1u_{11} - \frac{5}{9}u_{12}, \quad \Gamma_{43}^3 = -\frac{1}{9}u_{11}, \\ \Gamma_{13}^4 &= \frac{1}{9}(u_1^3 + 4u_1u_2)u_{12} - \frac{1}{9}(u_1^2u_2 + 5u_1u_3 + 4u_2^2)u_{11} - u_1u_{14} - \frac{1}{3}u_3u_{12} - u_{24}, \\ \Gamma_{23}^4 &= \frac{1}{9}(u_1^2 + 4u_2)u_{12} - \frac{2}{3}u_3u_{11} - u_{14}, \quad \Gamma_{33}^4 = -\frac{1}{9}(2u_1^2 + 7u_2)u_{11} - \frac{1}{9}u_1u_{12} - u_{13}, \\ \Gamma_{43}^4 &= -\frac{2}{9}u_1u_{11} - \frac{1}{3}u_{12}, \quad \Gamma_{14}^4 = \frac{1}{9}(u_1^4 + 5u_1^2u_2 + 6u_1u_3 + 4u_2^2)u_{12} - u_1u_{24} \\ &\quad - \frac{1}{9}(u_1^3u_2 + u_1^2u_3 + 4u_1u_2^2 + 12u_2u_3)u_{11} - (u_1^2 + u_2)u_{14} - u_{34}, \\ \Gamma_{24}^4 &= \frac{1}{9}(u_1^3 + 4u_1u_2 + 9u_3)u_{12} - \frac{1}{9}(u_1^2u_2 - 3u_1u_3 + 4u_2^2)u_{11} - u_1u_{14} - u_{24}, \\ \Gamma_{34}^4 &= -\frac{1}{9}u_1u_{22} - \frac{8}{9}u_1u_{13} + \frac{1}{3}u_2u_{12} - u_{23}, \quad \Gamma_{44}^4 = -\frac{4}{9}u_1^2u_{11} - \frac{4}{3}u_1u_{12} - u_{22}. \end{aligned}$$

**Normal (totally geodesic)  $GL(2, \mathbb{R})$  connection with trace-free torsion** is given by

$$\begin{aligned} \Gamma_{11}^2 &= \Gamma_{21}^3 = \Gamma_{31}^4 = -u_{11}, \quad \Gamma_{11}^3 = \Gamma_{21}^4 = -u_1u_{11} - u_{12}, \\ \Gamma_{11}^4 &= -u_1^2u_{11} - u_1u_{12} - u_2u_{11} - u_{13}, \quad \Gamma_{12}^1 = \Gamma_{22}^2 = \Gamma_{32}^3 = \Gamma_{42}^4 = -\frac{1}{3}u_{11}, \\ \Gamma_{12}^2 &= \Gamma_{22}^3 = \Gamma_{32}^4 = -u_{12}, \quad \Gamma_{12}^3 = \Gamma_{22}^4 = -u_2u_{11} - u_{13}, \\ \Gamma_{12}^4 &= -u_1u_2u_{11} - u_2u_{12} - u_3u_{11} - u_{14}, \\ \Gamma_{13}^1 &= \Gamma_{23}^2 = \Gamma_{33}^3 = \Gamma_{43}^4 = -\frac{1}{3}u_1u_{11} - \frac{2}{3}u_{12}, \quad \Gamma_{13}^2 = \Gamma_{23}^3 = \Gamma_{33}^4 = -u_{13}, \\ \Gamma_{13}^3 &= \Gamma_{23}^4 = -u_3u_{11} - u_{14}, \quad \Gamma_{13}^4 = -u_1u_3u_{11} - u_1u_{14} - u_2u_{13} - u_{33}, \\ \Gamma_{14}^1 &= \Gamma_{24}^2 = \Gamma_{34}^3 = \Gamma_{44}^4 = -\frac{1}{3}u_1^2u_{11} - \frac{1}{3}u_1u_{12} - \frac{2}{3}u_2u_{11} - u_{13}, \\ \Gamma_{14}^2 &= \Gamma_{24}^3 = \Gamma_{34}^4 = -u_{14}, \quad \Gamma_{14}^3 = \Gamma_{24}^4 = -u_1u_{14} - u_{24}, \\ \Gamma_{14}^4 &= -u_1^2u_{14} - u_1u_{24} - u_2u_{14} - u_{34}, \end{aligned}$$

note that  $\Gamma_{ij}^k = \Gamma_{i+a,j}^{k+a}$ , as long as the indices are in the range.

**Totally geodesic projective connection** is given by

$$\begin{aligned} \Gamma_{12}^1 &= \Gamma_{13}^2 = \Gamma_{14}^3 = -\frac{1}{2}u_{11}, \quad \Gamma_{13}^1 = -\frac{1}{2}u_1u_{11}, \quad \Gamma_{14}^1 = -\frac{1}{2}u_1^2u_{11} - \frac{1}{2}u_1u_{12} - \frac{1}{2}u_2u_{11}, \\ \Gamma_{14}^2 &= -\frac{1}{2}u_1u_{11} - \frac{1}{2}u_{12}, \quad \Gamma_{12}^1 = -u_{12}, \quad \Gamma_{23}^1 = -\frac{1}{2}u_2u_{11} - u_{13}, \\ \Gamma_{24}^1 &= -\frac{1}{2}u_1u_2u_{11} - \frac{1}{2}u_2u_{12} - \frac{1}{2}u_3u_{11} - u_{14}, \quad \Gamma_{24}^2 = -\frac{1}{2}u_2u_{11}, \\ \Gamma_{24}^3 &= -\frac{1}{2}u_{12}, \quad \Gamma_{33}^1 = -u_3u_{11} - u_{14}, \quad \Gamma_{33}^2 = -u_{13}, \quad \Gamma_{33}^3 = u_{12}, \\ \Gamma_{34}^1 &= -\frac{1}{2}u_1u_3u_{11} - u_1u_{14} - u_{24} - \frac{1}{2}u_3u_{12}, \quad \Gamma_{34}^2 = -\frac{1}{2}u_3u_{11} - u_{14}, \\ \Gamma_{34}^3 &= \frac{1}{2}u_{12}, \quad \Gamma_{44}^1 = -u_1^2u_{14} - u_1u_{24} - u_2u_{14} - u_{34}, \\ \Gamma_{44}^2 &= -u_1u_{14} - u_{24}, \quad \Gamma_{44}^3 = -u_{14}, \quad \Gamma_{44}^4 = u_{13}. \end{aligned}$$

B.3. *Connections associated with Adler–Shabat triples*

In what follows,  $i, j, k$  are pairwise distinct indices taking values 2, 3, 4.

**Torsion-free  $GL(2, \mathbb{R})$  connection** is given by (no summation unless specified):

$$\begin{aligned} \Gamma_{11}^1 &= \frac{2}{9}(3R_d - R_a) - \frac{1}{9} \sum_{i \neq 1} \sigma_{ijk} u_{ii}, \quad \Gamma_{1i}^1 = \frac{1}{3}(\gamma_{jk} u_{ij} + \gamma_{kj} u_{ik}) - \frac{1}{9} R_e, \\ \Gamma_{11}^i &= \frac{1}{9}(u_i - u_j)(u_i - u_k)(\gamma_{jk}(u_{jj} - 4u_{ij}) + \gamma_{kj}(u_{kk} - 4u_{ik})), \\ \Gamma_{1i}^i &= -\frac{2}{9} R_a - \frac{4}{9} u_{ii} + \frac{1}{9} \frac{\gamma_{jk}}{\gamma_{ji}}(u_{jj} - u_{ij}) + \frac{1}{9} \frac{\gamma_{kj}}{\gamma_{ki}}(u_{kk} - u_{ik}) + \frac{5}{9}(u_{ij} + u_{ik}), \\ \Gamma_{1i}^j &= \frac{1}{9} \frac{\gamma_{ki}}{\gamma_{kj}}(u_{ii} - 4u_{ij} + 4u_{jk} - u_{kk}), \quad \Gamma_{ii}^1 = -\frac{1}{9} R_f, \quad \Gamma_{ij}^1 = -\frac{1}{9} R_f, \\ \Gamma_{ii}^i &= -\frac{1}{3} R_e + \frac{\sigma_{ijk} + 2}{9}(\gamma_{jk} u_{jj} + \gamma_{kj} u_{kk}) + \frac{2}{9} \left( \gamma_{jk} - \gamma_{ij} + \gamma_{ik} - \frac{\gamma_{ij} \gamma_{ik}}{\gamma_{jk}} \right) (u_{ij} - u_{ik}), \\ \Gamma_{ii}^j &= -\frac{\gamma_{ij}}{9} \frac{\gamma_{ki}}{\gamma_{kj}}(u_{ii} - 4u_{ij} + 4u_{jk} - u_{kk}), \quad \Gamma_{ij}^k = \frac{\gamma_{ij}}{9}(u_{ii} - 4u_{ik} + 4u_{jk} - u_{jj}), \\ \Gamma_{ij}^i &= -\frac{2}{3} \gamma_{ij} R_a - \frac{1}{9}(\gamma_{jk} - 4\gamma_{ij})(u_{jj} + u_{ik}) + \frac{1}{9}(\gamma_{jk} + 5\gamma_{ij})(u_{kk} + u_{ij}), \end{aligned}$$

where

$$\begin{aligned} \gamma_{ij} &= \frac{1}{u_i - u_j}, \quad \sigma_{ijk} = \frac{\gamma_{ij}}{\gamma_{ik}} + \frac{\gamma_{ik}}{\gamma_{ij}} = \frac{u_i - u_k}{u_i - u_j} + \frac{u_i - u_j}{u_i - u_k}, \\ R_d &= \sum_{i \neq 1} (\gamma_{ij} + \gamma_{ik}) u_{1i}, \quad R_e = \sum_{i \neq 1} (\gamma_{ij} + \gamma_{ik}) u_{ii}, \quad R_f = \sum_{i \neq 1} \gamma_{ij} \gamma_{ik} u_{ii}. \end{aligned}$$

**Normal (totally geodesic)  $GL(2, \mathbb{R})$  connection with trace-free torsion** is given by

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{3}(2R_b - R_a), \quad \Gamma_{1i}^1 = R_c, \quad \Gamma_{i1}^1 = -\frac{1}{3} \left( \frac{u_{ii} - u_{jj}}{u_i - u_j} + \frac{u_{ii} - u_{kk}}{u_i - u_k} + 5R_c \right), \\ \Gamma_{1i}^i &= 2u_{kl} - \frac{1}{3}(R_a + R_b), \quad \Gamma_{i1}^i = -u_{ii} + \frac{u_i - u_k}{u_j - u_k} u_{ij} + \frac{u_i - u_j}{u_k - u_j} u_{ik}, \\ \Gamma_{ii}^i &= \Gamma_{i1}^1 + R_c, \quad \Gamma_{ij}^i = \frac{R_b - u_{ii} - 2u_{jk}}{u_i - u_j}, \\ \Gamma_{ij}^j &= \frac{2u_{ii} + u_{jj} - 3u_{ij} - u_{ik} + u_{jk}}{3(u_i - u_j)} - \frac{u_{ii} - u_{kk}}{3(u_i - u_k)}, \end{aligned}$$

where

$$\begin{aligned} R_a &= u_{22} + u_{33} + u_{44}, \quad R_b = u_{23} + u_{24} + u_{34}, \\ R_c &= \frac{u_2 u_{34}}{(u_2 - u_3)(u_2 - u_4)} + \frac{u_3 u_{24}}{(u_3 - u_2)(u_3 - u_4)} + \frac{u_4 u_{23}}{(u_4 - u_2)(u_4 - u_3)}. \end{aligned}$$

**Totally geodesic projective connection** is given by

$$\Gamma_{1i}^i = -\frac{1}{2} u_{ii}, \quad \Gamma_{ij}^i = \frac{2u_{ij} - u_{ii} - u_{jj}}{2(u_i - u_j)},$$

recall that  $\Gamma_{jk}^i = \Gamma_{kj}^i$ , all other Christoffel symbols are zero.

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