New characterisations of bicentric quadrilaterals

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Introduction

A *bicentric quadrilateral* is a convex quadrilateral that can have both an incircle (it is tangential) and a circumcircle (it is cyclic), see Figure 1. We know of only a dozen characterisations of bicentric quadrilaterals published before. In all of them the starting point is either a tangential or a cyclic quadrilateral, which then must satisfy some condition in order also to be of the other type. Before we proceed to prove seven new such necessary and sufficient conditions for bicentric quadrilaterals, we review one characterisation and one property of tangential quadrilaterals that we will apply later.

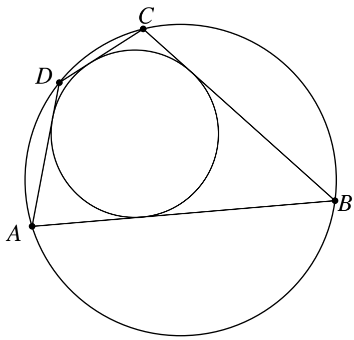

FIGURE 1:A bicentric quadrilateral with its incircle and circumcircle

It is quite well known that a convex quadrilateral $ABCD$ admits an incircle if, and only if, its sides satisfy the equation $AB + CD = BC + DA$, a condition called *Pitot's theorem*. The proof for the necessary condition is a direct consequence of the fact that the two tangents to a circle through an external point have equal lengths, sometimes called the *two tangent theorem*. If the incircle is tangent to AB, BC, CD, DA at W, X, Y, Z respectively, then $ZA = AW \equiv e$, $WB = BX \equiv f$, $XC = CY \equiv g$ and $YD = DZ \equiv h$. Hence we get

$$AB + CD = e + f + g + h = BC + DA.$$

The converse theorem is harder to prove and much more interesting. There have been about ten different proofs published over the last two centuries, for most of which references were given in [1]. A few months after the publication of that note we were contacted by Alan Beardon, who suggested another proof based on a dynamic argument. A variant of his proof will be given here, where we use his main idea of applying the cosines rule, which we have not seen among the previously published proofs. Thus, starting with a convex quadrilateral whose sides satisfy

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$$AB + CD = BC + DA,$$

we shall prove that all four sides are tangent to an internal circle. First draw the angle bisectors to two adjacent vertex angles. They intersect in a point I that is equidistant to three of the sides (see Figure 2). We shall prove that this point has the same distance, let us call it r, to the fourth side CD. Let a circle with centre I and radius r be tangent to DA, AB, BC at E, F, G respectively. By the two tangent theorem we have $EA = AF$ and $FB = BG$, so (1) is reduced to $CD = GC + DE$. This means that there is a point H on CD such that $DH = DE$ and $CH = CG$. Now label DE as u, CG as v and the angle DHI as φ.

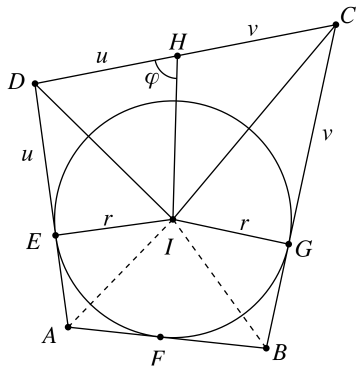

FIGURE 2: Proving the converse of Pitot's theorem

Applying the Pythagorean theorem and the cosine rule in triangles DEI and DHI yields

$$r^2 + u^2 = DI^2 = u^2 + HI^2 - 2u.HI \cos\varphi.$$

In the same way, in triangles CGI and CHI we have

$$r^2 + v^2 = CI^2 = v^2 + HI^2 + 2v.HI \cos\varphi$$

since $\cos(\pi - \varphi) = -\cos\varphi$. From these equalities we deduce

$$2u.HI \cos\varphi = HI^2 - r^2 = -2v.HI \cos\varphi,$$

which we rewrite as

$$2(u + v)HI \cos\varphi = 0.$$

The only possible solution to this equation is $\cos\varphi = 0$. Hence $\varphi = \frac{\pi}{2}$, which implies $HI = r$. This means that the fourth side CD is tangent to the circle at H. We assumed in the proof that CD did not intersect the circle, but even if it does, the proof holds as long as the circle is tangent to BC between B and C and tangent to DA between D and A. If both of the tangency points lie on the extensions of BC and DA, then (1) is reduced to $CD = -(GC + ED)$, which is clearly impossible, so this case cannot happen (see the left part of Figure 3). If one tangency point lies on an extended side, as in the right part of Figure 3, (1) implies $CL = CD + DL$,

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where L is a point on the extension of AD such that CL is tangent to the circle. This equality violates the triangle inequality, so that case is not possible either. This concludes the proof of the converse of Pitot's theorem.

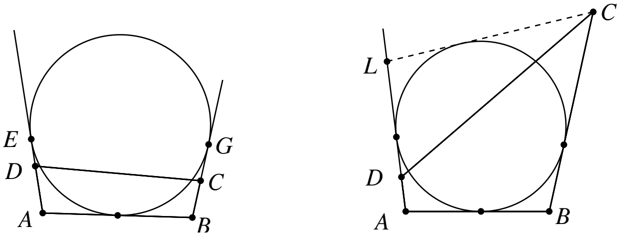

FIGURE 3: Two impossible cases

A property shared by all tangential quadrilaterals that is perhaps not so well known to all readers, is the concurrency of the two diagonals and the two *tangency chords*. The latter are the line segments connecting opposite points of tangency of the incircle with the sides. This property can be proved in several different ways; our proof (which is not original) uses Menelaus' theorem. We only consider the case when opposite sides intersect. Let BC and AD intersect at K, and BA and CD intersect at J. Suppose the incircle is tangent to AB, BC, CD, DA at W, X, Y, Z respectively, and that the tangency chords WY and ZX intersect diagonal BD in P_1 and P_2 respectively. Applying Menelaus' theorem in triangle BDK with transversal ZX (using non-directed distances), and also in triangle BDJ with transversal WY (see Figure 4), we get

$$\frac{DP_2}{P_2B} \cdot \frac{BX}{XK} \cdot \frac{KZ}{ZD} = 1 = \frac{DP_1}{P_1B} \cdot \frac{BW}{WJ} \cdot \frac{JY}{YD}.$$

Since $BW = BX$, $WJ = JY$, $YD = ZD$ and $XK = KZ$ according to the two tangent theorem, this is reduced to

$$\frac{DP_2}{P_2B} = \frac{DP_1}{P_1B},$$

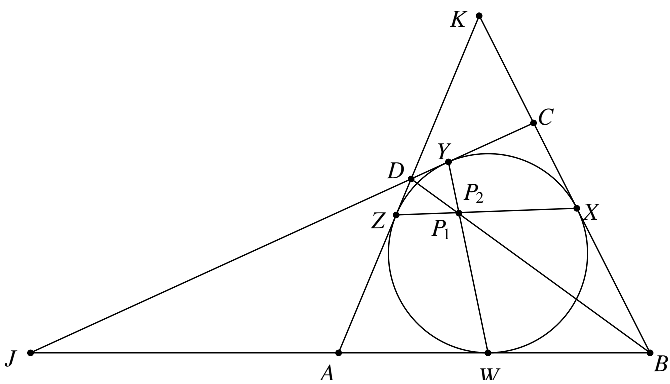

FIGURE 4: The diagonals and the tangency chords are concurrent

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which means that P_1 and P_2 divide BD in the same ratio. Thus $P_1 = P_2$, so WY, ZX and BD are concurrent at that point. In the same way it can be proved that WY, ZX and AC are concurrent. Hence all four of WY, ZX, AC and BD are concurrent at the diagonal intersection.

Seven new characterisations

We only consider *convex* quadrilaterals in all theorems. The first characterisation of bicentric quadrilaterals is a simple condition about angles related to the tangency chords.

Theorem 1

In a tangential quadrilateral, the tangency chords are angle bisectors to the angles between the diagonals if, and only if, the quadrilateral is also cyclic.

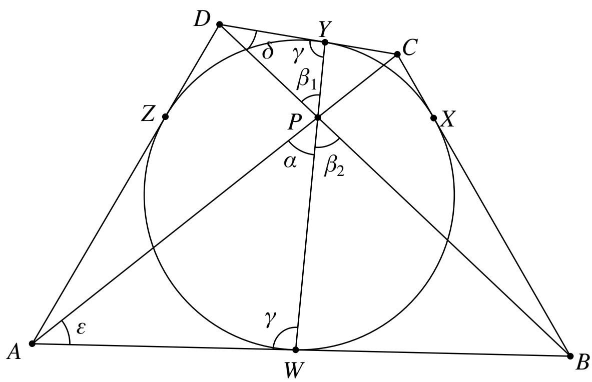

FIGURE 5: Tangential $ABCD$ is also cyclic if, and only if, $\alpha = \beta_2$

Proof

Besides that the diagonals and the tangency chords are concurrent, we use the well-known necessary and sufficient condition that a quadrilateral is cyclic if, and only if, the angle between one side and a diagonal is equal to the angle between the opposite side and the other diagonal. With notations as in Figure 5, the tangential quadrilateral is also cyclic if, and only if,

$$\varepsilon = \delta \qquad \Leftrightarrow \qquad \alpha = \beta_1 \qquad \Leftrightarrow \qquad \alpha = \beta_2,$$

which is equivalent to saying that the tangency chord WY is an angle bisector of angle APB. We used the fact that the two angles marked by γ are equal, which is true since they are exterior angles to the base angles of an isosceles triangle WYJ, where J is the point where the extensions of AB and CD intersect.

By symmetry the same result holds for the other tangency chord XZ.

Now we prove a necessary and sufficient condition for a cyclic quadrilateral to be tangential that involves the angle between the diagonals. To prove that this formula is valid in bicentric quadrilaterals was given as a problem in [2, p. 30].

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Theorem 2

The angle between the diagonals that is opposite side a in a cyclic quadrilateral with consecutive sides a, b, c, d satisfies

$$\tan\frac{\theta}{2} = \sqrt{\frac{bd}{ac}}$$

if, and only if, the quadrilateral is also tangential.

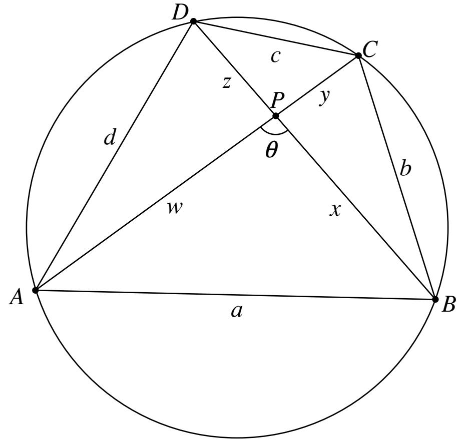

FIGURE 6: The diagonal parts in a cyclic quadrilateral

Proof

In a convex quadrilateral $ABCD$ with sides $a = AB$, $b = BC$, $c = CD$, $d = DA$ and with diagonal parts $w = AP$, $x = BP$, $y = CP$, $z = DP$, where P is the intersection of the diagonals, we have by the cosine rule (see Figure 6)

$$a^2 = w^2 + x^2 - 2wx\cos\theta,$$
$$b^2 = x^2 + y^2 + 2xy\cos\theta,$$
$$c^2 = y^2 + z^2 - 2yz\cos\theta,$$
$$d^2 = z^2 + w^2 + 2zw\cos\theta,$$

whence

$$a^2 - b^2 + c^2 - d^2 = -2(wx + xy + yz + zw)\cos\theta = -2pq\cos\theta \qquad (2)$$

where $w + y = p = AC$ and $x + z = q = BD$ are the lengths of the diagonals. Next we use the formula

$$\tan^2\frac{\theta}{2} = \frac{1 - \cos\theta}{1 + \cos\theta} = \frac{2pq - 2pq\cos\theta}{2pq + 2pq\cos\theta}.$$

In a cyclic quadrilateral we apply Ptolemy's theorem $pq = ac + bd$, as well as (2), to get

$$\tan^2\frac{\theta}{2} = \frac{2(ac + bd) + (a^2 - b^2 + c^2 - d^2)}{2(ac + bd) - (a^2 - b^2 + c^2 - d^2)} = \frac{(a + c)^2 - (b - d)^2}{(b + d)^2 - (a - c)^2}.$$

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If the quadrilateral is also tangential, then Pitot's theorem $a + c = b + d$ applies, giving

$$\tan^2 \frac{\theta}{2} = \frac{(b + d)^2 - (b - d)^2}{(a + c)^2 - (a - c)^2} = \frac{4bd}{4ac},$$

which simplifies to $\tan \frac{\theta}{2} = \sqrt{\frac{bd}{ac}}$.

Conversely, if the formula $\tan \frac{\theta}{2} = \sqrt{\frac{bd}{ac}}$ holds in a cyclic quadrilateral, then we get

$$\frac{(a + c)^2 - (b - d)^2}{(b + d)^2 - (a - c)^2} = \frac{bd}{ac},$$

which can be factorised as

$$(ac + bd)(a + c + b + d)(a + c - b - d) = 0.$$

This equality has only one possible solution, $a + c = b + d$, which according to the converse of Pitot's theorem implies that the quadrilateral is also tangential, completing the proof.

Let us consider what would happen if the cyclic and tangential quadrilateral were to change roles in the previous theorem. From Pitot's theorem we have $(a - d)^2 = (b - c)^2$, which when used together with (2) yields

$$2(bd - ac) = a^2 - b^2 + c^2 - d^2 = -2pq \cos \theta.$$

Thus

$$\tan^2 \frac{\theta}{2} = \frac{pq + bd - ac}{pq - bd + ac}.$$

Inserting Ptolemy's theorem $pq = ac + bd$, we have

$$\tan^2 \frac{\theta}{2} = \frac{2bd}{2ac},$$

which again gives the formula in the theorem (as expected, since this formula is valid in bicentric quadrilaterals). For the converse, however, solving the equation

$$\frac{pq + bd - ac}{pq - bd + ac} = \frac{bd}{ac}$$

we get

$$(bc - ac)(ac + bd - pq) = 0.$$

The second solution gives $ac + bd = pq$ and according to the converse of Ptolemy's theorem (a proof can be found in [3, pp. 20-21]), then the quadrilateral is cyclic. But the first solution $bd = ac$ combined with Pitot's

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theorem $a + c = b + d$ (remember this time we started with a tangential quadrilateral) yields

$$b(a + c) = b^2 + ac \qquad \Leftrightarrow \qquad (a - b)(b - c) = 0$$

with the solutions $a = b$ and thus $c = d$, or $b = c$ and then also $a = d$. In both of these cases we get a kite, which is always tangential but in general not cyclic. We conclude that it is not possible for the cyclic and tangential condition to be reversed.

Next we have a characterisation regarding the distances from the incentre to the vertices. The direct part of this theorem was proved in March 2003 at the geometry forum Hyancinthos (message number 6762) by Nikolaos Dergiades as a response to Juan Carlos Salazar, who stated that it is both a necessary and sufficient condition. However, neither he nor anybody else gave a proof of that claim. That forum is no longer available.

Theorem 3

In a tangential quadrilateral *ABCD* with incentre *I*,

$$\frac{1}{AI^2} + \frac{1}{CI^2} = \frac{1}{BI^2} + \frac{1}{DI^2}$$

if, and only if, the quadrilateral is also cyclic.

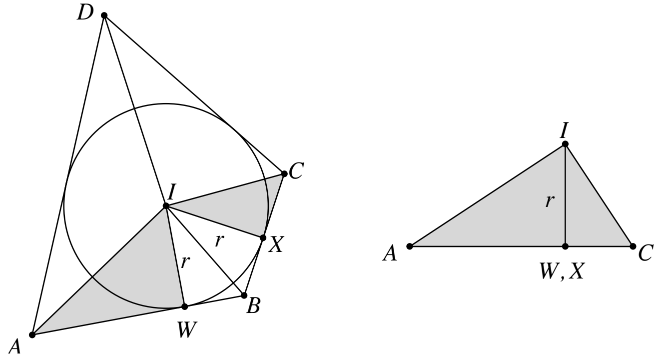

FIGURE 7: Creating triangle *AIC*

Proof

In a tangential quadrilateral *ABCD* that is not cyclic, we have $\angle A + \angle C = \pi + \varepsilon$ and $\angle B + \angle D = \pi - \varepsilon$ where ε is the deviation from being cyclic. Suppose the incircle is tangent to the sides *AB* and *BC* at *W* and *X* respectively. We make a new triangle *AIC* by joining *AIW* and *CIX* along the inradius $IW = IX$ (see Figure 7). In this new triangle, the angles are $\frac{1}{2}\angle A$, $\frac{1}{2}\angle C$ and $\angle AIC$, where $\angle A$ and $\angle C$ refer to the vertex angles at *A* and *C* in the tangential quadrilateral. Then

$$\angle AIC = \pi - \left(\frac{\angle A}{2} + \frac{\angle C}{2}\right) = \pi - \frac{\pi + \varepsilon}{2} = \frac{\pi}{2} - \frac{\varepsilon}{2}.$$

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Calculating the area of triangle AIC in two different ways yields

$$\frac{1}{2}r(AW + CX) = \frac{1}{2}AI \cdot CI \sin\left(\frac{\pi}{2} - \frac{\varepsilon}{2}\right)$$

where r is the inradius; hence

$$AW + CX = \frac{AI \cdot CI}{r} \cos\frac{\varepsilon}{2}. \tag{3}$$

Next we apply the cosine rule in triangle AIC to get

$$(AW + CX)^2 = AI^2 + CI^2 - 2AI \cdot CI \cos\left(\frac{\pi}{2} - \frac{\varepsilon}{2}\right)$$

and using (3) yields

$$\frac{1}{r^2} \cdot AI^2 \cdot CI^2 \cos^2\frac{\varepsilon}{2} = AI^2 + CI^2 - 2AI \cdot CI \sin\frac{\varepsilon}{2},$$

which we rewrite as

$$\frac{1}{r^2}\cos^2\frac{\varepsilon}{2} = \frac{1}{CI^2} + \frac{1}{AI^2} - \frac{2}{AI \cdot CI}\sin\frac{\varepsilon}{2}.$$

In the same way we have

$$\frac{1}{r^2}\cos^2\frac{\varepsilon}{2} = \frac{1}{DI^2} + \frac{1}{BI^2} + \frac{2}{BI \cdot DI}\sin\frac{\varepsilon}{2}.$$

Equating the last two expressions, we get

$$\frac{1}{AI^2} + \frac{1}{CI^2} - \frac{1}{BI^2} - \frac{1}{DI^2} = 2\sin\frac{\varepsilon}{2}\left(\frac{1}{BI \cdot DI} + \frac{1}{AI \cdot CI}\right)$$

which holds in all tangential quadrilaterals. Then

$$\frac{1}{AI^2} + \frac{1}{CI^2} = \frac{1}{BI^2} + \frac{1}{DI^2} \Leftrightarrow \sin\frac{\varepsilon}{2} = 0 \Leftrightarrow \varepsilon = 0,$$

since $0 \leqslant \varepsilon < \pi$. Thus $\varepsilon = 0$ is equivalent to $ABCD$ being cyclic.

We note that in a bicentric quadrilateral the inradius r satisfies

$$\frac{1}{r^2} = \frac{1}{AI^2} + \frac{1}{CI^2} = \frac{1}{BI^2} + \frac{1}{DI^2},$$

so just two opposite of the four distances from the incentre to the vertices are needed to calculate the inradius. In a tangential quadrilateral, all four of these distances are needed, and the formula is

$$r = 2\sqrt{\frac{(\sigma - uvx)(\sigma - vxy)(\sigma - xyu)(\sigma - yuv)}{uvxy(uv + xy)(ux + vy)(uy + vx)}}$$

where $u = AI$, $v = BI$, $x = CI$, $y = DI$ and $\sigma = \frac{1}{2}(uvx + vxy + xyu + yuv)$. This was derived in [4].

That the formula in the following theorem is valid in bicentric

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quadrilaterals was proved as Theorem 5 in [5]. Now let us prove that it is in fact both a necessary and sufficient condition for a tangential quadrilateral to be cyclic.

Theorem 4

A tangential quadrilateral $ABCD$ with incentre I has the area

$$K = AI \cdot CI + BI \cdot DI$$

if, and only if, it is also cyclic.

Proof

We start by deriving a similar formula for the area K of a tangential quadrilateral. Using the idea in the previous proof to form triangle AIC, and in the same way also a triangle BID, we get

$$\begin{aligned} K &= 2T_{AIC} + 2T_{BID} \\ &= AI \cdot CI \cdot \sin\left(\pi - \frac{A + C}{2}\right) + BI \cdot DI \cdot \sin\left(\pi - \frac{B + D}{2}\right) \\ &= (AI \cdot CI + BI \cdot DI) \sin\frac{A + C}{2}, \end{aligned}$$

where T_{AIC} denote the area of AIC and in the last step we used the angle sum in a quadrilateral. As a direct consequence, we have

$$K = AI \cdot CI + BI \cdot DI \Leftrightarrow \angle A + \angle C = \pi,$$

which is equivalent to the quadrilateral being cyclic.

The next characterisation was suggested to hold in [6], but no complete proof was given. An orthodiagonal quadrilateral is a quadrilateral with perpendicular diagonals.

Theorem 5

The intersections of the external angle bisectors to a tangential quadrilateral $ABCD$ create another quadrilateral $W'X'Y'Z'$, which is orthodiagonal if, and only if, $ABCD$ is also cyclic.

Proof

First we prove that the diagonals of $W'Y'X'Z'$ intersect at the incentre I of a tangential quadrilateral $ABCD$, where the internal angle bisectors intersect. Draw the line segments IW', IZ' and IY'. The quadrilaterals $AIBW'$, $DIAZ'$, $CIDY'$ are cyclic due to each having two opposite right angles (an internal angle bisector is perpendicular to the external angle bisector at the same vertex of $ABCD$). We get $\angle AIW' = \angle ABW' = \frac{1}{2}(\pi - \angle B)$ (see Figure 8) and similarly for angles in $DIAZ'$ and $CIDY'$. Thus

$$\angle Y'IW' = \frac{\pi - \angle C}{2} + \frac{\pi - \angle A}{2} + \frac{\pi - \angle D}{2} + \frac{\pi - \angle B}{2} = \pi,$$

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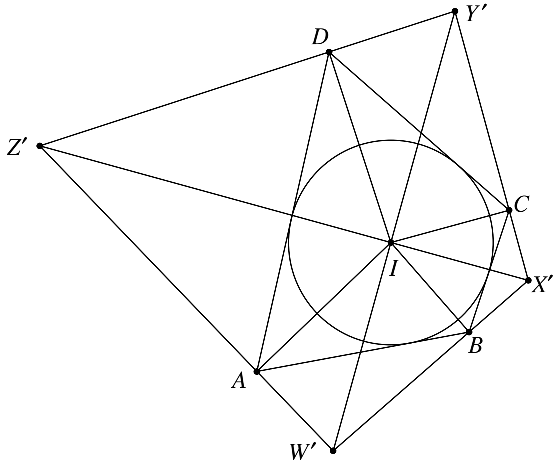

FIGURE 8: Tangential *ABCD* is also cyclic if, and only if, $W'Y' \perp X'Z'$

which proves that I lies on the diagonal $W'Y'$. By symmetry, I also lies on $Z'X'$. We further have

$$\angle Y'IZ' = \frac{\pi - \angle C}{2} + \frac{\pi - \angle A}{2} = \pi - \frac{\angle A + \angle C}{2}.$$

Hence

$$\angle Y'IZ' = \frac{\pi}{2} \Leftrightarrow \angle A + \angle C = \pi$$

and the conclusion that $Y'W'$ is perpendicular to $Z'X'$ if, and only if, *ABCD* is cyclic.

Using similar formulas as in the previous proof, it is an easy calculation to verify that $W'X'Y'Z'$ is always a cyclic quadrilateral for all convex quadrilaterals *ABCD*, which is a quite well-known property.

The next theorem has a close connection to Theorem 5 in [7].

Theorem 6

In a tangential quadrilateral *ABCD* where the incircle is tangent to the sides *AB*, *BC*, *CD*, *DA* at W, X, Y, Z respectively, let the external angle bisectors intersect outside of these sides at W', X', Y', Z' respectively. The line segments WY, XZ, $W'Y'$, $X'Z'$ create a quadrilateral *GHIJ*, which is a rectangle if, and only if, *ABCD* is also cyclic.

Proof

Suppose the extensions of *AB* and *CD* intersect at E. Let us first prove that the points E, Z', I, X' are collinear in all tangential quadrilaterals with incentre I (see Figure 9). The external angle bisectors of A and D intersect at Z'. Then EZ' is the angle bisector of the angle at E since the three angle bisectors in triangle *ADE* are concurrent at a point. Also *EI* is the bisector of

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angle E since I is the incentre in triangle BCE. Thus E, Z', I are collinear. In the proof of the previous theorem we concluded that Z', I, X' are collinear. Hence so are all four of E, Z', I, X' since there is only one line through the points Z' and I. In the same way, points F, W', I, Y' are collinear, where F is the intersection of the extensions of BC and DA.

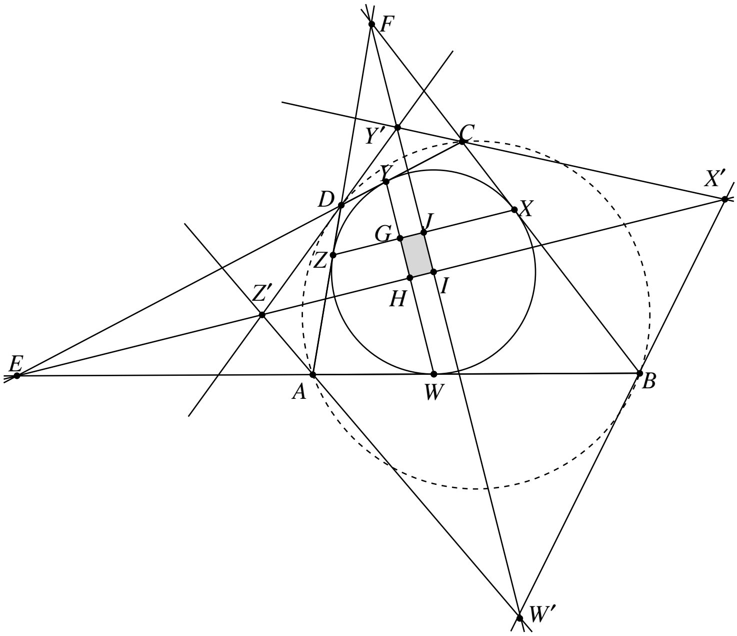

FIGURE 9: Tangential *ABCD* is also cyclic if, and only if, *GHIJ* is a rectangle

Next we conclude that EI is perpendicular to WY since it is an angle bisector in the isosceles triangle EWY. In the same way, FI is perpendicular to $X'Z'$. Thus quadrilateral $GHIJ$ always has right angles at H and J. Then it is a rectangle if, and only if, one of the other two angles is a right angle. According to the previous theorem, the angle at I is a right angle if, and only if, $ABCD$ is also cyclic, completing the proof.

The last necessary and sufficient condition is an extension of problem 3 on day 1 for grade level 10 from the All-Russian Olympiad in 2004 [8], which was about proving the necessary condition. Two solutions to that problem can be found at [9].

Theorem 7

In a tangential quadrilateral $ABCD$ where the incircle is tangent to the sides AB, BC, CD, DA at W, X, Y, Z respectively, let the external angle bisectors intersect outside of these sides at W', X', Y', Z' respectively. Then WW', XX', YY', ZZ' are concurrent if, and only if, $ABCD$ is also cyclic.

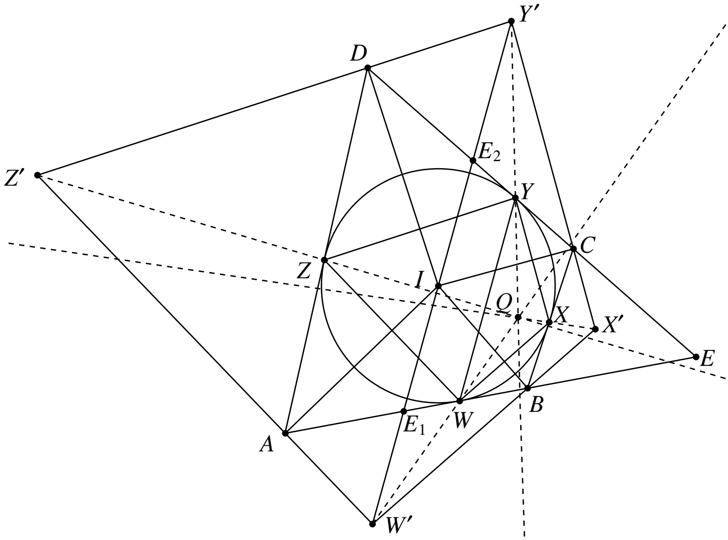

FIGURE 10: In search of isosceles triangles

Proof

We will use the following logic to prove this theorem: WW', XX', YY', ZZ' are concurrent at a point, say Q, which is equivalent to saying that the points in each of the triplets $\{W, W', Q'\}$, $\{X, X', Q\}$, $\{Y, Y', Q\}$, $\{Z, Z', Q\}$ are collinear. This is equivalent to that the two pairs of triangles WXY, $W'X'Y'$ and WXZ, $W'X'Z'$ each being homothetic, which in turn is equivalent to the corresponding sides in these pairs of triangles beind parallel, this being equivalent to having two pairs of isosceles triangles, which finally is equivalent to $ABCD$ also being cyclic. Thus the proof is reduced to proving this final equivalence; this is enough since it is trivial by simple angle properties that $WX \parallel W'X'$, $XY \parallel X'Y'$ and $YZ \parallel Y'Z'$, $ZW \parallel Z'W'$ in all tangential quadrilaterals (see Figure 10). So what isosceles triangles are we referring to?

We consider one of these pairs, the other being an identical argument with other letters. If the extensions of AB and CD intersect at E, then triangle WYE is isosceles since $WE = YE$, both being tangents to the incircle. For the second triangle, let the line segment $W'Y'$ intersect AB and CD at E_1 and E_2 respectively. We will prove that $E_1E = E_2E$ (triangle E_1E_2E being isosceles) if, and only if, $ABCD$ is cyclic (this is the same as saying that $WY \parallel W'Y'$ if, and only if, $ABCD$ is cyclic). Since

$$\angle BE_1I = \angle BAI + \angle W'IA = \frac{\angle A}{2} + \angle W'BA = \frac{\angle A}{2} + \frac{\pi}{2} - \frac{\angle B}{2}$$

and

$$\angle CE_2I = \angle CDI + \angle Y'ID = \frac{\angle D}{2} + \angle Y'CD = \frac{\angle D}{2} + \frac{\pi}{2} - \frac{\angle C}{2}$$

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we find that $E_1E = E_2E$ if, and only if,

$$\angle BE_1I = \angle CE_2I \Leftrightarrow \angle A - \angle B = \angle D - \angle C.$$

This is equivalent to

$$\angle A + \angle C = \angle B + \angle D \Leftrightarrow \angle A + \angle C = \pi,$$

by the angle sum in a quadrilateral, concluding the proof of this characterisation concerning four concurrent line segments.

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References

1. M. Josefsson, On Pitot's theorem, *Math. Gaz.* **103** (July 2019) pp. 333-337.
2. C. V. Durell and A. Robson, Advanced trigonometry, Dover (2003).
3. M. Josefsson, More characterizations of cyclic quadrilaterals, *Int. J. Geom.* **8**(2) (2019) pp. 14-32.
4. M. Josefsson, On the inradius of a tangential quadrilateral, *Forum Geom.* **10** (2010) pp. 27-34.
5. M. Josefsson, The area of a bicentric quadrilateral, *Forum Geom.* **11** (2011) pp. 155-164.
6. M. A. Covas, Problem 2209, *Crux Math.* **24**(2) (1998) pp. 112-113.
7. M. Josefsson, Characterizations of bicentric quadrilaterals, *Forum Geom.* **10** (2010) pp. 165-173.
8. 2004 All-Russian Olympiad, *AoPS Online*, available at https://artofproblemsolving.com/community/c5164
9. Quadrilateral which is a cyclic quadrilateral and tangent qu, *AoPS Online*, accessed March 2022 at https://artofproblemsolving.com/community/c6h5510

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