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# On critical point for two-dimensional holomorphic systems

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*Abstract*. Let $f : M \to M$ be a biholomorphism on a two-dimensional complex manifold, and let $X \subseteq M$ be a compact $f$-invariant set such that $f|_X$ is asymptotically dissipative and without periodic sinks. We introduce a solely dynamical obstruction to dominated splitting, namely critical point. Critical point is a dynamical object and captures many of the dynamical properties of a one-dimensional critical point.

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## 1. *Introduction*

In the study of complex dynamical systems in several variables, polynomial automorphisms of $\mathbb{C}^2$ are a first step in the global understanding of holomorphic dynamics in higher dimensions.

One of the first results in this direction was presented by Friedland and Milnor in [**10**]. They showed that for polynomial automorphisms in $\mathbb{C}^2$, the only systems (modulo conjugation by a polynomial automorphism) that exhibit rich dynamics are the so-called generalized Hénon maps (or complex Hénon maps). These maps are obtained as a finite composition of maps of the form $(y, p(y) - ax)$, where $p$ is a polynomial whose degree is at least two and $a \in \mathbb{C}^*$. In the early 1990s, complex Hénon maps were the subject of serious study, with foundational work carried out by Hubbard [**12**], Hubbard and Oberste-Vorth [**13**, **14**], Bedford and Smillie [**2–4**] and Bedford *et al* [**1**], among others.

In many of the studies mentioned, there are two-dimensional versions of classical one-dimensional results for rational maps. We are interested in obtaining, if possible, a two-dimensional version of the classic result that *a Julia set of a polynomial is hyperbolic if and only if the Julia set is disjoint from the postcritical set*. We shall return to this idea later.

In this paper we take the first steps in obtaining a result in this direction.

As in the case of rational maps, complex Hénon maps have a well-defined Julia set $J$. This set is a compact invariant set that contains the support of the unique measure of maximal entropy (see [**2**]). We will denote the support of the measure of maximal entropy by $J^*$. In this setting, hyperbolicity in the Julia set is the two-dimensional notion of expansiveness on the Julia set for rational maps. Moreover, hyperbolicity in $J$ or $J^*$ is one way to establish the equality $J = J^*$ that remains as an open question.

Recently in our paper [**22**], several equivalences of uniform hyperbolicity, under the hypothesis of dominated splitting for holomorphic map, were given. Therefore it is important to determine when the function has dominated splitting.

*Question.* When does the set $J$ or $J^*$ have dominated splitting?

Dominated splitting in one dimension is an empty notion. However, since domination is necessary to obtain hyperbolicity in dimension $n \geq 2$, we may 'assume' that these two notions are equal in dimension one. With this in mind, the answer to the previous question is already known in the one-dimensional context. In the real setting, Mañé showed that smooth and generic (Kupka–Smale) one-dimensional endomorphisms without critical points are either hyperbolic or conjugate to an irrational rotation (see [**17**]). In the complex case of rational maps, the Julia set $J$ is hyperbolic if and only if $J$ and the postcritical set are disjoint. Hence we would say that *for generic smooth one-dimensional endomorphisms, any compact invariant set is hyperbolic if, and only if, it does not contain or is not approximated to critical points*. In other words, in one-dimensional settings, critical points portray the dynamical obstruction for hyperbolicity.

Our main goal is to introduce the dynamical obstruction to accomplish dominated splitting for two-dimensional biholomorphisms in complex manifolds. This will allow us to introduce the notion of *critical point* for complex Hénon maps, which capture many of the dynamical properties of their one-dimensional counterpart.

The notion of critical point was given by Pujals and Rodrigues Hertz in their paper [**20**], for surfaces. The main result of [**20**] states that, under certain hypotheses, $C^2$-generic diffeomorphisms have dominated splitting if and only if the set of critical points is empty. From Theorem B of Pujals and Sambarino in [**21**], the authors of [**20**] conclude that, *generically, an invariant set is either a hyperbolic set or a normally hyperbolic closed curve whose dynamics is conjugated to an irrational rotation, if and only if the set of critical points is empty*. We remark that the authors of [**20**] prove their main result using [**21**, Theorem B]. Later, Crovisier [**8**] gave a new proof of the main result in [**20**], independent of the Pujals–Sambarino theorem. We also remark that our definition of critical points differs from the original one given in [**20**].

To introduce the notion of critical point for a polynomial automorphism $f$, we look for the projective action of the derivative. Let

$$df_x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix},$$

and let

$$F_x(z) = \frac{a_x z + b_x}{c_x z + d_x}$$

be the Möbius transformation induced by $df_x$. We denote the spherical norm of the derivative of $F_x$ at the point $\xi \in \overline{\mathbb{C}}_x$ by $\|F_x'(\xi)\|$. Notice that for a polynomial automorphism their Jacobian determinant is constant. In what follows, we will assume that $f$ is a dissipative map, that is, $|\det(df_x)| = b < 1$. Let $\beta = (\beta_-, \beta_+)$ with $b < \beta_+ \le \beta_- < 1$. We say that $x \in \mathbb{C}^2$ is a $\beta$-*critical point* if there exists $\xi \in \overline{\mathbb{C}}_x$ such that

$$\begin{cases} \|(F_x^{-n})'(\xi)\| \ge \beta_-^{-n} & \text{for each } n \ge 0, \\ \|(F_x^{n})'(\xi)\| \ge \beta_+^{n} & \text{for each } n \ge 0. \end{cases} \tag{1.1}$$

We denote the set of all $\beta$-critical points by $\mathrm{Crit}(\beta)$. The preceding definition asserts that a point is critical when there exists a (projective) direction that is expanded (in norm) to the past by the action of $F$, but the possible contraction to the future is weak.

MAIN THEOREM. *Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be a dissipative complex Hénon map, with $|\det(df_x)| = b < 1$. Then $J$ has dominated splitting if and only if* $\mathrm{Crit}(\beta) = \emptyset$ *for some $\beta = (\beta_-, \beta_+)$ with $b < \beta_+ \le \beta_- < 1$.*

This theorem is a consequence of our Theorem A stated for complex linear cocycles over a vector bundle with compact base. A hypothesis necessary both in the surfaces setting is the absence of sinks. In our version it is only necessary that the difference between Lyapunov exponents be greater than $|\log(b)|$. This last fact is a direct consequence of the absence of sinks plus dissipativity (see Lemma 2.8). Since Julia sets for complex Hénon maps only contain periodic saddle points, this hypothesis of absence of sinks does not appear in the statement of our main theorem.

For now, there is no Pujals–Sambarino theorem in the two-dimensional complex case. The way to prove Theorem A is to adapt in our context the main ideas of Crovisier in [**8**]. However, since the definition of critical point in [**20**] and our definition are distinct, these ideas, as adapted, are different from the original version in several respects.

At this point, we will explain several properties relating to the critical set. Firstly, the critical set is a compact set. Secondly, we will introduce a partial order in the index set

$$\Delta = \{\beta = (\beta_-, \beta_+) : b < \beta_+ \le \beta_- < 1\}.$$

For $\alpha, \beta \in \Delta$, we say that $\beta \ge \alpha$ if and only if $\alpha_+ \le \beta_+ \le \beta_- \le \alpha_-$. Then we have a monotony property: if $\beta \ge \alpha$ then $\mathrm{Crit}(\beta) \subseteq \mathrm{Crit}(\alpha)$. Thirdly, we have an invariance property. In fact, critical points remain critical both under a change of the hermitian metric and for conjugated systems, perhaps after finite bounded iterates to the past or to the future. We remark that the monotony and invariance properties do not hold in the original definition of critical points given in [**20**], and are the principal motive for changing the definition of critical point†. Another consequence of changing the definition is that there are more critical points than in the original definition. This allows us to have a real object that denies domination.

Another property is that the critical set is far from a compact dominated, hyperbolic or Pesin's block set. This follows directly from compactness of the critical set. Moreover, critical points are not a regular point in the Oseledets sense, and a more important property is that the orbit of a tangency between the stable manifold and the unstable manifold contains critical points.

We recall that the previous properties hold when the critical set is viewed over a compact invariant set.

Let us now return to our initial purpose: to obtain a two-dimensional characterization of hyperbolicity. Recall that a polynomial in $\mathbb{C}$ always has critical points. Moreover, the critical points determine the global dynamics of a polynomial. In fact, we can state the following. *Let $p$ be a polynomial over $\mathbb{C}$ whose degree is at least two. Then the Julia set $J_p \subset \mathbb{C}$ is hyperbolic if and only if the postcritical set given by*

$$PC(p) = \overline{\bigcup_{n \geq 1} p^n(\{z : p'(z) = 0\})}$$

*satisfies $PC(p) \cap J_p = \emptyset$.*

Following the previous result, we wonder if, for complex Hénon maps, there always exist critical points, even outside the Julia set. Moreover, if these 'critical points' exist, we wonder if they determine the global dynamics. Recall that our main theorem asserts that *a dissipative complex Hénon map $f$ has dominated splitting in the Julia set $J_f$ if and only if* $\mathrm{Crit} \cap J_f = \emptyset$. Then we can formulate the following question.

*Question A.* Does a critical point in $\mathbb{C}^2$ always exist? That is, can it happen that $\mathrm{Crit} \cap J_f^c \neq \emptyset$?

If we denote $K^+ = \{x \in \mathbb{C}^2 : \{f^n(x)\}_{n \geq 0}$ is bounded$\}$, another question arises.

*Question B.* If $K^+$ has an interior, is it possible that a critical point in $K^+$ always exists?

We can answer Question B positively only in a particular case, but we do not know how to give a positive answer in a general context. In fact, let

$$f_\delta(x, y) = (y, p(y) + (1 + \delta)y - \delta x),$$

where $\delta \in \mathbb{C}^*$ and $p$ is a polynomial with a zero of order $k + 1$ at the origin with $\deg(p) = d \geq 2$. Then for $|\delta| < 1$, $f_\delta$ has the origin of $\mathbb{C}^2$ as a semi-parabolic fixed point (*also called semi-attractive*), that is, $Df_\delta(0)$ has eigenvalues equal to 1 and $\delta$. Then, (see [**11**, Theorem 1.3]) for each $j = 1, \ldots, k$, there exists an open set $\mathcal{B}_j \subset \mathbb{C}^2$ such that the sets $\mathcal{B}_j$ are disjoint, each $\mathcal{B}_j$ is biholomorphic to $\mathbb{C}^2$, $0 \in \partial\mathcal{B}_j$ and, for each $q \in \mathcal{B}_j$, $f_\delta^n(q) \to 0$

† For example, a $\delta$-critical point in the sense of [**20**] does not necessarily remain $\delta'$-critical for any $\delta' \approx \delta$.

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when $n \to +\infty$. In other words, they are connected components of the basin of attraction of 0. Therefore $\mathcal{B}_j \subset (K^+(f_\delta))^\circ$ and each $\mathcal{B}_j$ is a component of the basin of attraction of $0 \in \mathbb{C}^2$.

On the other hand, each $\mathcal{B} = \mathcal{B}_j$ is endowed with a holomorphic strong stable foliation $\mathcal{F}^{ss}(\mathcal{B})$, whose leaves are characterized by the property that points in the same leaf approach one another exponentially fast under iteration (see [**9**, Proposition 4.1]). Let $w \in J_\delta = J_{f_\delta}$ be a saddle periodic point. We say that $\mathcal{B}$ has a tangency with $W^u(w)$ if there exists a tangency between $W^u(w)$ and some strong stable leaf of $\mathcal{F}^{ss}(\mathcal{B})$. We recall that the orbit of such tangency contains a critical point, and critical point accumulate on $J_\delta$ by forward/backward iterates. Then we have the following proposition.

PROPOSITION 1.1. [**9**] *Let $|\delta| < d^{-2}$ and let $w \in J_\delta$ be a periodic saddle point. Then each component of the basin of attraction of $0 \in \mathbb{C}^2$ has a tangency with $W^u(w)$.*

*Proof.* This follows from [**9**, Corollary 5.7 and Proposition 5.8]. □

Moreover, it follows from [**9**] that if $f$ has a semi-parabolic periodic point on $J$, and $|\det(Df)| < d^{-2}$, then there exist critical points $x$ in the interior of $K^+$ such that $\operatorname{dist}(f^n(x), J) \to 0$ where $n \to \pm\infty$.

On the other hand, we can take $p$ and $\delta$ such that the function $f_\delta$ has dominated splitting on $J_\delta$. Let $q(z) = p(z) + z$ such that the critical points of $q$ are far from $J(q) \subset \mathbb{C}$. Then

$$f_\delta(x, y) = (y, q(y) + \delta y - \delta x)$$

and the derivative

$$Df_\delta(x, y) = \begin{pmatrix} 0 & 1 \\ -\delta & q'(y) + \delta \end{pmatrix}$$

preserves the vertical cone

$$C^h = \{(u, v) \in \mathbb{C}^2 : |u| \le |v|\}$$

over $J_\delta$ for $|\delta|$ small enough. In fact, take $|\delta|$ small enough such that if $(x, y) \in J_\delta$ then $y$ is near $J(q)$. Hence, we may assume that if $(x, y) \in J_\delta$ then $|q'(y)| > 1$ and

$$\begin{aligned} \frac{|v|}{|q'(y)v + \delta(v-u)|} &\le \frac{|v|}{|q'(y)v| - |\delta| \cdot |v-u|} \\ &\le \frac{|v|}{|q'(y)v| - |\delta| \cdot (|v| + |u|)} \\ &\le \frac{1}{|q'(y)| - |\delta| \cdot (1 + |u/v|)}. \end{aligned}$$

Therefore, for $|\delta|$ small enough, $Df_\delta(x, y)(C^h) \subset \operatorname{int}(C^h) \cup \{0\}$. Then domination of $J_\delta$ follows from Proposition 2.15.

The previous analysis shows that it is possible that the Julia set $J$ does not contain critical points; however, there are critical points approaching $J$ under forward/backward iterates. In this case $J$ contains a semi-parabolic periodic point and therefore $J$ cannot be hyperbolic.

On the other hand, if $J$ is hyperbolic, it cannot contain any semi-parabolic periodic points, but we do not know if in this case the set $\mathrm{int}(K^+)$ contains critical points. With all the foregoing in mind, and defining

$$PC(f) = \overline{\bigcup_{n\in\mathbb{Z}} f^n(\mathrm{Crit}(f))},$$

we make the following conjecture.

CONJECUTRE. *If $J \cap PC(f) = \emptyset$, then $J$ is hyperbolic.*

The paper is organized as follows. In §2 we state preliminaries about linear and projective cocycles. We introduce the notion of multiplier of a projective cocycle. We also present a series of elementary tools that are used throughout the paper.

In §3 we introduce the notion of critical point and state its main properties. We will also show that the orbit of a tangency point contains critical points.

In §4 we introduce the notion of hyperbolic projective cocycle and relate this to dominated splitting.

In §5 we give the main tool used in the proof of our main theorem, the criterion of domination.

In §6 we prove our main theorem.

In Appendix A we state the relation between hermitian and spherical metrics, and how these are defined in a arbitrarily bundle.

## 2. *Preliminaries*

### 2.1. *Bundles and cocycles.*

Let $X$ be a compact metric space. We denote a *complex vector bundle* over $X$ of complex dimension 2 by $TX$ (cf. [**15**]). We also denote the bundle projection by $\mathrm{pr} : TX \to X$ and the fiber over $z \in X$ by $T_z = \mathrm{pr}^{-1}(\{z\})$.

Let $U \subseteq X$ be an open set. In what follows, a *section* over $U$ is a continuous function $\sigma : U \to TX$ such that $\mathrm{pr}(\sigma(x)) = x$ for all $x \in U$. We denote the set of all sections over $U$ by $\Gamma(U, TX)$.

A *projective bundle* over $X$ is a bundle $\pi : \mathbb{P}(X) \to X$ where $\pi$ denotes the bundle projection and the fiber $\overline{\mathbb{C}}_z = \pi^{-1}(\{z\})$ has a Riemann surface structure biholomorphic to the Riemann sphere $\overline{\mathbb{C}}$.

The following proposition establishes a bijection between linear and projective bundles.

PROPOSITION 2.1. *Given a vector bundle TX, the set*

$$\mathbb{P}(X) = \bigcup_{z\in X} \{z\} \times \mathbb{P}^1(T_z) \tag{2.1}$$

*has the structure of a projective bundle, called the projective bundle induced by TX.*

*Reciprocally, given a projective bundle $\mathbb{P}(X)$, there exists a vector bundle TX, such that $\mathbb{P}(X)$ is the projective bundle induced by TX.*

*Proof.* Let $\{(W_i, \psi_i, U_i); i = 1, \ldots, n\}$ be a system of trivialization functions for $TX$. The system satisfies the following properties.

(1) $\mathrm{pr}^{-1}(U_i) = W_i$, $X = \bigcup_{i=1}^n U_i$ and $TX = \bigcup_{i=1}^n W_i$.

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(2) For each $i$, the function $\psi_i : W_i \to U_i \times \mathbb{C}^2$ is a homeomorphism such that $\psi_{i,z} = \psi_i \mid T_z : T_z \to \{z\} \times \mathbb{C}^2$ is a linear isomorphism. Moreover, if $v \in T_z$ we can write $\psi_{i,z}(v) = (z, L_i(z)v)$.

(3) We denote $W_{ij} = W_i \cap W_j$ and $U_{ij} = U_i \cap U_j = \mathrm{pr}(W_{ij})$. The change of coordinate $\psi_{ij} = \psi_j \circ \psi_i^{-1} : U_{ij} \times \mathbb{C}^2 \to U_{ij} \times \mathbb{C}^2$ satisfies $\psi_{ij}(z, u) = (z, L_{ij}(z)u)$ where $L_{ij}(z) = L_j(z) \circ L_i^{-1}(z)$ and $L_{ij} : U_{ij} \to \mathbb{GL}(2, \mathbb{C})$ is continuous.

Let $\mathbb{P}(X)$ defined as in equation (2.1) and let $\pi : \mathbb{P}(X) \to X$ be the projection of the first coordinate. We claim that there exits a system of trivialization functions for $\mathbb{P}(X)$. More precisely, there exists a set $\{(P_i, \varphi_i, U_i); i = 1, \ldots, n\}$ satisfying the following properties.

(a) $\pi^{-1}(U_i) = P_i$, $X = \bigcup_{i=1}^n U_i$ and $\mathbb{P}(X) = \bigcup_{i=1}^n P_i$.

(b) For each $i$, the function $\varphi_i : P_i \to U_i \times \overline{\mathbb{C}}$ is a homeomorphism such that $\varphi_{i,z} = \varphi_i \mid \overline{\mathbb{C}}_z : \overline{\mathbb{C}}_z \to \{z\} \times \overline{\mathbb{C}}$ is a biholomorphism. Moreover, if $\xi \in \overline{\mathbb{C}}_z$ we can write $\varphi_{i,z}(\xi) = (z, H_i(z)\xi)$.

(c) Denoting $P_{ij} = P_i \cap P_j$ then $U_{ij} = \pi(P_i \cap P_j)$ and the change of coordinate $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1} : U_{ij} \times \overline{\mathbb{C}} \to U_{ij} \times \overline{\mathbb{C}}$ satisfies $\varphi_{ij}(z, u) = (z, H_{ij}(z)u)$ where $H_{ij}(z) = H_j(z) \circ H_i^{-1}(z)$ is a Möbius transformation whose coefficient varies continuously with respect to $z$.

To prove this claim, we denote by $[v] \in \mathbb{P}^1(T_z)$ the equivalence class of $v \in T_z$. Therefore, $\xi \in \overline{\mathbb{C}}_z = \{z\} \times \mathbb{P}^1(T_z)$ if and only if it has the form $\xi = (z, [v])$. Defining $P_i = \pi^{-1}(U_i)$, $H_i(z)\xi = [L_i(z)v]$ and $\varphi_i : P_i \to U_i \times \overline{\mathbb{C}}$ by $\varphi_i(\xi) = (z, H_i(z)\xi)$. It is not difficult to check that the object defined above, satisfies items (a)–(c). Therefore it is a system of trivialization functions for $\mathbb{P}(X)$.

For the reverse direction, we let $\{(P_i, \varphi_i, U_i); i = 1, \ldots, n\}$ satisfy items (a)–(c) as above. We take $i \in \{1, \ldots, n\}$ fixed and $z \in U_i$. We will induce an algebraic structure in each set $\overline{\mathbb{C}}_z$. In fact we denote $\lambda_{i,z} = H_i^{-1}(z)(\lambda)$ for $\lambda \in \overline{\mathbb{C}}$. For each $\xi, \zeta \in \overline{\mathbb{C}}_z \setminus \{\infty_{i,z}\}$ we define: $\xi + \zeta$ by $(H_i(z)\xi + H_i(z)\zeta)_{i,z}$ and $\xi \cdot \zeta$ by $(H_i(z)\xi \cdot H_i(z)\zeta)_{i,z}$. Now define

$$T_{i,z} = \{(\lambda_{i,z}, 0_{i,z}) : \lambda \in \mathbb{C}\} \cup \{(\lambda_{i,z} \cdot \xi, \lambda_{i,z}) : \xi \in \overline{\mathbb{C}} \setminus \{\infty_{i,z}\} \text{ and } \lambda \in \mathbb{C}\}.$$

The set $T_{i,z}$, with vector addition operation and scalar multiplication defined in the natural fashion, is a two-dimensional complex vector space. We define $L_i(z) : T_{i,z} \to \mathbb{C}^2$ by

$$L_i(z)(w_1, w_2) = \begin{cases} \lambda(H_i(z)\xi, 1), & (w_1, w_2) = (\lambda_{i,z} \cdot \xi, \lambda_{i,z}), \\ (\lambda, 0), & (w_1, w_2) = (\lambda_{i,z}, 0_{i,z}). \end{cases}$$

It is easy to see that $L_i(z)$ is a linear isomorphism.

On the other hand, for each $z \in U_{ij}$ we take the Möbius transformation

$$H_{ij}(z)u = \frac{a_{ij}(z)u + b_{ij}(z)}{c_{ij}(z)u + d_{ij}(z)}.$$

We recall that the coefficient of $H_{ij}(z)$ varies continuously with respect to $z$, and $a_{ij}(z)d_{ij}(z) - c_{ij}(z)b_{ij}(z) \neq 0$. So the map $L_{ij} : U_{ij} \to \mathbb{GL}(2, \mathbb{C})$ defined by

$$L_{ij}(z) = \begin{pmatrix} a_{ij}(z) & b_{ij}(z) \\ c_{ij}(z) & d_{ij}(z) \end{pmatrix}$$

is continuous. We set

$$TX = \Bigg( \bigsqcup_{\substack{i\in\{1,\ldots,n\}\\ z\in U_i}} \{z\} \times T_{i,z} \Bigg) \Big/ \sim$$

where $\sim$ is the equivalence relation given by $(z, v) \sim (w, u)$ if and only if $z = w$ and $L_j^{-1}(z) \circ L_{ij}(z) \circ L_i(z)v = u$.

Finally, we define $\mathrm{pr} : TX \to X$ in the natural fashion and $\{(W_i, \psi_i, U_i); i = 1, \ldots, n\}$ where $W_i = \mathrm{pr}^{-1}(U_i)$ and $\psi_i(z, v) = (z, L_i(z)v)$. It follows from construction, that the system above is a system of trivialization functions for $TX$ and that $\mathbb{P}(X)$ is the projective bundle induced by $TX$. □

Let $TX \otimes TX$ be the subset of $TX \times TX$ consisting of the pairs $(u, v)$ such that $u$ and $v$ are in the same fiber. A *hermitian metric* on $TX$ is a continuous function $(\cdot \mid \cdot) : TX \otimes TX \to \mathbb{C}$ such that $(\cdot \mid \cdot)|_{T_z\times T_z} = (\cdot \mid \cdot)_z$ is a hermitian product in $T_z$. Since $X$ is compact, there exists a hermitian metric on $TX$ (see [**15**]). In what follows, we denote $\|v\|_z = (v|v)_z$.

A *linear cocycle* $A : TX \to TX$ (respectively, *projective cocycle* $M : \mathbb{P}(X) \to \mathbb{P}(X)$) is an isomorphism in the category of the vector bundles (respectively, projective bundles) (cf. [**15**]). More precisely, $A$ (respectively, $M$) is continuous, there exists $f : X \to X$ a homeomorphism such that $\mathrm{pr} \circ A = f \circ \mathrm{pr}$ (respectively, $\pi \circ M = f \circ \pi$) and $A_z = A \mid T_z : T_z \to T_{f(z)}$ is a $\mathbb{C}$-linear isomorphism (respectively, $M_z = M \mid \overline{\mathbb{C}}_z : \overline{\mathbb{C}}_z \to \overline{\mathbb{C}}_{f(z)}$ is a biholomorphism). We say that the homeomorphism $f$ is the *base* of the cocycle and we write $A = (f, A_*)$ (respectively, $M = (f, M_*)$).

Let $A : TX \to TX$ be a linear cocycle. The *projective cocycle induced* by $A$ is the cocycle $M : \mathbb{P}(X) \to \mathbb{P}(X)$ given by $M_z([v]) = [A_z v]$, where $[v]$ denotes the class of $v$ in the projective space $\mathbb{P}^1(T_z)$. When $M$ is the projective cocycle induced by $A$, we write $M = [A]$.

*Remark 2.2.* It is clear that given $A$, the cocycle $M = [A]$ is uniquely defined. However, there are different linear cocycles defining the same projective cocycle.

The following proposition establishes that a projective cocycle is always induced by a linear cocycle. We recall that this linear cocycle is not necessarily uniquely defined.

PROPOSITION 2.3. *Let $\mathbb{P}(X)$ be the projective bundle induced by the vector bundle TX. Let $M : \mathbb{P}(X) \to \mathbb{P}(X)$ be a projective cocycle. Then there exists a linear cocycle $A : TX \to TX$ such that $M = [A]$.*

*Proof.* Let $\{U_i : i = 1, \ldots, n\}$ be a finite open cover of $X$. Assume that for each $i \in \{1, \ldots, n\}$ there exist $\sigma_i, \sigma_i' \in \Gamma(U_i, TX)$ such that $\{\sigma_i(z), \sigma_i'(z)\}$ is an orthonormal base of $T_z$. We let $L_i(z) : T_z \to \mathbb{C}^2$ be the linear map such that $L_i(z)(\sigma_i(z)) = (1, 0)$ and $L_i(z)(\sigma_i'(z)) = (0, 1)$ for each $z \in U_i$. From continuity of the base of $T_z$ it follows that the map $z \mapsto L_i(z)$ is continuous.

Define $W_i = \mathrm{pr}^{-1}(U_i)$ and $\psi_i : W_i \to U_i \times \mathbb{C}^2$ given by $\psi_i(z, v) = (z, L_i(z)v)$ where $z \in U_i$ and $v \in T_z$. Therefore the set $\{(W_i, \psi_i, U_i) : i = 1, \ldots, n\}$ is a system of trivialization functions for $TX$. The induced system of trivialization functions for $\mathbb{P}(X)$ is denoted by $\{(P_i, \varphi_i, U_i) : i = 1, \ldots, n\}$, and we will use the notation of Proposition 2.1.

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Let $M$ be a projective cocycle with base $f$ and let $V(i, j) = U_i \cap f^{-1}(U_j)$. From the definition of projective cocycles, it follows that there exists a continuous map from $V(i, j)$ into the Möbius transformations, $z \mapsto \widetilde{M}_{ij}(z)$, such that

$$\widetilde{M}_{ij}(z) = H_j(f(z)) \circ M_z \circ H_i^{-1}(z),$$

where

$$\widetilde{M}_{ij}(z)u = \frac{\widetilde{a}_{ij}(z)u + \widetilde{b}_{ij}(z)}{\widetilde{c}_{ij}(z)u + \widetilde{d}_{ij}(z)}$$

and $\widetilde{a}_{ij}(z)\widetilde{d}_{ij}(z) - \widetilde{c}_{ij}(z)\widetilde{b}_{ij}(z) = 1$. It follows from the foregoing that if $z \in V(i, j) \cap V(k, l)$ then

$$H_j^{-1}(f(z)) \circ \widetilde{M}_{ij}(z) \circ H_i(z) = H_l^{-1}(f(z)) \circ \widetilde{M}_{kl}(z) \circ H_k(z). \tag{2.2}$$

On the other hand, for each $z \in V(i, j)$ we define

$$\widetilde{A}_{ij}(z) = \begin{pmatrix} \widetilde{a}_{ij}(z) & \widetilde{b}_{ij}(z) \\ \widetilde{c}_{ij}(z) & \widetilde{d}_{ij}(z) \end{pmatrix}.$$

We recall that $\{V(i, j) : i, j = 1, \ldots, n\}$ is an open cover of $X$. For each $z \in X$ we define the cocycle $A : TX \to TX$ with base $f$ by

$$A_z = L_j^{-1}(f(z)) \circ \widetilde{A}_{ij}(z) \circ L_i(z) \quad \text{when } z \in V(i, j).$$

From equation (2.2) it follows that

$$L_j^{-1}(f(z)) \circ \widetilde{A}_{ij}(z) \circ L_i(z) = L_l^{-1}(f(z)) \circ \widetilde{A}_{kl}(z) \circ L_k(z).$$

Therefore from the equality above, we conclude that the definition of $A_z$ is not dependent on the indices $(i, j)$ and it is easy to see that $M = [A]$. Our proposition is proven. □

Given $l > 0$ we define the iterates of $A$ by the equation

$$A_z^l = A_{f^{l-1}(z)} \circ \cdots \circ A_{f(z)} \circ A_z \quad \text{and} \quad A_z^{-l} = A^{-1}_{f^{-l}(z)} \circ \cdots \circ A^{-1}_{f^{-2}(z)} \circ A^{-1}_{f^{-1}(z)},$$

and $A_z^0 = \mathrm{Id}$. We define the iterates $M_z^l$ for $l \in \mathbb{Z}$ similarly.

Choosing a hermitian metric on *TX*, we get an associated *spherical metric on the projective bundle* $\mathbb{P}(X)$. We refer the reader to Appendix A for details of the construction of this spherical metric.

2.2. *Oseledets's theorem.* We denote by $\mathcal{M}(X, f)$ the set of all $f$-invariant probability measures. We say that $z \in X$ is a *regular point of* $A$, if the fiber $T_z$ admits a splitting $T_z = E_z \oplus F_z$ into one-dimensional complex subspaces, and numbers $\lambda^-(x) \leq \lambda^+(z)$ satisfying

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A_z^n u\| = \lambda^-(z) \quad \text{and} \quad \lim_{n\to\pm\infty} \frac{1}{n} \log \|A_z^n v\| = \lambda^+(z),$$

where $u \in E_z\backslash\{0\}$ and $v \in F_z\backslash\{0\}$. Recall that a set $S \subset X$ has *total probability in* $X$, if for every $\mu \in \mathcal{M}(X, f)$, we have $\mu(S) = 1$.

THEOREM 2.4. (Oseledets) *The set of regular points of $A$ has total probability. Moreover, $z \mapsto E_z$ and $z \mapsto F_z$ are measurable subbundles and the functions $z \mapsto \lambda^{\pm}(z)$ are measurable.*

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For a proof of Oseledets's theorem in the setting of cocycles, see [**5**].

We denote the set consisting of all regular points of a cocycle $A$ by $\mathcal{R}(A)$. Oseledets's theorem asserts that if $\mu \in \mathcal{M}(X, f)$ then $\mu(\mathcal{R}(A) \cap \operatorname{supp}(\mu)) = 1$. We denote the set $\mathcal{R}(A) \cap \operatorname{supp}(\mu)$ by $\mathcal{R}(A, \operatorname{supp}(\mu))$.

In the original work of Pujals and Rodriguez Hertz (cf. [**20**]), an important hypothesis is the absence of periodic sinks. In our setting, we replace this hypothesis by the following notion.

*Definition 2.5.* We say that $\mu \in \mathcal{M}(X, f)$ is *partially hyperbolic for* $A$, if for any $z \in \mathcal{R}(A, \sup(\mu))$ the inequality $\lambda^-(z) < 0 \leq \lambda^+(z)$ holds.

We also say that $A$ *has no attractors* (in the broad sense) if all $f$-invariant probability measures are partially hyperbolic.

*Definition 2.6.* Given $0 < b < 1$, we say that $A$ is *b-asymptotically dissipative* if there exists a positive constant $C > 0$ such that for every $z \in X$, $|\det A_z^n| \leq C b^n$ for every $n \geq 0$.

*Definition 2.7.* We say that a measure $\mu \in \mathcal{M}(X, f)$ has the *exponent b-separated* if for every $z \in \mathcal{R}(A, \sup(\mu))$ the inequality $\lambda^+(z) - \lambda^-(z) \geq |\log(b)|$ holds.

We have the following lemma.

LEMMA 2.8. *Let $A$ be a b-asymptotically dissipative linear cocycle that has no attractors. Then each $\mu \in \mathcal{M}(X, f)$ has the exponent b-separated.*

*Proof.* Note that for each $z \in \mathcal{R}(A, \sup(\mu))$, $b$-dissipativity implies that

$$\lim_{n\to\infty} \frac{1}{n} \log |\det A_z^n| = \lambda^-(z) + \lambda^+(z) \leq \log(b) < 0,$$

and since $\lambda^+(x) \geq 0$ and $\lambda^-(x) < 0$,

$$\log(b) \geq \lambda^-(z) + \lambda^+(z) \geq \lambda^-(z) \geq \lambda^-(z) - \lambda^+(z). \qquad \square$$

2.3. *The multiplier.* In studies of rational maps in the Riemann sphere, an important tool to describe the dynamics near a fixed point is the notion of multiplier. By Böcher's theorem, the dynamics in a neighborhood of the periodic point is given (via conjugation) by the dynamics of the map $w \mapsto \lambda w$, where $\lambda$ is called the multiplier of the point. However, in many cases it is important to know the value of $|\lambda|$ instead of $\lambda$. So for convenience, we refer to $|\lambda|$ as the *multiplier*.

For a point $z \in \overline{\mathbb{C}}$ which is not periodic, it is possible to define a tool similar to the multiplier, using the spherical metric.

*Definition 2.9.* Let $U \subset \overline{\mathbb{C}}$ be an open set and $R : U \to \overline{\mathbb{C}}$ be a holomorphic map. We define the *multiplier of $R$ at the point $z$*, as the spherical norm of the derivative of $R$ at the point $z$. That is,

$$\|R'(z)\| = \sup\left\{ \frac{\|R'(z)\xi\|_{R(z)}}{\|\xi\|_z} : \xi \in T_z\overline{\mathbb{C}} \backslash \{0_z\} \right\}, \tag{2.3}$$

where $\|\cdot\|_z$ denotes the spherical norm in $T_z\overline{\mathbb{C}}$.

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Under the identification $T_z\overline{\mathbb{C}}$ with $\mathbb{C}$, an explicit expression for the spherical metric is

$$\|\xi\|_z = \frac{2|\xi|}{1+|z|^2}. \tag{2.4}$$

Thus, it is not difficult to see that

$$\|R'(z)\| = |R'(z)| \cdot \frac{1+|z|^2}{1+|R(z)|^2}. \tag{2.5}$$

The following lemma gives an explicit formula to calculate the multiplier for a Möbius transformation.

LEMMA 2.10. *Let $M$ be a Möbius transformations given by*

$$M(u) = \frac{au+b}{cu+d}.$$

*Then*

$$\|M'(z)\| = \frac{|\delta|}{\|Av_z\|^2}, \tag{2.6}$$

*where*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

*$v_z$ is a unitary vector in $\mathbb{C}^2$ with $[v_z] = z$, and $\delta = \det(A)$.*

*Proof.* From equation (2.5), we have that

$$\|M'(z)\| = |M'(z)| \cdot \frac{1+|z|^2}{1+|M(z)|^2} = \frac{|\delta|}{|cz+d|^2}\frac{1+|z|^2}{1+|M(z)|^2}.$$

If we take $v_z = (v_1, v_2) \in \mathbb{C}^2$ a unitary vector such that $z = v_1/v_2$, then

$$\begin{aligned}
\|M'(z)\| &= |\delta| \cdot \frac{1+|z|^2}{|az+b|^2+|cz+d|^2} \\
&= |\delta| \cdot \frac{1+|v_1/v_2|^2}{|a(v_1/v_2)+b|^2+|c(v_1/v_2)+d|^2} \\
&= |\delta| \cdot \frac{|v_1|^2+|v_2|^2}{|av_1+bv_2|^2+|cv_1+dv_2|^2} \\
&= \frac{|\delta|}{\|Av_z\|^2}.
\end{aligned}$$

□

Equation (2.6) gives an explicit expression for the norm of a multiplier for a projective cocycle $M$.

PROPOSITION 2.11. *Let $A$ be a linear cocycle on $TX$ and $M = [A]$. Given $n \in \mathbb{Z}$ and $\xi \in \mathbb{P}(X)$ with $\xi \in \overline{\mathbb{C}}_z$, the multiplier of $M^n$ at the point $\xi \in \overline{\mathbb{C}}_z$ is given by*

$$\|(M_z^n)'(\xi)\| = \frac{|\det(A_z^n)|}{\|A_z^n v_\xi\|^2_{f^n(z)}}$$

*where $v_\xi$ is chosen unitary and such that $[v_\xi] = \xi$.*

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*Proof.* The proposition follows from Definition 2.9 and Lemma 2.10. □

In what follows, we denote the *multiplier of $M^n$ at the point $\xi \in \overline{\mathbb{C}}_z$* by

$$\mathbf{m}(n, \xi) := \|(M_z^n)'(\xi)\| = \frac{|\det(A_z^n)|}{\|A_z^n v_\xi\|_{f^n(z)}^2}. \tag{2.7}$$

From equation (2.7) and the chain rule, it follows that

$$\mathbf{m}(n + m, \xi) = \mathbf{m}(n, M^m \xi) \cdot \mathbf{m}(m, \xi)$$

for each $n, m \in \mathbb{Z}$. This fact is elementary and its proof left to the reader.

2.4. *Pliss's lemma.* The following lemma is a remarkable result and is frequently used in this paper.

LEMMA 2.12. (Pliss's lemma) *Given $0 < \gamma_1 < \gamma_0$ and $a > 0$, there exist $N_0 = N_0(\gamma_0, \gamma_1, a)$ and $\delta_0 = \delta_0(\gamma_0, \gamma_1, a) > 0$ such that the following results hold.*

(1) *For any sequence of numbers $(a_l)_{l=0}^{n-1}$ with $n > N_0$ and $a^{-1} < a_l < a$, such that $\prod_{l=0}^{n-1} a_l \geq \gamma_0^n$, the set*

$$H = \left\{0 \leq k < n : \forall k < s < n, \prod_{l=k+1}^{s} a_l \geq \gamma_1^{s-k}\right\} \tag{2.8}$$

*satisfies $\#H \geq n \cdot \delta_0$.*

(2) *For any sequence of numbers $(a_l)_{l\geq 0}$ with $a^{-1} < a_l < a$ such that $\prod_{l=0}^{n-1} a_l \geq \gamma_0^n$ for each $n \geq N_0$, there exists a set of natural numbers $n_1 < n_2 < \cdots < n_l < \cdots$ with density greater than $\delta_0$, such that*

$$\prod_{i=n_j+1}^{n} a_i \geq \gamma_1^{n-n_j},$$

*for each $n_j < n$ and $j \geq 1$.*

As a corollary, we have the following result.

COROLLARY 2.13. *Let $M : \mathbb{P}(X) \to \mathbb{P}(X)$ be a projective cocycle, and let $0 < \gamma_1 < \gamma_0$. Then there exist $N_0 = N_0(\gamma_0, \gamma_1, M)$ and $\delta_0 = \delta_0(\gamma_0, \gamma_1, M)$ with the following properties.*

(1) *Given $z \in X$ and $\xi \in \overline{\mathbb{C}}_z$ satisfying $\mathbf{m}(N, \xi) \geq \gamma_0^N$ (respectively, $\mathbf{m}(-N, \xi) \geq \gamma_0^N$) for some $N \geq N_0$, there exists $0 \leq m < N$ such that $N - m > N\delta_0$ and*

$$\mathbf{m}(n, M^m(\xi)) \geq \gamma_1^n \quad \textit{for every } 0 < n \leq N - m,$$

*(respectively, $\mathbf{m}(-n, M^{-m}(\xi)) \geq \gamma_1^n$ for every $0 < n \leq N - m$).*

(2) *Given $z \in X$ and $\xi \in \overline{\mathbb{C}}_z$ satisfying $\mathbf{m}(n, \xi) \geq \gamma_0^n$ (respectively, $\mathbf{m}(-n, \xi) \geq \gamma_0^n$) for each $n \geq N_0$, for some $N_0 \geq 0$, there exist $m \geq N_0$ such that*

$$\mathbf{m}(n, M^m(\xi)) \geq \gamma_1^n \quad \textit{for every } 0 < n,$$

*(respectively, $\mathbf{m}(-n, M^{-m}(\xi)) \geq \gamma_1^n$ for every $0 < n$).*

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*Proof.* Note that

$$\mathbf{m}(N, \xi) = \prod_{n=0}^{N-1} \mathbf{m}(1, M^n(\xi)) \geq \gamma_0^N.$$

We will apply Pliss's lemma item (1) for $a_n = \mathbf{m}(1, M^n(\xi))$. Let $n_0$ be the lowest hyperbolic time. We have that $N - n_0 \geq N\delta_0$, and for every $n_0 < n < N$,

$$\gamma_1^{n-n_0} \leq \prod_{k=n_0+1}^{n} \mathbf{m}(1, M^k(\xi)) = \mathbf{m}(n - n_0, M^{n_0+1}(\xi)).$$

Hence our item (1) follows, taking $m = n_0 + 1$. The proof of item (2) is similar. □

2.5. *Dominated splitting.* We recall the notion of dominated splitting for linear cocycles.

*Definition 2.14.* We say that a linear cocycle $A : TX \to TX$ has *dominated splitting* if there exists an $A$-invariant splitting $TX = E \oplus F$ where $E$ and $F$ are one-dimensional complex planes, such that

$$\|A_z^l|_{E_z}\| \cdot \|A_{f^l(z)}^{-l}|_{F_{f^l(z)}}\| < \tfrac{1}{2} \tag{2.9}$$

for some $l \geq 1$.

Recall that $A$ has dominated splitting if and only if $A^k$ has dominated splitting for every $k \in \mathbb{N}$. We also recall that if $A$ has dominated splitting then the splitting $TX = E \oplus F$ is unique, continuous and $\angle(E_z, F_z) \geq \alpha > 0$ for all $z \in X$ (cf. [**6**]). The following result establishes equivalent conditions to dominated splitting.

PROPOSITION 2.15. *Let A be a linear cocycle on a vector bundle TX. Then the following statements are equivalent:*

(1) *The cocycle $A : TX \to TX$ has dominated splitting.*
(2) *There exist an $A$-invariant splitting $TX = E \oplus F$ where $E$ and $F$ are one-dimensional complex planes, a constant $0 < \lambda < 1$, and a $C > 0$ such that*

$$\|A_z^n|_{E_z}\| \cdot \|A_z^{-n}|_{F_{f^n(z)}}\| \leq C\lambda^n \tag{2.10}$$

*for every $z \in X$ and all $n > 0$.*
(3) *There exist a splitting $TX = \widetilde{E} \oplus \widetilde{F}$ (not necessarily $A$-invariant) where $\widetilde{E}$ and $\widetilde{F}$ are one-dimensional complex planes, a constant $l > 0$, and a cone field $C = C(\alpha, \widetilde{F})$, where*

$$C(z) = C(\alpha, \widetilde{F}_z) = \{u + v \in \widetilde{E}_z \oplus \widetilde{F}_z : \|u\| \leq \alpha\|v\|\}$$

*is $A^l$-invariant, that is,*

$$A_z^l(C(z)) \subset C(f^l(z))^\circ$$

*and*

$$\|A_z^l \mid C(z)^c\| \cdot \|A_{f^l(z)}^{-l} \mid C(f^l(z))\| < \tfrac{1}{2}, \tag{2.11}$$

*with $C(z)^\circ = \text{int}(C(z)) \cup \{0\}$ and $C(z)^c = (\widetilde{E}_z \oplus \widetilde{F}_z)\backslash C(z)$.*

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*Proof.* (1)⇒(2). Define

$$\Phi(z,n) = \|A^n_z|_{E_z}\| \cdot \|A^{-n}_{f^n(z)}|_{F_{f^n(z)}}\| \quad \text{and} \quad C' = \max_{0\le j<l} \sup_{z\in X} \Phi(z,j).$$

Note that $\Phi(z, n+m) = \Phi(f^m(z), n) \cdot \Phi(z, m)$. It follows from (2.9) that $\Phi(z, l) < 1/2$. Taking $\lambda = (1/2)^{1/l}$, $C = C'/\lambda^{l-1}$ and $n = sl + j$, we conclude that

$$\Phi(z,n) = \Phi(f^j(z), sl) \cdot \Phi(z, j) \le C'\lambda^{sl} \le C\lambda^n.$$

(2)⇒(1). This implication follows from taking $n$ large enough.

(2)⇒(3). Define $C(z) = \{u + v \in E_z \oplus F_z : \|u\| \le \|v\|\}$. Note that property (2.10) is equivalent to

$$\mathfrak{m}(A^n_z \mid F_z) \cdot \mathfrak{m}(A^{-n}_{f^n(z)} \mid E_{f^n(z)}) \ge C\mu^n$$

where

$$\mathfrak{m}(L \mid V) = \inf_{v\in V^*} \frac{\|Lv\|}{\|v\|}$$

is the minimum norm and $\mu > 1$. Note that if $L$ is a linear isomorphism then $\mathfrak{m}(L \mid V) = \|L^{-1} \mid L(V)\|^{-1}$.

Let $u \in E_z$ and $v \in F_z$ two unitary vectors. It is not difficult to see that for every $z \in X$ we have

$$\frac{\|A^n_z v\|}{\|A^n_z u\|} \ge \mathfrak{m}(A^n_z \mid F_z) \cdot \mathfrak{m}(A^{-n}_{f^n(z)} \mid E_{f^n(z)}) > 1$$

and therefore $A^n_z(C(z)) \subset C(f^n(z))^\circ$ for $n$ greater enough.

On the other hand, we claim that there exists $k \ge 1$ such that, for every $z \in X$,

$$\mathfrak{m}(A^k_z \mid C(z)) \cdot \mathfrak{m}(A^{-k}_{f^k(z)} \mid C(f^k(z))^c) > 1,$$

which is equivalent to inequality (2.11). In fact, suppose to the contrary that the opposite inequality occurs. Then for every $k$ there would exist $z_k \in X$ such that

$$\mathfrak{m}(A^k_{z_k} \mid C(z_k)) \cdot \mathfrak{m}(A^{-k}_{f^k(z_k)} \mid C(f^k(z_k))^c) \le 1.$$

Take $w_k \in C(z_k)$ and $w'_z \in \overline{C(f^k(z_k))^c}$ unitary vectors such that $\|A^k_{z_k} w_k\| = \mathfrak{m}(A^k_{z_k} \mid C(z_k))$ and $\|A^{-k}_{f^k(z_k)} w'_k\| = \mathfrak{m}(A^{-k}_{f^k(z_k)} \mid C(f^k(z_k))^c)$. Writing $w_k = u_k + v_k$ and $w'_k = u'_k + v'_k$, from domination it follows, for $k$ great enough, that $A^k_{z_k} w_k$ grow at the same rate as $A^k_{z_k} v_k$ and $A^{-k}_{f^k(z_k)} w'_k$ grow at the same rate as $A^{-k}_{f^k(z_k)} u'_k$. Then

$$\begin{aligned} 1 \ge \|A^k_{z_k} w_k\| \cdot \|A^{-k}_{f^k(z_k)} w'_k\| &\approx \frac{\|A^k_{z_k} v_k\|}{\|v_k\|} \cdot \frac{\|A^{-k}_{f^k(z_k)} u'_k\|}{\|u'_k\|} \\ &\ge \mathfrak{m}(A^n_z \mid F_z) \cdot \mathfrak{m}(A^{-n}_{f^n(z)} \mid E_{f^n(z)}) \\ &\ge C\mu^k, \end{aligned}$$

which is a contradiction.

(3)⇒(2). The author of [**19**] shows that there is a unique $A$-invariant splitting $TX = E \oplus F$ such that $F_z \subset C(z)$ and $E_z \subset C(z)^c$. Moreover, such subspaces are given by

$$E_z = \bigcap_{n\ge 0} A^{-n}_{f^n(z)} C(z)^c \quad \text{and} \quad F_z = \bigcap_{n\ge 0} A^n_{f^{-n}(z)} C(f^{-n}(z)). \tag{2.12}$$

Finally, the inequality

$$\|A^l_z|_{E_z}\| \cdot \|A^{-l}_{f^l(z)}|_{F_{f^l(z)}}\| \le \|A^l_z \mid C(z)^c\| \cdot \|A^{-l}_{f^l(z)} \mid C(f^l(z))\| < 1/2$$

is clear. □

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## 3. *Critical points*

Critical points are the main object of study of this work. In this section we explain the notion of critical point and we state Theorem A. In the next subsection we explain its basic properties.

In what follows, $A$ denotes a linear cocycle and $M$ is the projective cocycle induced by $A$.

Let $\Delta = \{\beta = (\beta_-, \beta_+) \in \mathbb{R}^2 : 0 < \beta_+ \leq \beta_- < 1\}$ be the set below the diagonal of the square $(0, 1) \times (0, 1)$.

*Definition 3.1.* Given $\beta \in \Delta$, we say that $x \in X$ is a *$\beta$-critical point* for $A$ if there exists a direction $\xi_x \in \overline{\mathbb{C}}_x$ such that

$$\begin{cases} \mathbf{m}(n, \xi_x) \geq \beta_+^n & \text{for all } n \geq 0, \\ \mathbf{m}(n, \xi_x) \geq \beta_-^n & \text{for all } n \leq 0. \end{cases}$$

The direction $\xi_x$ will be called the *critical direction*. We denote the set of all $\beta$-critical points by $\mathrm{Crit}(\beta)$.

Our main theorem is a corollary of the following result.

THEOREM A. *Let $A = (f, A_*)$ be a linear cocycle such that every $\mu \in \mathcal{M}(X, f)$ has the exponents b-separated. Then A has dominated splitting if and only if, for some $\beta \in \Delta$ such that $\beta_+ > b$, we have that* $\mathrm{Crit}(\beta) = \emptyset$.

This theorem is proven in §6. From Lemma 2.8 we conclude that Theorem A can be applied to a $b$-asymptotically dissipative linear cocycle that has no attractors. In what follows, we explain some properties of critical points.

### 3.1. *Main properties of critical points.*

In this section we introduce new notions related to critical points.

*Definition 3.2.*

(1) Let $\beta \in \Delta$ and $n_- \leq 0 \leq n_+$ integers. We say that $x \in X$ is a *$\beta$-critical point at the times* $(n_-, n_+)$ for $A$, if there exist a direction $\xi_x \in \overline{\mathbb{C}}_x$ such that

$$\begin{cases} \mathbf{m}(n, M^{n_+}\xi_x) \geq \beta_+^n & \text{for all } n \geq 0, \\ \mathbf{m}(n, M^{n_-}\xi_x) \geq \beta_-^n & \text{for all } n \leq 0. \end{cases}$$

The direction $\xi_x$ will be called the *critical direction*.

(2) We say that $y \in X$ is a *$\beta$-critical value* if $y$ is a $\beta$-critical point for the linear cocycle $A^{-1}$. We denote the set of all $\beta$-critical values by $\mathrm{CVal}(\beta)$.

(3) Given $x \in X$, we say that the *orbit of $x$ is critical* if there exists a $\beta$-critical point in the orbit of $x$.

We draw attention to the fact that a $\beta$-critical point at the times $(0, 0)$ is just a $\beta$-critical point. We also highlight that Lemma 4.8 (in the next section) asserts that the critical direction is unique.

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Given a direction $\xi_x \in \overline{\mathbb{C}}_x$ we denote the properties

$$(\dagger) \qquad \begin{cases} \mathbf{m}(n, \xi_x) \geq \beta_+^n & \text{for all } n \geq 0, \\ \mathbf{m}(n, \xi_x) \geq \beta_-^n & \text{for all } n \leq 0, \end{cases}$$

and

$$(\ddagger) \qquad \begin{cases} \mathbf{m}(n, \xi_x) \geq \beta_+^n & \text{for all } n \leq 0, \\ \mathbf{m}(n, \xi_x) \geq \beta_-^n & \text{for all } n \geq 0. \end{cases}$$

From the foregoing we have that

$$\mathrm{Crit}(\beta) = \{x \in X : \exists \xi_x \in \overline{\mathbb{C}}_x \text{ such that } (\dagger) \text{ hold}\}$$

and

$$\mathrm{CVal}(\beta) = \{x \in X : \exists \xi_x \in \overline{\mathbb{C}}_x \text{ such that } (\ddagger) \text{ hold}\}.$$

*Definition 3.3.* We say that $x \in X$ is a *$\beta$-postcritical point of order $N \in \mathbb{Z}^+$* for $A$, if there exists $n \in \mathbb{Z}$ with $|n| \leq N$ such that $f^n(x) \in \mathrm{Crit}(\beta)$.

Note that a postcritical point of order 0 is just a critical point.

In the definition above, when $n$ is positive, it is more natural to replace the term 'postcritical' with 'precritical'. For the sake of simplicity, we choose the term 'postcritical' given the sense that this point is an iterate (positive or negative) of a critical point.

Our following result explains that we really only have postcritical points.

THEOREM 3.4. *If $x \in X$ is a $\beta$-critical point at the times $(n_-, n_+)$, then $x$ is a $\beta$-postcritical point of order $N = |n_+ - n_-|$.*

*Proof.* Let $x \in X$ be a critical point at the times $(n_-, n_+)$ with critical direction $\xi_x \in \overline{\mathbb{C}}_x$. Without loss of generality, in what follows we will assume that $n_- = 0$. Let $0 \leq k \leq n_+$ be the minimal number satisfying $\mathbf{m}(n, M^k \xi_x) \geq \beta_+^n$ for every $n \geq 0$. Define $y = f^k(x)$ and $\varpi_y = M^k \xi_x$. We assert that $y$ is a critical point with critical direction $\varpi_y$. In fact, if $k = 0$ then this is obvious. If $0 < k$, we claim that for each $0 \leq l \leq k$ we have $\mathbf{m}(l, M^{-l} \varpi_y) < \beta_+^l$. Therefore, for each $0 \leq l \leq k$ we have

$$\mathbf{m}(-l, \varpi_y) > \beta_+^{-l} \geq \beta_-^{-l},$$

and for each $l > k$ we show that

$$\mathbf{m}(-l, \varpi_y) = \mathbf{m}(-k, \varpi_y) \cdot \mathbf{m}(-(l-k), \xi_x) > \beta_+^{-k} \cdot \beta_-^{k-l} \geq \beta_-^{-l}$$

which implies our theorem.

Our claim remains to be proven, and this follows from an induction on $0 \leq l \leq k$. This is true for $l = 1$ (otherwise we contradict the minimality of $k$). We will assume that our assertion is true for every $1 \leq s \leq l - 1$. For $l$, suppose to the contrary that $\mathbf{m}(l, M^{-l} \varpi_y) \geq \beta_+^l$; then if $0 \leq s \leq l$ would have the equality

$$\mathbf{m}(l, M^{-l} \varpi_y) = \mathbf{m}(s, M^{-l} \varpi_y) \cdot \mathbf{m}(l - s, M^{s-l} \varpi_y)$$

which would imply

$$\mathbf{m}(s, M^{-l} \varpi_y) = \mathbf{m}(l, M^{-l} \varpi_y) \cdot (\mathbf{m}(l - s, M^{s-l} \varpi_y))^{-1} > \beta_+^l \cdot \beta_+^{s-l} = \beta_+^s,$$

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and if $s > l$ we would have

$$\mathbf{m}(s, M^{-l}\varpi_y) = \mathbf{m}(l, M^{-l}\varpi_y) \cdot \mathbf{m}(s-l, \varpi_y) \geq \beta_+^l \cdot \beta_+^{s-l} = \beta_+^s$$

which contradicts the minimality of $k$. □

We now explain the main properties of $\mathrm{Crit}(\beta)$. These properties justify the notion of critical point and show how it is an intrinsic notion of the dynamics.

PROPOSITION 3.5. (Compactness) *Given $\beta \in \Delta$, the set $\mathrm{Crit}(\beta)$ is compact.*

*Proof.* Let $(x_k)_{\mathbb{N}} \subset \mathrm{Crit}(\beta)$ such that $x_k \to x$, and denote the critical direction of $x_k$ by $\xi_k \in \overline{\mathbb{C}}_{x_k}$. Taking a subsequence, if necessary, there exists a direction $\xi \in \overline{\mathbb{C}}_x$ such that $\xi_k \to \xi$. Since we have

$$\begin{cases} \mathbf{m}(n, \xi_k) \geq \beta_+^n & \text{for all } n \geq 0, \\ \mathbf{m}(n, \xi_k) \geq \beta_-^n & \text{for all } n \leq 0, \end{cases}$$

then for each $n \geq 0$ fixed and letting $k$ tend to infinity, we conclude that $x \in \mathrm{Crit}(\beta)$. □

On $\Delta$ we define a partial order: for $\alpha, \beta \in \Delta$ we say that $\beta \geq \alpha$ if and only if $\beta_+ \geq \alpha_+$ and $\beta_-^{-1} \geq \alpha_-^{-1}$. It follows easily from the definition that if $\alpha, \beta \in \Delta$ such that $\beta \geq \alpha$ then $\mathrm{Crit}(\beta) \subset \mathrm{Crit}(\alpha)$.

PROPOSITION 3.6. (Invariance by metric) *A critical orbit remains critical under change of metric.*

*More precisely, if $(\cdot \mid \cdot)_0$ and $(\cdot \mid \cdot)_1$ are hermitian metrics in $TX$, then for all $\alpha, \beta \in \Delta$ with $\alpha < \beta$ there exists a positive number $N = N(\alpha, \beta)$ such that each $\beta$-critical point of $A$ in the metric $(\cdot \mid \cdot)_0$, is an $\alpha$-postcritical point of order $N$ of $A$ in the metric $(\cdot \mid \cdot)_1$. In other words,*

$$\mathrm{Crit}_0(\beta) \subset \bigcup_{j=-N}^{N} f^j(\mathrm{Crit}_1(\alpha)).$$

*Proof.* We denote by $\|\cdot\|_i$ be the spherical metric in $\mathbb{P}(X)$ induced by the hermitian metric $(\cdot \mid \cdot)_i$ (see Appendix A), and

$$\mathbf{m}_i(n, \xi) = \|(M^n)'(\xi)\|_i.$$

It is enough to show that there exists $N$ such that every $x \in \mathrm{Crit}_0(\beta)$ is an $\alpha$-critical point at the times $(n_-, n_+)$ with $|n_+ - n_-| \leq N$ (cf. Theorem 3.4).

Let $a > 1$ such that $a^{-1}\|\cdot\|_1 \leq \|\cdot\|_0 \leq a\|\cdot\|_1$. It follows from equation (2.7) that for every $x \in X$, every $\xi \in \overline{\mathbb{C}}_x$, $n \in \mathbb{Z}$ and every $w \in T_\xi\overline{\mathbb{C}}_x$ we have

$$\frac{\mathbf{m}_1(n, \xi)}{\mathbf{m}_0(n, \xi)} = \frac{\|(M^n)'(\xi)w\|_{1,R^n(\xi)}}{\|(M^n)'(\xi)w\|_{0,R^n(\xi)}} \cdot \frac{\|w\|_{0,\xi}}{\|w\|_{1,\xi}}.$$

Hence we conclude that $C^{-1} \leq \mathbf{m}_1/\mathbf{m}_0 \leq C$ where $C = a^2$, and therefore $\mathbf{m}_1 \geq C^{-1}\mathbf{m}_0$.

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Let $x \in \mathrm{Crit}_0(\beta)$ and $\xi \in \overline{\mathbb{C}}_x$ be its critical direction. We claim that there exists $n_- \le 0$ such that $\mathbf{m}_1(n, M^{n_-}\xi) \ge \alpha_-^n$ for every $n \le 0$. In fact, since $\beta_- < \alpha_- < 1$ there exists $k_0 < 0$ maximal such that $C^{-1}\beta_-^n \ge \alpha_-^n$ for all $n \le k_0$. Therefore for each $n \le k_0$ we have that

$$\mathbf{m}_1(n, \xi) \ge C^{-1}\beta_-^n \ge \alpha_-^n.$$

If $\mathbf{m}_1(n, \xi) \ge \alpha_-^n$ for every $n \le 0$ our claim follows. If not, there exists $k_0 < n_- \le 0$ maximal such that $\alpha_-^{n_-} > \mathbf{m}_1(n_-, \xi)$ and therefore, for every $n < n_-$ the inequality $\mathbf{m}_1(n, \xi) \ge \alpha_-^n$ holds. From the foregoing we conclude that for $n \le 0$ we have that

$$\mathbf{m}_1(n, M^{n_-}\xi) = \frac{\mathbf{m}_1(n + n_-, \xi)}{\mathbf{m}_1(n_-, \xi)} \ge \frac{\alpha_-^{n+n_-}}{\alpha_-^{n_-}} = \alpha_-^n.$$

Similarly, from the inequality $\alpha_+ < \beta_+$ we let $k_1 > 0$ minimal such that $C^{-1}\beta_+^n \ge \alpha_+^n$ for all $n \ge k_1$ and there exists $k_1 > n_+ \ge 0$ minimal with the property $\mathbf{m}_1(n, M^{n_+}\xi) \ge \alpha_+^n$ for all $n \ge 0$. Thus $x$ is an $\alpha$-critical point at the times $(n_-, n_+)$ for the multiplier $\mathbf{m}_1$ with $|n_+ - n_-| \le N := |k_1 - k_0|$. □

As the notion of dominated splitting is invariant by conjugation, we can expect a similar property for the notion of critical point.

*Definition 3.7.* Given two linear cocycles $A : TX \to TX$ and $B : TY \to TY$, we say that $A$ and $B$ are conjugated if there exists a linear cocycle $L : TX \to TY$ such that $L \circ A = B \circ L$.

If we write $A = (f, A_*)$, $B = (g, B_*)$ and $L = (h, L_*)$ then the foregoing asserts that $h \circ f(z) = g \circ h(z)$ and

$$L_{f(z)} \circ A_z(v) = B_{h(z)} \circ L_z(v) \tag{3.1}$$

for every $z \in X$ and $v \in T_z$.

PROPOSITION 3.8. (Invariance by conjugacy) *A critical orbit remains critical under conjugation.*

*More precisely, let $A$ and $B$ be two conjugated linear cocycles where $L \circ A = B \circ L$ with $L = (h, L_*)$, let $\alpha, \beta \in \Delta$ with $\alpha < \beta$ and let $x \in X$ be a $\beta$-critical point of $A$. Then there exists $N = N(\alpha, \beta, L)$ such that $h(x) \in Y$ is an $\alpha$-postcritical point of order $N$ of $B$.*

*Proof.* Let $(\cdot \mid \cdot)_0$ and $(\cdot \mid \cdot)_1$ be Hermitian metrics over $TX$ and $TY$, respectively. Denote $M = [A]$, $N = [B]$ and $H = [L]$ and the respective multiplier of $M$ and $N$ by

$$\mathbf{m}_0(n, \xi) = \|(M_z^n)'(\xi)\|_0 \quad \text{and} \quad \mathbf{m}_1(n, \varpi) = \|(N_w^n)'(\varpi)\|_1.$$

From equation (3.1) we conclude that

$$H_{f^n(z)} \circ M_z^n = N_{h(z)}^n \circ H_z$$

for each $z \in X$ and $n \in \mathbb{Z}$. Since $(H_z)^{-1} = H_{h(z)}^{-1}$ we deduce that $(H_{f^n(z)})^{-1} = H_{g^n(h(z))}^{-1}$. Thus

$$M_z^n = H_{g^n(h(z))}^{-1} \circ N_{h(z)}^n \circ H_z$$

and therefore

$$\begin{aligned}\mathbf{m}_0(n,\xi) &= \|(M_z^n)'(\xi)\|_0\\ &\le \|(H^{-1}_{g^n(h(z))})'(N^n_{h(z)}(H_z(\xi)))\|_0\cdot\|(N^n_{h(z)})'(H_z(\xi))\|_1\cdot\|H_z'(\xi)\|_1\\ &= \|(H^{-1}_{g^n(h(z))})'(N^n_{h(z)}(H_z(\xi)))\|_0\cdot\|H_z'(\xi)\|_1\cdot\mathbf{m}_1(n,H(\xi)).\end{aligned}\tag{3.2}$$

Taking

$$C_0=\sup\{\|(H_x^{-1})'(\xi)\|_0 : x\in X \text{ and } \xi\in\overline{\mathbb{C}}_x\},$$
$$C_1=\sup\{\|H_y'(\varpi)\|_1 : y\in Y \text{ and } \varpi\in\overline{\mathbb{C}}_y\},$$

$C=C_0\cdot C_1\ge 1$ and the inequality (3.2) we obtain that $\mathbf{m}_1(n,H(\xi))\ge C^{-1}\mathbf{m}_0(n,\xi)$.

Then arguing in a similar fashion to the previous proposition we conclude our assertion. □

PROPOSITION 3.9. *If the exponents are b-separated, then the $\beta$-critical points with $b<\beta_+$ are not Oseledets regular points.*

*Proof.* Let $x$ be a $\beta$-critical point with critical direction $\xi$ and let $b<\beta_+\le\beta_-$. We assume that $x$ is a regular point. Let $T_x=E_x^+\oplus E_x^-$ be the splitting related to the Lyapunov exponents. The next section (in particular, Lemma 4.8) asserts that $\xi$ is the unique point uniformly contracted by $M$. This implies that for all $v\in E_x^+$ we have that $[v]=\xi$ and therefore

$$\lambda^+(x)=\lim_n\frac{1}{n}\log\|A_z^n v\|.$$

Then from equation (2.7) it follows that

$$\begin{aligned}\lim_n\frac{1}{n}\log\mathbf{m}(n,\xi) &= \lim_n\frac{1}{n}(\log|\det A_x^n| - 2\log\|A_x^n v\|)\\ &= \lambda^+(x)+\lambda^-(x)-2\lambda^+(x)\\ &= \lambda^-(x)-\lambda^+(x)\\ &\le \log(b).\end{aligned}$$

However, since $\mathbf{m}(n,\xi)\ge\beta_+^n$ we conclude hat

$$\log(b)\ge\log(\beta_+),$$

a contradiction. □

3.2. *Tangencies of a periodic point contain a critical point.* Let $f$ be a Hénon map with $b=|\det(Df)|<1$. Let $p$ be a periodic point of $f$ and $q$ be a point of tangency between the stable and the unstable manifolds of $p$. *We assert that in the orbit of $\mathcal{O}(p)$ there exists a $\beta$-critical point, when $\beta\in\Delta$ and $b_+>b$.*

In fact, without loss of generality, we can assume that $p$ is a fixed point. We denote the local stable/unstable manifold of $p$ of size $\varepsilon$ by $W^s_\varepsilon(p)$ and $W^u_\varepsilon(p)$, respectively. Let $\lambda^s$ and $\lambda^u$ bs the eigenvalues of $Df$ in $p$. Then $b=|\lambda^s|\cdot|\lambda^u|$.

Note that for each $n\ge 0$,

$$\mathbf{m}(-n,[v_p^u])=\frac{b^{-n}}{\|Df^{-n}v_p^u\|^2}=\left(\frac{|\lambda^u|^2}{b}\right)^n>b^{-n}>\beta_-^{-n}$$

and

$$\mathbf{m}(n, [v_p^s]) = \frac{b^n}{\|Df^n v_p^s\|^2} = \frac{|\lambda^u|^n}{|\lambda^s|^n} = \left(\frac{|\lambda^u|^2}{b}\right)^n > b^{-n} > 1 > \beta_+^n$$

where $v_p^\sigma \in E_p^\sigma$ is a unitary vector, with $\sigma = s, u$.

We can take $\varepsilon > 0$ small enough such that for every $z \in W_\varepsilon^s(p)$ (respectively, $z \in W_\varepsilon^u(p)$) we have that $z \approx p$ and $T_z W_\varepsilon^s(p) \approx E_p^s$ (respectively, $T_z W_\varepsilon^u(p) \approx E_p^u$). We can conclude that for each $z \in W_\varepsilon^u(p)$ (respectively, $z \in W_\varepsilon^s(p)$) we have that $\mathbf{m}(-n, [v]) \geq \beta_-^{-n}$ (respectively, $\mathbf{m}(n, [w]) \geq \beta_+^n$) for each $n \geq 0$, where $v \in T_z W_\varepsilon^u(p)$ (respectively, $w \in T_z W_\varepsilon^s(p)$) is a unitary vector.

Finally, let $q_u$ be the first iterate to the past of $q$ that is inside of $W_\varepsilon^u(p)$ and let $n_+ > 0$ such that $f^{n_+}(q_u)$ is the first iterate to the future of $q$ that is inside of $W_\varepsilon^s(p)$. Since $q$ is a tangency point we have that $Df^{n_+}(T_{q_u} W_\varepsilon^u(p)) = T_{f^{n_+}(q_u)} W_\varepsilon^s(p)$, hence we conclude that $q_u$ is a $\beta$-critical point at the times $(0, n_+)$. From Theorem 3.4 we conclude that there exists $0 \leq n_0 \leq n_+$ such that $f^{n_0}(q_u)$ is a $\beta$-critical point.

## 4. *Linear domination and projective hyperbolicity*

The main goal of this section is to characterize the notion of dominated splitting for a linear cocycle in terms of its action in the projective bundle. We introduce the notion of *hyperbolic projective cocycle*. Roughly speaking, hyperbolic projective cocycle are those that have the same dynamics as a hyperbolic Möbius transformation. In Theorem 4.2 we prove that a linear cocycle has dominated splitting if and only if its projective cocycle is hyperbolic. Moreover, in the same theorem we state that the continuity of the section is not necessary to obtain domination.

The main idea is that the invariant splitting determines (in each fiber) two special points in the sphere, and the cone fields are related to disks in the Riemann sphere that are asymptotically contracted/expanded by the projective cocycle.

We recall that a section over $X$ is a continuous function $\sigma : X \to TX$ such that $\mathrm{pr}(\sigma(x)) = x$ for all $x \in X$, and $\Gamma(X, TX)$ denotes the set of all sections over $X$.

*Definition 4.1.*

(1) We say that a section $\sigma \in \Gamma(X, \mathbb{P}(X))$ is $M$-*invariant* if $M(\sigma(z)) = \sigma(f(z))$.

(2) We say that a section $\sigma$ is *contractive* if it is $M$-invariant and there exist constants $C > 0$ and $0 < \lambda < 1$, such that $\mathbf{m}(n, \sigma(z)) \leq C\lambda^n$ for every $z \in X$ and all $n \geq 1$. Similarly, we say that a section is *expansive* if it is contractive for $M^{-1}$.

(3) We say that a cocycle $M$ is *hyperbolic* if there exists two disjoint sections $\tau$ and $\sigma$ in $\Gamma(X, \mathbb{P}(X))$ (i.e. $\tau(z) \neq \sigma(z)$ for every $z \in X$) such that $\tau$ is expansive and $\sigma$ is contractive.

THEOREM 4.2. *Let A be a linear cocycle on TX and M be the projective cocycle induced by A. Then the following statements are equivalent.*

(1) *The cocycle A has dominated splitting.*

(2) *The cocycle M is hyperbolic.*

(3) *There exists a contractive section for M (equivalently, there exists an expansive section).*

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(4) *There exist $C > 0$ and $\mu > 1$ such that for every $z \in X$ there is one direction $\tau_z \in \overline{\mathbb{C}}_z$ such that $\mathbf{m}(n, \tau_z) \geq C\mu^n$ for every $n > 0$ (equivalently, $\sigma_z \in \overline{\mathbb{C}}_z$ such that $\mathbf{m}(-n, \sigma_z) \geq C\mu^n$ for every $n > 0$).*

Theorem 4.2 will be proved in §4.4. In the following subsections we explain a series of results necessary for its proof.

A corollary of item (4) of the previous theorem is the following result.

COROLLARY 4.3. *Assume that there exist $k > 0$ and $0 < \eta < 1$ such that for each $z \in X$ there exists $\xi_z \in \overline{\mathbb{C}}_z$ satisfying*

$$\mathbf{m}(k, M^n(\xi_z)) < \eta \quad \textit{for each } n \geq 1. \tag{4.1}$$

*Then $A$ has dominated splitting.*

*Proof.* Let $z \in X$ and let $n = sk + r$ with $0 \leq r < k$. Then it is not difficult to see that

$$\mathbf{m}(n, \xi_z) = \mathbf{m}(r, \xi_z) \cdot \prod_{j=0}^{s-1} \mathbf{m}(k, M^{jk+r}(\xi_z)) < c\eta^s,$$

where $c = \max_{0 \leq r < k} \sup\{\mathbf{m}(r, \varpi) : \varpi \in \mathbb{P}(X)\}$. Taking $\eta_0 = \eta^{1/k}$ and $c_0 = c/\eta_0^{k-1}$, we conclude that $\mathbf{m}(n, \xi_z) < c_0\eta_0^n$ and hence

$$\mathbf{m}(-n, \xi_z) \geq c_0^{-1}(\eta_0^{-1})^n$$

for every $n \geq 1$. Therefore our assertion follows from Theorem 4.2 item (4). □

4.1. *Equivalence of contractive sections.* Denote the unit disk in $\mathbb{C}$ by $\mathbb{D}$. Given $\xi \in \overline{\mathbb{C}}$ and $r > 0$, we denote the *ball with center at $\xi$ and radius $r$ in the spherical metric* by $B(\xi, r)$. Let $L$ be an isometry in the Riemann sphere with $L(0) = \xi \in \overline{\mathbb{C}}$. We recall that the set $L(r\overline{\mathbb{D}})$ does not depend on $L$. We denote the set $L(r\overline{\mathbb{D}})$ for any $L$ as above by $D_r(\xi)$; this is called the *disk of radius $r$ centered at $\xi$*. On the other hand, we have that for any $r$, the disk $D_r(\xi)$ is equal to $B(\xi, \varepsilon)$ where $\varepsilon$ satisfies the equation (cf. [**7**])

$$\frac{r}{\sqrt{1 + r^2}} = \sin\left(\frac{\varepsilon}{2}\right).$$

PROPOSITION 4.4. *Let $\sigma \in \Gamma(X, \mathbb{P}(X))$ be a $M$-invariant section. Then the following statements are equivalent.*

(i) *Section $\sigma$ is contractive.*
(ii) *There exist $0 < \eta < 1$ and $k > 0$ such that $\mathbf{m}(k, \sigma(z)) < \eta$ for all $z$ in $X$.*
(iii) *There exist $k > 0$ and $r > 0$ such that $M_z^k(\overline{D}_r(\sigma(z))) \subset D_r(\sigma(f^k(z)))$.*
(iv) *There exist $k > 0$ and $R > 0$ such that for all $0 < r \leq R$,*

$$M_z^k(\overline{D}_r(\sigma(z))) \subset D_r(\sigma(f^k(z))).$$

*Proof.* Clearly (i) and (ii) are equivalent and (iv) formally implies (iii). On the other hand, from the Schwartz lemma it follows that (iii) implies (ii).

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To prove that (ii) implies (iv), we let $\beta_z = \{v_1(z), v_2(z)\} \subset T_z$ denote an orthonormal basis such that $[v_2(z)] = \sigma(z)$. Using homogeneous coordinates in the basis $\beta_z$, we can identify $\sigma(z)$ with $0 \in \overline{\mathbb{C}}$. It follows from the foregoing that

$$M_z(\xi) = \frac{\xi}{\beta(z)\xi + \alpha(z)},$$

and for $n \geq 1$,

$$\mathbf{m}(n, \sigma(z)) = \|(M_z^n)'(0)\|_0 = |(M_z^n)'(0)| = \prod_{j=0}^{n-1} |\alpha(f^j(z))|^{-1}.$$

On the other hand, our hypothesis asserts that there exist constants $0 < \eta < 1$ and $k \geq 0$ such that $\mathbf{m}(k, \sigma(z)) < \eta$ for every $z$ in $X$. It follows from the foregoing that there exists $R' > 0$ such that if $|\xi| \leq R'$ we can write

$$M_z^k(\xi) = (\mathbf{m}(k, \sigma(z)) + g_z(\xi))\xi$$

where $g_z$ is holomorphic in $R'\overline{\mathbb{D}}$ for all $z \in X$ and $g_z(0) = g_z'(0) = 0$. Notice that the map $z \mapsto g_z$ is continuous†. Let $\eta < \eta' < 1$. There exists $0 < R < R'$ such that $|\mathbf{m}(k, \sigma(z)) + g_z(\xi)| < \eta'$ when $|\xi| \leq R$ for all $z \in X$. We conclude that if $0 < r \leq R$ and $|\xi| < r$ we have

$$|M_z^k(\xi)| < \eta' \cdot r < r,$$

for every $z \in X$, and our assertion is proved. □

COROLLARY 4.5. *Let $\sigma$ and $R > 0$ satisfy item (iv) of Proposition 4.4. Then for all $0 < r \leq R$ and $\xi \in D_r(\sigma(z))$ we have*

$$\lim_{n \to +\infty} \rho(M_z^n(\xi), M_z^n(\sigma(z))) = 0$$

*where $\rho$ is the spherical metric.*

We define the *stable set* of $\xi \in \overline{\mathbb{C}}_z$ for the cocycle $M$ as the set

$$W^s(\xi) = \left\{w \in \overline{\mathbb{C}}_z : \lim_{n \to \infty} \rho(M_z^n(w), M_z^n(\xi)) = 0\right\}$$

and the *local stable set* of $\xi \in \overline{\mathbb{C}}_z$ of size $\varepsilon > 0$ by

$$W_\varepsilon^s(\xi) = \{w \in W^s(\xi) : \rho(M_z^n(w), M_z^n(\xi)) < \varepsilon, \forall\, n \in \mathbb{N}\}.$$

The unstable set is defined as the stable set of $\xi$ for the cocycle $M^{-1}$.

We can write the stable (respectively, unstable) set in terms of backward (respectively, forward) iterates of the local stable (respectively, local unstable) sets. In fact, given $\varepsilon > 0$, we have

$$W^s(\xi) = \bigcup_{n=0}^{\infty} M_z^{-n}(W_\varepsilon^s(\xi_n)) \quad \text{and} \quad W^u(\xi) = \bigcup_{n=0}^{\infty} M_{f^{-n}(z)}^n(W_\varepsilon^u(\xi_{-n})) \tag{4.2}$$

where $\xi_n = M^n(\xi)$ for each $n \in \mathbb{Z}$.

† The continuity is regarded as meaning that under a change of chart the map is continuous.

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LEMMA 4.6. *Let $\sigma$ be a contractive section for $M$. Then there exists a constant $r > 0$ such that*

$$W^s(\sigma(z)) = \bigcup_{n\geq 0} M_z^{-n}(D_r(\sigma(f^n(z)))).$$

*Proof.* From Corollary 4.5 it follows that there exist $0 < \varepsilon_1 < \varepsilon_2$ such that

$$W^s_{\varepsilon_1}(\sigma(z)) \subseteq D_r(\sigma(z)) \subseteq W^s_{\varepsilon_2}(\sigma(z))$$

for every $z \in X$. The lemma follows directly from the foregoing and equation (4.2). □

4.2. *Uniqueness of the expansive/contractive direction.* In this subsection we explain some properties of the multiplier function $\mathbf{m}$. The main goal is to establish the uniqueness of the expansive direction.

LEMMA 4.7. *If $\xi_i$ with $i = 1, 2$ are two different directions in $\overline{\mathbb{C}}_z$ and $u_i$ is a unitary vector that generates the direction $\xi_i$ for $i = 1, 2$, then*

$$\mathbf{m}(n, \xi_1) \cdot \mathbf{m}(n, \xi_2) = \left(\frac{\sin(\measuredangle(A_z^n u_1, A_z^n u_2))}{\sin(\measuredangle(u_1, u_2))}\right)^2,$$

*for any $n \in \mathbb{Z}$.*

*Proof.* Let $x, y \in \mathbb{C}^2$ and denote the area of the polygon formed by the vertices $0, x, x + y$ and $y$ by $\phi(x, y)$. Then we have the equality

$$\phi(x, y) = \|x\| \cdot \|y\| \cdot \sin(\measuredangle(x, y)) = \sqrt{\det([x\ y]^* \cdot [x\ y])} = |\det([x\ y])|, \tag{4.3}$$

where $[x\ y]$ is a column matrix and $[x\ y]^*$ denotes its transposed conjugate. Then it is easy to see that $\phi(Ax, Ay) = |\det(A)|\phi(x, y)$ for any linear map $A$ in $\mathbb{C}^2$.

Then from equation (4.3) we have that

$$\begin{aligned}\left(\frac{\sin(\measuredangle(A_z^n u_1, A_z^n u_2))}{\sin(\measuredangle(u_1, u_2))}\right)^2 &= \frac{\phi(A_z^n u_1, A_z^n u_2)^2/\|A_z^n u_1\|^2 \cdot \|A_z^n u_2\|^2}{\phi(u_1, u_2)^2/\|u_1\|^2 \cdot \|u_2\|^2} \\ &= \frac{|\det(A_z^n)|^2}{\|A_z^n u_1\|^2 \cdot \|A_z^n u_2\|^2}.\end{aligned}$$

According to equation (2.7) and the equality above, it follows that

$$\mathbf{m}(n, \xi_1) \cdot \mathbf{m}(n, \xi_2) = \frac{|\det(A_z^n)|^2}{\|A_z^n u_1\|^2 \cdot \|A_z^n u_2\|^2} = \left(\frac{\sin(\measuredangle(A_z^n u_1, A_z^n u_2))}{\sin(\measuredangle(u_1, u_2))}\right)^2.$$ □

LEMMA 4.8. *An expansive (contractive) direction is unique.*

*Proof.* Let $z \in X$ and $\xi_i \in \overline{\mathbb{C}}_z$, for $i = 1, 2$, such that $\xi_1 \neq \xi_2$. Assume that there exist constants $C > 0$ and $\lambda > 1$ with the property $\mathbf{m}(n, \xi_i) \geq C\lambda^n$ for all $n \geq 0$. If $u_i$ is a unitary vector such that $[u_i] = \xi_i$ then $\measuredangle(u_1, u_2) > 0$. From the previous lemma, we conclude that

$$C^2\lambda^{2n} \leq \mathbf{m}(n, \xi_1) \cdot \mathbf{m}(n, \xi_2) = \left(\frac{\sin(\measuredangle(A_z^n u_1, A_z^n u_2))}{\sin(\measuredangle(u_1, u_2))}\right)^2 < \frac{1}{\sin(\measuredangle(u_1, u_2))},$$

which is a contradiction. For the case where we have expansion for the past, the same proof holds. □

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COROLLARY 4.9. *Suppose that, given a projective cocycle $M$, there exist $C > 0$ and $\mu > 1$ such that for every $z \in X$ there exists one direction $\tau_z \in \overline{\mathbb{C}}_z$ satisfying $\mathbf{m}(n, \tau_z) \geq C\mu^n$ for every $n > 0$. Then $\tau(z) := \tau_z$ defines a section that is an expansion.*

*Proof.* Since we have uniqueness of an expansive direction (Lemma 4.8), we conclude that $M(\tau_{f^{-1}(z)}) = \tau_z$.

Now let $z_n \to z$ in $X$. Then $\tau_{z_n} \to \tau_z$. In fact, by compactness there exists some adherence point for the sequence $(\tau_{z_n})_n$, named $\tau' \in \overline{\mathbb{C}}_z$, that is expansive for the future. From Lemma 4.8, it follows that $\tau'$ is equal to $\tau_z$. Therefore the map $z \mapsto \tau_z$ is continuous and $\tau$ is an expansion. □

4.3. *Module.* A *double connected domain* in $\overline{\mathbb{C}}$ is a open connected set $U$ such that its complement has two connected components. The definition of the module of a double connected domain is based on the following mapping theorem: *every double connected domain $U$ is biholomorphic to a ring domain of the form*

$$A(r_1, r_2) = \{z \in \mathbb{C} : 0 \leq r_1 < |z| < r_2 \leq \infty\}$$

*and is called a canonical image of $U$.*

If $r_1 > 0$ and $r_2 < \infty$ for one canonical image of $U$, then the ratio of the radii $r_2/r_1$ is the same for all canonical images of $U$. The number

$$\operatorname{mod}(U) = \log\left(\frac{r_2}{r_1}\right)$$

determines the conformal equivalence class of $U$ and is called the module of $U$. Otherwise we define $\operatorname{mod}(U) = \infty$, and this happens if and only if at least one boundary component of $U$ consists of a single point.

The following proposition will be crucial in the proof of Theorem 4.2.

PROPOSITION 4.10. *Let $D_1, D_2, D_3, \ldots$ be conformal disks in $\overline{\mathbb{C}}$ such that for every $i \geq 1$ we have $\overline{D_i} \subset D_{i+1}$. If there exists $\kappa > 0$ such that* $\operatorname{mod}(\operatorname{int}(D_{i+1} \backslash \overline{D_i})) \geq \kappa$, *then the set $D = \bigcup_n D_n$ is biholomorphic to $\mathbb{C}$.*

See [**16**, **18**] for details. As a corollary we show the following result.

COROLLARY 4.11. *Let $\sigma$ be a contractive section of $M$. Then for every $z \in X$, $W^s(\sigma(z))$ is biholomorphic to $\mathbb{C}$. Moreover, the complement of $W^s(\sigma(z))$ is an expansive section.*

*A similar result holds from changing 'contractive' to 'expansive' and 'stable set' to 'unstable set'.*

*Proof.* Fix $z \in X$. Take $k > 0$ and $r > 0$ as in Proposition 4.4 item (3). Define

$$D_n = M_z^{-kn}(D_r(\sigma(f^{kn}(z)))).$$

Then for every $n \geq 0$ we have that $D_{n-1} \subsetneq D_n$. Recall that the function $M_z^{kn}$ maps biholomorphically $D_n \backslash D_{n-1}$ on the set

$$A_n = D_r(\sigma(f^{kn}(z))) \backslash M^k_{f^{(n-1)k}(z)}(D_r(\sigma(f^{(n-1)k}(z)))),$$

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so the modules $\mathrm{mod}(D_n \backslash D_{n-1})$ and $\mathrm{mod}(A_n)$ are equal. Taking $k$ large enough, if necessary, we can assume that $\mathbf{m}(k, \sigma(z)) \leq \eta$, for every $z \in X$ and some $0 < \eta < 1$.

Notice that a Möbius transformation maps circles to circles. Therefore, in appropriate homogeneous coordinates (see, for instance, the proof of Proposition 4.4) we can write, for all $z_0 \in X$,

$$M^k_{z_0}(\xi) = \mathbf{m}(k, \sigma(z_0))\xi + o(|\xi|^2).$$

It follows that $\mathrm{mod}(A_n) \approx \log(1/\eta)$ and the difference between these two values is bounded for every $n \in \mathbb{N}$. Our assertion follows from Proposition 4.10.

To prove our second statement, note that from the foregoing and the uniformization theorem the complement of $W^s(\sigma(z))$ contains only one point. Denote such a point by $\tau(z)$. Since $W^s$ varies continuously and is $M$-invariant, it follows that $\tau$ is a $M$-invariant section. By definition of $\tau$, it follows that small disks around $\tau$ are contracted uniformly by $M^{-1}$. From Proposition 4.4 it follows that $\tau$ is an expansion. □

4.4. *Proof of Theorem 4.2.*

*Proof.* We have the following claim.

CLAIM. *A linear cocycle $A$ has dominated splitting if and only if the cocycle induced by $A$, $M([v]) = [Av]$, is hyperbolic.*

*Proof of Claim.* Suppose that $A$ has dominated splitting.

For each $z \in X$ we let $u(z) \in E_z$ and $v(z) \in F_z$. Since the splitting $TX = E \oplus F$ is continuous and $A$-invariant, it follows that both $\tau(z) = (z, [u(z)])$ and $\sigma(z) = (z, [v(z)])$ are well-defined continuous sections in $\mathbb{P}(X)$, which are $M$-invariant.

We recall that (see Proposition 2.15 and its proof) there exists $l \geq 0$ such that the cone

$$C(z) = \{u + v \in E_z \oplus F_z : \|u\| \leq \|v\|\}$$

is $A^l$-invariant. Let $D(z)$ be the set of all $[x]$ with $x \in C(z)\backslash\{0\}$. We recall that $D(z)$ is a closed conformal disk. Notice that we have:

(1) $\sigma(z) \in D(z)$;
(2) $M^l_z(D(z)) \subset \mathrm{int}(D(f^l(z)))$;
(3) $\sigma(z) = \bigcap_{n\geq 0} M^n_{f^{-n}(z)}(D(f^{-n}(z)))$.

In fact, items (1) and (2) follow directly from the foregoing. From equation (2.12) we obtain item (3). From compactness of $X$ there exists $r > 0$ such that $D(\sigma(z), r) \subset D(z)$ for every $z \in X$. A typical compactness argument using items (2) and (3) and the continuous dependence of the objects implies that there exists $k > l$ such that

$$M^k_{f^{-k}(z)}(D(\sigma(f^{-k}(z)), r)) \subset \mathrm{int}(D(\sigma(z), r))$$

for every $z \in X$, and Proposition 4.4 implies that $\sigma$ is contractive. A similar argument applied to $\tau$ and $M^{-1}$ implies the hyperbolicity of $M$.

On the other hand, we assume that $M$ is hyperbolic. We need to prove that $A$ is a linear cocycle with dominated splitting. Let $\sigma$ (respectively, $\tau$) be the contractive (respectively, expansive) direction of $M$. For any $z \in X$ we take $v(z) \in T_z$ (respectively, $u(z) \in T_z$) a unitary vector such that $[v(z)] = \sigma(z)$ (respectively, $[u(z)] = \tau(z)$). Let $F_z$ (respectively, $E_z$) be the subspace of all vectors of the form $\alpha v(z)$ (respectively, $\alpha u(z)$) where $\alpha \in \mathbb{C}$.

On the other hand, there exist $k \geq 0$ and $0 < \eta < 1$ such that

$$\mathbf{m}(k, \sigma(z)) < \eta \quad \text{and} \quad \mathbf{m}(-k, \tau(f^k(z))) < \eta,$$

and therefore $\sqrt{\mathbf{m}(k, \sigma(z)) \cdot \mathbf{m}(-k, \tau(f^k(z)))} < \eta$. It follows from the equation (2.7) that

$$\begin{aligned}
\sqrt{\mathbf{m}(k, \sigma(z)) \cdot \mathbf{m}(-k, \tau(f^k(z)))} &= \sqrt{\frac{|\det(A_z^k)|}{\|A_z^k v(z)\|^2} \frac{|\det((A_z^k)^{-1})|}{\|(A_z^k)^{-1} u(z)\|^2}} \\
&= \frac{1}{\|A_z^k v(z)\|} \frac{1}{\|(A_z^k)^{-1} u(z)\|} \\
&= \frac{1}{\|A_z^k|_{F_z}\|} \frac{1}{\|(A_z^k)^{-1}|_{E_{f^k(z)}}\|} \\
&= \frac{\|A_z^k|_{E_z}\|}{\|A_z^k|_{F_z}\|} \\
&< \eta,
\end{aligned}$$

as required. □

The previous claim asserts that item (1) is equivalent to item (2). Clearly item (2) formally implies items (3) and (4). The implication (4) ⇒ (3) follows from Corollary 4.9 and (3) ⇒ (2) follows from Corollary 4.11. □

## 5. *Criterion for domination*

In this section we present a criterion to establish when a linear cocycle has dominated splitting. Recall that Theorem 4.2 asserts that a linear cocycle has dominated splitting if, in each fiber, there exists a contractive direction. The following theorem asserts that it is enough to assume that in each fiber there exists a direction such that the (possible) expansion is not too strong. This criterion is essential for proving our main theorem.

THEOREM 5.1. (Criterion of domination) *Let $A = (f, A_*)$ be a linear cocycle such that every $\mu \in \mathcal{M}(X, f)$ has the exponents b-separated. Let $b < \eta < 1$ and assume that there exists $k_0 > 0$ such that for all $z \in X$ there exists $\xi_z \in \overline{\mathbb{C}}_z$ satisfying*

$$\mathbf{m}(k_0, M^m(\xi_z)) \leq \eta^{-k_0} \quad \textit{for every } m \geq 1. \tag{5.1}$$

*Then A has dominated splitting.*

The following lemma is the main tool we use to prove the criterion of domination. This establishes that if there exists one direction which is not strongly contractive, then in many iterates this direction is strongly expansive.

LEMMA 5.2. (Strong expansion) *Let $A = (f, A_*)$ be a linear cocycle such that each $\mu \in \mathcal{M}(X, f)$ supported in $\omega(x, f)$ has the exponents b-separated. Let $b < \eta_0 < 1$, $\xi_x \in \overline{\mathbb{C}}_x$ and $n_0 \in \mathbb{N}$ such that*

$$\eta_0^n \leq \mathbf{m}(n, \xi_x) \quad \textit{for each } n \geq n_0.$$

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*Then we have the following property:*

(†) *for each* $b < \eta_1 < 1$ *and* $k \geq 0$*, there exists* $m_k \geq 0$ *such that*

$$\eta_1^{-n} \leq \mathbf{m}(n, M^{m_k}(\xi_x))$$

*for any* $0 \leq n \leq k$.

*Proof.* First we will prove that the property (†) is equivalent to the following property:

(‡) for each $b < \eta_2 < 1$ and $l \geq 0$ there exist $m_l \geq 0$ and $k_l \geq l$ such that

$$\eta_2^{-k_l} \leq \mathbf{m}(k_l, M^{m_l}(\xi_x)).$$

*Proof of the Equivalence.* Clearly property (†) implies property (‡). Assuming that property (‡) holds, we let $b < \eta_2 < \eta_1 < 1$ and denote $\gamma_0 = \eta_2^{-1}$ and $\gamma_1 = \eta_1^{-1}$ as in Corollary 2.13. Let $N_0$ and $\delta_0$ be the constants given by this corollary. Property (‡) asserts that for every $l \geq 0$ there exist constants $m_l \geq 0$ and $k_l \geq l$ such that $\eta_2^{-k_l} \leq \mathbf{m}(k_l, M^{m_l}\xi_x)$. For $k \geq 0$ fixed, we can take $l \geq 0$ large enough such that $k_l \geq N_0$ and $k_l\delta_0 > k$. It follows from Corollary 2.13 that there exists $0 \leq m \leq k_l$ with $k_l - m \geq k_l\delta_0 > k$ such that $\eta_1^{-n} \leq \mathbf{m}(n, M^{m_l+m}\xi_x)$ for each $0 \leq n \leq k_l - m$. Therefore, taking $m_k = m_l + m$, we obtain property (†). □

In what follows we prove that property (‡) holds. We will use the following notation. Given $\nu \in \mathcal{M}(\mathbb{P}(X))$, we denote $\hat{\nu} = \mathrm{pr}^*(\nu)$, that is, given a Borelian $A \subset X$,

$$\hat{\nu}(A) = \mathrm{pr}^*(\nu)(A) = \nu(\mathrm{pr}^{-1}(A)) = \nu(\mathbb{P}(A)).$$

Clearly $\hat{\nu} \in \mathcal{M}(X, f)$ when $\nu \in \mathcal{M}(\mathbb{P}(X), M)$. Moreover, if $\nu$ is ergodic then $\hat{\nu}$ is ergodic. Let $\Sigma \subset \mathbb{P}(X)$ be the set given by the ergodic decomposition theorem. Then $\Sigma$ is a full probability set, and for all $\varpi \in \Sigma$ the measure

$$\mu_\varpi = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{M^j(\varpi)} \tag{5.2}$$

lies in $\mathcal{M}(\mathbb{P}(X), M)$ and is ergodic. Moreover, given $\nu \in \mathcal{M}(\mathbb{P}(X), M)$ and $h \in \mathcal{L}^1(\nu)$, the function $h$ is $\mu_\varpi$-integrable for $\nu$-almost every $\varpi \in \Sigma$ and

$$\int_{\mathbb{P}(X)} h(\varpi)\, d\nu(\varpi) = \int_\Sigma \left( \int_{\mathbb{P}(X)} h(\zeta)\, d\mu_\varpi(\zeta) \right) d\nu(\varpi). \tag{5.3}$$

If $\Sigma(\nu) = \Sigma \cap \mathbb{P}(\mathcal{R}(A, \mathrm{supp}(\hat{\nu})))$ then $\nu(\Sigma(\nu)) = 1$. In fact, this follows from the facts that $\Sigma$ and $\mathcal{R}(A, \mathrm{supp}(\hat{\nu}))$ have full probability and that $1 = \hat{\nu}(\mathcal{R}(A, \mathrm{supp}(\hat{\nu}))) = \nu(\mathbb{P}(\mathcal{R}(A, \mathrm{supp}(\hat{\nu}))))$.

*Proof of property (‡).* Suppose the contrary, that is, there exists $l \geq 0$ such that for each $m \geq 0$ we have that

$$\mathbf{m}(n, M^m(\xi_x)) < \eta_2^{-n} \tag{5.4}$$

for each $n \geq l$. Let $\varphi(\zeta) = \log \mathbf{m}(1, \zeta)$.

CLAIM 1. *For each* $\varpi \in \Sigma \cap \omega(\xi_x, M)$ *the inequality*

$$\mu_\varpi(\varphi) = \int \varphi(\zeta)\, d\mu_\varpi(\zeta) < |\log(b)|$$

*holds.*

*Proof of Claim 1.* Let $\varpi \in \Sigma \cap \omega(\xi_x, M)$ and let $(n_k)_k \nearrow \infty$ such that $M^{n_k}(\xi_x) \to \varpi$ when $k \to \infty$. From equation (5.4) we have that

$$\begin{aligned}\frac{1}{n}\sum_{j=0}^{n-1} \varphi \circ M^j(\varpi) &= \frac{1}{n} \log \mathbf{m}(n, \varpi)\\ &= \lim_{k\to\infty} \frac{1}{n} \log \mathbf{m}(n, M^{n_k}(\xi_x))\\ &< -\log(\eta_2)\\ &< |\log(b)|\end{aligned}$$

and our assertion follows from equation (5.2). □

CLAIM 2. *Let* $\varpi \in \Sigma$ *such that* $z = \mathrm{pr}(\varpi)$ *is a regular point. Then*

$$\mu_\varpi(\varphi) = \pm(\lambda^+(z) - \lambda^-(z)).$$

*Proof of Claim 2.* Let $v \in T_z$ such that $[v] = \varpi$ and denote

$$\lambda(v) = \lim_n \frac{1}{n} \log \|A_z^n v\|.$$

Recall that if $T_z = E_z \oplus F_z$ is the Oseledets splitting, then $\lambda(v) = \lambda^-(z)$ when $v \in E_z$ and $\lambda(v) = \lambda^+(z)$ in other cases. Then from equation (2.7) it follows that

$$\begin{aligned}\mu_\varpi(\varphi) = \lim_n \frac{1}{n} \log \mathbf{m}(n, \varpi) &= \lim_n \frac{1}{n}(\log|\det A_z^n| - 2\log\|A_z^n v\|)\\ &= \lambda^+(z) + \lambda^-(z) - 2\lambda(v)\\ &= \pm(\lambda^+(z) - \lambda^-(z)).\end{aligned}$$ □

Note that when $z = \mathrm{pr}(\varpi)$,

$$\hat{\mu}_\varpi = \mu_z = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)}.$$

We recall from hypotheses that $\mu_z$ has the exponent $b$-separated.

CLAIM 3. *For any measure* $\mu \in \mathcal{M}(\mathbb{P}(X), M)$ *and* $\varpi \in \Sigma(\mu)$, $\mu_\varpi(\varphi) \leq \log(b)$.

*Proof of Claim 3.* Otherwise from Claim 1 we have that

$$\log(b) < \mu_\varpi(\varphi) < |\log(b)|.$$

Since $\lambda^-(z) \leq \lambda^+(z)$ and Claim 2 then either
(a) $0 \leq \mu_\varpi(\varphi) = \lambda^+(z) - \lambda^-(z) < |\log(b)|$ or
(b) $\log(b) < \mu_\varpi(\varphi) = \lambda^-(z) - \lambda^+(z) < 0$.

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In both cases we conclude that $\lambda^+(z) - \lambda^-(z) < |\log(b)|$, which is a contradiction with the fact that the exponents of $\mu_z$ are $b$-separated. □

Let $(\mu_n)_n \subset \mathcal{M}(\mathbb{P}(X))$ be the sequence given by

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{M^j(\xi_x)}.$$

Taking a subsequence, if necessary, we may assume that $\mu_n \to \mu$ in the weak* topology. Therefore $\mu \in \mathcal{M}(\mathbb{P}(X), M)$ and $\operatorname{supp}(\mu) \subset \omega(\xi_x, M)$.

CLAIM 4.

$$\log(b) < \int_{\mathbb{P}(X)} \varphi(\zeta)\, d\mu(\zeta). \tag{5.5}$$

*Proof of Claim 4.* Recall that from hypotheses $\eta_2^n \leq \mathbf{m}(n, \xi_x)$ for each $n \geq n_0$, therefore

$$\begin{aligned}\int_{\mathbb{P}(X)} \varphi(\zeta)\, d\mu(\zeta) = \lim_n \int_{\mathbb{P}(X)} \varphi(\zeta)\, d\mu_n(\zeta) &= \lim_n \frac{1}{n} \log \mathbf{m}(n, \xi_x)\\ &\geq \log(\eta_0)\\ &> \log(b).\end{aligned}$$ □

To conclude our proof, it follows from equation (5.3), Claims 3 and 4 that

$$\log(b) < \int_{\mathbb{P}(X)} \varphi(\zeta)\, d\mu(\zeta) = \int_\Sigma \mu_\varpi(\varphi)\, d\mu(\varpi) \leq \log(b),$$

which is a contradiction. Therefore, property (‡) holds. □

□

*Proof of the Criterion of Domination.* Let $b < \eta < \widetilde{\eta} < 1$. We claim that there exist $k$ and $m$ such that $\mathbf{m}(k, M^n(\xi_z)) \leq \widetilde{\eta}$ for each $n \geq m$, where $\xi_z$ satisfies equation (5.1). Then our theorem follows from Corollary 4.3 applied to $M^{-m}(\xi_z)$.

Our claim remains to be proven. In fact, suppose to the contrary that for every $k$ there exist $z_k \in X$ and $n_k > k$ such that

$$\mathbf{m}(k, M^{n_k}(\xi_k)) \geq \widetilde{\eta} \geq \widetilde{\eta}^k,$$

where $\xi_k := \xi_{z_k}$.

In what follows, we will construct a point $z \in X$ and a direction $\varpi \in \overline{\mathbb{C}}_z$ satisfying the hypothesis of the strong expansion lemma.

Let $1 > \widetilde{\eta} > \eta_0 > \eta > b$ and let $N_0 := N_0(\widetilde{\eta}, \eta_0, M)$ and $\delta_0 := \delta_0(\widetilde{\eta}, \eta_0, M)$ be the constants given by Corollary 2.13. For each $s \geq 1$, let $k(s)$ be the smallest $n_k$ satisfying $n_{k(s)}\delta_0 > s$. It follows from Corollary 2.13 item (1) that there exists $0 \leq m_s < n_{k(s)}$ such that

$$\mathbf{m}(l, M^{m_s+n_{k(s)}}(\xi_{k(s)})) \geq \eta_0^l \quad \text{for every } 0 < l \leq s.$$

Taking a subsequence, if necessary, we may assume that if $s \to \infty$ then $f^{m_s+n_{k(s)}}(z_{k(s)}) \to z$ and $M^{m_s+n_{k(s)}}(\xi_{k(s)}) \to \varpi$. It follows that $\varpi \in \overline{\mathbb{C}}_z$ and

$$\mathbf{m}(n, \varpi) \geq \eta_0^n \quad \text{for every } n \geq 1,$$

and then $z$ and $\varpi$ satisfy the hypothesis of the strong expansion lemma.

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On the other hand, our hypothesis asserts that for every $z \in X$ we have $\mathbf{m}(k_0, M^m(\xi_z)) \leq \eta^{-k_0}$ for each $z \in X$ and $m \geq 1$. Then there exists a constant $C > 0$ such that $\mathbf{m}(n, M^m(\xi_z)) \leq C\eta^{-n}$ for each $n \geq 0$. This fact is shown using an adaptation of the proof of Corollary 4.3.

Taking $\eta > \eta_1 > b$ and $n_0$ large enough, we obtain that $\mathbf{m}(n, M^m(\xi_k)) < \eta_1^{-n}$ for each $k \geq 1$ and $n \geq n_0$. Letting $k$ go to infinity in the previous inequality implies that for all $m \geq 1$ and $n \geq n_0$,

$$\mathbf{m}(n, M^m(\varpi)) < \eta_1^{-n}.$$

Strong expansion lemma item (a) asserts that there exists $m_0 \geq 1$ such that $\eta_1^{-n} \leq \mathbf{m}(n, M^{m_0}(\varpi))$ for each $0 \leq n \leq n_0$, but from the previous inequality we have

$$\eta_1^{-n_0} \leq \mathbf{m}(n_0, M^{m_0}(\varpi)) < \eta_1^{-n_0},$$

which is a contradiction. □

## 6. *Proof of main theorem*

This section is based on the ideas of Crovisier in [**8**] for the proof of the same result in the context of $C^2$ generic diffeomorphisms in compact manifolds (see [**20**]). Our proof presents significant changes compared with that of Silvan, including a different critical point definition than [**20**].

Here and subsequently, $A = (f, A_*)$ denotes a linear cocycle such that for every $\mu \in \mathcal{M}(X, f)$ has the exponents $b$-separated and let $M$ be the projective cocycle induced by $A$.

### 6.1. *Absence of critical points implies domination.*

In this subsection we will prove the following statement.

THEOREM 6.1. *If $A$ does not have dominated splitting, then for each $\beta = (\beta_-, \beta_+) \in \Delta$ with $b < \beta_+$ we have* $\mathrm{Crit}(\beta) \neq \emptyset$.

We begin with a notion that allows us to prove this theorem.

*Definition 6.2.* Given $0 < \beta_0 < 1$, we say that a projective cocycle $M$ satisfies the property $P(\beta_0)$ if there exists $k_0 > 0$, such that for every $k \geq k_0$ there exist $x_k \in X$, $\xi_k \in \overline{\mathbb{C}}_{x_k}$ and $m_k \geq 0$ so that:

(1) $\mathbf{m}(-n, \xi_k) \geq \beta_0^{-n}$, for every $1 \leq n \leq k$;

(2) $\mathbf{m}(k, M^{m_k}(\xi_k)) \geq 1$.

PROPOSITION 6.3. *Let* $0 < \beta_0 < 1$. *If $M$ satisfies the property $P(\beta_0)$, then for every $\beta \in \Delta$ with $(\beta_0, \beta_0) \geq \beta$ the set* $\mathrm{Crit}(\beta)$ *is not empty.*

*Proof.* Let $(\beta_0, \beta_0) \geq \beta = (\beta_-, \beta_+)$. Recall that $\beta_+ \leq \beta_0 \leq \beta_-$ so that for each $0 \leq n \leq k$ we have that $\mathbf{m}(-n, \xi_k) \geq \beta_0^{-n} \geq \beta_-^{-n}$, when $k \geq k_0$.

Let $\gamma_0 = 1$, $\gamma_1 = \beta_+$ and let $N_0 := N_0(\gamma_0, \gamma_1, M)$ and $\delta_0 := \delta_0(\gamma_0, \gamma_1, M)$ be the constants given by Corollary 2.13. Let $k \geq k_0$ and let $s > N_0$ such that $s\delta_0 > k$. Since $\mathbf{m}(s, M^{m_s}\xi_s) \geq 1$ it follows from Corollary 2.13 that there exists $0 \leq j < s$ such that $s - j > s\delta_0 > k$ and

$$\mathbf{m}(i, M^{m_s+j}\xi_s) \geq \beta_+^i \quad \text{for every } 0 < i \leq s - j.$$

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Therefore taking $y_k = x_s$, $\upsilon_k = \xi_s$ and $n_k = m_s + j$, we conclude that for every $k > 0$ there exist $y_k \in X$, $\upsilon_k \in \overline{\mathbb{C}}_{y_k}$ and $n_k \geq 0$ such that

$$\mathbf{m}(-n, \upsilon_k) \geq \beta_-^{-n} \quad \text{and} \quad \mathbf{m}(n, M^{n_k}\upsilon_k) \geq \beta_+^n$$

for every $0 < n \leq k$.

Arguing as in the proof of Theorem 3.4, we conclude that there exists $0 \leq l_k \leq n_k$ such that

$$\mathbf{m}(-n, M^{l_k}\upsilon_k) \geq \beta_-^{-n} \quad \text{and} \quad \mathbf{m}(n, M^{l_k}\upsilon_k) \geq \beta_+^n$$

for every $0 < n \leq k$.

Finally, if we take $z_k = f^{l_k}(y_k)$ and $\omega_k = M^{l_k}(\upsilon_k)$, we have that for each $0 < n \leq k$,

$$\mathbf{m}(-n, \omega_k) \geq \beta_-^{-n} \quad \text{and} \quad \mathbf{m}(n, \omega_k) \geq \beta_+^n.$$

Let $(z, \omega)$ be an adherence point of $(z_k, \omega_k)$. Then for $n \geq 0$,

$$\mathbf{m}(-n, \omega) \geq \beta_-^{-n} \quad \text{and} \quad \mathbf{m}(n, \omega) \geq \beta_-^n,$$

therefore $\mathrm{Crit}(\beta)$ is non-empty as asserted. □

We denote

$$\mathrm{supp}(X) = \overline{\bigcup \{\mathrm{supp}(\nu) : \nu \in \mathcal{M}(X, f)\}}.$$

LEMMA 6.4. *If there exists $b < \beta_0 < 1$ such that the property $P(\beta_0)$ is not satisfied, then $A|_{\mathrm{supp}(X)}$ has dominated splitting.*

*Proof.* The proof proceeds via a series of claims.

We denote

$$H = \{z \in X : \exists \xi \in \overline{\mathbb{C}}_z \text{ such that } \mathbf{m}(-n, \xi) \geq \beta_0^{-n}, \text{ for every } n \geq 0\}.$$

CLAIM 1. *Let $\nu \in \mathcal{M}(X, f)$ and $x \in \mathcal{R}(A, \nu)$. Then $x$ has infinitely many iterates in $H$.*

*Proof of Claim 1.* Let $v \in F_x$ be a unitary vector and let $\xi = [v]$. Then

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{n} \log \mathbf{m}(-n, \xi) &= \lim_{n\to+\infty} \frac{1}{n}(\log |\det A_z^{-n}| - 2 \log \|A_z^{-n} v\|) \\ &= -(\lambda^+(x) + \lambda^-(x)) + 2\lambda^+(x) \\ &= \lambda^+(x) - \lambda^-(x) \geq -\log(b).\end{aligned}$$

The last inequality follows from Lemma 2.8. Therefore, for $n$ great enough we have that $\mathbf{m}(-n, \xi) \geq b^{-n}$. Our assertion follows from Corollary 2.13 item (2) applied to the constants $b^{-1} > \beta_0^{-1}$. □

We denote

$$\omega(H) = \overline{\bigcup_{x \in H} \omega(x, f)}.$$

CLAIM 2. *We have* $\mathrm{supp}(X) \subset \omega(H)$.

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*Proof of Claim 2.* Let $x \in \text{supp}(\nu) = \overline{\mathcal{R}(A, \text{supp}(\nu))}$, let $V \ni x$ be an open set and let $V_0 = V \cup \mathcal{R}(A, \text{supp}(\nu))$. We recall that $\nu(V_0) > 0$. From the Poincaré recurrence theorem we have that the set $V_1$ consisting of all points with infinitely many iterates on $V_0$ satisfies $\nu(V_0) = \nu(V_1)$. The previous claim asserts that each $z \in V_0$ has infinitely many iterates on $H$, and since this holds for every point of $V$, we conclude that $x \in \omega(H)$. □

CLAIM 3. *The restriction $A|_{\omega(H)}$ has dominated splitting.*

*Proof of Claim 3.* We will prove that every $z \in \omega(H)$ satisfies the hypotheses of the criterion of domination.

Since the property $P(\beta_0)$ is not satisfied, then there exists $k_0$ arbitrarily large such that for every $x \in X$ and $\varpi \in \overline{\mathbb{C}}_x$:

(1) either $\mathbf{m}(-n, \varpi) < \beta_0^{-n}$ for some $1 \leq n \leq k_0$;

(2) or $\mathbf{m}(k_0, M^m \varpi) < 1$ for every $m \geq 1$.

Let $x \in H$ and let $\xi \in \overline{\mathbb{C}}_x$ such that $\mathbf{m}(-n, \xi) \geq \beta_0^{-n}$, for $n \geq 0$. Therefore we conclude that $\mathbf{m}(k_0, M^m \xi) < 1 < \beta_0^{-k_0}$ for every $m \geq 1$. Let $y \in \omega(x, f)$ with $f^{m_l}(x) \to y$ and $M^{m_l}\xi \to \xi_y \in \overline{\mathbb{C}}_y$. Therefore for each $m \geq 0$ we have that

$$\mathbf{m}(k_0, M^m(\xi_y)) = \lim_{l\to\infty} \mathbf{m}(k_0, M^m(M^{m_l}\xi)) = \lim_{l\to\infty} \mathbf{m}(k_0, M^{m+m_l}\xi) \leq \beta_0^{-k_0}.$$

Given $z \in \omega(H)$, there exist $x_n \in H$ and $y_n \in \omega(x_n, f)$ such that $y_n \to z$. From the foregoing, for each $n \geq 0$ there exists $\xi_n \in \overline{\mathbb{C}}_{y_n}$ such that $\mathbf{m}(k_0, M^m(\xi_n)) \leq \beta_0^{-k_0}$. Taking a subsequence, if necessary, we may assume that there exists $\xi_z \in \overline{\mathbb{C}}_z$ such that $\xi_n \to \xi_z$, and passing to the limit we conclude that $\mathbf{m}(k_0, M^m(\xi_n)) \leq \beta_0^{-k_0}$ as desired. □

Since the set supp$(X)$ is a compact $f$-invariant and supp$(X) \subset \omega(H)$, our lemma follows. □

PROPOSITION 6.5. *Assume that $A|_{\text{supp}(X)}$ has dominated splitting but $A$ does not have dominated splitting. Then there exists $\varepsilon_0 > 0$ such that for all $1 - \varepsilon_0 < \beta_0 < 1$, the property $P(\beta_0)$ is satisfied on $X$.*

*Proof.* Recall that Theorem 4.2 item (4), applied to supp$(X)$, asserts the existence of constants $C > 0$ and $\mu > 1$ such that for every $z \in \text{supp}(X)$ there exists $\sigma_z \in \overline{\mathbb{C}}_z$ satisfying $\mathbf{m}(-n, \sigma_z) \geq C\mu^n$, for every $n \geq 0$. We let $0 < \varepsilon_0 < 1$ such that $\mu = (1 - \varepsilon_0)^{-1}$. Let $1 - \varepsilon_0 < \beta' < \beta_0 < 1$.

Since $X$ does not have dominated splitting, it follows from the criterion of domination that for every $k \geq 0$ there exist $x_k \in X$ and $m_k > 0$ such that for all $\omega \in \overline{\mathbb{C}}_{x_k}$ we have

$$\mathbf{m}(k, M^{m_k}\omega) \geq 1. \tag{6.1}$$

On the other hand, the $\alpha$-limit of $x_k$ supports an $f$-invariant measure, hence there exists $z_0 \in \alpha(x_k, f) \cap \text{supp}(X)$. From the foregoing, there exists $\sigma_0 \in \overline{\mathbb{C}}_{z_0}$ such that

$$\mathbf{m}(-n, \sigma_0) > C(\beta')^{-n} \quad \text{for every } n \geq 1,$$

where $1 - \varepsilon_0 < \beta' < 1$.

Let $k \geq 0$ fixed and let $(n_t)_t \nearrow \infty$ such that $f^{-n_t}(x_k) \to z_0$. For every $s \geq 0$, there exists a neighborhood $U_s \subset \mathbb{P}(X)$ of $\sigma_0$, such that for every $\xi \in U_s$ the inequality

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$\mathbf{m}(-n, \xi) \geq C(\beta')^{-n}$ holds for all $1 \leq n \leq s$. Taking $t \geq 0$ large enough, $f^{-n_t}(x_k)$ is inside the projection in $X$ of the neighborhood $U_s$. Hence, there exists $\xi_s \in \overline{\mathbb{C}}_{f^{-n_t}(x_k)}$ such that $\mathbf{m}(-n, \xi_s) \geq C(\beta')^{-n}$ for all $1 \leq n \leq s$, and for $s \geq 0$ large enough, $\mathbf{m}(-s, \xi_s) \geq C(\beta')^{-s} \geq \beta_0^{-s}$, hence we are in the hypothesis of Corollary 2.13.

We conclude that there exist constants $s \geq 0$ and $l_s \geq 0$ such that $s - l_s > k$ and

$$\mathbf{m}(-n, M^{-l_s}\xi_s) \geq \beta_0^{-n} \quad \text{for every } 0 < n \leq s - l_s;$$

in particular, writing $\upsilon_k = M^{-l_s}(\xi_s)$, we conclude that

$$\mathbf{m}(-n, \upsilon_k) \geq \beta_0^{-n} \quad \text{for every } 0 < n \leq k.$$

From equation (6.1), we have that there exists $m_k$ such that $\mathbf{m}(k, M^{m_k}\upsilon_k) \geq 1$, so the property $P(\beta_0)$ is satisfied. □

*Proof of Theorem 6.1.* We claim that there exists $b < \beta_0 < 1$ such that the property $P(\beta_0)$ holds. In fact, if we assume to the contrary that for each $b < \beta_1 < 1$ the property $P(\beta_1)$ does not hold, then from Lemma 6.4 we conclude that $A|_{\mathrm{supp}(X)}$ has dominated splitting. Since $A$ does not have dominated splitting, Proposition 6.5 asserts that the property $P(\beta_1)$ holds for every $1 - \varepsilon_0 < \beta_1 < 1$, which is a contradiction.

Therefore, from Proposition 6.3 we conclude that for each $(\beta_0, \beta_0) \geq \beta$ the set $\mathrm{Crit}(\beta)$ is not empty. □

6.2. *Critical points do not allow domination.* In this subsection we will prove the following statement.

THEOREM 6.6. *Let $\beta = (\beta_-, \beta_+) \in \Delta$ with $b < \beta_+$. If $\mathrm{Crit}(\beta) \neq \emptyset$, then $A$ does not have dominated splitting.*

We begin with a notion that allows us to prove this theorem.

*Definition 6.7.* Let $\beta \in \Delta$. We say that a pair $(x, y) \in X \times X$ is a $\beta$-critical pair if:

(1) $x \in \mathrm{Crit}(\beta)$, with critical direction $\xi$;
(2) $y \in \mathrm{CVal}(\beta)$, with critical direction $\varpi$;
(3) there exists a sequence of positive integers $l_k$ such that

$$f^{l_k}(x) \to y \quad \text{and} \quad M^{l_k}\xi \to \varpi.$$

PROPOSITION 6.8. *Let $\beta = (\beta_-, \beta_+) \in \Delta$ with $b < \beta_+$. Then for every $\beta$-critical point $x$, there exists a $\beta$-critical value $y$ such that $(x, y)$ is a $\beta$-critical pair.*

*Proof.* From the strong expansion lemma applied to the critical point $x$, for every $k \geq 0$ there exists $m_k \geq 0$ such that $\beta_-^{-n} \leq \mathbf{m}(n, M^{m_k}\xi_x)$ for each $0 \leq n \leq k$. Therefore

$$\mathbf{m}(n, M^{m_k}\xi_x) \geq \beta_-^{-n}$$

for each $0 \leq n \leq k$ and

$$\mathbf{m}(-k, \xi_x) \geq \beta_-^{-k} > 1.$$

Let $\gamma_0 = 1$, $\gamma_1 = \beta_+$ and let $N_0 := N_0(\gamma_0, \gamma_1, M)$ and $\delta_0 := \delta_0(\gamma_0, \gamma_1, M)$ be the constants given by Corollary 2.13. Let $k \geq 0$ and let $s > N_0$ such that $s\delta_0 > k$. Since $\mathbf{m}(-s, \xi_x) \geq 1$ it follows from Corollary 2.13 that there exists $0 \leq j_k < s$ such that $s - j_k > s\delta_0 > k$ and

$$\mathbf{m}(-n, M^{-j_k}\xi_x) \geq \beta_+^n \quad \text{for every } 0 < n \leq s - j_k$$

and so for $0 \leq n \leq k$.

For each $k \geq 0$, take $-j_k \leq l_k \leq m_k$ maximal with the property

$$\mathbf{m}(-n, M^{l_k}\xi_x) \geq \beta_+^n$$

for each $0 \leq n \leq k$. By an induction argument, it is not difficult to see that $\mathbf{m}(-l, M^{l_k+l}\xi_x) < \beta_+^l$ for all $0 \leq l \leq m_k - l_k$ (see the proof of Theorem 3.4 for a similar argument). Therefore, we conclude that for each $0 \leq n \leq k$,

$$\mathbf{m}(-n, M^{l_k}\xi_x) \geq \beta_+^n \quad \text{and} \quad \mathbf{m}(n, M^{l_k}\xi_x) \geq \beta_-^{-n}.$$

Taking a subsequence of $(l_k)_k$ if necessary, there exist $y \in X$ and $\varpi_y \in \overline{\mathbb{C}}_y$ such that

$$f^{l_k}(x) \to y \quad \text{and} \quad M^{l_k}(\xi_x) \to \varpi,$$

and therefore $y \in \mathrm{CVal}(\beta)$. □

*Proof of Theorem 6.6.* Assume that $A$ has dominated splitting $TX = E \oplus F$. We recall that there exists $\alpha > 0$ such that for each $x \in X$ the angle $\angle(F(x), E(x)) > \alpha$.

Let $x \in \mathrm{Crit}(\beta)$ with critical direction $\xi$. The previous proposition asserts that there exists a $\beta$-critical value $y$ such that $(x, y)$ is a $\beta$-critical pair. Let $\varpi$ be the critical direction of $y$. From uniqueness of the expansive/contractive direction, we conclude that there exist vectors $u \in F_x$ and $v \in E_y$ such that $\xi = [u]$ and $\varpi = [v]$. The convergence $M^{l_k}(\xi) \to \varpi$ implies that

$$\alpha < \angle(F(f^{l_k}(x)), E(y)) \to 0,$$

which is a contradiction. □

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## A. *Appendix*

A.1. *Hermitian and spherical metrics.* This appendix is devoted to proving the existence of a spherical metric in a projective bundle, given previously a hermitian metric. For this purpose, it suffices to carry out this construction in $\mathbb{C}^2$.

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The Riemann sphere is the projective space consisting of all one-dimensional subspaces of $\mathbb{C}^2$ or complex lines. The complex line that goes through point $v \neq 0$ is the set $[v] = \{\lambda v : \lambda \in \mathbb{C}\backslash\{0\}\}$. In homogeneous coordinates the point $[v]$ has the form $[z_1 : z_2]$, where $v = (z_1, z_2) \neq 0$. So we obtain that

$$\overline{\mathbb{C}} = \{[z_1 : z_2] : (z_1, z_2) \in \mathbb{C}^2\}.$$

Each $z \in \mathbb{C}$ is related to $[z_1 : z_2]$ if and only if $[z_1 : z_2] = [z_1/z_2 : 1] = [z : 1]$. The point at infinity is related to the class $[1 : 0]$, and we can write $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In these coordinates the *standard spherical metric*

$$d\rho = \frac{2|dz|}{1 + |z|^2} \tag{A.1}$$

and has constant Gaussian curvature $+1$.

The previous construction was made under the representation in homogenous coordinates in the canonical base. Now, we will repeat this construction, but considering an arbitrary base, and we will find the relationship between these different representations.

Let $\beta = \{v_1, v_2\}$ be a base of $\mathbb{C}^2$ and write $v = w_1 v_1 + w_2 v_2 = (w_1, w_2)_\beta \neq 0$. We write the homogeneous coordinate in the base $\beta$ of the vector $v$ as $[w_1 : w_2]_\beta$. Also we relate each $[w_1 : w_2]_\beta$ to the point $w \in \mathbb{C}$ if and only if $w = w_1/w_2$. Finally, we denote

$$\overline{\mathbb{C}}_\beta = \{[w_1 : w_2]_\beta : w_1 v_1 + w_2 v_2 \in \mathbb{C}^2\},$$

and define the *spherical metric in the base* $\beta$ *on* $\overline{\mathbb{C}}_\beta$ by the equation

$$d\rho_\beta = \frac{2|dw|}{1 + |w|^2}.$$

On the other hand, let $L$ be the linear transformation satisfying $Lv_i = e_i$ with $i = 1, 2$ and where $\{e_1, e_2\}$ denote the canonical base. It is not difficult to see that, denoting the Möbius transformation related to $L$ by $N$, we have that $N(z) = w$.

Let $v_z = (z_1, z_2)$ such that $z = z_1/z_2$. From equation (A.1) we conclude that

$$d\rho = 2|dz| \frac{|(v_z \mid e_2)|^2}{(v_z \mid v_z)}$$

where $(\cdot \mid \cdot)$ denotes the standard hermitian metric. Similarly, if $(\cdot \mid \cdot)_0$ is a hermitian metric in $\mathbb{C}^2$ such that $\beta$ is a orthonormal bases, then

$$d\rho_\beta = 2|dw| \frac{|(v_w \mid v_2)_0|^2}{(v_w \mid v_w)_0} \tag{A.2}$$

where $v_w = (w_1, w_2)_\beta$ such that $w = w_1/w_2$, and this definition depends only on the hermitian metric. In fact, we make the following claim:

(†) If $\beta$ is an orthonormal base (different of the canonical base) in the standard hermitian metric, then $d\rho = d\rho_\beta$.

Let $\beta$ be an orthonormal base standard hermitian metric. Let $L$ be the isometry in the hermitian metric

$$L = \begin{pmatrix} \overline{b} & -a \\ \overline{a} & b \end{pmatrix}$$

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and let $N$ be the induced Möbius transformation by $L$. Then

$$N(z) = \frac{\overline{b}z - a}{\overline{a}z + b}$$

is an isometry in the standard spherical metric.

Let $v_z = (z_1, z_2)$ with $z = z_1/z_2$. Since $N(z) = w$, we can take $v_w = (w_1, w_2)_\beta = w_1 v_1 + w_2 v_2 = (\overline{b}z - a, \overline{a}z + b)_\beta$. From the foregoing and equation (A.2) we conclude that

$$\begin{aligned} d\rho = 2|dz|\frac{|(v_z \mid e_2)|^2}{(v_z \mid v_z)} &= 2|dz|\frac{|z_2|^2}{|z_1|^2 + |z_2|^2} \\ &= 2|dz|\frac{|z_2|^2}{(Lv_z \mid Lv_z)} \\ &= 2|dz|\frac{|z_2|^2}{|\overline{b}z_1 - az_2|^2 + |\overline{a}z_1 + bz_2|^2} \\ &= 2|dz|\frac{1}{|\overline{b}z - a|^2 + |\overline{a}z + b|^2} \\ &= \frac{2|dz|}{1 + |(\overline{b}z - a)/(\overline{a}z + b)|^2} \cdot \left|\frac{1}{(\overline{a}z + b)^2}\right| \\ &= \frac{2|dz|}{1 + |N(z)|^2} \cdot |N'(z)| \\ &= 2\frac{|dw|}{1 + |w|^2}, \end{aligned}$$

and note that

$$\frac{|(v_w \mid v_2)|^2}{(v_w \mid v_w)} = \frac{|w_2|^2}{|w_1|^2 + |w_2|^2} = \frac{1}{1 + |w_1/w_2|^2} = \frac{1}{1 + |w|^2}.$$

Hence (†) holds.

Finally, equation (A.2) allows us to define *the spherical metric* as an intrinsic object of the hermitian metric (prefixing an orthonormal base, but not depending on this base). With this, we can justify the existence of a spherical metric in a projective bundle in terms of the hermitian metric defined in the fibre bundle.

## References


**[1]** E. Bedford, M. Lyubich and J. Smillie. Polynomial diffeomorphisms of $C^2$. IV. The measure of maximal entropy and laminar currents. *Invent. Math.* **112**(1) (1993), 77–125.

**[2]** E. Bedford and J. Smillie. Polynomial diffeomorphisms of $C^2$: currents, equilibrium measure and hyperbolicity. *Invent. Math.* **103**(1) (1991), 69–99.

**[3]** E. Bedford and J. Smillie. Polynomial diffeomorphisms of $C^2$. II. Stable manifolds and recurrence. *J. Amer. Math. Soc.* **4**(4) (1991), 657–679.

**[4]** E. Bedford and J. Smillie. Polynomial diffeomorphisms of $\mathbb{C}^2$. III. Ergodicity, exponents and entropy of the equilibrium measure. *Math. Ann.* **294**(3) (1992), 395–420.

**[5]** J. Bochi. The Multiplicative ergodic theorem of Oseledets http://www.mat.uc.cl/ jairo.bochi/docs/oseledet s.pdf.

[6] C. Bonatti, L. Díaz and M. Viana. Dynamics beyond uniform hyperbolicity. *A Global Geometric and Probabilistic Perspective (Encyclopaedia of Mathematical Sciences, 102)*. Springer, Berlin, 2005, Mathematical Physics, III.

[7] J. Conway. *Functions of One Complex Variable (Graduate Texts in Mathematics, 11)*, 2nd edn. Springer, New York, 1978.

[8] S. Crovisier. Sur la notion de Criticalité de Pujals–Rodriguez Hertz. *Preprint*.

[9] R. Dujardin and M Lyubich. Stability and bifurcations for dissipative polynomial automorphisms of $\mathbb{C}^2$. *Invent. Math.* **200** (2015), 439–511.

[10] S. Friedland and J. Milnor. Dynamical properties of plane polynomial automorphisms. *Ergod. Th. & Dynam. Sys.* **9**(1) (1989), 67–99.

[11] M. Hakim. Attracting domains for semi-attractive transformations of $\mathbb{C}^p$. *Publicacions Matemàtiques* **38** (1994), 479–499.

[12] J. H. Hubbard. The Hénon mapping in the complex domain. *Chaotic Dynamics and Fractals (Atlanta, GA, 1985) (Notes Reports on Mathematical Science and Engineering, 2)*. Academic Press, Orlando, FL, 1986, pp. 101–111.

[13] J. Hubbard and R. Oberste-Vorth. Hénon mappings in the complex domain. I. The global topology of dynamical space. *Publ. Math. Inst. Hautes Études Sci.* **79** (1994), 5–46.

[14] J. H. Hubbard and R. W. Oberste-Vorth. Hénon mappings in the complex domain. II. Projective and inductive limits of polynomials. *Real and Complex Dynamical Systems (Hillerød, 1993) (NATO Advanced Science Institutes Series C: Mathematical and Physical Sciences, 464)*. Kluwer Academic Publications, Dordrecht, 1995, pp. 89–132.

[15] D. Husemoller. *Fibre Bundles (Graduate Texts in Mathematics, 20)* 3rd edn. Springer, New York, 1994.

[16] O. Lehto and K. I. Virtanen. *Quasiconformal mappings in the plane (Grundlehren der mathematischen Wissenschaften, 126)*, 2nd edn. Springer, New York, 1973.

[17] R. Mañé. Hyperbolicity, sinks and measure in one-dimensional dynamics. *Comm. Math. Phys.* **100**(4) (1985), 495–524.

[18] J. Milnor. *Dynamics in One Complex Variable (Annals of Mathematics Studies, 160)*, 3rd edn. Princeton University Press, Princeton, NJ, 2006.

[19] S. Newhouse. Cone-fields, domination, and hyperbolicity. *Modern Dynamical Systems and Applications*. Cambridge University Press, Cambridge, 2004, pp. 419–432.

[20] E. R. Pujals and F. Rodriguez Hertz. Critical points for surface diffeomorphisms. *J. Mod. Dyn.* **1**(4) (2007), 615–648.

[21] E. R. Pujals and M. Sambarino. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. *Ann. of Math. (2)* **151** (2000), 961–1023.

[22] F. Valenzuela-Henriquez. Equivalent conditions for hyperbolicity on partially hyperbolic holomorphic map. *Indiana Univ. Math. J.* **60** (2011), 1363–1392.