J. Austral. Math. Soc. (*Series A*) 31 (1981), 276–288

REMARKS ON MULTIPLIERS ON CERTAIN FUNCTION SPACES ON GROUPS

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(Received 14 February 1979; revised 1 March 1980)

Communicated by E. Strzelecki

Abstract

Let G be a LCA group with an algebraically ordered dual $\hat{G}$. Suppose also that the semigroup P of positive elements in $\hat{G}$ is not dense in $\hat{G}$. Subspaces $H_P^s(G)$ $(1 < s < \infty)$ are defined analogous to the Hardy spaces on the circle group, and the question whether every multiplier from $H_P^s(G)$ into $H_P^q(G)$ can be extended to a multiplier from $L^s(G)$ into $L^q(G)$ is investigated. If we suppose that $s \neq \infty$, then it is shown that such an extension is possible if and only if $(s, q) \in (1, \infty) \times [1, \infty] \cup \{(1, \infty)\}$. (The negative result for (1, 1) was obtained in a previous paper.)

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 43 A 22.

1. Introduction

Let G be a LCA group with the dual group $\hat{G}$. For $1 \leqslant p \leqslant \infty$ $L^p(G)$ denotes the usual Banach space. Let $M(G)$ be the Banach algebra consisting of all bounded regular measures on G. For a subset E of $\hat{G}$, $L_E^1(G)$ ($M_E(G)$) denotes the subspace of $L^1(G)$ ($M(G)$) consisting of functions (measures) whose Fourier transforms vanish off E. Let $C_0(G)$ be the space of all continuous functions on G which vanish at infinity and $C_c(G)$ the space of all continuous functions on G which have compact supports, '$\hat{\ }$' and '$\check{\ }$' denote the Fourier transform and the inverse Fourier transform respectively. When $\hat{G}$ is algebraically ordered, we define subspaces $H_P^s(G)$ of $L^s(G)$ $(1 \leqslant s \leqslant \infty)$ which are analogous to the Hardy spaces on the circle group. In a previous paper (Yamaguchi (1980)), the author showed that there exists a multiplier on $H_P^1(G)$ which can not be represented by convolution with a measure (the compact case is due to Caudry), that is there exists a multiplier on $H_P^1(G)$ which can not be extended to a multiplier on $L^1(G)$. In Section 2, if we suppose that $s \neq \infty$, then it is shown that a multiplier from $H_P^s(G)$ into $H_P^q(G)$ can be extended to a multiplier from $L^s(G)$ into $L^q(G)$ whenever $(s, q) \in (1, \infty) \times [1, \infty] \cup \{(1, \infty)\}$.

Next, in Section 3, we show that a linear operator on $H_P^\infty(G)$ which is continuous with respect to the weak* topology and commutes with translations is given by convolution with a bounded regular measure on G. We also construct a multiplier on $H_P^\infty(G)$ which is not given by convolution with a bounded regular measure on G. Let β be the canonical map from $C_0(G)$ onto $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$. In Section 4, we show that there exists a multiplier S on $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$ such that $\beta \circ S' \neq S \circ \beta$ for every multiplier S' on $C_0(G)$.

DEFINITION 1.1. Let Γ be a LCA group. Γ is called an algebraically ordered group if and only if there exists a subsemigroup P of Γ with the (AO)-condition, that is (i) $P \cup (-P) = \Gamma$ and (ii) $P \cap (-P) = \{0\}$ (see Doss (1968), p. 257). We do not assume the closedness of P.

Let G be a LCA group such that $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of G with the (AO)-condition such that it is not dense in $\hat{G}$. We define $H_P(G)$ $(1 \leqslant s \leqslant \infty)$ and $M_P^a(G)$ as follows:

$$H_P^1(G) = \{ f \in L^1(G); \hat{f}(\gamma) = 0 \text{ on } P^c \},$$

$$H_P^\infty(G) = \left\{ f \in L^\infty(G); \int_G f(x) g(x)\, dx = 0 \text{ for } g \in H_P^1(G) \right\},$$

$$H_P^s(G) = \{ f \in L^s(G); \hat{f}(\gamma) = 0 \text{ a.e. on } (P^-)^c \} \quad \text{for } 1 < s \leqslant 2,$$

where P^- denotes the closure of P,

$$H_P^q(G) = \left\{ f \in L^q(G); \int_G f(x) g(x)\, dx = 0 \text{ for every } g \in H_P^{q'}(G) \right\}$$

for $2 < q < \infty$, where $1/q + 1/q' = 1$,

$$M_P^a(G) = \{ \mu \in M(G); \hat{\mu}(\gamma) = 0 \text{ on } P^c \}.$$

REMARK 1.1. (i) If $G = T$ and $P = Z^+$ (the semigroup of nonnegative integers), $H_P^1(T)$ is the usual $H^1(T)$ and $H_P^\infty(T) = H_0^\infty(T)$ $(= \{ f \in H^\infty(T); \hat{f}(0) = 0 \})$. If $G = R$ and $P = R^+$ (the semigroup of nonnegative real numbers), $H_P^1(R)$ and $H_P^\infty(R)$ are $H^1(R)$ and $H^\infty(R)$ respectively.

(ii) Let G be a compact connected abelian group. Then there exists a subsemigroup P of $\hat{G}$ with the (AO)-condition. Let $H^s(G)$ $(1 \leqslant s \leqslant \infty)$ be the

Hardy spaces defined in (Rudin (1962), 8.1.8, p. 197). Then the following relations are satisfied:

$$H_P^s(G) = H^s(G) \quad \text{for } 1 \leqslant s \leqslant 2,$$

$$H_P^q(G) = H_0^q(G) \left(= \{ f \in H^q(G); \hat{f}(0) = 0 \}\right) \quad \text{for } 2 < q \leqslant \infty.$$

DEFINITION 1.2. Let A and B be subspaces of $L^p(G)$ and $L^q(G)$ respectively. Let T be a continuous linear operator from A into B. T is called a multiplier if T commutes with τ_a ($a \in G$), where $\tau_a f(x) = f(x - a)$.

Doss proved in (Doss (1970), Theorem 2, p. 64) that there exists an analytic projection U from $L^s(G)$ onto $H_P^s(G)$ for $1 < s \leqslant 2$. Let U^* be the adjoint of U. Then, since U is a translation invariant projection, $I - U^*$ is also. Hence, for $1 < s < \infty$, every multiplier from $H_P^s(G)$ into $H^q(G)$ ($1 \leqslant q \leqslant \infty$) can be extended to a multiplier from $L^s(G)$ into $L^q(G)$.

DEFINITION 1.3. For a LCA group Γ, the coset ring of Γ_d is the smallest ring of sets consisting of cosets of arbitrary subgroups of Γ.

PROPOSITION 1.4. *Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Then $(P^c)^-$ does not belong to the coset ring of $\hat{G}_d$ (the discrete dual of G).*

PROOF. By the structure theorem of LCA groups, $\hat{G} \cong R^n \oplus F$, where n is a nonnegative integer and F is a LCA group which contains a compact open subgroup F_0.

Case 1. Suppose $n = 0$.

Since F_0 is compact, P is dense in F_0 (see Otaki (1977), Lemma 1, p. 307). Since P is not dense in $\hat{G}$ and $\hat{G} = \bigcup \{\gamma + F_0; \gamma \in \hat{G}\}$, there exists $\gamma_0 \in \hat{G}$ such that P is not dense in $\gamma_0 + F_0$. By $[\gamma_0 + F_0]$, we denote the open subgroup of $\hat{G}$ generated by γ_0 and F_0. Then $[\gamma_0 + F_0]$ is isomorphic to $Z \oplus F_0$, and by (Yamaguchi (1980), Lemma 7), we have $P \cap [\gamma_0 + F_0] \cong \{(n, f) \in Z \oplus F_0; n > 0, \text{ or } n = 0 \text{ and } f \geqslant_P 0\}$ (because P is not dense in $[\gamma_0 + F_0]$ and P is dense in F_0), where '$\leqslant_P$' denotes the order on F_0 induced by P. Hence we have $(P^c)^- \cap [\gamma_0 + F_0] = \{(n, f) \in Z \oplus F_0; n \leqslant 0\}$. Suppose that $(P^c)^-$ belongs to the coset ring of $\hat{G}_d$. Then $\{n\gamma_0; n \leqslant 0\}$ also belongs to the coset ring of $\hat{G}_d$. This is a contradiction.

Case 2. Suppose $n \geqslant 1$.

First we consider the case that P is dense in $R^n \oplus F_0$. Then, since P is not dense in $\hat{G}$, there exists $\gamma_0 \in \hat{G}$ such that P is not dense in $[\gamma_0 + R^n \oplus F_0] \cong Z \oplus R^n \oplus F_0$ (see Yamaguchi (1980), Lemma 9). Hence we can prove that $(P^c)^-$ does not belong to the coset ring of $\hat{G}_d$ by the same method in Case 1.

Next we consider the case that P is not dense in $R^n \oplus F_0$. Then P is not dense in R^n (because P is dense in F_0). From (Yamaguchi (1980), Lemma 5 and Lemma 10), we may assume that $(P^c)^- \cap R^n \oplus F_0 = \{(x_1, x_2, \ldots, x_n, f) \in R^n \oplus F_0; x_1 \geqslant 0\}$. Hence $(P^c)^-$ does not belong to the coset ring of $\hat{G}_d$.

Therefore, in each case, $(P^c)^-$ does not belong to the coset ring of $\hat{G}_d$, and the proof is completed.

By Proposition 1.4 and (Birtel (1966), Theorem 3, p. 268), there is no analytic projection from $L^1(G)$ onto $H_P^1(G)$. However the following proposition is satisfied.

PROPOSITION 1.5. *Let G be a nondiscrete LCA group whose dual $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Then a multiplier from $H_P^1(G)$ into $H_P^\infty(G)$ can be extended to a multiplier from $L^1(G)$ into $L^\infty(G)$.*

PROOF. Let T be a multiplier from $H_P^1(G)$ into $H_P^\infty(G)$. For $f \in H_P^1(G)$, we have

$$\|\tau_x Tf - Tf\|_\infty \leqslant \|T\| \, \|\tau_x f - f\|_1.$$

Hence we have $\lim_{x\to 0}\|\tau_x Tf - Tf\|_\infty = 0$. Therefore, Tf belongs to $C_u(G)$ (the space of all bounded uniformly continuous functions on G). We define a bounded linear functional A on $H_P^1(G)$ as follows;

$$A(f) = Tf(0) \quad \text{for } f \in H_P^1(G).$$

By Hahn-Banach's extension theorem, there exists $g \in L^\infty(G)$ such that

$$A(f) = \int_G f(-y)g(y)\,dy \quad \text{for } f \in H_P^1(G).$$

Since T commutes with translations, we have

$$Tf(x) = f * g(x) \quad \text{for } f \in H_P^1(G) \text{ and } x \in G.$$

Therefore T can be extended to a multiplier from $L^1(G)$ into $L^\infty(G)$.

2. Multipliers from $H_P^1(G)$ into $H_P^q(G)$

From (Yamaguchi (1980), Theorem 11), there exists a multiplier on $H_P^1(G)$ which cannot be extended to a multiplier on $L^1(G)$. In this section, we prove that there exists a multiplier from $H_P^1(G)$ into $H_P^q(G)$ $(1 < q < \infty)$ such that it cannot be extended to a multiplier from $L^1(G)$ into $L^q(G)$.

https://doi.org/10.1017/S144678870001942X Published online by Cambridge University Press

THEOREM 2.1. *Let G be a nondiscrete LCA group whose dual $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Let q be in $[1, \infty)$. Then there exists a multiplier from $H_P^1(G)$ into $H_P^q(G)$ such that it can not be extended to a multiplier from $L^1(G)$ into $L^q(G)$.*

PROOF. Let T_Φ be the multiplier on $H_P^1(G)$ constructed in (Yamaguchi (to appear), Theorem 11). We note that Φ is constructed as follows:

(1) $L^1_{\mathrm{supp}(\Phi)}(G) \subset \bigcap \{L^p(G); 1 \leqslant p < \infty\}$,

(2) $P^- \cap (-\mathrm{supp}(\Phi)) = \varnothing$,

(3) there exist an open set $(0 \in)\ U$ in $\hat{G}$ with the compact closure, a sequence $\{\gamma_m\}$ in $\hat{G}$ and a nonzero function Δ in $A(\hat{G})$ $(= L^1(G)\hat{})$ which satisfy the following properties:

$(3)_{\mathrm{I}}$ $\bigcup_1^\infty(\gamma_m + U) \subset P^0$ and $(\gamma_m + U) \cap (\gamma_n + U) = \varnothing$ if $m \neq n$,

$(3)_{\mathrm{II}}$ $\Delta \geqslant 0$ and $\mathrm{supp}(U) \subset U$,

$(3)_{\mathrm{III}}$ $\Phi(\gamma + \gamma_n) = \Delta(\gamma)$ on U $(n = 1, 2, 3, \ldots)$.

First we prove that $T_\Phi(H_P^1(G)) \subset \bigcap_{1 \leqslant s < \infty} H_P^s(G)$. It is easy to check that $T_\Phi(H_P^1(G)) \subset \bigcap_{1 \leqslant s \leqslant 2} H_P^s(G)$. Let s be in $(2, \infty)$. Let f be a function in $H_P^1(G)$. Then there exists a sequence $\{h_m\}$ in $H_P^1(G)$ such that

(4) $\mathrm{supp}(\hat{h}_n)$ has a compact support $(n = 1, 2, \ldots)$,

(5) $\mathrm{supp}(\hat{h}_n) \subset \mathrm{supp}(\Phi)$,

(6) $\lim_{n\to\infty}\|T_\Phi f - h_n\|_1 = 0$.

Hence, by (1) and (6), we have $\lim_{n\to\infty}\|T_\Phi f - h_n\|_s = 0$. For each $g \in H_P^q(G)$ $(1/s + 1/q = 1)$, we have

$$\begin{aligned}\int_G T_\Phi f(x) g(x)\,dx &= \lim_{n\to\infty}\int_G h_n(x) g(x)\,dx \\ &= \lim_{n\to\infty}\int_G \hat{h}_n(-\gamma)\hat{g}(\gamma)\,d\gamma \\ &= 0 \quad (\text{by } (2)).\end{aligned}$$

Thus $T_\Phi f$ belongs to $H_P^s(G)$.

If T_Φ can be extended to a multiplier from $L^1(G)$ into $L^q(G)$ $(1 < q < \infty)$, there exists a function g in $L^q(G)$ such that $T_\Phi f = f * g$ for $f \in H_P^1(G)$. But this is impossible by $(3)_{\mathrm{I}} \sim (3)_{\mathrm{III}}$. Hence T_Φ can not be extended to a multiplier from $L^1(G)$ into $L^q(G)$. This completes the proof.

3. Multipliers on $H_P^\infty(G)$

If G is a compact abelian group whose dual is ordered, each multiplier on $H^\infty(G)$ is given by convolution with a bounded regular measure on G (Larsen

https://doi.org/10.1017/S144678870001942X Published online by Cambridge University Press

(1971), Theorem 7.3.1, p. 222). In this section we prove that a linear operator on $H_P^\infty(G)$ which is continuous with respect to the weak* topology and commutes with translations is given by convolution with a bounded regular measure on G. Moreover we construct a multiplier on $H_P^\infty(G)$ which is not given by convolution with a measure.

LEMMA 3.1. *Let G be a nondiscrete, noncompact LCA group whose dual $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Then, for each neighborhood V of 0 in $\hat{G}$, $P^- \cap V$ has a nonempty interior. Moreover there exists a function h_V in $L^1(G) \cap H_P^\infty(G)$ such that $\hat{h}_V \geqslant 0$, $\operatorname{supp}(\hat{h}_V) \subset P^- \cap V$ and $\int_{\hat{G}} \hat{h}_V(\gamma)\, d\gamma = 1$.*

PROOF. By the structure theorem of LCA groups, $\hat{G} = R^n \oplus F\mathrm{d}$, where n is a nonnegative integer and F is a LCA group which contains a compact open subgroup F_0. If $n = 0$, F_0 is a nontrivial compact open subgroup of $\hat{G}$. Hence, in this case, P^- contains F_0 (see Yamaguchi (1980), Lemma 4). Therefore we may assume that $n \geqslant 1$.

Case 1. Suppose $P \cap R^n$ is dense in R^n.

Since P is dense in F_0, we have $P^- \supset R^n \oplus F_0$. Hence $P^- \cap V$ has a nonempty interior. It is easy to construct such a function $h_V \in L^1(G) \cap H_P^\infty(G)$.

Case 2. Suppose $P \cap R^n$ is not dense in R^n.

By (Yamaguchi (to appear), Lemma 5), we may assume that $(P \cap R^n)^0 \supset \{x = (x_1, x_2, \ldots, x_n) \in R^n; x_1 > 0\}$ without loss of generality. Since $P^- \supset \{x = (x_1, x_2, \ldots, x_n) \in R^n; x_1 \geqslant 0\} \times F_0$, $P^- \cap V$ has a nonempty interior. In this case, we can construct such a function $h_V \in L^1(G) \cap H_P^\infty(G)$. This completes the proof of Lemma 3.1.

LEMMA 3.2. *Let G be a nondiscrete LCA group whose dual $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Then $(C_0(G) \cap H_P^\infty(G))^- = H_P^\infty(G)$, where 'bar' denotes weak* closure.*

PROOF. If G is compact, we can prove this lemma easily. Hence we may assume that G is not compact. Let γ_0 be in $P \setminus \{0\}$. For each compact neighborhood V of 0 in $\hat{G}$, by Lemma 3.1, there exists a function $h_V \in L^1(G) \cap H_P^\infty(G)$ such that $\hat{h}_V \geqslant 0$, $\operatorname{supp}(\hat{h}_V) \subset V \cap P^-$ and $\int_{\hat{G}} \hat{h}_V(\gamma)\, d\gamma = 1$. We note that $h_V \in C_0(G)$ and $H_P^\infty(G)$ is a subalgebra of $L^\infty(G)$.

Claim. $\lim_V \gamma_0 h_V = \gamma_0$ (weak*).

Indeed, for $f \in L^1(G)$, we have

$$\begin{aligned}\langle \gamma_0 h_V, f\rangle - \langle \gamma_0, f\rangle &= \int_G (x, \gamma_0) h_V(x) f(-x)\, dx - \int_G (x, \gamma_0) f(-x)\, dx \\ &= \int_G (x, \gamma_0) \int_{\hat{G}} \hat{h}_V(\gamma)(x, \gamma)\, d\gamma f(-x)\, dx - \int_{\hat{G}} \hat{h}_V(\gamma) \int_G (x, \gamma_0) f(-x)\, dx\, d\gamma \\ &= \int_{\hat{G}} \hat{h}_V(\gamma)\big(\hat{f}(\gamma + \gamma_0) - \hat{f}(\gamma)\big)\, dd\gamma.\end{aligned}$$

Since $\hat{f}(\gamma)$ is uniformly continuous, we have $\lim_V \langle \gamma_0 h_V, f\rangle = \langle \gamma_0, f\rangle$. Hence we have (a) $P \setminus \{0\} \subset (C_0(G) \cap H_P^\infty(G))^-$. We note that $(L^1(G)/H^1_{(-P)}(G))^* \cong H_P^\infty(G)$. Suppose that $(C_0(G) \cap H_P(G))^- \subsetneq H_P^\infty(G)$. Then there exists $[f] \in L^1(G)/H^1_{(-P)}(G)$ and $g \in H_P^\infty(G) \setminus (C_0(G) \cap H_P^\infty(G))^-$ such that (b) $\langle [f], g\rangle = 1$ and (c) $[f] \perp (C_0(G) \cap H_P^\infty(G))^-$. From (a) and (c), we have $[f] = 0$. This contradicts (b). Hence the proof is complete.

We can prove the following proposition by using Lemma 3.2 and the ideas in the proof of (Glicksberg and Wik (1971), Theorem 1.1, p. 620).

PROPOSITION 3.3. *Let G be a nondiscrete LCA group whose dual $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Let T be a linear operator on $H_P^\infty(G)$ such that it is continuous with respet to the weak* topology and commutes with translations. Then there exists a bounded regular measure μ on G such that $Tg = g * \mu$ for $g \in H_P^\infty(G)$.*

PROOF. Since $(L^1(G)/H^1_{(-P)}(G))^* \cong H_P^\infty(G)$ and T is continuous with respect to the weak* topology, there exists a multiplier S on $L^1(G)/H^1_{(-P)}(G)$ such that $S^* = T$. For $g \in C_0(G) \cap H_P^\infty(G)$ and $x \in G$, we have $\|Tg - \tau_x Tg\|_\infty \leq \|T\|\, \|g - \tau_x g\|_\infty$. Hence Tg is a bounded uniformly continuous function for each $g \in C_0(G) \cap H_P^\infty(G)$. We define a bounded linear functional A on $C_0(G) \cap H_P^\infty(G)$ as follows:

$$A(g) = Tg(0).$$

Then, by the Hahn-Banach and the Riesz theorem, there exists a measure μ in $M(G)$ such that

$$Tg(x) = g * \mu(x) \quad \text{for } g \in C_0(G) \cap H_P^\infty(G) \text{ and } x \in G.$$

Hence we have

$$\begin{aligned}\langle g, S([f]) - [f] * \mu\rangle &= \langle Tg, [f]\rangle - \langle g * \mu, [f]\rangle \\ &= 0 \quad \text{for } [f] \in L^1(G)/H^1_{(-P)}(G) \text{ and } g \in C_0(G) \cap H_P^\infty(G).\end{aligned}$$

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Hence, by Lemma 3.2, we have

$$S([f]) = [f] * \mu \quad \text{for } [f] \in L^1(G)/H^1_{(-P)}(G).$$

Therefore $Tg = g * \mu$ for $g \in H_P^\infty(G)$. This completes the proof.

We denote by $AP(G)$ the space of all almost periodic functions on G. We can identify $AP(G)$ with $C(\overline{G})$, where $\overline{G}$ denotes the Bohr compactification of G. Since $AP(G)$ is a translation invariant closed subspace of $L^\infty(G)$ containing constant functions, we can prove the following lemma by the same method in (Power (1977), Theorem 4, p. 74).

LEMMA 3.4. *Let m be the invariant mean on $AP(G)$ defined by $\int_{\overline{G}} f(\bar{x})\, d\bar{x}$ for $f \in AP(G)$. Then there exists an invariant mean M on $L^\infty(G)$ which is an extension of m.*

PROPOSITION 3.5. *Let G be a nondiscrete, noncompact LCA group whose dual $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Then there exists a multiplier T on $H_P^\infty(G)$ such that it is not given by convolution with a bounded regular measure on G.*

PROOF. Let M be the invariant mean on $L^\infty(G)$ in Lemma 3.4. We define an operator on $H_P^\infty(G)$ as follows:

$$Tf(x) = M(f) \quad \text{for } f \in H_P^\infty(G) \text{ and } x \in G.$$

Then T is a multiplier on $H_P^\infty(G)$. Suppose $Tf = f * \mu$ for some $\mu \in M(G)$. Let γ be in $P \setminus \{0\}$. Then $T(\gamma)(x) = M(\gamma) = m(\gamma) = 0$. On the other hand, $\gamma * \mu(x) = \int_G (x - y, \gamma)\, d\mu(y) = (x, \gamma)\hat{\mu}(\gamma)$. Hence we have $\hat{\mu}(\gamma) = 0$ on $P \setminus \{0\}$. Since $\hat{G}$ is not discrete, 0 is an accumulation point of $P \setminus \{0\}$. Hence we have $\hat{\mu}(0) = 0$. On the other hand, since $\hat{G}$ is not discrete, constant functions belong to $H_P^\infty(G)$. Hence we have $T(1)(x) = M(1) = 1$. This is a contradiction.

4. Some multiplier on $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$.

Let β be the canonical map from $C_0(G)$ onto $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$. It is well known that a multiplier on $C_0(G)$ is given by convolution with a bounded regular measure on G (Larsen (1971), Theorem 3.3.2, p. 74). Hence, for each multiplier S' on $C_0(G)$, there exists a multiplier S on $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$ such that $\beta \circ S' = S \circ \beta$. In this section, we prove that there exists a multiplier S on $C_0(G)C_0(G) \cap H_{(-P)}(G)$ such that $S \circ \beta \neq \beta \circ S'$ for every multiplier S' on $C_0(G)$ (see diagram).

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$$\begin{array}{ccc} C_0(G) & \xrightarrow{\beta} & C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G) \\ \downarrow S' & & \downarrow S \\ C_0(G) & \xrightarrow{\beta} & C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G) \end{array}$$

DEFINITION 4.1. A bounded linear operator T on $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$ is called a multiplier if it commutes with translations.

PROPOSITION 4.2. *Let G be a LCA group such that $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Then we have $M^a_P(G) \cong (C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G))^*$.*

PROOF. Let f be a function in $C_0(G) \cap H_{(-P)}(G)$ and μ a measure in $M^a_P(G)$. For each neighborhood V of 0 in G with the compact closure, we choose a nonnegative continuous function g_V such that $\operatorname{supp}(g_V) \subset V$ and $\int_G g_V(x)\,dx = 1$. Then we have $\lim_V g_V * \mu = \mu$ in the weak* topology. Therefore, since $g_V * \mu \in H^1_P(G)$ and $f(-x) \in H^\infty_P(G)$, we have

$$\begin{aligned} \langle f, \mu \rangle &= \int_G f(-x)\,d\mu(x) \\ &= \lim_V \int_G f(-x) g_V * \mu(x)\,dx \\ &= 0. \end{aligned}$$

Thus $M^a_P(G)$ is included in $(C_0(G) \cap H^\infty_{(-P)}(G))^\perp$.

Conversely, let μ be a measure in $M(G)$ such that $\mu \perp C_0(G) \cap H^\infty_{(-P)}(G)$. By Lemma 3.2, for each $\gamma_0 \in (-P) \setminus \{0\}$, there exists a net $\{k_\alpha\}$ in $C_0(G) \cap H^\infty_{(-P)}(G)$ such that $\lim_\alpha k_\alpha = \gamma_0$ in the weak* topology. Let g be a function in $L^1(G)$ such that $\hat{g}(\gamma_0) = 1$. For $\varepsilon > 0$, since $g * \mu \in L^1(G)$, there exists $k_\beta \in \{k_\alpha\}$ such that

$$|\langle k_\beta, g * \mu \rangle - \langle \gamma_0, g * \mu \rangle| < \varepsilon.$$

On the other hand, since $k_\beta * g \in C_0(G) \cap H_{(-P)}(G)$, we have

$$\langle k_\beta, g * \mu \rangle = \langle k_\beta * g, \mu \rangle = 0.$$

Hence we have

$$|\langle \gamma_0, \mu \rangle| = |\langle \gamma_0, g * \mu \rangle| < \varepsilon.$$

Since ε (> 0) is arbitrary, μ belongs to $M^a_P(G)$. Therefore we have

$$(C_0(G) \cap H^\infty_{(-P)}(G))^\perp \subset M^a_P(G).$$

This completes the proof.

https://doi.org/10.1017/S144678870001942X Published online by Cambridge University Press

THEOREM 4.3. *Let G be a nondiscrete LCA group such that $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in $\hat{G}$. Then there exists a linear operator T on $M_P^a(G)$ as follows:*

(i) *T is continuous with respect to the weak* topology,*

(ii) *T commutes with translations,*

(iii) *T is not given by convolution with a bounded regular measure on G.*

PROOF. From (Yamaguchi (1980), Theorem 11), there exists a multiplier T_Φ on $H_P^1(G)$ with the following properties:

(a) $T_\Phi(f)\hat{} = \Phi\hat{f}$ for $f \in H_P^1(G)$,

(b) T_Φ is not given by convolution with a bounded regular measure on G, where Φ is a bounded continuous function on P^0. We define a bounded Borel measurable function Φ' on $\hat{G}$ as follows:

$$\Phi'(\gamma) = \begin{cases} \Phi(\gamma) & \text{for } \gamma \in P^0, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mu \in M_P^a(G)$, $\Phi'\hat{\mu}$ belongs to $M_P^a(G)\hat{}$. Indeed, let γ_i be in $\hat{G}$ and $c_i \in \mathbf{C}$ (complex numbers) $(i = 1, 2, \dots, n)$. For $\varepsilon > 0$, we choose a function f in $L^1(G)$ such that $\|f\|_1 < 1 + \varepsilon$ and $\hat{f}(\gamma_i) = 1$ $(i = 1, 2, \dots, n)$. Put $q(x) = \sum_1^n c_i(x, \gamma_i)$. Then, since $f * \mu \in H_P^1(G)$, we have

$$\begin{aligned} \left|\sum_1^n c_i\Phi'(\gamma_i)\hat{\mu}(\gamma_i)\right| &= \left|\sum_1^n c_i\Phi'(\gamma_i)(f*\mu)\hat{}(\gamma_i)\right| \\ &= \left|\sum_1^n c_i T_\Phi(f*f)\hat{}(\gamma_i)\right| \\ &\leqslant \|q\|_\infty \|T_\Phi(f*\mu)\|_1 \\ &\leqslant (1 + \varepsilon)\|T_\Phi\|\ \|\mu\|\ \|q\|_\infty. \end{aligned}$$

Since ε (> 0) is arbitrary, we haves

$$\left|\sum_1^n c_i\Phi'(\gamma_i)\hat{\mu}(\gamma_i)\right| \leqslant \|T_\Phi\|\ \|\mu\|\ \|q\|_\infty.$$

Hence, by Bochner-Eberlein's theorem, $\Phi'\hat{\mu}$ belongs to $M_P^a(G)\hat{}$.

Let T be a linear operator on $M_P^a(G)$ such that $T(\mu)\hat{} = \Phi'\hat{\mu}$. Then, by the closed graph theorem, T is continuous with respect to the norm topology. Since T_Φ is not given by convolution with a bounded regular measure on G, (iii) is satisfied. It is trivial that T commutes with translations.

Next we prove that T is continuous with respect to the weak* topology. Let μ be a measure in $M_P^a(G)$ and $\{\mu_\alpha\}_{\alpha\in\Lambda}$ a net in $M_P^a(G)$ such that $\|\mu_\alpha\| \leqslant C$ and μ_α converges weak* to μ, where C is a positive constant such that $\|\mu\| \leqslant C$. Let f be a function in $C_0(G)$ and $\varepsilon > 0$. Since $L^1(\hat{G})\hat{}$ is dense in $C_0(G)$, there exists a

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function g in $C_c(\hat{G})$ such that $\|f - \hat{g}\|_\infty < \varepsilon/4(1 + C\|T\|)$. Put $F(\gamma) = \Phi'(\gamma)g(\gamma)$. Then F belongs to $L^1(\hat{G})$. Hence there exists $\alpha_0 \in \Lambda$ such that

$$\left|\int_G \hat{F}(x)\, d\mu_\alpha(x) - \int_G \hat{F}(x)\, d\mu(x)\right| < \frac{\varepsilon}{2} \quad \text{for } \alpha \geqslant \alpha_0.$$

Therefore, if $\alpha \geqslant \alpha_0$, we have

$$\begin{aligned}
\left|\int_G f(x)\, dT(\mu_\alpha)(x) - \int_G f(x)\, dT(\mu)(x)\right| &< \left|\int_G (f(x) - \hat{g}(x))\, dT(\mu_\alpha)(x)\right| \\
&\quad + \left|\int_G \hat{g}(x)\, dT(\mu_\alpha)(x) - \int_G \hat{g}(x)\, dT(\mu)(x)\right| \\
&\quad + \left|\int_G (\hat{g}(x) - f(x))\, dT(\mu)(x)\right| \\
&< \varepsilon/4(1 + C\|T\|) \times \|T(\mu_\alpha)\| + \varepsilon/4(1 + C\|T\|) \times \|T(\mu)\| \\
&\quad + \left|\int_G \hat{g}(x)\, dT(\mu_\alpha)(x) - \int_G \hat{g}(x)\, dT(\mu)(x)\right| \\
&< \frac{\varepsilon}{2} + \left|\int_{\hat{G}} G(\mu_\alpha)\hat{}(\gamma)g(\gamma)\, d\gamma - \int_{\hat{G}} T(\mu)\hat{}(\gamma)g(\gamma)\, d\gamma\right| \\
&= \frac{\varepsilon}{2} + \left|\int_{\hat{G}} F(\gamma)\hat{\mu}_\alpha(\gamma)\, d\gamma - \int_{\hat{G}} F(\gamma)\hat{\mu}(\gamma)\, d\gamma\right| \\
&= \frac{\varepsilon}{2} + \left|\int_G \hat{F}(x)\, d\mu_\alpha(x) - \int_G \hat{F}(x)\, d\mu(x)\right| \\
&< \varepsilon.
\end{aligned}$$

That is, $T(\mu_\alpha)$ converges to $T(\mu)$ in the weak* topology. Hence, by (Larsen (1971), Appendix D.4.2 Theorem), T is continuous with respect to the weak* topology. This completes the proof of Theorem 4.3.

REMARK 4.4. By (Koshi and Yamaguchi (1979), Theorem 1 and Theorem 2, pp. 295, 296), $M_P^a(G) = H_P^1(G)$ if and only if G is one of the following:

$$(1) \quad G = T \oplus D, \qquad (2) \quad G = R \oplus D,$$

where D is a discrete abelian group whose dual is torsion-free or $D = \{0\}$.

Suppose $M_P^a(G) = H_P^1(G)$, then for each multiplier S on $M_P^a(G)$, there exists a bounded continuous function ψ on P^0 such that $S(f)\hat{} = \psi\hat{f}$. Hence, in this case, we can verify that every multiplier on $M_P^a(G)$ is continuous with respect to the weak* topology by the same method used in the proof of Theorem 4.3. However, if $H_P^1(G) \subsetneqq M_P^a(G)$, there exists a multiplier on $M_P^a(G)$ such that it is not continuous with respect to the weak* topology. Indeed, we define a linear operator S from $M_P^a(G)$ into $M(G)$ as follows:

$$S(\mu) = \mu_s, \quad \text{where } \mu_s \text{ is the singular part of } \mu.$$

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Then, by (Doss (1968), Lemma 1, p. 258), μ_s belongs to $M_P^a(G)$ if $\mu \in M_P^a(G)$. Hence S is a multiplier on $M_P^a(G)$. Since $H_P^1 \subsetneq M_P^a(G)$, there exists a nonzero measure μ_0 in $M_P^a(G)$ which is singular with respect to the Haar measure on G. Let $\dot{V}$ be the net consisting of all neighborhoods of 0 in $\hat{G}$. For each $V \in \dot{V}$, let f_V be a nonnegative continuous function on G with the compact support such that $\operatorname{supp}(f_V) \subset V$ and $\int_G f_V(x)\,dx = 1$. Then $f_V * \mu_0$ converges to μ_0 in the weak* topology. But, since $f_V * \mu_0 \in H_P^1(G)$, $S(f_V * \mu_0)$ $(= 0)$ does not converge to μ_0 in the weak* topology.

COROLLARY 4.5. *Let G be a nondiscrete LCA group such that $\hat{G}$ is algebraically ordered. Let P be a subsemigroup of $\hat{G}$ with the (AO)-condition such that it is not dense in G. Then there exists a multiplier S on $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$ such that it is not given by convolution with a bounded regular measure on G.*

PROOF. Let T be the multiplier on $M_P^a(G)$ obtained in Theorem 4.3, and put $S = T^*$. Then, since T is continuous with respect to the weak* topology, S is a multiplier on $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$. Suppose there exists a measure $\mu \in M(G)$ such that $S([f]) = [f] * \mu$ for $[f] \in C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$. Then we have $T(\nu) = \nu * \mu$ for $\nu \in M_P^a(G)$. This is a contradiction.

COROLLARY 4.6. *Let G and P be as in Corollary 4.5. Let β be the canonical map from $C_0(G)$ onto $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$. Then there exists a multiplier S on $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$ such that $S \circ \beta \neq \beta \circ S'$ for every multiplier S' on $C_0(G)$.*

PROOF. Let S be the multiplier on $C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$ obtained in Corollary 4.5. Suppose there exists a multiplier S' on $C_0(G)$ such that $S \circ \beta = \beta \circ S'$. Then there exists a measure $\mu \in M(G)$ such that $S'(f) = f * \mu$ for $f \in C_0(G)$. Hence, since $g * \mu \in C_0(G) \cap H^\infty_{(-P)}(G)$ for $g \in C_0(G) \cap H^\infty_{(-P)}(G)$, S is given by $S([f]) = [f] * \mu$ for $[f] \in C_0(G)/C_0(G) \cap H^\infty_{(-P)}(G)$. This contradicts the fact that S is not given by convolution with a measure.

Finally I wish to express my thanks to the referee for his valuable advice.

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