Probability in the Engineering and Informational Sciences, **29**, 2015, 527–546.
doi:10.1017/S0269964815000170

STATIC STOCHASTIC KNAPSACK PROBLEMS

Kai Chen and Sheldon M. Ross
Epstein Department of Industrial and Systems Engineering, University of Southern California, Los Angeles, CA 90089, USA
E-mails: kaic@usc.edu; smross@usc.edu

Two stochastic knapsack problem (SKP) models are considered: the static broken knapsack problem (BKP) and the SKP with simple recourse and penalty cost problem. For both models, we assume: the knapsack has a constant capacity; there are n types of items and each type has an infinite supply; a type i item has a deterministic reward v_i and a random weight with known distribution F_i. Both models have the same objective to maximize expected total return by finding the optimal combination of items, that is, quantities of items of each type to be put in knapsack. The difference between the two models is: if knapsack is broken when total weights of items put in knapsack exceed the knapsack's capacity, for the static BKP model, all existing rewards would be wiped out, while for the latter model, we could still keep the existing rewards in knapsack but have to pay a fixed penalty plus a variant cost proportional to the overcapacity amount. This paper also discusses the special case when knapsack has an exponentially distributed capacity.

1. INTRODUCTION

The knapsack problem is a classic and widely studied one in the field of combinatorial optimization (see Kellerer, Pferschy, and Pisinger [12]). In its original form, there are multiple items whose weights and values are given in advance, and we want to choose a subset of these items to maximize their total values with the constraint of their total weights under a preset ceiling. Knapsack problems arise naturally in forms of resource allocation or budget planning problems where one aims to extract the maximum economic value from candidate projects while keep resource/budget in check.

For a knapsack problem, in its optimization formulation, each item has a decision variable, either 1 or 0, indicating whether an item is selected or not to be included in the subset which would later be put to the knapsack. Knapsack problem is NP-hard even in its deterministic version where items' weights and values are all known. Fortunately, for deterministic knapsack problems, dynamic programming (see Toth [29]) and the branch and bound algorithm (see Kolesar [16]) both provide efficient ways to locate problem solutions in pseudo-polynomial time.

In real-life circumstances, however, the deterministic assumption on items' weights and values is often violated due to the fact that an item's weight and value information may not be available before making any decision. For instance, when we decide to enroll a project, its true budget requirement cannot be given in advance and we would rather assume the final value has certain distribution patterns. The reward from completing the project may

© Cambridge University Press 2015 0269-9648/15 $25.00
https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

also vary due to uncertain factors in the execution process. Stochastic knapsack problems (SKP) were introduced to accommodate these practical constraints where either or both of items' weights and values are randomly distributed. For the two models discussed in this paper, the static broken knapsack problem (static BKP) and the SKP with simple recourse and penalty, we assume tems' weights to be randomly distributed and their values to be deterministic.

There are two groups of stochastic knapsack problems: adaptive and static. In adaptive problems, items are put in the knapsack one by one and decisions are made sequentially to take advantage of the information feedback from the system. Ilhan & Daskin [10] considered the adaptive SKP with random rewards, which only become known when items are put in the knapsack, and deterministic weights, with the objective being to maximize the probability of reaching a predetermined reward target. Derman, Lieberman, and Ross [6] and Dean, Goemans, and Vondrdk [5] considered the adaptive SKP with random weights but deterministic rewards with the objective to maximize the total values in the knapsack before it is broken. Kleywegt and Papastavrou [14,15] discussed a version of adaptive SKP in which items arrive to the system sequentially in time, with each arriving item having a value and weight which is assumed to have a specified joint distribution. Upon an item's arrival, its value and weight become known, and the decision maker has to decide whether to accept or reject the item, with the objective being to maximize the expected total value earned over a given time period. Some other papers dealing with adaptive stochastic knapsack models include Ross and Tsang [25], Van and Young [30], Lin Lu, and Yao [19], Lu, Chiu, and Cox [20]s.

In the static SKP, one decides at the beginning of the problem which subset of items to put in the knapsack. After putting in the selected set of items, outcome would be evaluated based on revealed information on items' values and weights. There exist at least two directions in studying the static SKP: to maximize the probability of achieving a given reward target while the capacity constraint is strictly satisfied; or to maximize the total rewards while the overload probability is restricted to a certain level, or there is a penalty incurred by any overload.

Problems on the first direction assume stochastic rewards and deterministic weights. Steinberg and Parks [28] and Sniedovich [27] considered the problems where all weights are integral numbers. A preference ordering was proposed on the distributions of the selected items' rewards, and this preference ordering was used in conjunction with dynamic programming to facilitate the search for the optimal solution. Henig [9] presented a search policy to locate the optimal solution given random rewards and known weights. Morton and Wood [22] used a new dynamic programming method and an integer programming method in the case of normally distributed rewards and deterministic weights, and provided a simulation based procedure to approximate the optimal solution in the case of more general reward distributions.

Problems on the second direction assume items' weights are stochastic with known distributions. They usually impose no assumption on items' rewards since only the mean rewards play a role when the objective is to maximize the total (expected) reward subject either to a bound on the probability of overload or the incurrence of penalty costs when the capacity is exceeded. One example of this is the SKP with recourse model, which supposes that a penalty cost proportional to the amount of overload is incurred. Kosuch and Lisser [17,18] gave a good review on the SKP with recourse model as well as chance constrained knapsack problems. They proposed stochastic methods that provide upper and lower bounds as a way to solve the problems using branch and bound techniques, similar to what Cohn and Barnhart did in [4]. Concerning approximation methods, Kleinberg, Rabani, and Tardos [13] proved that there exist polynomial time approximation schemes in static SKP models which

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

return total values at least as good as the optimal one when the overload constraint is relaxed in a small fractional manner. Goel and Indyk [8] presented approximation schemes for static SKP models when the items' weights follow Poisson, exponential or Bernoulli distributions respectively. Merzifonluoğlu Geunes, and Romeijn [21] considered a static SKP model with normally distributed items' weights where penalty cost is proportional to overflow amount and salvage value is allowed for unused capacity. Ağralı & Geunes [1] considered a model with a fixed overflow penalty cost and items' weights follow Poisson distributions. Both papers develop customized branch-and-bound algorithms for optimal solutions respectively. They also provide heuristics to solve relaxations and prove high-quality. Fortz et al. [7] discussed several classes of capacity-constraint problems which can be solvable in pseudo-polynomial time.

Static SKP models either avoid breaking the knapsack by assuming deterministic weights, or tolerate the knapsack being broken within a certain probability, or suppose that a penalty cost is incurred in the event of overflow. This paper considers two model settings: the static BKP model where if the knapsack is broken, all the items' values in the knapsack will be wiped out; and the SKP with simple recourse and penalty model, where any overflow incurs a fixed penalty cost c plus a variant cost proportional to the overcapacity. We discuss the unimodality of the expected return (ER) function and the monotonicity of the marginal optimal decision function for these models. In Section 2, we define the static BKP and prove these two properties. In Section 3, the SKP with simple recourse and penalty is defined. We show that the two properties hold either if $c = 0$ or if all weight distributions are exponential. In Section 4, we consider for both models the case when the knapsack capacity is exponentially distributed. For the first model, we show that with exponential capacity, the two properties hold for general distributions on items' weights. For the second model, we give a sufficient condition and similar results as in constant capacity model. In Section 5, we present a search algorithm for the optimal solutions based on these properties. We show the performance of our algorithm by giving numerical examples in Section 6. Further remarks and the conclusion are in Section 7.

2. THE STATIC BKP MODEL

2.1. Problem Definition

Consider a knapsack with a deterministic capacity w. There are n types of items and each type has infinite supply of items available. For a type i item, $1 \leq i \leq n$, its weight is a non-negative random variable with a known distribution function F_i; its value is v_i, a positive deterministic number given in advance. We want to determine in the beginning of the problem the quantities on each type of items to be put to the knapsack. The total weights of items in the knapsack are revealed only after our decision has been executed. We assume that the weights of the items put in the knapsack are independent. If the knapsack is not broken after we execute our decision, we take all the values in the knapsack; otherwise, we get nothing. The aim is to maximize the expected total returns. In this paper for the static BKP model, we make the following assumption:

All F_i, $1 \leq i \leq n$, have decreasing reversed hazard rate (DRH), that is, if F_i is a continuous (discrete) distribution, then $(f_i(x))/(F_i(x))$ $((f_i(k))/(F_i(k)))$ is non-increasing in x for $x \geq 0$ (in k for $k \in \mathbb{N}$) where $f_i(x)$ $(f_i(k))$ is the probability density (mass) function w.r.t. F_i.

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

Remark 1: If a distribution function is log-concave, then it is obviously DRH. As shown in An [2] and Bagnoli and Bergstrom [3], examples of continuous log-concave distributions include normal, exponential, uniform over convex domain, gamma with shape parameter $\geq$1, etc.; examples of discrete log-concave distributions include Bernoulli, binomial, Poisson and geometric.

Mathematical Notations

A decision vector, denoted as $(k_1, k_2, \ldots, k_n)$, is the instruction to put k_i type i items, $i = 1, \ldots, n$, in the knapsack.

The ER by choosing $(k_1, k_2, \ldots, k_n)$ is

$$R(k_1, \ldots, k_n) = \left(\sum_{i=1}^{n} k_i v_i\right) P\left(\sum_{i=1}^{n}\sum_{h=1}^{k_i} W_{ih} \leq w\right),$$

where $W_{ih}, h = 1, \ldots, k_i,\ i = 1, \ldots, n$ are independent and $W_{ih} \sim F_i$.

The marginal optimal decision function is defined as follows:

$$h(k_1, \ldots, k_{j-1}, \cdot, k_{j+1}, \ldots, k_n) = \arg\max_{k \in \mathbb{N}} R(k_1, \ldots, k_{j-1}, k, k_{j+1}, \ldots, k_n),$$

which decides the quantity of type j items to be put to the knapsack in order to maximize the ER while fixing quantities of all other types. (The function returns the smallest one if there are multiple integers that maximize the ER function.)

2.2. Solution Structures

We first want to delete the types that will never be used in the optimal decision. A natural criteria is to filter out any type which has both lower value and stochastically larger weight at the same time compared with another type. The following proposition, whose proof is immediate, shows the validity of this criteria.

Proposition 1: *For two types i and j, if $v_i \leq v_j$, and $F_i \succeq_{st} F_j$, then type i is dominated by type j, that is, replacing a type i item by a type j item does not decrease the ER.*

The preceding proposition says that the optimal decision never puts in any dominated types. In the following discussions, we assume that none of the n types falls into the dominated types category. To start exploring solution structures for the static BKP, we need the following observation based on the assumption that all weight distributions are *DRH*.

Lemma 1: *Suppose $\{X_i\}$ is a sequence of independent non-negative DRH random variables, and let $S_m = \sum_{i=1}^{m} X_i, \forall m \geq 1$. Then for any constant $c > 0$, the conditional random variable $[S_m | S_m \leq c]$ is stochastically increasing in m.*

Proof: Follows directly from Theorem 1.C.12 (p47) and the equation 1.B.43 (p37) in Shaked and Shanthikumar [26]. ■

Theorem 1 Unimodality of the ER function: *Assume that all F_i, $1 \leq i \leq n$, are DRH. For any j, and fixed values $k_i, i \neq j$, the function $R(k_1, \ldots, k_{j-1}, k, k_{j+1}, \ldots, k_n)$ is unimodal in k, first increasing and then decreasing in k.*

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

PROOF: We have to prove that if

$$R(k_1,\ldots,k_{j-1},k,k_{j+1},\ldots,k_n) \geq R(k_1,\ldots,k_{j-1},k+1,k_{j+1},\ldots,k_n),$$

then

$$R(k_1,\ldots,k_{j-1},k+1,k_{j+1},\ldots,k_n) \geq R(k_1,\ldots,k_{j-1},k+2,k_{j+1},\ldots,k_n).$$

Let us define

$$S(k) = \sum_{h=1}^{k} W_{jh} + \sum_{i\neq j}\sum_{h=1}^{k_i} W_{ih},$$
$$v(k) = kv_j + \sum_{i\neq j} k_i v_i,$$

where $W_{ih} \sim F_i \; \forall\, i \in [1,n]$, and all these random variables ($r.v.s$) are independent. Here $S(k)$ and $v(k)$ are the total items' weights and values respectively by applying the decision $(k_1,\ldots,k,\ldots,k_n)$.

Now,

$$\begin{aligned} &R(k_1,\ldots,k,\ldots,k_n) \geq R(k_1,\ldots,k+1,\ldots,k_n)\\ &\quad\Leftrightarrow v(k)P\left(S(k)\leq w\right) \geq v(k+1)P\left(S(k+1)\leq w\right)\\ &\quad\Leftrightarrow \frac{v(k)}{v(k+1)} \geq P\left(S(k+1)\leq w | S(k)\leq w\right). \end{aligned}$$

Hence, if $R(k_1,\ldots,k,\ldots,k_n) \geq R(k_1,\ldots,k+1,\ldots,k_n)$, then

$$\frac{v(k+1)}{v(k+2)} \geq \frac{v(k)}{v(k+1)} \geq P\left(S(k+1)\leq w|S(k)\leq w\right) \geq P\left(S(k+2)\leq w|S(k+1)\leq w\right),$$

where the first inequality follows because $(v(k))/(v(k+1))$ increases in k and the final inequality follows from Lemma 1. ∎

COROLLARY 1: *For any $j\in[1,n]$, fixed k_i for all $i\neq j$, there exists $k^*<\infty$ such that:*

$$R(k_1,\ldots,k_{j-1},k^*,k_{j+1},\ldots,k_n) \geq R(k_1,\ldots,k_{j-1},k^*+1,k_{j+1},\ldots,k_n).$$

PROOF: From Theorem 1, it suffices to show:

$$R(k_1,\ldots,k_{j-1},1,k_{j+1}\cdots,k_n) \geq \lim_{m\to+\infty} R(k_1,\ldots,k_{j-1},m,k_{j+1},\ldots,k_n).$$

Because the left-hand side is non-negative, the above inequality can be proved by showing:

$$\lim_{m\to+\infty} R(k_1,\ldots,k_{j-1},m,k_{j+1},\ldots,k_n) = 0, \tag{1}$$

for which the proof is in the Appendix. ∎

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

Remark 2: The unimodality property does not hold for the static BKP model under general weight distributions. For instance, suppose $n = 1,\ v_1 = 1, w = 3,$ and that W, the weight of a type 1 item is such that

$$W = \begin{cases} 1 & \text{with probability } \frac{\sqrt{17}}{6}, \\ 3 & \text{w.p. } 1 - \frac{\sqrt{17}}{6}. \end{cases}$$

Then

$$R(1) = 1, \quad R(2) = \frac{17}{18}, \quad R(3) = \frac{17}{18}\frac{\sqrt{17}}{4}.$$

Because $R(1) > R(2) < R(3)$, the unimodality property is violated.

Proposition 2 Monotonicity of the marginal optimal decision function: *For any* $\mathbf{r} = (k_1, \ldots, k_{n-1})$, *the marginal optimal decision function for type n items is defined as*

$$h(\mathbf{r}) = \arg\max_{k \in \mathbb{N}} R(\mathbf{r}, k),$$

where

$$R(\mathbf{r}, k) \triangleq R(k_1, \ldots, k_{n-1}, k).$$

Then, $h(\mathbf{r})$ *is non-increasing in* $\mathbf{r}$.

Proof: Let $\mathbf{I}_i$ be the unit vector whose ith element equals 1 and others equal 0 for $i = 1, \ldots, n-1$. By Theorem 1, it suffices to show:

$$R(\mathbf{r}, m) \geq R(\mathbf{r}, m+1) \Rightarrow R(\mathbf{r} + \mathbf{I}_i, m) \geq R(\mathbf{r} + \mathbf{I}_i, m+1).$$

The proof of the preceding implication is similar to the proof in Theorem 1 and it immediately follows by the fact that:

$$[S(\mathbf{r} + \mathbf{I}_i, m) | S(\mathbf{r} + \mathbf{I}_i, m) < w] \geq_{st} [S(\mathbf{r}, m) | S(\mathbf{r}, m) < w],$$

where $S(\mathbf{r} + \mathbf{I}_i, m)$ and $S(\mathbf{r}, m)$ are the total weights by taking decisions $(\mathbf{r} + \mathbf{I}_i, m)$ and $(\mathbf{r}, m)$, respectively. ■

Note that as there is no specific assumption on the order of the n types imposed on Proposition 2, it follows that the above monotonicity property holds for the marginal optimal value function of any type. An immediate result from Proposition 2 bounds the search space for the optimal solutions.

Corollary 2 Bounded search space: *Let* $(k_1^*, \ldots, k_n^*)$ *be the optimal solution for the static BKP with n types available, and* k_i^1 *be the optimal solution if only type i items are available. Then,*

$$k_i^* \leq k_i^1, \ \forall i = 1, \ldots, n.$$

Proof: From Proposition 2. ■

With Corollary 2, we know there are at most $\prod_{i=1}^{n} (k_i^1 + 1)$ decision vectors to check for the optimal solutions. Indeed, combining the results in Theorem 1 and Proposition 2, a great portion of all these vectors can be skipped in the search process to locate one optimal solution. We describe a search scheme in Section 5.

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

3. THE SKP WITH SIMPLE RECOURSE AND PENALTY MODEL

3.1. Problem Definition with Preliminaries

This model has the same problem setting as for the static BKP model. However, after we decide the quantity for each type and put them to the knapsack, instead of losing everything when the knapsack is broken as in static BKP, in SKP with simple recourse and penalty, any overflow incurs a recourse cost that is proportional to the amount of overcapacity with a constant factor d, as well as a constant penalty $c \geq 0$. Therefore, given a decision vector $\mathbf{k} = (k_1, \ldots, k_n)$, we have the following ER function for this model:

$$R(\mathbf{k}) = R(k_1, \ldots, k_n) = \sum_{i=1}^{n} k_i v_i - d \cdot E[(W_{\text{total}} - w)^+] - c \cdot P(W_{\text{total}} > w), \tag{2}$$

where $W_{\text{total}} = \sum_{i=1}^{n} \sum_{h=1}^{k_i} W_{ih}$, and $W_{ih} \sim F_i$.

Remark 3: When $c = 0$, the above model is called SKP with simple recourse model. Problems of SKP with simple recourse under different problem settings were discussed in Cohn and Barnhart [4]; Kosuch and Lisser [17].

Proposition 1 is still true for this model: if one type has less value but stochastically greater weight, then it should never be used. We also have the following immediate observation.

PROPOSITION 3: *Let w_i be the mean weight of a type i item. If $d < \max_i(v_i/w_i)$ then*

$$\sup_{\boldsymbol{k}} R(\boldsymbol{k}) = \infty.$$

PROOF: Suppose $(v_1/w_1) = \max_i(v_i/w_i) > d$. Then,

$$R(n, 0, \ldots, 0) \geq nv_1 - dE[W_{\text{total}}] - c = n(v_1 - dw_1) - c.$$

Hence, $\lim_{n\to\infty} R(n, 0, \ldots, 0) = \infty$. ∎

We will assume in this paper that $d > \max_i(v_i/w_i)$.

Now, for a fixed j, we shall assume in the following that the decision vector is $(k_1, \ldots, k_{j-1}, k, k_{j+1}, \ldots, k_n)$, where values $k_i, i \neq j$ are fixed. Define

$$S(k) \triangleq \sum_{h=1}^{k} W_{jh} + \sum_{i=1, i\neq j}^{n} \sum_{h=1}^{k_i} W_{ih}, \tag{3}$$

$$v(k) \triangleq kv_j + \sum_{i=1, i\neq j}^{n} k_i v_i,$$

where $W_{ih} \sim F_i \, \forall i, h$. $S(k)$ and $v(k)$ are the total weight and the total items' values respectively by putting k type j items with the fixed number of items in other types. We want to see whether $R(k_1, \ldots, k, \ldots, k_n)$ is unimodal in k.

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

Using the notation I_A to be the indicator variable of the event A, note that

$$\begin{aligned} &R(k_1,\ldots,k,\ldots,k_n) \geq R(k_1,\ldots,k+1,\ldots,k_n) \\ &\quad \Leftrightarrow E[v(k) - d\cdot(S(k)-w)^+ - c\cdot I_{S(k)>w}] \geq E[v(k+1) \\ &\qquad - d\cdot(S(k+1)-w)^+ - c\cdot I_{S(k+1)>w}] \\ &\quad \Leftrightarrow d\cdot E[(S(k+1)-w)^+ - (S(k)-w)^+] + c\cdot E[I_{S(k+1)>w} - I_{S(k)>w}] \geq v_j. \end{aligned} \tag{4}$$

The following is the equivalent of Corollary 1 for the current model, which implies the existence of marginal optimal values given fixed quantities of all types except one.

Corollary 3: *In the SKP with simple recourse and penalty model, for any $j \in [1,n]$, fixed k_i for all $i \neq j$, there exists $k^* < \infty$ such that:*

$$R(k_1,\ldots,k_{j-1},k^*,k_{j+1},\ldots,k_n) \geq R(k_1,\ldots,k_{j-1},k^*+1,k_{j+1},\ldots,k_n).$$

Proof: Because $d > \max_{i\in[1,n]}(v_i/w_i)$, it follows that

$$\lim_n R(k_1,\ldots,k_{j-1},n,k_{j+1},\ldots,k_n) = -\infty.$$

■

3.2. Unimodality and Monotonicity

Theorem 2: *If $c=0$, the ER function in SKP with simple recourse and penalty has the unimodality property.*

Proof: Given $c=0$, from the equivalence relation (4), we only have to show:

$$d\cdot E[(S(k+1)-w)^+ - (S(k)-w)^+] \geq v_j \Rightarrow d\cdot E[(S(k+2)-w)^+ - (S(k+1)-w)^+] \geq v_j.$$

Let us assume two r.v.s $W_1 \sim F_j, W_2 \sim F_j$, where $W_1, W_2, S(k)$ are independent. Since the function $g(x) = x^+$ is a convex function, we have:

$$g(S(k)+W_2+W_1-w) - g(S(k)+W_2-w) \geq g(S(k)+W_1-w) - g(S(k)-w), \tag{5}$$

where W_1, W_2 are always non-negative by our assumption on F_j. Hence,

$$\begin{aligned} &d\cdot E[(S(k+2)-w)^+ - (S(k+1)-w)^+] \\ &\quad = d\cdot E[(S(k)+W_1+W_2-w)^+ - (S(k)+W_2-w)^+] \\ &\quad \geq d\cdot E[(S(k)+W_1-w)^+ - (S(k)-w)^+] \\ &\quad = d\cdot E[(S(k+1)-w)^+ - (S(k)-w)^+], \end{aligned}$$

which concludes the proof. ■

Proposition 4: *When $c=0$, the marginal optimal decision function has the monotonicity property.*

Proof: The proof is similar to that of Theorem 2. ■

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

The following example shows that the unimodality property need not hold when $c > 0$.

Counter Example: Assume that $n = 1, v_1 = 1, w = 199.9$ and that W, the weight of an item, is such that

$$W = \begin{cases} 1 & \text{with probability } 1-10^{-4}, \\ 100 & \text{with probability } 10^{-4}. \end{cases}$$

With $d = 1$ and $c = 200$, we have:

$$R(100) \approx 100, \quad R(101) \approx 98, \quad R(199) \approx 193.$$

which contradicts unmodality.

We now prove that the unimodality property holds for the SKP when F_i is exponential with mean w_i, $i = 1, \ldots, n$.

THEOREM 3: *If all F_i, $i = 1, \ldots, n$, are exponential, then the ER function in the SKP with simple recourse and penalty model has the unimodality property.*

PROOF: First we need to define a discrete random variable N as follows:,

$$N = \begin{cases} 0 & \text{if } S(0) > w, \\ k & \text{if } S(k-1) \le w < S(k), \text{ for all } k \ge 1, \end{cases} \tag{6}$$

where $S(k)$ is defined in equation (3).

Let

$$f_N(k) = P(N = k), \quad k \ge 0.$$

Note,

$$P(S(k) > w) = \sum_{h=0}^{k} f_N(h), \text{ and } \sum_{h=0}^{\infty} f_N(h) = 1.$$

Given all items' weights are exponentially distributed, from the equivalence relation (4), we have

$$\begin{aligned} &R(k_1, \ldots, k, \ldots, k_n) \ge R(k_1, \ldots, k+1, \ldots, k_n) \\ &\quad \Leftrightarrow d \cdot E[(S(k+1) - w)^+ - (S(k) - w)^+] + c \cdot E[I_{S(k+1)>w} - I_{S(k)>w}] \ge v_j, \end{aligned}$$

where

$$\begin{aligned} E[(S(k+1) - w)^+ - (S(k) - w)^+] &= w_j \left(P(S(k) > w) + P(S(k) \le w, S(k+1) > w) \right), \\ E[I_{S(k+1)>w} - I_{S(k)>w}] &= P(S(k) \le w, S(k+1) > w). \end{aligned}$$

Therefore, the preceding inequality is equivalent to

$$\begin{aligned} &dw_j P(S(k) > w) + (dw_j + c) P(S(k) \le w, S(k+1) > w) \ge v_j \\ &\quad \Leftrightarrow dw_j \sum_{h=0}^{k+1} f_N(h) + c f_N(k+1) \ge v_j. \end{aligned}$$

To prove the theorem, we only need to show

$$dw_j \sum_{h=0}^{k+1} f_N(h) + c f_N(k+1) \ge v_j \Rightarrow dw_j \sum_{h=0}^{k+2} f_N(h) + c f_N(k+2) \ge v_j. \tag{7}$$

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

Let us define:

$$k_u \triangleq \min\left\{k \geq 0 : dw_j \sum_{h=0}^{k+1} f_N(h) - v_j \geq 0\right\},$$

$$k_0 \triangleq \min\left\{k \geq 0 : dw_j \sum_{h=0}^{k+1} f_N(h) + cf_N(k+1) - v_j \geq 0\right\}.$$

Because $\sum_{h=0}^{\infty} f_N(h) = 1$, $dw_j > v_j$ and $c > 0$, we know k_u must exist and $k_0 \leq k_u$. We want to show the claim (7) holds for all $k \geq k_0$. When $k \geq k_u - 1$, the claim (7) follows from the definition of k_u. If $k_u = 0$ or $k_0 = k_u$, the claim (7) obviously holds. Now we only have to check the claim (7) when $k \in [k_0, k_u - 1)$ given $k_u > 0$ and $k_0 < k_u$.

For any $k \in [k_0, k_u - 1)$ given $k_u > 0$ and $k_0 < k_u$, because

$$dw_j \sum_{h=0}^{k_u} f_N(h) < v_j \leq dw_j \sum_{h=0}^{k_u+1} f_N(h),$$

we have

$$\begin{aligned} & dw_j \sum_{h=0}^{k+1} f_N(h) + cf_N(k+1) \geq v_j \\ & \Leftrightarrow c \geq \frac{v_j - dw_j \sum_{h=0}^{k+1} f_N(h)}{f_N(k+1)} \\ & \Rightarrow c > \frac{dw_j \sum_{h=0}^{k_u} f_N(h) - dw_j \sum_{h=0}^{k+1} f_N(h)}{f_N(k+1)}; \end{aligned}$$

on the other hand,

$$\begin{aligned} & c \geq \frac{dw_j \sum_{h=0}^{k_u+1} f_N(h) - dw_j \sum_{h=0}^{k+2} f_N(h)}{f_N(k+2)} \\ & \Rightarrow c \geq \frac{v_j - dw_j \sum_{h=0}^{k+2} f_N(h)}{f_N(k+2)} \\ & \Leftrightarrow dw_j \sum_{h=0}^{k+2} f_N(h) + cf_N(k+2) \geq v_j. \end{aligned}$$

Therefore, to prove the claim (7) for $k \in [k_0, k_u - 1)$, we only need to show:

$$\frac{dw_j \sum_{h=0}^{k_u} f_N(h) - dw_j \sum_{h=0}^{k+1} f_N(h)}{f_N(k+1)} \geq \frac{dw_j \sum_{h=0}^{k_u+1} f_N(h) - dw_j \sum_{h=0}^{k+2} f_N(h)}{f_N(k+2)},$$

which is equivalent to show:

$$\frac{f_N(k+2)}{f_N(k+1)} \geq \frac{\sum_{h=k+3}^{k_u+1} f_N(h)}{\sum_{h=k+2}^{k_u} f_N(h)},$$

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

and this inequality follows if we can show:

$$\frac{f_N(k+2)}{f_N(k+1)} \geq \frac{f_N(k+3)}{f_N(k+2)}, \quad \forall k \geq 0. \tag{8}$$

The inequality (8) indeed says that the discrete random variable $[N|N>0]$ is log-concave, which is showed in Lemma 2 below. ■

LEMMA 2: *The discrete random variable $[N|N>0]$ where N is defined in (6) is log-concave.*

PROOF: See the proof in the Appendix. ■

PROPOSITION 5: *When $c>0$ and items' weights are exponentially distributed, the marginal optimal value function has the monotonicity property.*

PROOF: The proof is similar as for Theorem 3. ■

4. WITH EXPONENTIALLY DISTRIBUTED CAPACITY FOR THE TWO MODELS

In the preceding discussions, we assumed that the knapsack capacity is a known constant. In this section, we show the same results still hold when the knapsack has exponential capacity for the two models. Since the proof of monotonicity of the marginal optimal decision function applies the similar logic as in the proof for the unimodality property, in the following, we only sketch the arguments for the unimodality.

4.1. The Static BKP Model

With exponential capacity W, the ER function for the static BKP model is

$$\begin{aligned} R(k_1,\ldots,k_n) &= \left(\sum_{i=1}^{n} k_i v_i\right) P\left(\sum_{i=1}^{n}\sum_{h=1}^{k_i} W_{ih} \leq W\right) \\ &= \left(\sum_{i=1}^{n} k_i v_i\right) \prod_{i=1}^{n}\prod_{h=1}^{k_i} P(W_{ih} \leq W), \end{aligned}$$

where $W_{ih} \sim F_i$.

With exponential capacity, a result similar to Lemma 1 holds with no need of assuming DRH rate on items' weights.

LEMMA 3: *Suppose $\{X_i\}$ is a sequence of independent non-negative random variables, and let $S_m = \sum_{i=1}^{m} X_i$, $\forall m \geq 1$. Then the conditional random variable $[S_m|S_m \leq W]$ is stochastically increasing in m, where W is an exponential random variable that is independent with $X_1, X_2, \ldots$.*

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

Proof: We want to prove

$$[S_m|S_m \le W] \le_{st} [S_{m+1}|S_{m+1} \le W],$$

which is equivalent to

$$\begin{aligned}
&P(S_m \le s|S_m \le W) \ge P(S_{m+1} \le s|S_{m+1} \le W),\ \forall s \ge 0.\\
&\qquad \Leftrightarrow \frac{P(S_m \le s, S_m \le W)}{P(S_m \le W)} \ge \frac{P(S_{m+1} \le s, S_{m+1} \le W)}{P(S_{m+1} \le W)}\\
&\qquad \Leftrightarrow \frac{P(S_{m+1} \le W)}{P(S_m \le W)} \ge \frac{P(S_{m+1} \le s, S_{m+1} \le W)}{P(S_m \le s, S_m \le W)}\\
&\qquad\quad (\text{let } H(W) = \min\{s, W\} \text{ and } X = S_{m+1} - S_m,)\\
&\qquad \Leftrightarrow P(S_{m+1} \le W|S_m \le W) \ge P(S_{m+1} \le h(W)|S_m \le h(W))\\
&\qquad \Leftrightarrow P(W \ge X) \ge P(X \le h(W) - S_m|S_m \le h(W)).
\end{aligned}$$

Let $W_h =_{st} h(W) - S_m|h(W) \ge S_m$, where we use the notation $V =_{st}$, if random variables V and U have the same distribution. Now from the preceding equivalent inequality, it suffices to show

$$W \ge_{st} W_h,$$

which is from

$$\begin{aligned}
W_h &=_{st} \min\{W - S_m, s - S_m\}|W \ge S_m, s \ge S_m\\
&\le_{st} W - S_m|W \ge S_m, s \ge S_m\\
&=_{st} W.
\end{aligned}$$

■

With the above lemma, as in Section 2.2, the following theorem is immediate.

Theorem 4: *In the static BKP model with exponential capacity, for any F_i, $1 \le i \le n$, both unimodality of the ER function and monotonicity of the marginal optimal decision function hold.*

4.2. The SKP with Simple Recourse and Penalty Model

When the knapsack capacity W is an exponential random variable and it is independent with items' weights, the ER function for the SKP with simple recourse and penalty model is

$$R(k_1, \ldots, k_n) = \sum_{i=1}^{n} k_i v_i - d \cdot E[(W_{\text{total}} - W)^+] - c \cdot P(W_{\text{total}} > W), \tag{9}$$

where $W_{\text{total}} = \sum_{i=1}^{n} \sum_{h=1}^{k_i} W_{ih}$, and $W_{ih} \sim F_i$.

We still have to assume that $d > \max_i(v_i/w_i)$. For the general model with no constraints on c and no specific assumptions on F_i, there exists a sufficient condition for the unimodality property, which as well implies the monotonicity property.

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

PROPOSITION 6: *Let $X_i \sim F_i$, $1 \le i \le n$, where $w_i = E[X_i]$; and the capacity W is an exponential r.v. and independent with all F_i. A sufficient condition for the unimodality is:*

$$c \le \min_{1\le i\le n} \frac{dw_i - d\cdot E[(X_i - W)^+]}{P(X_i > W)}. \quad \textbf{(10)}$$

PROOF: Using the same notations as in equation (4), for any $j \in \{1,\ldots,n\}$, given

$$d\cdot E[(S(k+1)-W)^+ - (S(k)-W)^+] + c\cdot E[I_{S(k+1)>W} - I_{S(k)>W}] \ge v_j, \quad \textbf{(11)}$$

to prove the unimodality, we have to show

$$d\cdot E[(S(k+2)-W)^+ - (S(k+1)-W)^+] + c\cdot E[I_{S(k+2)>W} - I_{S(k+1)>W}] \ge v_j. \quad \textbf{(12)}$$

Let $X_j \sim F_j$, $w_j = E[X_j]$, then

$$\begin{aligned}
&d\cdot E[(S(k+1)-W)^+ - (S(k)-W)^+] + c\cdot E[I_{S(k+1)>W} - I_{S(k)>W}]\\
&\quad = dw_j \cdot P(S(k)\ge W) + d\cdot E[(S(k+1)-W)^+ | S(k) < W]\cdot P(S(k)<W)\\
&\qquad + c\cdot P(S(k+1) > W | S(k) < W)\cdot P(S(k) < W)\\
&\quad = dw_i \cdot P(S(k)\ge W) + \big(d\cdot E[(X_j - W)^+] + c\cdot P(X_j > W)\big)\cdot P(S(k) < W).
\end{aligned}$$

Similarly, the left-hand side of the inequality (12) is equal to

$$dw_i \cdot P(S(k+1)\ge W) + \big(d\cdot E[(X_j - W)^+] + c\cdot P(X_j > W)\big)\cdot P(S(k+1) < W).$$

Now with condition (10),

$$\begin{aligned}
&dw_i \cdot P(S(k+1)\ge W) + \big(d\cdot E[(X_j - W)^+] + c\cdot P(X_j > W)\big)\cdot P(S(k+1) < W)\\
&\qquad - dw_i \cdot P(S(k)\ge W) - \big(d\cdot E[(X_j - W)^+] + c\cdot P(X_j > W)\big)\cdot P(S(k) < W)\\
&\quad = dw_i \cdot P(S(k+1)\ge W, S(k) < W)\\
&\qquad - \big(d\cdot E[(X_j - W)^+] + c\cdot P(X_j > W)\big)\cdot P(S(k) < W, S(k+1)\ge W)\\
&\quad = \big(dw_i - d\cdot E[(X_j - W)^+] - c\cdot P(X_j > W)\big)\cdot P(S(k) < W, S(k+1)\ge W) \ge 0.
\end{aligned}$$

■

Remark 4: When $c = 0$, condition (10) in Proposition 6 holds because the left-hand side of condition (10) is always non-negative.

When $c > 0$ but items have exponential weights, we can prove the unimodality similarly as in Theorem 3: by re-defining N in definition (6) with W replacing w, the new r.v. N satisfies Lemma 2, for which we need to observe the fact that the r.v. Y (in Proof of Lemma 2 in the Appendix), for $Y = W - S(0)$, still has log-concave probability density function (pdf) when W is an exponential r.v.

THEOREM 5: *In the SKP with simple recourse and penalty model with exponential capacity, when $c = 0$ or all types have exponential weights, both the unimodality and the monotonicity properties hold.*

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

5. SEARCH SCHEME FOR THE OPTIMAL SOLUTION

In the following, we assume that the unimodality property holds for the ER function and the monotonicity property holds for the marginal optimal decision functions. We use these two properties to design a search algorithm for the two static SKP models described in this paper.

Given distribution F_i of item' s weight, $\forall i \in [1, n]$, and all other parameters, the value of $R(\cdots)$ on a decision vector is difficult to compute for most distributions (The case of normal distribution has been discussed in other papers, see the Introduction.). We will use simulation to approximate ER value for a decision vector input.

To facilitate the simulation process, a large set of random variables for each type of distribution are generated in the beginning of the program. Then whenever random variables are needed on each iteration of the simulation, they can be randomly selected from these sets respectively.

For type i, if F_i has well-defined inverse function F_i^{-1}, then the set corresponding to the type i items' weights can be generated through stratified simulation, that is, choose a large enough integer N, then the set is: $\{F_i^{-1}((i-(1/2))/N) : i = 1, 2, \ldots, N\}$. For the cases where all F_i, $i \in [1, n]$, are exponential distributions or normal distributions, we only have to generate one large set of standard random variables instead of n sets for every type, then whenever an instance of an item's weight is needed, we randomly select one from the set and transform the standard r.v. to our desired one. In the following, given a decision vector, the ER function $R(\cdots)$ represents the simulation-generated value.

As shown in Corollary 2, we first have to find all these k_i^1, $\forall i \in [1, n]$ to bound our search space. If only one type of items available, say type i, we compute $R(2^l)$ for $l = 0, 1, 2, \ldots$, respectively, until the first l, denoted as l_u, such that $R(2^{l_u - 1}) > R(2^{l_u})$. From the unimodality property of the function $R(\cdot)$, 2^{l_u} must be an upper bound of the optimal solution. Then we can take a bisection search in $[0, 2^{l_u}]$, which applies the unimodality property, to find the optimal solution k_i^1.

For general n, a heuristic search algorithm similar to simulated annealing is presented. In this heuristic algorithm, for each $i \in [1, n]$, we start with the decision vector $(0, \ldots, k_i^1, \ldots, 0)$. For each of the n starting vectors, on the first round, we try to find the best marginal decision for the first element while keeping the remaining $n-1$ elements fixed. We update each vector respectively by replacing the first element with the calculated best marginal decision on the first element. Then on the next round, we find the best marginal decision on the second element of each updated vector while keeping other elements fixed, update the vector, and move to the next round so that on the rth round, we consider the element on position $(r-1) \bmod n + 1$. The update process for each vector stops when no more improvement can be made after n rounds. The decision vector that has the highest ER among the n final vectors is selected to be returned by the program.

Search Algorithm

Compute k_i^1 for each $i \in [1, n]$. Let S be an empty set.
For i from 1 to n.
    Set $\mathbf{d_i} = (0, \ldots, k_i^1, \ldots, 0)$.
    Set $c_i = 0$, and pos $= 0$.
    While $c_i \leq n$.
        Update pos $=$ pos $(\mod n)$;
        Keep all elements in decision vector $\mathbf{d_i}$ fixed except for that in position
        pos $+ 1$; apply the unimodality property in bisection search to find the local

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

optimal decision for type pos + 1 in the range $[0, k_i^1]$.
If the newly-found quantity differs the old one for type pos + 1 in $\mathbf{d_i}$, set $c_i = 0$; otherwise $c_i = c_i + 1$.
Update pos = pos + 1.
End While.
Add $(\mathbf{d_i}, R(\mathbf{d_i}))$ to the set S.
End For.
In the set S, find $(\mathbf{d}, R(\mathbf{d}))$ which has the largest $R(\mathbf{d})$, output $\mathbf{d}$.

The decision vector returned by the above search algorithm is not guaranteed to be the optimal vector; however, as we will show in the following numerical examples, the ER from our solution is very close to the optimal ER.

6. NUMERICAL EXAMPLES

In this part, we give numerical examples to illustrate the implementation of the search algorithm presented this paper for the two models with constant capacity. The programs are written in C + + codes, and the value of the ER function on each decision vector is approximated by Monte Carlo simulation.

6.1. Example for the Static BKP Model

We assume that the items' weights are absolute values of normal r.v.s. (note: the r.v.s are in DRH). The parameters of the distributions are randomly generated, that is, for each type $i \in [1,3]$, we set $v_i = u_1$, $F_i \sim |N(0, u_2^2)|$ where u_1, u_2 are independent uniform r.v.s in $(0,1)$.

On Table 1 of $n = 3$, the running time (RT) in our algorithm (ALG) becomes an even smaller percentage of RT in the exhaustive search (ES) as the problem size goes up in terms of the increasing knapsack capacity. The computation efficiency achieved by the algorithm for $n = 3$ is evident compared with ES. As we can see in this example, with capacity 20, our algorithm runs nearly a thousand times faster than in purely ES without compromising the result. All the ERs from our algorithm under different problem sizes in this example are close to the optimal values computed from ES.

6.2. Example for the SKP with Simple Recourse and Penalty Model with $c > 0$ and Exponential Items' Weights

We consider an example of $n = 3$ and for a type i item, $\forall i \in [1,3]$, its weight is exponentially distributed with mean w_i. In this example as shown in Table 2, we set $d = 20$

Table 1. $n = 3$, with parameters: $v_1 = 0.305025$, $\sigma_1 = 0.313816$, $v_2 = 0.334888$, $\sigma_2 = 0.466047$, $v_3 = 0.68152$, $\sigma_3 = 0.562881$

Capacity (w)	RT (ALG)	RT (ES)	ER (ALG)	ER (ES)
1	0.013508	0.016692	0.858716	0.868598
5	0.05416	1.08355	4.95937*	4.94351
20	0.244282	236.822	23.0249	23.0105

*ER from our algorithm is higher than the optimal one due to the sample errors in simulation processes.

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

TABLE 2. $n = 3$, with parameters: $c = 5.0$, $d = 20.0$, $v_1 = 2.0$, $w_1 = 0.32$, $v_2 = 3.0$, $w_2 = 0.40$, $v_3 = 4.0$, $w_3 = 0.52$

Capacity (w)	RT (ALG)	RT (ES)	ER (ALG)	ER (ES)
5	0.405688	5.35432	25.3228	25.3396
10	0.936152	124.067	58.5908	59.033
15	2.03097	$\gg 10^3$	92.3793	NA

where $d > \max_i\{v_i/w_i\}$. When the total capacity equals 15, the ES runs more than 15 min compared to less than 4 s in our heuristic algorithm. The performance and efficiency of the presented algorithm which takes advantage of the unimodality property for the second model is illustrated on Table 2.

7. CONCLUSION

This paper discusses two static SKP models: the static BKP and the SKP with simple recourse and penalty. In both models, there are n types of items with each type having infinite supply; an item's value is deterministic and its weight is stochastic; only one decision has to be made in the beginning on the quantities of each type of items to be put to the knapsack; the objective is to achieve the highest ER. The difference is that the event of overflow, that is, broken knapsack, wipes out all the existing values in the static BKP model; whereas in the SKP with simple recourse and penalty, the overflow incurs a fixed penalty plus a variant cost proportional to the overcapacity amount. When the knapsack capacity is constant, we explored the solution structures of the static BKP by showing the unimodality property of the ER function under the assumption that all items' weights have *DRH* rate . We proved the unimodality property always holds even without the *DRH* assumption if the knapsack capacity is exponentially distributed. In the second model, the SKP with simple recourse and penalty, we showed the unimodality property always holds for the SKP with simple recourse but no fixed penalty. We also proved that this property holds for the general model when all items' weights are exponentially distributed. With exponential capacity, we gave a sufficient condition of the unimodality property and proved the same results hold in this case. Based on the unimodality and monotonicity properties, we develop a search algorithm in the two models. The advantage of the search algorithm compared to the ES in performance and efficiency is manifested in numerical examples.

Acknowledgement

This material is based upon work supported by the U.S. Army Research Laboratory and the U.S. Army Research Office under grant number W911NF-11-1-0115.

References

1. Ağralı, S. & Geunes, J. (2009). A single-resource allocation problem with Poisson resource requirements. *Optimization Letters* 3(4): 559–571.
2. An, M. (1997). Log-concave probability distributions: theory and statistical testing. *Duke University Department of Economics Working Paper*, (95-03).
3. Bergstrom, T. & Bagnoli, M. (2005). Log-concave probability and its applications. *Economic Theory* 26: 445–469.

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

4. Cohn, A.M. & Mit, C.B. (1998). The stochastic knapsack problem with random weights: a heuristic approach to robust transportation planning. In *Proceedings of the Triennial Symposium on Transportation Analysis*. Citeseer.
5. Dean, B.C., Goemans, M.X., & Vondrdk, J. (2004). Approximating the stochastic knapsack problem: the benefit of adaptivity. In *2004 Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science*, IEEE, pp. 208–217.
6. Derman, C., Lieberman, G.J., & Ross, S.M. (1978). A renewal decision problem. *Management Science* 24(5): 554–561.
7. Fortz, B., Labbé, M., Louveaux, F., & Poss, M. (2013). Stochastic binary problems with simple penalties for capacity constraints violations. *Mathematical Programming* 138(1–2): 199–221.
8. Goel, A. & Indyk, P. (1999). Stochastic load balancing and related problems. In *1999, 40th Annual Symposium on Foundations of Computer Science*, IEEE, pp. 579–586.
9. Henig, M.I. (1990). Risk criteria in a stochastic knapsack problem. *Operations Research* 38(5): 820–825.
10. S.M.R. Iravani Ilhan, T. & Daskin, M.S. (2011). The adaptive knapsack problem with stochastic rewards. *Operations Research* 59(1): 242–248.
11. Karlin, S. (1968). *Total positivity*. vol. 1. Stanford University Press.
12. Kellerer, H., Pferschy, U., & Pisinger, D. (2004). *Knapsack problems*. Berlin, Heidelberg: Springer.
13. Kleinberg, J., Rabani, Y., & Tardos, É. (1997). Allocating bandwidth for bursty connections. In *Proceedings of the 29th Annual ACM Symposium on Theory of Computing*, ACM, pp. 664–673.
14. Kleywegt, A.J. & Papastavrou, J.D. (1998). The dynamic and stochastic knapsack problem. *Operations Research* 46: 17–35.
15. Kleywegt, A.J. & Papastavrou, J.D. (2001). The dynamic and stochastic knapsack problem with random sized items. *Operations Research* 49(1): 26–41.
16. Kolesar, P.J. (1967). A branch and bound algorithm for the knapsack problem. *Management Science* 13(9): 723–735.
17. Kosuch, S. & Lisser, A. (2010). Upper bounds for the 0–1 stochastic knapsack problem and a branch-and-bound algorithm. *Annals of Operations Research* 176: 77–93.
18. Kosuch, S. & Lisser, A. (2011). On two-stage stochastic knapsack problems. *Discrete Applied Mathematics* 159(16): 1827–1841.
19. Lin, G.Y., Lu, Y., & Yao, D.D. (2008). The stochastic knapsack revisited: switch-over policies and dynamic pricing. *Operations Research* 56(4): 945–957.
20. Lu, L.L., Chiu, S.Y., & Cox, L.A., Jr. (1999). Optimal project selection: stochastic knapsack with finite time horizon. *Journal of the Operational Research Society* 50(6): 645–650.
21. Merzifonluoğlu, Y., Geunes, J., & Romeijn, H.E. (2012). The static stochastic knapsack problem with normally distributed item sizes. *Mathematical Programming* 134(2): 459–489.
22. Morton, D.P. & Wood, R.K. (1998). On a stochastic knapsack problem and generalizations. *Advances in Computational and Stochastic Optimization, Logic Programming, and Heuristic Search: Interfaces in Computer Science and Operations Research* 9: 149–168.
23. Nanda, A.K. & Sengupta, D. (2005). Discrete life distributions with decreasing reversed hazard. *Sankhyā: The Indian Journal of Statistics* 67: 106–125.
24. Prekopa, A. (1973). On logarithmic concave measures and functions. *Acta Scientiarum Mathematicarum (Szeged)* 34: 335–343.
25. Ross, K.W. & Tsang, D.H.K. (1989). The stochastic knapsack problem. *IEEE Transactions on Communications* 37(7): 740–747.
26. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders*. Springer Science & Business Media.
27. Sniedovich, M. (1980). Preference order stochastic knapsack problems: methodological issues. *Journal of the Operational Research Society* 31(11): 1025–1032.
28. Steinberg, E. & Parks, M.S. (1979). A preference order dynamic program for a knapsack problem with stochastic rewards. *Journal of the Operational Research Society* 30(2): 141–147.
29. Toth, P. (1980). Dynamic programming algorithms for the zero-one knapsack problem. *Computing* 25(1): 29–45.
30. Van Slyke, R. & Young, Y. (2000). Finite horizon stochastic knapsacks with applications to yield management. *Operations Research* 48(1): 155–172.

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

APPENDIX

Proof for inequality (1) in Corollary 1.

Denote:

$$S = \sum_{i \neq j} \sum_{h=1}^{k_i} W_{ih},$$

$$v = \sum_{i \neq j} k_i v_i,$$

where $W_{ih} \sim F_i \; \forall\, i \in [1, n]$, that is, S and v are the total weights and the total values of all other types excluding type j. We also assume:

$$E[S] = \mu, Var(S) = \sigma^2.$$

Because,

$$R(k_1, \ldots, m, \ldots, k_n) = (v + mv_j) P\left(S + \sum_{h=1}^{m} W_{jh} \le w\right),$$

after applying the central limit theorem, we have:

$$\begin{aligned}
\lim_{m \to +\infty} R(k_1, \ldots, m, \ldots, k_n) &= \lim_{m \to +\infty} (v + mv_j) \Phi\left(\frac{w - v - mw_j}{\sqrt{\sigma^2 + mw_j^2}}\right) \\
&= \lim_{m \to +\infty} (v + mv_j) \Phi(-\sqrt{m}) \\
&= \lim_{m \to +\infty} mv_j \Phi(-\sqrt{m}),
\end{aligned}$$

where $\Phi(\cdot)$ is the cdf for the standard normal distribution $N(0, 1)$.

We have the following upper-bound for the term $\Phi(-\sqrt{m})$:

$$\begin{aligned}
\Phi(-\sqrt{m}) &= 1 - \Phi(\sqrt{m}) \\
&= \int_{\sqrt{m}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \\
&\le \int_{\sqrt{m}}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{x}{\sqrt{m}} e^{-\frac{x^2}{2}} dx \\
&= \frac{e^{\frac{-m}{2}}}{\sqrt{2\pi m}}.
\end{aligned}$$

Therefore,

$$\lim_{m \to +\infty} mv_j \Phi(-\sqrt{m}) \le \lim_{m \to +\infty} mv_j \frac{e^{\frac{-m}{2}}}{\sqrt{2\pi m}} = 0.$$

Proof of Lemma 2.

Denote: $Y = w - S(0)$; $\{W_h\}_{h \ge 1}$ is a sequence of i.i.d exponentially distributed random variable with mean w_j. We can rewrite the definition of the random variable N as follows:

$$N = \begin{cases} 0 & \text{if } Y < 0, \\ k & \text{if } \sum_{h=1}^{k-1} W_h \le Y \text{ and } \sum_{h=1}^{k} W_h > Y. \; \forall k \ge 1 \end{cases}$$

Let us denote $\hat{N} = [N | N > 0]$, let $f(y)$ be the pdf of Y, and $\hat{f}(y)$ as the pdf for $[Y | Y \ge 0]$. If $S(0) \equiv 0$, then $\hat{N}$ is a Poisson distribution which implies the log-concavity of $\hat{N}$. Otherwise,

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

$S(0)$ is the sum of some independent exponentially distributed random variables whose pdf are all log-concave, from the fact that the convolution of log-concave distributions preserves the log-concavity (see Theorem 7 in Prekopa [24]), we know $f(y)$ must be log-concave in the domain $(-\infty, w]$. Therefore, $\hat{f}(y)$ is log-concave in $[0, w]$. Now we want to show $\hat{N}$ is log-concave, which follows Lemma 4 below.

We now want to prove Lemma 4. The idea of the proof here is from Nanda and Sengupta [23]. The authors in that paper proved that: the number of events which arrive according to a Poisson process during a stochastic time period is in discrete *DRH* if the distribution of the length of this time period is in continuous *DRH*. In Lemma 4, we prove: the distribution of the number of events is discrete log-concave if the distribution of the time length is continuous log-concave.

LEMMA 4: *If $\{N(t), t \geq 0\}$ is a Poisson process with rate λ and is independent of T, a positive continuous log-concave random variable, (It is required that the pdf of T has first-order derivative in its domain.) then the discrete random variable $N(T)$ is log-concave.*

PROOF: Let $p(k)$, $k \in \mathbb{N}$, be the pmf of $N(T)$. To prove $N(T)$ is log-concave, it is equivalent to show: $((p(k+1))/p(k))$ is non-increasing for $k \geq 0$.

Assuming the continuous r.v. T has pdf $f(t)$, and denote:

$$w = \sup_{t \geq 0}\{t : f(t) > 0\}.$$

Let us define the following functional for $k \geq 0$,

$$\Gamma(\lambda, g, k) = \int_0^w e^{-\lambda x} \frac{(\lambda x)^k}{k!} g(x) dx,$$

where g is any function which makes the above integration well defined.

It is easy to verify:

$$p(k) = \Gamma(\lambda, f, k), \; \forall k \geq 0.$$

Let $f'(t)$ be the first derivative of $f(t)$ on t, then for $k \geq 0$, we have:

$$\Gamma(\lambda, f', k) = C(k) + \lambda \Gamma(\lambda, f, k) - \lambda \Gamma(\lambda, f, k-1),$$

where $C(k) = e^{-\lambda w} \frac{(\lambda w)^k}{k!} f(w) - f(0) I_{k=0}$, and $\Gamma(\lambda, f, -1) = 0$. Hence,

$$\begin{aligned}
&p(k+1)/p(k) \text{ is non-increasing in } k \text{ for } k \geq 0 \\
&\quad \Leftrightarrow \Gamma(\lambda, f, k+1)/\Gamma(\lambda, f, k) \text{ is non-increasing in } k \text{ for } k \geq 0 \\
&\quad \Leftrightarrow (\Gamma(\lambda, f, k) - \Gamma(\lambda, f, k-1))/\Gamma(\lambda, f, k) \text{ is non-increasing in } k \text{ for } k \geq 1 \\
&\quad \Leftrightarrow \left(\Gamma(\lambda, f', k) - C(k)\right)/\Gamma(\lambda, f, k) \text{ is non-increasing in } k \text{ for } k \geq 1 \\
&\quad \Leftarrow \Gamma(\lambda, f', k)/\Gamma(\lambda, f, k) \text{ is non-increasing in } k \text{ for } k \geq 1, \\
&\qquad \text{and} C(k)/\Gamma(\lambda, f, k) \text{ is non-decreasing in } k \text{ for } k \geq 1.
\end{aligned}$$

Now we only have to prove the following two claims:
(A1) $\Gamma(\lambda, f', k)/\Gamma(\lambda, f, k)$is non-increasing in k for $k \geq 1$;
(A2) $C(k)/\Gamma(\lambda, f, k)$ is non-decreasing in k for $k \geq 1$.

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press

Let us first prove the claim (A2), for $k \geq 1$:

$$\begin{aligned}
&C(k)/\Gamma(\lambda, f, k) \text{ is non-decreasing in } k \\
&\quad \Leftrightarrow e^{-\lambda w}\frac{(\lambda w)^k}{k!}f(w)/\int_0^w e^{-\lambda x}\frac{(\lambda x)^k}{k!}f(x)dx \text{ is non-decreasing in } k \\
&\quad \Leftrightarrow \int_0^w e^{-\lambda x}\frac{(\lambda x)^k}{(\lambda w)^k}f(x)dx \text{ is non-increasing in } k \\
&\quad \Leftarrow \frac{(\lambda x)^k}{(\lambda w)^k} \text{ is non-increasing in } k \text{ for all } x \in [0, w].
\end{aligned}$$

To prove the claim (A1), we will utilize the log-concave property of $f(y)$ on $[0, w]$. Because $f(t)$ is log-concave on $[0, w]$, $((f'(t))/f(t))$ is monotone deceasing (see Remark1 in Bagnoli and Bergstrom [3]). Therefore, for any $\theta > 0$, $f'(t) - \theta f(t)$ changes sign at most once from positive to negative as t goes from 0 to w. Denote

$$K(x, k) = e^{-\lambda x}\frac{(\lambda x)^k}{k!},$$

then $K(x, k)$ is TP_2 (total positivity of order 2) over $[0, w] \times \mathbb{N}$. From Chapter 5 in Karlin [11], the variation diminishing property of $K(x, k)$ implies that $\Gamma(\lambda, f', k) - \theta\Gamma(\lambda, f, k)$ changes sign at most once from positive to negative as k goes from 0 to ∞, which implies that $\Gamma(\lambda, f', k)/\Gamma(\lambda, f, k)$ is monotone decreasing. Therefore, we have proved the claim (A1). ■

https://doi.org/10.1017/S0269964815000170 Published online by Cambridge University Press