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# SPECTRAL INTEGRATION OF MARCINKIEWICZ MULTIPLIERS

NAKHLÉ ASMAR, EARL BERKSON AND T. A. GILLESPIE

ABSTRACT Let $X$ be a closed subspace of $L^p(\mu)$, where $\mu$ is an arbitrary measure and $1 < p < \infty$ By extending the scope of spectral integration, we show that every invertible power-bounded linear mapping of $X$ into $X$ has a functional calculus implemented by the algebra of complex-valued functions on the unit circle satisfying the hypotheses of the Strong Marcinkiewicz Multiplier Theorem This result expands the framework of the Strong Marcinkiewicz Multiplier Theorem to the setting of abstract measure spaces

1. **Introduction.** Harmonic analysis can be broadly described as the study of the ways that spaces decompose under the actions of groups. Multiplier theory often serves this purpose by its ability to treat weakened forms of orthogonality. For example, the boundedness of the Hilbert transform on $L^p$-spaces of the unit circle $\mathbb{T}$, $1 < p < \infty$, sets up the convergence in $L^p(\mathbb{T})$ of Fourier series (a convergence which is unconditional only when $p = 2$ [7, p. 12]), and thereby leads to the M. Riesz decomposition and its ramifications in Macaev's results for the von Neumann-Schatten $p$-classes [10]. As this conditional convergence and similar examples suggest, an operator-theoretic approach to the weakened forms of orthogonality in general analysis must take account of the delicacy involved by forgoing reliance on strongly countably additive spectral measures in formulating the spectral decomposability of operators. The following weakening of the notion of spectral measure has been useful in this regard (see, *e.g.* [1], [3], [4], [5]), and, in Theorem 1.4 below, we shall take up the link it provides between operator theory and the Strong Marcinkiewicz Multiplier Theorem ([7, Theorem 8.4.2]).

DEFINITION. Let $\mathfrak{B}(Y)$ denote the Banach algebra of all bounded linear mappings of a Banach space $Y$ into itself, and let $I$ be the identity operator on $Y$. A *spectral family* in $Y$ is a projection-valued function $F(\cdot)$ mapping the real line $\mathbb{R}$ into $\mathfrak{B}(Y)$, and having the following properties:

(i) $\sup\{\|F(\lambda)\| : \lambda \in \mathbb{R}\} < \infty$;

(ii) $F(\lambda)F(\tau) = F(\tau)F(\lambda) = F(\lambda)$ whenever $\lambda \leq \tau$;

(iii) $F(\cdot)$ is right-continuous on $\mathbb{R}$ with respect to the strong operator topology of $\mathfrak{B}(Y)$;

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(iv) at each $\lambda \in \mathbb{R}$, $F(\cdot)$ has a left-hand limit $F(\lambda^-)$ in the strong operator topology of $\mathfrak{B}(Y)$;

(v) with respect to the strong operator topology of $\mathfrak{B}(Y)$, $F(\lambda) \longrightarrow I$ as $\lambda \longrightarrow +\infty$, and $F(\lambda) \longrightarrow 0$ as $\lambda \longrightarrow -\infty$.

If there is a compact interval $[a, b]$ such that $F(\lambda) = 0$ for $\lambda < a$ and $F(\lambda) = I$ for $\lambda \geq b$, then we say that $F(\cdot)$ is *concentrated* on $[a, b]$.

Corresponding to any spectral family $F(\cdot)$ of projections in $Y$, a Riemann-Stieltjes notion of spectral integration with respect to $F(\cdot)$ can be defined as follows. Given a bounded complex-valued function $f$ on a compact interval $J = [\alpha, \beta]$ of $\mathbb{R}$, for each partition $\mathcal{P} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ of $J$ we put

$$\mathcal{S}(\mathcal{P}; f, F) = \sum_{k=1}^{n} f(\lambda_k)\{F(\lambda_k) - F(\lambda_{k-1})\}.$$

If the net $\{\mathcal{S}(\mathcal{P}; f, F)\}$ converges in the strong operator topology of $\mathfrak{B}(Y)$ as $\mathcal{P}$ increases through the partitions of $J$ directed by inclusion, then we denote the strong limit by $\int_{[\alpha,\beta]} f\, dF$, and we further define $\int_{[\alpha,\beta]}^{\oplus} f\, dF$ by writing

$$\int_{[\alpha,\beta]}^{\oplus} f\, dF = f(\alpha)F(\alpha) + \int_{[\alpha,\beta]} f\, dF.$$

Until now, the only general class of functions known to be integrable with respect to $F(\cdot)$ over $J$ has been the algebra $\mathbf{BV}(J)$ consisting of all complex-valued functions $f$ on $J$ whose total variation $\operatorname{var}(f, J)$ is finite (see [6, Chapter 17] or the abbreviated account of spectral integration in [3]). In fact, the mapping $f \in \mathbf{BV}(J) \longrightarrow \int_{[\alpha,\beta]}^{\oplus} f\, dF$ is an algebra homomorphism of $\mathbf{BV}(J)$ into $\mathfrak{B}(Y)$ such that

$$\left\| \int_{[\alpha,\beta]}^{\oplus} f\, dF \right\| \leq \|f\|_J \sup\{\|F(\lambda)\| : \lambda \in J\},$$

where $\|\cdot\|_J$ denotes the Banach algebra norm on $\mathbf{BV}(J)$ specified by

$$\|f\|_J = \sup\{|f(\lambda)| : \lambda \in J\} + \operatorname{var}(f, J).$$

Having attended to the basic facts of spectral integration, we now pass to the setting wherein our main result (Theorem 1.4) expands the scope of spectral integration to a broader class of integrands. The notation established here will be in effect henceforth. Let $X$ be a closed subspace of $L^p(\mu)$, where $(\mathcal{M}, \mu)$ is an arbitrary measure space, and $1 < p < \infty$. Denote the set of all integers by $\mathbb{Z}$, and suppose that $U \in \mathfrak{B}(X)$ is an invertible operator such that

$$c \equiv \sup\{\|U^n\| : n \in \mathbb{Z}\} < \infty. \tag{1.1}$$

Under these circumstances, it is known [3, (2.17), (2.18), and Theorem (4.8)(ii)] that there is a unique spectral family $E(\cdot)$ of projections in $X$ such that $E(\cdot)$ is concentrated on $[0, 2\pi]$, $E\big((2\pi)^-\big) = I$, and

$$U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE(\lambda).$$

This unique spectral family $E(\cdot)$ is called the *spectral decomposition* of $U$. It has the further property [3, Theorem (4.8)(iii)] that:

$$\sup\{\|E(\lambda)\| : \lambda \in \mathbb{R}\} \leq c^2 C_p, \tag{1.2}$$

where, here and henceforth, $C_p$ denotes a positive real constant depending only on $p$ which may change in value from one occurrence to another.

REMARKS. By virtue of [8, Theorem 2(ii)], the operator $U$ can easily fail to be a spectral operator in the sense of Dunford. Consequently its spectral decomposition $E(\cdot)$ is, in general, not induced by a countably additive spectral measure on the Borel sets of $\mathbb{R}$.

Our main result in Theorem 1.4 establishes integrability with respect to $E(\cdot)$ for the complex-valued functions on $\mathbb{T}$ satisfying the hypotheses of the Strong Marcinkiewicz Multiplier Theorem [7, Theorem 8.4.2]. Such functions are characterized in terms of the *dyadic decomposition* of $\mathbb{T}$, which is described in the following manner. For $j \in \mathbb{Z}$, let $t_j$ be the *j-th dyadic point* of $(0, 2\pi)$ specified by

$$t_j = \begin{cases} 2^{j-1}\pi, & \text{if } j \leq 0, \\ 2\pi - 2^{-j}\pi & \text{if } j > 0; \end{cases}$$

and put

$$\omega_j = e^{it_j} \text{ for } j \in \mathbb{Z}.$$

Also, for $j \in \mathbb{Z}$, let $\Gamma_j$ be the arc specified by

$$\Gamma_j = \{e^{it} : t_j < t < t_{j+1}\},$$

and denote the closure in $\mathbb{T}$ of $\Gamma_j$ by $\Delta_j$. With this notation, the functions $\phi$ mapping $\mathbb{T}$ into the complex numbers $\mathbb{C}$ which satisfy the hypotheses of the Strong Marcinkiewicz Multiplier Theorem can be characterized by the condition

$$\|\phi\|_{\mathfrak{M}} \equiv \sup\{|\phi(z)| : z \in \mathbb{T}\} + \sup\{\operatorname{var}(\phi, \Delta_j) : j \in \mathbb{Z}\} < \infty. \tag{1.3}$$

We shall call the functions satisfying (1.3) *Marcinkiewicz multipliers*, and shall denote the set of all Marcinkiewicz multipliers by $\mathfrak{M}$. Notice that with pointwise operations and the norm $\|\cdot\|_{\mathfrak{M}}$ in (1.3), $\mathfrak{M}$ is a Banach algebra. Our main result can now be stated as follows.

THEOREM 1.4. *Let $\mu, X, U \in \mathfrak{B}(X)$, and $E(\cdot)$ be as described above. Then for each Marcinkiewicz multiplier $\phi$ defined on $\mathbb{T}$, the integral $\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$ exists in the strong operator topology of $\mathfrak{B}(X)$. Furthermore, the mapping from $\mathfrak{M}$ to $\mathfrak{B}(X)$ specified by*

$$\phi \longrightarrow \int_{[0,2\pi]}^{\oplus} \phi(e^{i\lambda})\, dE(\lambda)$$

*is an identity-preserving algebra homomorphism such that*

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi(e^{i\lambda})\, dE(\lambda) \right\| \leq c^6 C_p \|\phi\|_{\mathfrak{M}}, \text{ for all } \phi \in \mathfrak{M}, \tag{1.5}$$

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*where $C_p$ is a positive real constant depending only on p, and c is the constant defined in (1.1).*

After a preliminary review of some needed tools in §2, the demonstration of Theorem 1.4 will be carried out in §3. In §4 we consider circumstances under which the bound in (1.5) can be improved.

REMARKS. The proof of Theorem 1.4 in §3 will rely on a certain "dyadic spectral measure" associated with $U$ (described in Theorem 2.1 below). Since the existence and properties of this dyadic spectral measure are established in [2] with the aid of the Strong Marcinkiewicz Multiplier Theorem, the latter is an ingredient in the proof of Theorem 1.4. Nevertheless, it should be noted that Theorem 1.4 reflects and generalizes its classical antecedents by including the Strong Marcinkiewicz Multiplier Theorem as a special case. Specifically, when $U$ is the left shift on $\ell^p(\mathbb{Z})$, the spectral decomposition $E(\cdot)$ of $U$ is determined as follows (see [8, proof of Theorem 1] or [3, Proposition (4.23)(ii)]): for each $\lambda \in [0, 2\pi)$, $E(\lambda)$ is the Fourier multiplier transform on $\ell^p(\mathbb{Z})$ corresponding to the characteristic function of the arc $\{e^{is} : 0 \leq s \leq \lambda\}$. Consequently, it is easy to see that in this special case Theorem 1.4 states that each $\phi \in \mathfrak{M}$ is an $\ell^p(\mathbb{Z})$-Fourier multiplier, with corresponding multiplier transform $\int_{[0,2\pi]}^{\oplus} \phi(e^{i\lambda})\, dE(\lambda)$.

2. **Background items.** In this section we assemble a few required facts from [2] regarding estimates for square functions. We continue with the notation established in §1. Let $\Sigma_d$ denote the sigma-algebra of subsets of $\mathbb{T}$ generated by the class $\mathcal{D}_{\mathbb{T}}$ consisting of the sets $\Gamma_j$ ($j \in \mathbb{Z}$) together with the singleton sets $\{\omega_j\}$ ($j \in \mathbb{Z}$) and $\{1\}$. This *dyadic sigma-algebra* $\Sigma_d$ has the following obvious description.

For each $\sigma \in \Sigma_d$, there is a unique subclass $\mathcal{A}_\sigma$ of $\mathcal{D}_{\mathbb{T}}$ such that

$$\sigma = \bigcup \{\alpha : \alpha \in \mathcal{A}_\sigma\}.$$

We next indicate how the spectral decomposition $E(\cdot)$ of $U$ produces a strongly countably additive spectral measure on $\Sigma_d$. The preliminary step is to define the projection-valued function $\mathcal{E}_0(\cdot)$ on $\mathcal{D}_{\mathbb{T}}$ as follows.

DEFINITION. For each $j \in \mathbb{Z}$, define $\mathcal{E}_0(\Gamma_j)$ by putting

$$\mathcal{E}_0(\Gamma_j) = E(t_{j+1}^-) - E(t_j);$$

and let

$$\mathcal{E}_0(\{1\}) = E(0); \quad \mathcal{E}_0(\{\omega_j\}) = E(t_j) - E(t_j^-), \text{ for } j \in \mathbb{Z}.$$

The *dyadic spectral measure* $\mathcal{E}(\cdot)$ is now described as follows.

THEOREM 2.1 ([2, THEOREM (2.12)]). *For each $\sigma \in \Sigma_d$, we can define*

$$\mathcal{E}(\sigma) \equiv \sum \{\mathcal{E}_0(\alpha) : \alpha \in \mathcal{A}_\sigma\},$$

*where the sum on the right represents a series which is unconditionally convergent in the strong operator topology of $\mathfrak{B}(X)$. The projection-valued function $\mathcal{E}(\cdot)$ thereby obtained on $\Sigma_d$ is a strongly countably additive spectral measure in $X$ such that*

$$\sup\{\|\mathcal{E}(\sigma)\| : \sigma \in \Sigma_d\} \leq c^2 C_p. \tag{2.2}$$

By applying Khintchine's inequality [9, Theorem 2.b.3] in a standard way to Theorem 2.1, one obtains the following analogue for $X$ of the Littlewood-Paley theorem for $\mathbb{Z}$ [7, Theorem 7.2.1].

COROLLARY 2.3 ([2, COROLLARY (2.14)]). *There is a positive real constant $C_p$, depending only on p, such that whenever $f \in X$ and $\{\sigma_j\}_{j\geq 1}$ is a sequence of mutually disjoint elements of $\Sigma_d$ satisfying $\mathbb{T} = \bigcup_{j\geq 1} \sigma_j$, then*

$$c^{-2}C_p^{-1}\|f\|_{L^p(\mu)} \leq \left\| \left\{ \sum_{j\geq 1} |\mathcal{E}(\sigma_j)f|^2 \right\}^{1/2} \right\|_{L^p(\mu)} \leq c^2 C_p \|f\|_{L^p(\mu)}.$$

The following transferred version of the Vector-valued M. Riesz theorem [7, Theorem 6.5.2] is the last item of the preliminary machinery for §3 below. It is a special case of the methods in [2, §3] for transferring the bounds of square functions defined by sequences of multiplier transforms on $\ell^p(\mathbb{Z})$.

THEOREM 2.4 ([2, THEOREM (3.15)]). *There is a positive real constant $C_p$, depending only on p, such that*

$$\left\| \left\{ \sum_{j=1}^{\infty} |E(a_j)g_j|^2 \right\}^{1/2} \right\|_{L^p(\mu)} \leq c^2 C_p \left\| \left\{ \sum_{j=1}^{\infty} |g_j|^2 \right\}^{1/2} \right\|_{L^p(\mu)},$$

*for all sequences $\{a_j\}_{j=1}^{\infty} \subseteq [0, 2\pi)$ and all sequences $\{g_j\}_{j=1}^{\infty} \subseteq X$.*

3. **Proof of Theorem 1.4.** The proof of Theorem 1.4 will rest on the following two lemmas. Given a bounded function $\phi: \mathbb{T} \longrightarrow \mathbb{C}$, we shall take the liberty of writing $\mathcal{S}(\mathcal{P}; \phi, E)$ for the Riemann-Stieltjes approximating sum corresponding to the function $\lambda \in [0, 2\pi] \longrightarrow \phi(e^{i\lambda})$ and a partition $\mathcal{P}$ of $[0, 2\pi]$. Recall from §1 that $\{t_j\}_{j=-\infty}^{\infty}$ denotes the sequence of dyadic points in $(0, 2\pi)$.

LEMMA 3.1. *Let $\phi: \mathbb{T} \longrightarrow \mathbb{C}$ be a Marcinkiewicz multiplier, and let $f \in X$. Then, in the notation of §1, we have:*

$$\left\| \left\{ \sum_{j\in\mathbb{Z}} |[E(t_{j+1}) - E(t_j)]\mathcal{S}(\mathcal{P}; \phi, E)f|^2 \right\}^{1/2} \right\|_{L^p(\mu)} \leq c^4 C_p \|\phi\|_{\mathfrak{M}} \|f\|_{L^p(\mu)}, \tag{3.2}$$

*for all partitions $\mathcal{P}$ of $[0, 2\pi]$.*

PROOF. Let $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{n+1} = 2\pi$, where $n \geq 0$, be a partition $\mathcal{P}$ of $[0, 2\pi]$. If $n = 0$, then the Marcinkiewicz-Zygmund inequality [7, p. 203], together with

(1.2) and Corollary 2.3, easily gives (3.2) in this case. So we can assume without loss of generality that $n$ is a positive integer. Temporarily fix $j \in \mathbb{Z}$, and observe that if

$$\lambda_k \notin (t_j, t_{j+1}] \text{ for } 1 \le k \le n, \tag{3.3}$$

then,

$$[E(t_{j+1}) - E(t_j)]\mathcal{S}(\mathcal{P}; \phi, E) = \phi(e^{\imath\lambda_{\varkappa_j}})[E(t_{j+1}) - E(t_j)], \text{ for some } \varkappa_j \in \{1, 2, \ldots, n+1\}.$$

Suppose, on the other hand, that $j$ is such that

$$\begin{gathered}\{k : 1 \le k \le n \text{ and } \lambda_k \in (t_j, t_{j+1}]\} \text{ is a non-empty set} \\ \text{having minimum element } k_1 \text{ and maximum element } k_2.\end{gathered} \tag{3.4}$$

Then, upon putting $g_j = [E(t_{j+1}) - E(t_j)]f$, we easily find after a summation by parts that:

$$\begin{aligned}[E(t_{j+1}) - E(t_j)]\mathcal{S}(\mathcal{P}; \phi, E)f = &\sum_{k=k_1}^{k_2-1} \{\phi(e^{\imath\lambda_k}) - \phi(e^{\imath\lambda_{k+1}})\}E(\lambda_k)g_j \\ &+ \{\phi(e^{\imath\lambda_{k_2}}) - \phi(e^{\imath\lambda_{k_2+1}})\}E(\lambda_{k_2})g_j \\ &+ \phi(e^{\imath\lambda_{k_2+1}})g_j.\end{aligned} \tag{3.5}$$

It follows by applying the Cauchy-Schwarz inequality to the sum constituting the first member on the right of (3.5) that $\mu$-a.e. on $\mathcal{M}$, we have:

(3.6)

$$\begin{aligned}&|[E(t_{j+1}) - E(t_j)]\mathcal{S}(\mathcal{P}; \phi, E)f| \\ &\quad\le \left\{\sum_{k=k_1}^{k_2-1} |\phi(e^{\imath\lambda_k}) - \phi(e^{\imath\lambda_{k+1}})|\right\}^{1/2} \left\{\sum_{k=k_1}^{k_2-1} |\phi(e^{\imath\lambda_k}) - \phi(e^{\imath\lambda_{k+1}})|\,|E(\lambda_k)g_j|^2\right\}^{1/2} \\ &\qquad + \|\phi\|_{\mathfrak{M}}\{2|E(\lambda_{k_2})g_j| + |g_j|\} \\ &\quad\le \|\phi\|_{\mathfrak{M}}^{1/2}\left\{\sum_{k=k_1}^{k_2-1} |\phi(e^{\imath\lambda_k}) - \phi(e^{\imath\lambda_{k+1}})|\,|E(\lambda_k)g_j|^2\right\}^{1/2} \\ &\qquad + \|\phi\|_{\mathfrak{M}}\{2|E(\lambda_{k_2})g_j| + |g_j|\}.\end{aligned}$$

Let $J$ denote the set of $j \in \mathbb{Z}$ such that $\{k : 1 \le k \le n \text{ and } \lambda_k \in (t_j, t_{j+1}]\}$ is non-void. As in (3.4), for $j \in J$, let $k_{1_j}$ and $k_{2_j}$ be, respectively, the minimum and maximum elements of the set $\{k : 1 \le k \le n \text{ and } \lambda_k \in (t_j, t_{j+1}]\}$. Also, for $k \in \{1, \ldots, n\}$, let $j_k$ be the unique $j \in \mathbb{Z}$ such that $\lambda_k \in (t_j, t_{j+1}]$. In particular, $j_k \in J$, and $J$ is a finite, non-empty set. By (3.6) we see with the aid of the triangle inequality for finite-dimensional $\ell^2$ that $\mu$-a.e. on $\mathcal{M}$:

$$\begin{aligned}&\left\{\sum_{j\in J} |[E(t_{j+1}) - E(t_j)]\mathcal{S}(\mathcal{P}; \phi, E)f|^2\right\}^{1/2} \\ &\quad\le \|\phi\|_{\mathfrak{M}}^{1/2}\left\{\sum_{j\in J}\sum_{k=k_{1_j}}^{k_{2_j}-1} |\phi(e^{\imath\lambda_k}) - \phi(e^{\imath\lambda_{k+1}})|\,|E(\lambda_k)g_{j_k}|^2\right\}^{1/2} \\ &\qquad + \|\phi\|_{\mathfrak{M}}\left[2\left\{\sum_{j\in J} |E(\lambda_{k_{2_j}})g_j|^2\right\}^{1/2} + \left\{\sum_{j\in J} |g_j|^2\right\}^{1/2}\right].\end{aligned} \tag{3.7}$$

Hence by the triangle inequality in $L^p(\mu)$ and an application of Theorem 2.4, we see from the outer terms in (3.7) that

(3.8)
$$\Big\| \Big\{ \sum_{j\in J} |[E(t_{j+1}) - E(t_j)]\mathcal{S}(\mathcal{P};\phi,E)f|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$\leq \|\phi\|_{\mathfrak{M}}^{1/2} \Big\| \Big\{ \sum_{j\in J} \sum_{k=k_{1_j}}^{k_{2_j}-1} |\phi(e^{i\lambda_k}) - \phi(e^{i\lambda_{k+1}})|\, |E(\lambda_k)g_{j_k}|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$+ c^2 C_p \|\phi\|_{\mathfrak{M}} \Big\| \Big\{ \sum_{j\in J} |g_j|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}.$$

Let $h_k = |\phi(e^{i\lambda_k}) - \phi(e^{i\lambda_{k+1}})|^{1/2} g_{j_k}$ for $k = 1, \ldots, n$. Again using Theorem 2.4, we find that:

(3.9)
$$\Big\| \Big\{ \sum_{j\in J} \sum_{k=k_{1_j}}^{k_{2_j}-1} |\phi(e^{i\lambda_k}) - \phi(e^{i\lambda_{k+1}})|\, |E(\lambda_k)g_{j_k}|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$= \Big\| \Big\{ \sum_{j\in J} \sum_{k=k_{1_j}}^{k_{2_j}-1} |E(\lambda_k)h_k|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$\leq c^2 C_p \Big\| \Big\{ \sum_{j\in J} \sum_{k=k_{1_j}}^{k_{2_j}-1} |h_k|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$= c^2 C_p \Big\| \Big\{ \sum_{j\in J} \sum_{k=k_{1_j}}^{k_{2_j}-1} |\phi(e^{i\lambda_k}) - \phi(e^{i\lambda_{k+1}})|\, |g_{j_k}|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$= c^2 C_p \Big\| \Big\{ \sum_{j\in J} \sum_{k=k_{1_j}}^{k_{2_j}-1} |\phi(e^{i\lambda_k}) - \phi(e^{i\lambda_{k+1}})|\, |g_j|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$\leq c^2 C_p \|\phi\|_{\mathfrak{M}}^{1/2} \Big\| \Big\{ \sum_{j\in J} |g_j|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}.$$

Employing (3.9) on the right of (3.8) and recalling the definition of $g_j$, we infer that:

$$\Big\| \Big\{ \sum_{j\in J} |[E(t_{j+1}) - E(t_j)]\mathcal{S}(\mathcal{P};\phi,E)f|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$\leq c^2 C_p \|\phi\|_{\mathfrak{M}} \Big\| \Big\{ \sum_{j\in J} |g_j|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}$$
$$= c^2 C_p \|\phi\|_{\mathfrak{M}} \Big\| \Big\{ \sum_{j\in J} |[E(t_{j+1}) - E(t_j)]f|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)}.$$

An application of Corollary 2.3 to the majorant in this inequality shows that:

$$(3.10) \qquad \Big\| \Big\{ \sum_{j\in J} |[E(t_{j+1}) - E(t_j)]\mathcal{S}(\mathcal{P};\phi,E)f|^2 \Big\}^{1/2} \Big\|_{L^p(\mu)} \leq c^4 C_p \|\phi\|_{\mathfrak{M}} \|f\|_{L^p(\mu)}.$$

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Taking account of (3.3) for $j \in \mathbb{Z} \setminus J$, we readily observe with the aid of Corollary 2.3 that

$$\left\| \left\{ \sum_{j \in \mathbb{Z} \setminus J} |[E(t_{j+1}) - E(t_j)] \mathcal{S}(\mathcal{P}; \phi, E) f|^2 \right\}^{1/2} \right\|_{L^p(\mu)} \leq c^2 C_p \|\phi\|_{\mathfrak{M}} \|f\|_{L^p(\mu)}. \tag{3.11}$$

Since it is elementary that

$$\left\{ \sum_{j \in \mathbb{Z}} |[E(t_{j+1}) - E(t_j)] \mathcal{S}(\mathcal{P}; \phi, E) f|^2 \right\}^{1/2} \leq \left\{ \sum_{j \in J} |[E(t_{j+1}) - E(t_j)] \mathcal{S}(\mathcal{P}; \phi, E) f|^2 \right\}^{1/2} + \left\{ \sum_{j \in \mathbb{Z} \setminus J} |[E(t_{j+1}) - E(t_j)] \mathcal{S}(\mathcal{P}; \phi, E) f|^2 \right\}^{1/2},$$

the conclusion of Lemma 3.1 now follows immediately from recourse to the triangle inequality in $L^p(\mu)$, (3.10), and (3.11). ∎

Lemma 3.1 has the following lemma as a consequence.

LEMMA 3.12. *Under the hypotheses of Theorem 1.4 we have*

$$\|\mathcal{S}(\mathcal{P}; \phi, E)\| \leq c^6 C_p \|\phi\|_{\mathfrak{M}},$$

*for every Marcinkiewicz multiplier* $\phi : \mathbb{T} \longrightarrow \mathbb{C}$, *and every partition* $\mathcal{P}$ *of* $[0, 2\pi]$.

PROOF. Let $f \in X$, and let $\mathcal{P}$ be a partition of $[0, 2\pi]$. Since $E(0)\mathcal{S}(\mathcal{P}; \phi, E) = 0$, it follows from the first inequality of Corollary 2.3 and Lemma 3.1 that

$$\begin{aligned} \|\mathcal{S}(\mathcal{P}; \phi, E) f\|_{L^p(\mu)} &\leq c^2 C_p \left\| \left\{ \sum_{j \in \mathbb{Z}} |[E(t_{j+1}) - E(t_j)] \mathcal{S}(\mathcal{P}; \phi, E) f|^2 \right\}^{1/2} \right\|_{L^p(\mu)} \\ &\leq c^6 C_p \|\phi\|_{\mathfrak{M}} \|f\|_{L^p(\mu)}, \end{aligned}$$

as required. ∎

PROOF OF THEOREM 1.4. We first establish the existence of $\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$ for an arbitrary Marcinkiewicz multiplier $\phi$. Fix $f \in X$, and suppose that $\varepsilon > 0$. By virtue of properties (iii) and (iv) in the definition of spectral family, coupled with the fact that $E\big((2\pi)^-\big) = I$, the series

$$\sum_{j \in \mathbb{Z}} [E(t_{j+1}) - E(t_j)] f$$

converges in the norm topology of $X$ to $f - E(0)f$. (This also follows from the strong countable additivity of the dyadic spectral measure $\mathcal{E}(\cdot)$ of Theorem 2.1.) Hence there is a positive integer $K$ such that $\|h_K\|_{L^p(\mu)} < \varepsilon$, where

$$h_K = \sum_{|j| > K} [E(t_{j+1}) - E(t_j)] f.$$

Putting $g_k = \sum_{|j| \leq K} [E(t_{j+1}) - E(t_j)] f = \{E(t_{K+1}) - E(t_{-K})\} f$, we have

$$f = E(0)f + g_K + h_K.$$

Consequently for an arbitrary partition $\mathcal{P}$ of $[0, 2\pi]$,

$$\mathcal{S}(\mathcal{P}; \phi, E)f = \mathcal{S}(\mathcal{P}; \phi, E)g_K + \mathcal{S}(\mathcal{P}; \phi, E)h_K. \tag{3.13}$$

Applying Lemma 3.12 to $h_K$, we see that:

$$\|\mathcal{S}(\mathcal{P}; \phi, E)h_K\|_{L^p(\mu)} \leq c^6 C_p \|\phi\|_{\mathfrak{M}} \varepsilon.$$

From this and (3.13) we infer that for any two partitions $\mathcal{P}_1$, $\mathcal{P}_2$ of $[0, 2\pi]$,

(3.14)

$$\|\mathcal{S}(\mathcal{P}_1; \phi, E)f - \mathcal{S}(\mathcal{P}_2; \phi, E)f\|_{L^p(\mu)} \leq \|\mathcal{S}(\mathcal{P}_1; \phi, E)g_K - \mathcal{S}(\mathcal{P}_2; \phi, E)g_K\|_{L^p(\mu)} + c^6 C_p \|\phi\|_{\mathfrak{M}} \varepsilon.$$

As a function of $\lambda$, $\phi(e^{i\lambda})$ is obviously of bounded variation on every compact interval contained in $(0, 2\pi)$. In particular, $\int_{[t_{-K}, t_{K+1}]} \phi(e^{i\lambda})\, dE(\lambda)$ exists in the strong operator topology of $\mathfrak{B}(X)$. Moreover, if a partition $\mathcal{P}$ of $[0, 2\pi]$ contains the dyadic points $t_{-K}$ and $t_{K+1}$, then $\mathcal{S}(\mathcal{P}; \phi, E)g_K$ is the Riemann-Stieltjes approximating sum for

$$\int_{[t_{-K}, t_{K+1}]} \phi(e^{i\lambda})\, dE(\lambda)f$$

corresponding to the partition of $[t_{-K}, t_{K+1}]$ given by $\mathcal{P} \bigcap [t_{-K}, t_{K+1}]$. In view of (3.14) and the last observation, we can apply the Cauchy Criterion to deduce that the net $\{\mathcal{S}(\mathcal{P}; \phi, E)f\}$ converges in the norm topology of $X$, as $\mathcal{P}$ runs through the partitions of $[0, 2\pi]$ directed by inclusion. This fact together with Lemma 3.12 establishes the existence, in the strong operator topology, of $\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$, and also gives the estimate

$$\left\| \int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda) \right\| \leq c^6 C_p \|\phi\|_{\mathfrak{M}}.$$

It is obvious from this and (1.2) that the desired conclusion in (1.5) holds.

To complete the proof of Theorem 1.4, notice that for each partition $\mathcal{P}$ of $[0, 2\pi]$, the mapping

$$\phi \in \mathfrak{M} \longrightarrow \phi(1)E(0) + \mathcal{S}(\mathcal{P}; \phi, E)$$

is an identity-preserving algebra homomorphism. Taking limits in the strong operator topology as $\mathcal{P}$ varies, we immediately conclude, with the aid of Lemma 3.12 for multiplicativity, that

$$\phi \in \mathfrak{M} \longrightarrow \int_{[0,2\pi]}^{\oplus} \phi(e^{i\lambda})\, dE(\lambda)$$

is also an identity-preserving algebra homomorphism. ∎

4. **Spectral integrals bounded by mutliplier norms.** The Strong Marcinkiewicz Multiplier Theorem states that if $\phi \in \mathfrak{M}$, then $\|\phi\|_{M_p(\mathbb{T})}$, the norm of $\phi$ as an $\ell^p(\mathbb{Z})$-Fourier multiplier, satisfies:

$$\|\phi\|_{M_p(\mathbb{T})} \leq C_p \|\phi\|_{\mathfrak{M}}.$$

It is a well-known elementary fact that the roles of the two norms in this inequality cannot be reversed, and so, in a sense, on $\mathfrak{M}$ the norm $\|\cdot\|_{M_p(\mathbb{T})}$ has a lower order of magnitude than $\|\cdot\|_{\mathfrak{M}}$. As a concrete illustration, for each $n \in \mathbb{Z}$ we can use the dyadic points $t_0 = \frac{\pi}{2}$ and $t_1 = \frac{3\pi}{2}$ to define $\phi_n \in \mathbf{BV}(\mathbb{T}) \subseteq \mathfrak{M}$ by:

$$\phi_n(e^{i\lambda}) = \begin{cases} e^{in\lambda} & \text{for } t_0 \le \lambda \le t_1; \\ 0, & \text{for } \lambda \in [0, 2\pi] \setminus [t_0, t_1]. \end{cases}$$

It is obvious that

$$\sup\{\|\phi_n\|_{M_p(\mathbb{T})} : n \in \mathbb{Z}\} < \infty,$$

whereas

$$\|\phi_n\|_{\mathfrak{M}} \longrightarrow +\infty, \text{ as } |n| \longrightarrow +\infty.$$

Since the spectral integrals of functions in $\mathfrak{M}$, as treated in Theorem 1.4, can be viewed as transferring the actions of multipliers to the space $X$, we shall now seek conditions under which the bounds for such integrals can be estimated by the "smaller" norm $\|\cdot\|_{M_p(\mathbb{T})}$ rather than by $\|\cdot\|_{\mathfrak{M}}$.

Similar comments can be made in regard to $\mathbf{BV}(\mathbb{T})$ in place of $\mathfrak{M}$. The corresponding state of affairs for $\mathbf{BV}(\mathbb{T})$ has the following outcome [3, Theorems (3.10)(ii) and (4.14)], which we shall generalize to the framework of $\mathfrak{M}$.

THEOREM 4.1. *For each $\psi \in \mathbf{BV}(\mathbb{T})$, let $\Psi$: $[0, 2\pi] \longrightarrow \mathbb{C}$ be defined by*

$$\Psi(\lambda) \equiv 2^{-1}\Big\{\lim_{s\longrightarrow\lambda^+} \psi(e^{is}) + \lim_{s\longrightarrow\lambda^-} \psi(e^{is})\Big\}.$$

*Then $\Psi \in \mathbf{BV}([0, 2\pi])$, and*

$$\Big\|\int_{[0,2\pi]}^{\oplus} \Psi\, dE\Big\| \le c^2\|\psi\|_{M_p(\mathbb{T})}.$$

Our extension of Theorem 4.1 will require some consequences of Theorem 1.4, and these will be discussed first in order to avoid digressions later on. Given a function $\phi \in \mathfrak{M}$, and an interval $[a, b]$, where $0 \le a < b \le 2\pi$, it is easy to see from Theorem 1.4 and definitions that $\int_{[a,b]} \phi(e^{i\lambda})\, dE(\lambda)$ exists and can be expressed by

$$\int_{[a,b]} \phi(e^{i\lambda})\, dE(\lambda) = \Big\{\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)\Big\}\{E(b) - E(a)\}. \tag{4.2}$$

It follows from (4.2), the strong right-continuity of $E(\cdot)$, and the property $E\big((2\pi)^-\big) = E(2\pi) = I$ that, in the strong operator topology of $\mathfrak{B}(X)$, we have:

$$\lim_{a\longrightarrow 0^+} \int_{[0,a]} \phi(e^{i\lambda})\, dE(\lambda) = 0, \tag{4.3}$$

and

$$\lim_{b\longrightarrow(2\pi)^-} \int_{[b,2\pi]} \phi(e^{i\lambda})\, dE(\lambda) = 0. \tag{4.4}$$

The next lemma follows immediately from (4.3) and (4.4).

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LEMMA 4.5. *Let $\phi \in \mathfrak{M}$, and suppose that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of real numbers such that $0 < a_n < b_n < 2\pi$, for all n, and $a_n \longrightarrow 0$, $b_n \longrightarrow 2\pi$. Then*

$$\int_{[a_n,b_n]} \phi(e^{i\lambda})\, dE(\lambda) \longrightarrow \int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda),$$

*in the strong operator topology of $\mathfrak{B}(X)$. In particular, the value of $\int_{[0,2\pi]} \phi(e^{i\lambda})\, dE(\lambda)$ is completely determined by the restriction of $\phi$ to $\mathbb{T} \setminus \{1\}$.*

Given $\phi \in \mathfrak{M}$, let $\Phi\colon (0, 2\pi) \longrightarrow \mathbb{C}$ be defined by putting

$$\Phi(\lambda) = 2^{-1}\big\{\lim_{s\longrightarrow\lambda^+} \phi(e^{is}) + \lim_{s\longrightarrow\lambda^-} \phi(e^{is})\big\}, \text{ for } 0 < \lambda < 2\pi. \tag{4.6}$$

Simple examples show that the expression on the right in (4.6) need not make sense for $\lambda = 0$ or $\lambda = 2\pi$. However, if we extend $\Phi$ to $[0, 2\pi]$ by taking $\Phi(0) = \Phi(2\pi) = \alpha$, where $\alpha$ is an unspecified complex number, then it is clear that there is a corresponding function $\vartheta \in \mathfrak{M}$ such that $\Phi(\lambda) = \vartheta(e^{i\lambda})$, for all $\lambda \in [0, 2\pi]$. In particular, $\int_{[0,2\pi]} \Phi(\lambda)\, dE(\lambda)$ exists, and, by Lemma 4.5, has value independent of $\alpha$. With this understanding, we can now state our analogue for $\mathfrak{M}$ of Theorem 4.1.

THEOREM 4.7. *Suppose that $\phi \in \mathfrak{M}$, and let $\Phi$ be as above. Then*

$$\left\|\int_{[0,2\pi]} \Phi(\lambda)\, dE(\lambda)\right\| \leq c^2 C_p \|\phi\|_{M_p(\mathbb{T})}. \tag{4.8}$$

PROOF. For each positive integer $n$, put $a_n = \frac{1}{n}$, $b_n = 2\pi - \frac{1}{n}$. Let $\Phi$ and $\vartheta$ be as above. With $\vartheta$ playing an intermediary role, we see from Lemma 4.5 that, in the strong operator topology of $\mathfrak{B}(X)$,

$$\int_{[a_n,b_n]} \Phi(\lambda)\, dE(\lambda) \longrightarrow \int_{[0,2\pi]} \Phi(\lambda)\, dE(\lambda), \text{ as } n \longrightarrow \infty. \tag{4.9}$$

Let $\chi_n$ be the characteristic function, defined on $\mathbb{T}$, of the arc $\{e^{i\lambda} : a_n \leq \lambda \leq b_n\}$, and define $\psi_n \in \mathbf{BV}(\mathbb{T})$ to be the pointwise product $\chi_n\phi$. Let $\Psi_n$ correspond to $\psi_n$ as in the statement of Theorem 4.1. It is easy to see by direct calculation that for each positive integer $n$ we have:

$$\begin{aligned}\int_{[0,2\pi]}^{\oplus} \Psi_n(\lambda)\, dE(\lambda) = {} & 2^{-1}\big\{\lim_{s\longrightarrow a_n^+} \phi(e^{is})\big\}\{E(a_n) - E(a_n^-)\} + \int_{[a_n,b_n]} \Phi(\lambda)\, dE(\lambda) \\ & - 2^{-1}\big\{\lim_{s\longrightarrow b_n^+} \phi(e^{is})\big\}\{E(b_n) - E(b_n^-)\}.\end{aligned} \tag{4.10}$$

Since, in the strong operator topology, $E(a_n)$ and $E(a_n^-)$ separately converge to $E(0)$, while $E(b_n)$ and $E(b_n^-)$ separately converge to $I$, an application of (4.9) on the right of (4.10) shows that as $n \longrightarrow \infty$

$$\int_{[0,2\pi]}^{\oplus} \Psi_n(\lambda)\, dE(\lambda) \longrightarrow \int_{[0,2\pi]} \Phi(\lambda)\, dE(\lambda), \tag{4.11}$$

in the strong operator topology. However, by Theorem 4.1 and a standard theorem of

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M. Riesz [7, p. 104],

$$\left\| \int_{[0,2\pi]}^{\oplus} \Psi_n(\lambda)\, dE(\lambda) \right\| \leq c^2 \|\psi_n\|_{M_p(\mathbb{T})} = c^2 \|\chi_n \phi\|_{M_p(\mathbb{T})} \leq c^2 C_p \|\phi\|_{M_p(\mathbb{T})}.$$

Using this in (4.11), we obtain (4.8), as required. ∎

COROLLARY 4.12. *Let $\phi\colon \mathbb{T} \longrightarrow \mathbb{C}$ be a continuous function as well as a Marcinkiewicz multiplier. Then*

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi(e^{i\lambda})\, dE(\lambda) \right\| \leq c^2 C_p \|\phi\|_{M_p(\mathbb{T})}.$$

PROOF. In this case we can take $\Phi(\lambda) = \phi(e^{i\lambda})$, for all $\lambda \in [0, 2\pi]$. From Theorem 4.7 we see that

$$\left\| \int_{[0,2\pi]}^{\oplus} \phi(e^{i\lambda})\, dE(\lambda) \right\| \leq |\phi(1)|\, \|E(0)\| + c^2 C_p \|\phi\|_{M_p(\mathbb{T})}. \tag{4.13}$$

However, from the continuity of $\phi$ and a standard application of the M. Riesz Convexity Theorem [7, 1.2.2(ii)], we obviously have

$$|\phi(1)| \leq \|\phi\|_{M_2(\mathbb{T})} \leq \|\phi\|_{M_p(\mathbb{T})}.$$

Using this together with (1.2) in (4.13) completes the proof of Corollary 4.12. ∎

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*Department of Mathematics*
*University of Missouri-Columbia*
*Columbia Missouri 65211*
*U S A*

*University of Illinois*
*Department of Mathematics*
*1409 West Green Street*
*Urbana, Illinois 61801*
*U S A*

*Department of Mathematics*
*University of Edinburgh*
*James Clerk Maxwell Building*
*Edinburgh EH9 3JZ*
*Scotland*