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Well-posedness of the Navier–Stokes–Maxwell equations

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We study the local and global well-posedness of a full system of magnetohydrodynamic equations. The system is a coupling of the incompressible Navier–Stokes equations with the Maxwell equations through the Lorentz force and Ohm's law for the current. We show the local existence of mild solutions for arbitrarily large data in a space similar to the scale-invariant spaces classically used for Navier–Stokes. These solutions are global if the initial data are small enough. Our results not only simplify and unify the proofs for the space dimensions 2 and 3, but also refine those in [8]. The main simplification comes from an *a priori* $L^2_t(L^\infty_x)$ estimate for solutions of the forced Navier–Stokes equations.

1. Introduction

The purpose of this paper is the study of the full magnetohydrodynamics (MHD) system

$$\left.\begin{aligned} \frac{\partial v}{\partial t} + v\cdot\nabla v - \nu\Delta v + \nabla p &= j\times B,\\ \partial_t E - \operatorname{curl} B &= -j,\\ \partial_t B + \operatorname{curl} E &= 0,\\ \operatorname{div} v = \operatorname{div} B &= 0,\\ \sigma(E + v\times B) &= j, \end{aligned}\right\} \tag{1.1}$$

subject to the initial data

$$v|_{t=0} = v^0, \qquad B|_{t=0} = B^0, \qquad E|_{t=0} = E^0.$$

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Here, $v, E, B\colon \mathbb{R}_t^+ \times \mathbb{R}_x^d \to \mathbb{R}^3$ are vector fields defined on $\mathbb{R}^d$ ($d = 2$ or 3). The vector field $v = (v_1, \dots, v_d)$ is the velocity of the fluid, ν is its viscosity and the scalar function p stands for the pressure. The vector fields E and B are the electric and magnetic fields, respectively, and j is the electric current given by Ohm's law (the fifth equation of the system, where σ is the electric resistivity). The force term $j \times B$ in the Navier–Stokes equations comes from the Lorentz force under a quasi-neutrality assumption of the net charge carried by the fluid. Note that the pressure p can be recovered from v and $j \times B$ via an explicit Caldéron–Zygmund operator (see, for example, [4]). The second equation in (1.1) is the Ampère–Maxwell equation for an electric field E. The third equation is simply Faraday's law. For a detailed introduction to MHD, we refer the reader to [2, 7].

Note that in the two-dimensional case, the functions v, E, B and j are defined on the whole space $\mathbb{R}^2$ with values in $\mathbb{R}^3$. In this case, the operator ∇ is given by

$$\nabla = (\partial_{x_1}, \partial_{x_2}, 0)^{\mathrm{T}}.$$

Thus,

$$\operatorname{div} v := \partial_{x_1} v_1 + \partial_{x_2} v_2, \qquad \nabla p := (\partial_{x_1} p, \partial_{x_2} p, 0)^{\mathrm{T}}$$

and

$$\operatorname{curl} F := (\partial_{x_2} F_3, -\partial_{x_1} F_3, \partial_{x_1} F_2 - \partial_{x_2} F_1)^{\mathrm{T}}.$$

In the following, we take $\sigma = \nu = 1$ to simplify the notation.

Multiply the Navier–Stokes equations in (1.1) by v and the Ampère–Maxwell equations by $(B, E)^{\mathrm{T}}$ and integrate (using the divergence-free condition on the velocity); this gives the formal energy identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}[\|v\|_{L^2}^2 + \|B\|_{L^2}^2 + \|E\|_{L^2}^2] + \|j\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 = 0.$$

This identity shows that the energy is dissipated by the viscosity and the electric resistivity. It also suggests that one should be able to construct a global finite-energy weak solution (à la Leray) for data lying in $L^2(\mathbb{R}^d)$. However, this intuitive expectation remains an interesting open problem for (1.1) in both the dimensions $d = 2, 3$. Indeed, given a standard approximating scheme, it is hard to obtain the compactness of the solutions, especially for the magnetic field due to the hyperbolicity of Maxwell's equations. In dimension 2, the equation is energy critical, but running a fixed-point argument for the data (v^0, E^0, B^0) only in $L^2(\mathbb{R}^d)^3$ seems very difficult due to the term $E \times B$, which appears after writing $j \times B = \sigma(E + v \times B) \times B$.

Existence results are known in the case where more regularity is imposed on the initial electromagnetic field. Recently, for the initial data $(v^0, E^0, B^0) \in L^2(\mathbb{R}^2) \times (H^s(\mathbb{R}^2))^2$ with $s > 0$, Masmoudi [11] proved the existence and uniqueness of global strong solutions to (1.1). His proof relied on the use of the energy inequality combined with a logarithmic inequality that enabled him to upper estimate the L^∞-norm of the velocity field by the energy norm and higher Sobolev norms. It is also interesting to note that the proof in [11] does not use the divergence-free condition of the magnetic field or the decay property of the linear part coming from

Maxwell's equations, namely,

$$\left.\begin{aligned} \frac{\partial E}{\partial t} - \operatorname{curl} B + E = f, \\ \frac{\partial B}{\partial t} + \operatorname{curl} E = 0, \\ \nabla \cdot B = 0. \end{aligned}\right\} \tag{1.2}$$

Another line of research was pursued by Ibrahim and Keraani [8], who considered the data $(v_0, E_0, B_0) \in \dot{B}^{1/2}_{2,1}(\mathbb{R}^3) \times (\dot{H}^{1/2}(\mathbb{R}^3))^2$ in dimension $d = 3$, and $(v_0, E_0, B_0) \in \dot{B}^0_{2,1}(\mathbb{R}^2) \times (L^2_{\log}(\mathbb{R}^2))^2$ in dimension $d = 2$ (see below for the definition of these functional spaces). These authors built up strong solutions by using parabolic regularization arguments giving control of the L^∞-norm of the velocity field of the solution. More recently, Ibrahim and Yoneda constructed a local-in-time solution for non-decaying initial data on the torus. See [9] for more details.

In this paper, we follow up on the work of Ibrahim and Keraani by running a fixed-point argument to obtain mild solutions, but taking the initial velocity field in the natural Navier–Stokes space $H^{d/2-1}$. Our main theorem extends the aforementioned earlier results in many respects: the regularity of the initial velocity and electromagnetic fields is lowered, and we unify the proofs in the cases of space dimensions 2 and 3. One of the key ingredients will be to use an L^2L^∞-estimate on the velocity field, which greatly simplifies the fixed-point argument.

Before stating our main result, we need a few definitions.

Definition 1.1. First, let $\mathcal{P}$ denote the Leray projection on divergence-free vector fields.

A function $\Gamma := (v, E, B)$, with $\operatorname{div}(v) = \operatorname{div}(B) = 0$, is said to be a mild solution on a time interval $[0, T]$ of the full MHD problem (1.1) if $\Gamma \in \mathcal{C}([0, T], \dot{H}^{d/2-1})$ and satisfies the integral equation

$$\Gamma(t) = \mathrm{e}^{t\mathcal{A}}\Gamma(0) + \int_0^t \mathrm{e}^{(t-t')\mathcal{A}}\mathcal{N}(\Gamma(t'))\,\mathrm{d}t',$$

with

$$\mathcal{A} = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & -I & \operatorname{curl} \\ 0 & -\operatorname{curl} & 0 \end{pmatrix}$$

and $\mathcal{N}(\Gamma) = (\mathcal{P}(-\nabla(v \otimes v) + E \times B + (v \times B) \times B), -v \times B, 0)^{\mathrm{T}}$.

We use the following functional analytic framework.

Definition 1.2. Let Δ_q denote the dyadic frequency localization operator defined in § 2. For $s, t \in \mathbb{R}$ and $\alpha \geqslant 0$ define the space $\dot{H}^{s,t}_\alpha$ by its norm

$$\|\phi\|^2_{\dot{H}^{s,t}_\alpha} := \sum_{q \leqslant 0} 2^{2qs}\|\Delta_q\phi\|^2_{L^2} + \sum_{q>0} q^\alpha 2^{2qt}\|\Delta_q\phi\|^2_{L^2}.$$

We also use the shorthand notation

$$\dot{H}^s = \dot{H}^{s,s}_0, \qquad \dot{H}^s_{\log} := \dot{H}^{s,s}_1 \quad \text{and} \quad \dot{H}^{s,t} := \dot{H}^{s,t}_0.$$

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Finally, define $\tilde{L}^r_T\dot{H}^{s,t}_\alpha$ by its norm

$$\|\phi\|^2_{\tilde{L}^r_T\dot{H}^{s,t}_\alpha} := \sum_{q\leqslant 0} 2^{2qs}\|\Delta_q\phi\|^2_{L^r_T L^2} + \sum_{q>0} q^\alpha 2^{2qt}\|\Delta_q\phi\|^2_{L^r_T L^2},$$

with obvious generalizations to $\tilde{L}^r_T\dot{H}^s$, etc.

The space $\dot{H}^s_{\log}$ is articulated on the standard homogeneous Sobolev space $\dot{H}^s$ with an extra logarithmic weight for the high-frequency part. The space $\dot{H}^{s,t}$ is nothing but the usual Sobolev space $\dot{H}^t$ for high frequencies, while it behaves like $\dot{H}^s$ for low frequencies. If $s > t$, it is not difficult to see that $\dot{H}^{s,t} = \dot{H}^s + \dot{H}^t$. The $\tilde{L}$ spaces were first used by Chemin and Lerner [6].

Our main result can be stated as follows.

THEOREM 1.3.

(i) *In dimension 2, and for any*

$$\Gamma^0 := (v^0, E^0, B^0) \in L^2(\mathbb{R}^2) \times L^2_{\log}(\mathbb{R}^2) \times L^2_{\log}(\mathbb{R}^2),$$

there exist $T > 0$ and a unique mild solution $\Gamma = (v, E, B)$ of (1.1) with initial data Γ^0 and

$$\begin{aligned} v &\in \tilde{L}^\infty(0,T;L^2) \cap L^2(0,T;\dot{H}^1 \cap L^\infty), \\ E &\in \tilde{L}^\infty(0,T;L^2_{\log}) \cap L^2(0,T;L^2_{\log}), \\ B &\in \tilde{L}^\infty(0,T;L^2_{\log}) \cap L^2(0,T;\dot{H}^{1,0}). \end{aligned}$$

Moreover, the solution is global (i.e. $T = \infty$) if the initial data is sufficiently small in $L^2 \times L^2_{\log} \times L^2_{\log}$.

(ii) *In dimension 3, and for any*

$$\Gamma^0 := (v^0, E^0, B^0) \in \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{1/2}(\mathbb{R}^3) \times \dot{H}^{1/2}(\mathbb{R}^3),$$

there exist $T > 0$ and a unique mild solution $\Gamma = (v, E, B)$ of (1.1) with initial data Γ^0 and

$$\begin{aligned} v &\in \tilde{L}^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2} \cap L^\infty), \\ E &\in \tilde{L}^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{1/2}), \\ B &\in \tilde{L}^\infty(0,T;\dot{H}^{1/2}) \cap L^2(0,T;\dot{H}^{3/2,1/2}). \end{aligned}$$

Moreover, the solution is global (i.e. $T = \infty$) if the initial data is sufficiently small in $\dot{H}^{1/2} \times \dot{H}^{1/2} \times \dot{H}^{1/2}$.

In dimension 2, the extra logarithmic regularity is needed to estimate the term $E \times B$ appearing in the Navier–Stokes equations.

In dimension 3, the control of B in $L^2(0,T;\dot{H}^{3/2,1/2})$ is not needed to close the fixed-point estimate, but we added it for completeness.

The paper has the following structure. In § 2 we define some further tools needed in the proof. In § 3 we detail the linear (parabolic regularity) and nonlinear (product law) estimates needed in the proof of the main theorem. The main theorem is then proved in § 4. Finally, the proofs of some technical estimates are given in the appendix.

2. Notation and functional spaces

Throughout this work we use the following notation.

1. For any positive A and B the notation $A \lesssim B$ means that there exists a positive constant C such that $A \leqslant CB$.

2. c always denotes an absolute constant $0 < c < 1$.

3. For any tempered distribution u, both $\hat{u}$ and $\mathcal{F}u$ denote the Fourier transform of u.

4. For every $p \in [1, \infty]$, $\|\cdot\|_{L^p}$ denotes the norm in the Lebesgue space L^p.

5. For any normed space $\mathcal{X}$, the mixed space-time Lebesgue space $L^p([0,T],\mathcal{X})$ denotes the space of functions f such that, for almost all $t \in (0,T)$, $f(t) \in \mathcal{X}$ and $\|f(t)\|_{\mathcal{X}} \in L^p(0,T)$. The notation $L^p([0,T],\mathcal{X})$ is often shortened to $L^p_T\mathcal{X}$.

We recall the well-known Littlewood–Paley decomposition and the corresponding cut-off operators. There exists a radial positive function $\varphi \in \mathcal{D}(\mathbb{R}^d \setminus \{0\})$ such that

$$\sum_{q\in\mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \forall \xi \in \mathbb{R}^d \setminus \{0\},$$
$$\operatorname{supp}\varphi(2^{-q}\cdot) \cap \operatorname{supp}\varphi(2^{-j}\cdot) = \emptyset, \quad \forall |q-j| \geqslant 2.$$

For every $q \in \mathbb{Z}$ and $v \in \mathcal{S}'(\mathbb{R}^d)$, we set

$$\Delta_q v = \mathcal{F}^{-1}[\varphi(2^{-q}\xi)\hat{v}(\xi)] \quad \text{and} \quad S_q = \sum_{j=-\infty}^{q-1} \Delta_j.$$

Bony's decomposition [3] consists of splitting the product uv into three parts:

$$uv = T_u v + T_v u + R(u,v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \qquad R(u,v) = \sum_q \Delta_q u \tilde{\Delta}_q v \quad \text{and} \quad \tilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

(It should be said that this decomposition is true in the class of distributions for which $\sum_{q\in\mathbb{Z}} \Delta_q = I$. For example, polynomial functions do not belong to this class.) For $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$, we define the homogeneous Besov space $\dot{B}^s_{p,r}$ as the set of $u \in \mathcal{S}'(\mathbb{R}^d)$ such that $u = \sum_q \Delta_q u$ and

$$\|u\|_{\dot{B}^s_{p,r}} = \|(2^{qs}\|\Delta_q u\|_{L^p})_{q\in\mathbb{Z}}\|_{\ell^r(\mathbb{Z})} < \infty.$$

In the case $p = r = 2$, the space $\dot{B}^s_{2,2}$ turns out to be the classical homogeneous Sobolev space $\dot{H}^s$. Finally, the definition of $\tilde{L}^q_T \dot{B}^s_{p,r}$ is given by distributions u such that

$$\|u\|_{L^q_T \dot{B}^s_{p,r}} = \|(2^{qs}\|\Delta_q u\|_{L^q_T L^p})_{q\in\mathbb{Z}}\|_{\ell^r(\mathbb{Z})} < \infty.$$

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3. Linear and nonlinear estimates

We make extensive use of Bernstein's inequalities (see, for example, [4]).

Lemma 3.1 (Bernstein's lemma). *There exists a constant C such that, for any $q, k \in \mathbb{N}$, $1 \leqslant a \leqslant b$ and $f \in L^a(\mathbb{R}^d)$,*

$$\sup_{|\alpha|=k} \|\partial^\alpha S_q f\|_{L^b} \leqslant C^k 2^{q(k+d(1/a-1/b))} \|S_q f\|_{L^a},$$

$$C^{-k} 2^{qk} \|\Delta_q f\|_{L^a} \leqslant \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q f\|_{L^a} \leqslant C^k 2^{qk} \|\Delta_q f\|_{L^a}.$$

The parabolic regularity result we need reads as the following lemma.

Lemma 3.2 (parabolic regularization [1]). *Let u be a smooth divergence-free vector field solving*

$$\partial_t u - \Delta u + \nabla p = f,$$
$$u|_{t=0} = u^0$$

on some time interval $[0, T]$. Then, for every $p \geqslant r \geqslant 1$ and $s \in \mathbb{R}$ and $j \geqslant 1$,

$$\|u\|_{C([0,T];\dot{B}^s_{q,j}) \cap \tilde{L}^p_T \dot{B}^{s+2/p}_{q,j}} \lesssim \|u^0\|_{\dot{B}^s_{q,j}} + \|f\|_{\tilde{L}^r_T \dot{B}^{s-2+2/r}_{q,j}}.$$

The following result is an $L^2_T L^\infty$-estimate, which was originally proved in [5, 10] in dimension 2.

Lemma 3.3 ($L^2 L^\infty$-estimate). *Let $d = 2, 3$ and let u be a smooth divergence-free vector field solving*

$$\partial_t u - \Delta u + \nabla p = f_1 + f_2,$$
$$u|_{t=0} = u^0$$

on some time interval $[0, T]$. Assume that $f_1 \in L^1_T \dot{H}^{d/2-1}$ and $f_2 \in \tilde{L}^2_T \dot{B}^{d/2-2}_{2,1}$. Then,

$$\|u\|_{L^2_T L^\infty} \lesssim \|u_0\|_{\dot{H}^{d/2-1}} + \|f_1\|_{L^1_T \dot{H}^{d/2-1}} + \|f_2\|_{\tilde{L}^2_T \dot{B}^{d/2-2}_{2,1}}. \tag{3.1}$$

Proof. Due to lemma 3.2, and using the embeddings

$$\tilde{L}^2 \dot{B}^{d/2}_{2,1} \hookrightarrow L^2 \dot{B}^{d/2}_{2,1} \hookrightarrow L^2 L^\infty,$$

we can assume that $f_2 = 0$. Duhamel's formula gives that

$$u(t) = \mathrm{e}^{t\Delta} u_0 + \int_0^t \mathrm{e}^{(t-t')\Delta} \mathcal{P} f_1(t')\,\mathrm{d}t',$$

and thus

$$\|u(t)\|_{L^2_T L^\infty} \leqslant \|\mathrm{e}^{t\Delta} u_0\|_{L^2_T L^\infty} + \int_0^T \|\mathrm{e}^{t\Delta} \mathcal{P} f_1(t')\|_{L^2_T L^\infty}\,\mathrm{d}t'. \tag{3.2}$$

Using the embedding $\dot{H}^{d/2-1} \hookrightarrow \dot{\mathcal{B}}^{-1}_{\infty,2}$ and the characterization of Besov spaces of negative regularity (see, for example, [1]),

$$\|u\|_{\dot{\mathcal{B}}^{-1}_{\infty,2}} \sim \|\|\mathrm{e}^{t\Delta}u\|_{L^\infty}\|_{L^2(0,\infty)};$$

thus we obtain (3.1), as desired. □

We now focus on Maxwell's equations. The first result is an energy-type estimate.

LEMMA 3.4. *Let $\alpha \geqslant 0$, let $G_1 \in L^2_T \dot{H}^{d/2-1}_\alpha$ and let (E, B) be a smooth solution of*

$$\begin{aligned} \partial_t E - \operatorname{curl} B + E &= G, \\ \partial_t B + \operatorname{curl} E &= 0, \\ (E, B)|_{t=0} &= (E_0, B_0) \end{aligned}$$

on some time interval $[0, T]$. The following estimate then holds (with constants independent of T):

$$\|E\|_{\tilde{L}^\infty_T \dot{H}^{d/2-1}_\alpha \cap L^2_T \dot{H}^{d/2-1}_\alpha} + \|B\|_{\tilde{L}^\infty_T \dot{H}^{d/2-1}_\alpha} \lesssim \|(E_0, B_0)\|_{\dot{H}^{d/2-1}_\alpha} + \|G\|_{L^2_T \dot{H}^{d/2-1}_\alpha}. \tag{3.3}$$

Moreover, B satisfies the decay estimate

$$\|B\|_{L^2 \dot{H}^{d/2,d/2-1}_\alpha} \lesssim \|(E_0, B_0)\|_{\dot{H}^{d/2-1}_\alpha} + \|G\|_{L^2_T \dot{H}^{d/2-1}_\alpha}. \tag{3.4}$$

We emphasize that, for the existence and uniqueness part of theorem 1.3, in dimension 3, estimate (3.4) is irrelevant.

Proof. Only the estimate of $\|B\|_{L^2 \dot{H}^{d/2,d/2-1}_\alpha}$ requires a proof, which is given in the appendix. All other estimates can be derived by a standard energy estimate: apply Δ_q to the system, derive an energy inequality, multiply both members of that inequality by $2^{q(d/2-1)}\sqrt{\max(1, q^\alpha)}$ and take the $\ell^2(\mathbb{Z})$-norm. □

The following is a series of nonlinear estimates needed for the contraction argument.

PROPOSITION 3.5. *For all smooth functions u, E and B defined on some interval $[0, T]$, we have the following estimates, with constants independent of T. In space dimension 2,*

$$\|\nabla(u \otimes v)\|_{L^1_T L^2(\mathbb{R}^2)} \lesssim \|u\|_{L^2_T(L^\infty(\mathbb{R}^2)\cap \dot{H}^1(\mathbb{R}^2))}\|v\|_{L^2_T(L^\infty(\mathbb{R}^2)\cap \dot{H}^1(\mathbb{R}^2))}, \tag{3.5}$$

$$\|E \times B\|_{\tilde{L}^2_T \dot{\mathcal{B}}^{-1}_{2,1}(\mathbb{R}^2) + L^1_T L^2(\mathbb{R}^2)} \lesssim \|E\|_{L^2_T L^2_{\log}(\mathbb{R}^2)}\|B\|_{\tilde{L}^\infty_T L^2_{\log}(\mathbb{R}^2)\cap L^2_T \dot{H}^{1,0}}, \tag{3.6}$$

$$\|u \times B\|_{L^2_T L^2_{\log}(\mathbb{R}^2)} \lesssim \|u\|_{L^2(L^\infty(\mathbb{R}^2)\cap \dot{H}^1(\mathbb{R}^2))}\|B\|_{\tilde{L}^\infty_T L^2_{\log}(\mathbb{R}^2)}. \tag{3.7}$$

In space dimension 3,

$$\|\nabla(u \otimes v)\|_{L^1_T \dot{H}^{1/2}(\mathbb{R}^3)} \lesssim \|u\|_{L^2_T(L^\infty(\mathbb{R}^3)\cap \dot{H}^{3/2}(\mathbb{R}^3))}\|v\|_{L^2_T(L^\infty(\mathbb{R}^3)\cap \dot{H}^{3/2}(\mathbb{R}^3))}, \tag{3.8}$$

$$\|E \times B\|_{\tilde{L}^2_T \dot{\mathcal{B}}^{-1/2}_{2,1}(\mathbb{R}^3)} \lesssim \|E\|_{L^2_T \dot{H}^{1/2}(\mathbb{R}^3)}\|B\|_{\tilde{L}^\infty_T \dot{H}^{1/2}(\mathbb{R}^3)}, \tag{3.9}$$

$$\|u \times B\|_{L^2_T \dot{H}^{1/2}(\mathbb{R}^3)} \lesssim \|u\|_{L^2_T(L^\infty(\mathbb{R}^3)\cap \dot{H}^{3/2}(\mathbb{R}^3))}\|B\|_{\tilde{L}^\infty_T \dot{H}^{1/2}(\mathbb{R}^3)}. \tag{3.10}$$

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Estimates (3.5) and (3.8) enable us to control the advection term in the Navier–Stokes component of the system in dimensions 2 and 3, respectively. Estimates (3.6) and (3.9) are needed to control the Maxwell part in the Navier–Stokes component. To estimate the term $(u \times B) \times B$, we use (3.6), (3.7) in two space dimensions and (3.9), (3.10) in three space dimensions.

REMARK 3.6. Ignoring the time variable, (3.9) is a particular case of the product law

$$H^{s_1}(\mathbb{R}^d) \cdot H^{s_2}(\mathbb{R}^d) \hookrightarrow \dot{B}^{s_1+s_2-d/2}_{2,1}(\mathbb{R}^d),$$

with $s_1, s_2 \in]-d/2, d/2[$ and $s_1 + s_2 > 0$. Indeed, it corresponds to the admissible choice $s_1 = s_2 = \frac{1}{2}$. However, this product law becomes critical in two space dimensions. Estimate (3.6) shows that an extra logarithmic loss is needed in this case.

We give the proof of the above proposition in the appendix.

4. Proof of theorem 1.3

4.1. Small data and global existence

Let $\alpha = 1$ if $d = 2$ and let $\alpha = 0$ if $d = 3$. Let $\mathcal{Z}$ be the set of $\Gamma := (u, E, B)^{\mathrm{T}}$ such that

$$\begin{aligned} u \in \mathcal{Z}^u &:= L^2(0, \infty, \dot{H}^{d/2} \cap L^\infty) \cap \tilde{L}^\infty(0, \infty, \dot{H}^{d/2-1}), \\ E \in \mathcal{Z}^E &:= (\tilde{L}^\infty \cap L^2)(0, \infty, \dot{H}^{d/2-1}_\alpha), \\ B \in \mathcal{Z}^B &:= \tilde{L}^\infty(0, \infty, \dot{H}^{d/2-1}_\alpha) \cap L^2(0, \infty, H^{d/2, d/2-1}_\alpha). \end{aligned}$$

Endow $\mathcal{Z}$, $\mathcal{Z}^u$, $\mathcal{Z}^E$ and $\mathcal{Z}^B$ with the natural norms. Recall that we seek a solution to (1.1) in the integral form

$$\Gamma(t) = \mathrm{e}^{t\mathcal{A}}\Gamma(0) + \int_0^t \mathrm{e}^{(t-t')\mathcal{A}} \mathcal{N}(\Gamma(t'))\,\mathrm{d}t',$$

with

$$\mathcal{A} = \begin{pmatrix} \Delta & 0 & 0 \\ 0 & -I & \mathrm{curl} \\ 0 & -\mathrm{curl} & 0 \end{pmatrix}$$

and $\mathcal{N}(\Gamma) = (\mathcal{P}(-\nabla(u \otimes u) + E \times B + (u \times B) \times B), -u \times B, 0)^{\mathrm{T}}$. Let B_δ be the ball of the space $\mathcal{Z}_\infty$ centred at 0 and with radius $\delta > 0$ to be chosen. Define the map Φ on that ball as

$$\begin{aligned} \Phi \colon B_\delta \subset \mathcal{Z} &\to \mathcal{Z} \\ \Gamma &\mapsto \Phi(\Gamma) := \int_0^t \mathrm{e}^{(t-t')\mathcal{A}} \mathcal{N}(\mathrm{e}^{t'\mathcal{A}}\Gamma^0 + \Gamma(t'))\,\mathrm{d}t'. \end{aligned} \tag{4.1}$$

CLAIM 4.1. *If $\|\Gamma^0\|_{\dot{H}^{d/2-1} \times \dot{H}^{d/2-1}_\alpha \times \dot{H}^{d/2-1}_\alpha} \leqslant \kappa\delta$, with $\delta > 0$ and $\kappa > 0$ sufficiently small, then the map Φ is a contraction on B_δ.*

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The theorem follows immediately from the claim: Picard's theorem gives the existence of a fixed point of Φ; call it Γ. Then, $\mathrm{e}^{t\mathcal{A}}\Gamma^0 + \Gamma(t)$ is the desired solution.

Proof of the claim. First, note that $\Phi(-\mathrm{e}^{t\mathcal{A}}\Gamma_0) = 0$, while, by lemmas 3.2–3.4,

$$\|\mathrm{e}^{t\mathcal{A}}\Gamma_0\|_{\mathcal{Z}} \leqslant C\|\Gamma_0\|_{\dot{H}^{d/2-1}\times\dot{H}^{d/2-1}_\alpha\times\dot{H}^{d/2-1}_\alpha} \leqslant C\kappa\delta \leqslant \tfrac{1}{2}\delta \tag{4.2}$$

for κ small enough. On the other hand, we prove below that if Γ_1 and Γ_2 belong to B_δ,

$$\|\Phi(\Gamma_1) - \Phi(\Gamma_2)\|_{\mathcal{Z}} \leqslant \tfrac{1}{2}\|\Gamma_1 - \Gamma_2\|_{\mathcal{Z}} \tag{4.3}$$

under the assumptions of the claim.

The estimates (4.2) and (4.3) easily yield the claim.

To prove (4.3), let $\Gamma_j := (u_j, E_j, B_j)^{\mathrm{T}} \in B_\delta$ for $j = 1, 2$. Further, write

$$\mathrm{e}^{t\mathcal{A}}\Gamma^0 + \Gamma_j(t) = (\bar{u}_j, \bar{E}_j, \bar{B}_j)^{\mathrm{T}},$$

set $\Gamma := \Gamma_1 - \Gamma_2 := (u, E, B)^{\mathrm{T}}$ and let $\Phi(\Gamma_j) := \tilde{\Gamma}_j = (\tilde{u}_j, \tilde{E}_j, \tilde{B}_j)^{\mathrm{T}}$ be given by (4.1). Let $\tilde{\Gamma} := \tilde{\Gamma}_1 - \tilde{\Gamma}_2 := (\tilde{u}, \tilde{E}, \tilde{B})^{\mathrm{T}}$. We decompose $\tilde{u}$ into $\tilde{u} = \tilde{u}^{\mathrm{NS}} + \tilde{u}^{\mathrm{M}}$, with $\tilde{u}^{\mathrm{NS}}$ accounting for the convection term

$$\tilde{u}^{\mathrm{NS}} := -\int_0^t \mathrm{e}^{(t-t')\Delta}\mathcal{P}\nabla(u\otimes\bar{u}_1 + \bar{u}_2\otimes u)\,\mathrm{d}t',$$

and $\tilde{u}^{\mathrm{M}}$ accounting for the Lorentz force

$$\begin{aligned}\tilde{u}^{\mathrm{M}} := &\int_0^t \mathrm{e}^{(t-t')\Delta}\mathcal{P}(E\times\bar{B}_1 + \bar{E}_2\times B)\,\mathrm{d}t' \\ &+ \int_0^t \mathrm{e}^{(t-t')\Delta}\mathcal{P}((u\times\bar{B}_1)\times\bar{B}_1 + [\bar{u}_2\times B]\times\bar{B}_1 + [\bar{u}_2\times\bar{B}_2]\times B)\,\mathrm{d}t'.\end{aligned}$$

Moreover, the electromagnetic field $(\tilde{E}, \tilde{B})$ satisfies

$$\partial_t\tilde{E} - \operatorname{curl}\tilde{B} + \tilde{E} = u\times\bar{B}_1 + \bar{u}_2\times B, \tag{4.4}$$

$$\partial_t\tilde{B} + \operatorname{curl}\tilde{E} = 0 \tag{4.5}$$

with 0 data. First, by lemmas 3.2 and 3.3 and the embedding

$$L^1\dot{H}^{d/2-1} \hookrightarrow \tilde{L}^1\dot{H}^{d/2-1},$$

we have that

$$\|\tilde{u}^{\mathrm{NS}}\|_{\mathcal{Z}^u} \lesssim \|\mathcal{P}\nabla(u\otimes\bar{u}_1 + \bar{u}_2\otimes u)\|_{L^1\dot{H}^{d/2-1}}$$

and

$$\begin{aligned}&\|\tilde{u}^{\mathrm{M}}\|_{\mathcal{Z}^u} \\ &\quad\lesssim \|\mathcal{P}(E\times\bar{B}_1 + \bar{E}_2\times B)\|_{\tilde{L}^2\dot{B}^{d/2-2}_{2,1}+L^1\dot{H}^{d/2-1}} \\ &\qquad + \|\mathcal{P}((u\times\bar{B}_1)\times\bar{B}_1 + [\bar{u}_2\times B]\times\bar{B}_1 + [\bar{u}_2\times\bar{B}_2]\times B)\|_{\tilde{L}^2\dot{B}^{d/2-2}_{2,1}+L^1\dot{H}^{d/2-1}}.\end{aligned}$$

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Second, applying (3.5) and (3.8), we obtain for the convection term that

$$\begin{aligned}\|\tilde{u}^{\mathrm{NS}}\|_{\mathcal{Z}^u} &\lesssim \|u\|_{L^2(L^\infty\cap\dot{H}^{d/2})}\sum_{j=1,2}\|\bar{u}_j\|_{L^2(L^\infty\cap\dot{H}^{d/2})}\\ &\lesssim \|\Gamma\|_{\mathcal{Z}}\sum_{j=1,2}(\|\Gamma_j\|_{\mathcal{Z}}+\|\mathrm{e}^{t\mathcal{A}}\Gamma^0\|_{\mathcal{Z}}),\end{aligned} \tag{4.6}$$

whereas the Lorentz force term can be estimated by (3.6), (3.7), (3.9) and (3.10):

$$\begin{aligned}\|\tilde{u}^{\mathrm{M}}\|_{\mathcal{Z}^u} \lesssim{}& \|E\|_{L^2\dot{H}_\alpha^{d/2-1}}\|\bar{B}_1\|_{\tilde{L}^\infty\dot{H}_\alpha^{d/2-1}\cap L^2\dot{H}^{d/2,d/2-1}}\\ &+\|\bar{E}_2\|_{L^2\dot{H}_\alpha^{d/2-1}}\|B\|_{\tilde{L}^\infty\dot{H}_\alpha^{d/2-1}\cap L^2\dot{H}^{d/2,d/2-1}}\\ &+\|u\times\bar{B}_1\|_{L^2\dot{H}_\alpha^{d/2-1}}\|\bar{B}_1\|_{\tilde{L}^\infty\dot{H}_\alpha^{d/2-1}\cap L^2\dot{H}^{d/2,d/2-1}}\\ &+\|\bar{u}_2\times B\|_{L^2\dot{H}_\alpha^{d/2-1}}\|\bar{B}_1\|_{\tilde{L}^\infty\dot{H}_\alpha^{d/2-1}\cap L^2\dot{H}^{d/2,d/2-1}}\\ &+\|\bar{u}_2\times\bar{B}_2\|_{L^2\dot{H}_\alpha^{d/2-1}}\|\bar{B}\|_{\tilde{L}^\infty\dot{H}_\alpha^{d/2-1}}\\ \lesssim{}& \|E\|_{\mathcal{Z}^E}\|\bar{B}_1\|_{\mathcal{Z}^B}+\|\bar{E}_2\|_{\mathcal{Z}^B}\|B\|_{\mathcal{Z}^B}+\|u\|_{\mathcal{Z}^u}\|\bar{B}_1\|_{\mathcal{Z}^B}\|\bar{B}_1\|_{\mathcal{Z}^B}\\ &+\|\bar{u}_2\|_{\mathcal{Z}^u}\|B\|_{\mathcal{Z}^B}\|\bar{B}_1\|_{\mathcal{Z}^B}+\|\bar{u}_2\|_{\mathcal{Z}^u}\|\bar{B}_2\|_{\mathcal{Z}^B}\|B\|_{\mathcal{Z}^B}\\ \lesssim{}& \|\Gamma\|_{\mathcal{Z}}\sum_{j=1,2}[\|\mathrm{e}^{t\mathcal{A}}\Gamma^0\|_{\mathcal{Z}}+\|\mathrm{e}^{t\mathcal{A}}\Gamma^0\|_{\mathcal{Z}}^2+\|\Gamma_j\|_{\mathcal{Z}}+\|\Gamma_j\|_{\mathcal{Z}}^2].\end{aligned} \tag{4.7}$$

It remains to estimate the electromagnetic field components of Γ. Applying the energy and the decay estimates (3.3) to the system (4.4), we get that

$$\|\tilde{E}\|_{\mathcal{Z}^E}+\|\tilde{B}\|_{\mathcal{Z}^B}\lesssim\|\Gamma\|_{\mathcal{Z}}\sum_{j=1}^{2}(\|\mathrm{e}^{t\mathcal{A}}\Gamma^0\|_{\mathcal{Z}}+\|\Gamma_j\|_{\mathcal{Z}}). \tag{4.8}$$

Gathering the estimates (4.6)–(4.8) gives that

$$\|\tilde{\Gamma}\|_{\mathcal{Z}}\lesssim\|\Gamma\|_{\mathcal{Z}}(\|\mathrm{e}^{t\mathcal{A}}\Gamma^0\|_{\mathcal{Z}}+\|\mathrm{e}^{t\mathcal{A}}\Gamma^0\|_{\mathcal{Z}}^2+\|\Gamma_j\|_{\mathcal{Z}}+\|\Gamma_j\|_{\mathcal{Z}}^2). \tag{4.9}$$

Choosing δ small enough gives (4.3). □

4.2. The local existence

Decompose the initial data $(u^0,E^0,B^0)=(u_r^0,E_r^0,B_r^0)+(u_s^0,E_s^0,B_s^0)$, where (u_r^0,E_r^0,B_r^0) is regular (say, in H^2) and (u_s^0,E_s^0,B_s^0) is small in

$$\dot{H}^{d/2-1}\times\dot{H}_\alpha^{d/2-1}\times\dot{H}_\alpha^{d/2-1}$$

(this can be done using a Fourier cut-off). We look for a solution Γ of (1.1) of the form $(u,E,B)=(u_s,E_s,B_s)+(u_r,E_r,B_r)$, with

$$\left.\begin{aligned}\frac{\partial u_r}{\partial t}+u_r\cdot\nabla u_r-\Delta u_r+\nabla p_r&=j_r\times B_r,\\ \partial_t E_r-\operatorname{curl}B_r&=-j_r,\\ \partial_t B_r+\operatorname{curl}E_r&=0,\\ \operatorname{div}u_r=\operatorname{div}B_r&=0,\\ (E_r+u_r\times B_r)&=j_r\end{aligned}\right\} \tag{4.10}$$

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subject to the initial data

$$u_r|_{t=0} = u_r^0, \qquad B_r|_{t=0} = B_r^0, \qquad E_r|_{t=0} = E_r^0.$$

Arguing as in [9], we know that (4.10) has a unique regular solution. We now solve for (u_s, E_s, B_s). We have

$$\left.\begin{aligned} \frac{\partial u_s}{\partial t} + u_s \cdot \nabla u_s - \Delta u_s + u_s \cdot \nabla u_r + u_r \cdot \nabla u_s + \nabla p_s &= j \times B - j_r \times B_r, \\ \partial_t E_s - \operatorname{curl} B_s &= j - j_r, \\ \partial_t B_s + \operatorname{curl} E_s &= 0, \\ \operatorname{div} u_s = \operatorname{div} B_s &= 0 \end{aligned}\right\} \tag{4.11}$$

subject to the initial data

$$u_s|_{t=0} = u_s^0, \qquad B_s|_{t=0} = B_s^0, \qquad E|_{t=0} = E_s^0.$$

Proceeding in a similar way as for the small data result, set

$$\Phi(\Gamma) := \int_0^t \mathrm{e}^{(t-t')\mathcal{A}} \begin{pmatrix} \mathcal{P}[-u_s \cdot \nabla u_s - u_s \cdot \nabla u_r - u_r \cdot \nabla u_s + j \times B - j_r \times B_r] \\ j - j_r \end{pmatrix} \mathrm{d}t',$$

where Γ is defined by

$$(u_s, B_s, E_s)^{\mathrm{T}} = \mathrm{e}^{t\mathcal{A}}(u_s^0, B_s^0, E_s^0)^{\mathrm{T}} + \Gamma.$$

Applying the same proof as for the small data existence, we can show that the map Φ is a contraction if we choose a time of existence T sufficiently small. The main difference is that new linear terms (in Γ) appear in Φ. These linear terms need to be small (as linear maps) for Φ to be a contraction; this can be achieved by using the smoothness of B and by choosing T small enough.

For instance,

$$\begin{aligned} \left\| \int_0^t \mathrm{e}^{(t-t')\Delta} \mathcal{P} E_s \times B_r \, \mathrm{d}t' \right\|_{\mathcal{Z}_T^u} &\lesssim \|E_s \times B_r\|_{L_T^1 \dot{H}^{d/2-1}} \\ &\lesssim \|E_s\|_{L_T^\infty \dot{H}^{d/2-1}} \|B_r\|_{L_T^1 H^2} \\ &\lesssim \|E_s\|_{\mathcal{Z}_T^E} \|B_r\|_{L_T^1 H^2}. \end{aligned} \tag{4.12}$$

The key point is, of course, that $\|B_r\|_{L_T^1 H^2}$ can be made arbitrarily small by choosing T small enough.

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Appendix A. Proof of the decay for B and proposition 3.5

Recall the main part of lemma 3.3. We emphasize the fact that this property is extra information about a weak decay that the magnetic field satisfies, and it has no impact on the well-posedness result.

Lemma A.1. *Let* $\alpha \geqslant 0$, *let* $G \in L^2_T \dot{H}^{d/2-1}_\alpha$ *and let* (E, B) *be a solution of*

$$\partial_t E - \operatorname{curl} B + E = G,$$
$$\partial_t B + \operatorname{curl} E = 0$$

on some time interval $[0, T]$. *The following estimate then holds with constants not depending upon time:*

$$\|B\|_{L^2 \dot{H}^{d/2, d/2-1}_\alpha} \lesssim \|(E_0, B_0)\|_{\dot{H}^{d/2-1}_\alpha} + \|G\|_{L^2_T \dot{H}^{d/2-1}_\alpha}. \tag{A 1}$$

Proof. Because of the divergence-free property of B, we have that

$$\partial_{tt} B - \Delta B + \partial_t B = \operatorname{curl} G, \qquad (B, \partial_t B)|_{t=0} = (B^0, B^1). \tag{A 2}$$

Thus, the magnetic field B satisfies an inhomogeneous damped wave equation (A 2). In the following we denote by $\mathcal{L}_1(t)$ and $\mathcal{L}_2(t)$, respectively, the propagators associated with the Fourier multiplier functions

$$\varPhi_1(t, \xi) = \mathrm{e}^{-t/2} \cosh\left(\sqrt{\tfrac{1}{4} - |\xi|^2} t\right), \qquad \varPhi_2(t, \xi) = \mathrm{e}^{-t/2} \frac{\sinh(\sqrt{\frac{1}{4} - |\xi|^2} t)}{\sqrt{\frac{1}{4} - |\xi|^2}}.$$

A direct computation gives the Duhamel-type formula

$$B(t, x) = \mathcal{L}_1(t) B^0(x) + \mathcal{L}_2(t)(B^0/2 + B^1)(x) + \int_0^t \mathcal{L}_2(t-s) \operatorname{curl}(G)(s, x) \, \mathrm{d}s,$$

with $B^1 = \partial_t B(t = 0) = -\operatorname{curl}(E(t = 0)) = -\operatorname{curl}(E^0)$. As this was observed in [8], there exists $0 < c < 1$ such that we have the following bounds.

- For $|\xi| \geqslant 2$,

$$|\varPhi_1(t, \xi)| \lesssim \mathrm{e}^{-ct}, \qquad |\varPhi_2(t, \xi)| \lesssim \frac{\mathrm{e}^{-ct}}{|\xi|}.$$

- For $\frac{1}{4} \leqslant |\xi| < 2$,

$$|\varPhi_1(t, \xi)| + |\varPhi_2(t, \xi)| \lesssim \mathrm{e}^{-ct}.$$

- For $2^{q-1} \leqslant |\xi| \leqslant 2^{q+1}$, with $q \leqslant -3$,

$$|\varPhi_1(t, \xi)| \leqslant \varPhi^1_q(t) := \mathrm{e}^{-t/2} \cosh\left(t \sqrt{\tfrac{1}{4} - 2^{2(q-1)}}\right),$$

$$|\varPhi_2(t, \xi)| \leqslant \varPhi^2_q(t) := \mathrm{e}^{-t/2} \frac{\sinh(t\sqrt{1 - 2^{2(q-1)}})}{\sqrt{\frac{1}{4} - 2^{2(q-1)}}}.$$

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On the one hand, for $q \geqslant -1$, one has that

$$\|\Delta_q B(t)\|_{L^2} \lesssim \mathrm{e}^{-ct}\|\Delta_q B^0\|_{L^2} + \mathrm{e}^{-ct}2^{-q}(\|\Delta_q B^0\|_{L^2} + \|\Delta_q B^1\|_{L^2}) + \int_0^t \mathrm{e}^{-c(t-s)}\|\Delta_q G\|_{L^2}\,\mathrm{d}s. \tag{A 3}$$

Multiplying both sides of (A 3) by $2^{q(d/2-1)}$, applying Young's inequality (in time) and summing in q yields

$$\|(I-S_0)B\|_{L^2_T\dot H^{d/2-1}_\alpha} \lesssim \|(I-S_0)B^0\|_{\dot H^{d/2-1}_\alpha} + \|(I-S_0)B^1\|_{H^{d/2-2}_\alpha} + \|(I-S_0)G\|_{L^2_T\dot H^{d/2-1}_\alpha}. \tag{A 4}$$

On the other hand, for $q \leqslant 0$, one has that

$$\|\Delta_q B(t)\|_{L^2} \leqslant \Phi^1_q(t)\|\Delta_q B^0\|_{L^2} + \Phi^2_q(t)(\|\Delta_q B^1\|_{L^2} + \|\Delta_q B^0\|_{L^2}) + 2^q\int_0^t \Phi^2_q(t-s)\|\Delta_q G(s)\|_{L^2}\,\mathrm{d}s.$$

Taking the L^2_T-norm in time and applying Young's inequality we get that

$$\begin{aligned}\|\Delta_q B\|_{L^2L^2} \lesssim{}& \|\Phi^1_q\|_{L^2(\mathbb{R}^+)}\|\Delta_q B^0\|_{L^2}\\ &+ \|\Phi^2_q\|_{L^2(\mathbb{R}^+)}(\|\Delta_q B^1\|_{L^2} + \|\Delta_q B^0\|_{L^2})\\ &+ 2^q\|\Phi^2_q\|_{L^1(\mathbb{R}^+)}\|\Delta_q G\|_{L^2_TL^2}.\end{aligned}$$

But since, for every $q \leqslant 0$ and $r \in [1,+\infty]$, Φ^i_q satisfies

$$\|\Phi^i_q\|_{L^r(\mathbb{R}^+)} \lesssim 2^{-2q/r}, \quad i = 1, 2,$$

multiplying both sides by $2^{qd/2}q^{\alpha/2}$ and taking the ℓ^2-norm gives

$$\|S_0B\|_{L^2_T\dot H^{d/2}} \lesssim \|S_0B^0\|_{\dot H^{d/2-1}_\alpha} + \|S_0B^1\|_{\dot H^{d/2-2}_\alpha} + \|S_0G\|_{L^2_T\dot H^{d/2-1}_\alpha}. \tag{A 5}$$

Putting together (A 4) and (A 5) gives (3.3), as desired. □

Proof of proposition 3.5. The proof is based on the paraproduct decomposition. We choose to prove only (3.6) and (3.7) in detail. The other estimates are easier, or classical, and left to the reader.

Proof of (3.6). We decompose EB into

$$EB = T_EB + T_BE + S_2R(E,B) + (I-S_2)R(E,B),$$

and show the following estimates:

$$\|T_EB + T_BE\|_{\tilde L^2_T\dot B^{-1}_{2,1}(\mathbb{R}^2)} \lesssim \|E\|_{L^2_TL^2(\mathbb{R}^2)}\|B\|_{\tilde L^\infty_TL^2(\mathbb{R}^2)}, \tag{A 6}$$

$$\|S_2R(E,B)\|_{L^1_TL^2} \lesssim \|E\|_{L^2_TL^2(\mathbb{R}^2)}\|B\|_{L^2_T\dot H^{1,0}(\mathbb{R}^2)}, \tag{A 7}$$

$$\|(I-S_2)R(E,B)\|_{\tilde L^2_T\dot{\mathcal{B}}^{-1}_{2,1}(\mathbb{R}^2)} \lesssim \|E\|_{L^2_TL^2_{\log}(\mathbb{R}^2)}\|B\|_{\tilde L^\infty_TL^2_{\log}}. \tag{A 8}$$

First, we prove (A 6). Since the term $T_B E$ can be treated in a very similar way, we focus on $T_E B$. First,

$$\Delta_q(T_E B) = \sum_{|\tilde{q}-q|\leqslant 1} \Delta_q(\Delta_{\tilde{q}} B S_{\tilde{q}} E).$$

Since Δ_q is uniformly bounded on L^2, we have that

$$\sum_q 2^{-q}\|\Delta_q(T_E B)\|_{L^2_T L^2} \lesssim \sum_q 2^{-q} \sum_{|\tilde{q}-q|\leqslant 1} \|\Delta_{\tilde{q}} B S_{\tilde{q}} E\|_{L^2_T L^2}.$$

We are going to deal with the term $\tilde{q} = q$ only (the two other terms $\tilde{q} = q \pm 1$ can be estimated similarly). Successively applying Hölder's inequality (in the variables t and x), Bernstein's lemma, Young's inequality (in the variable q) and Hölder's inequality (in the variable q) gives that

$$\begin{aligned}
\sum_q 2^{-q}\|\Delta_q B S_q E\|_{L^2_T L^2} &\leqslant \sum_q 2^{-q}\|\Delta_q B\|_{L^\infty_T L^2}\|S_q E\|_{L^2_T L^\infty} \\
&\leqslant \sum_q 2^{-q} \sum_{j\leqslant q} \|\Delta_q B\|_{L^\infty_T L^2}\|\Delta_j E\|_{L^2_T L^\infty} \\
&\leqslant \sum_q \sum_{j\leqslant q} 2^{j-q}\|\Delta_q B\|_{L^\infty_T L^2}\|\Delta_j E\|_{L^2_T L^2} \\
&\leqslant \biggl(\sum_q \|\Delta_q B\|^2_{L^\infty_T L^2}\biggr)^{1/2} \biggl(\sum_j \|\Delta_j E\|^2_{L^2_T L^2}\biggr)^{1/2}.
\end{aligned}$$

We next prove (A 7). Applying Bernstein's lemma (see lemma 3.1) and the Cauchy–Schwarz inequality (in j) gives that

$$\begin{aligned}
\|S_2 R(B,E)\|_{\tilde{L}^1_T L^2} &\lesssim \sum_{q\leqslant 0} \|\Delta_q R(B,E)\|_{L^1_T L^2} \\
&\lesssim \sum_{q\leqslant 0} 2^q \|\Delta_q R(B,E)\|_{L^1_T L^1} \\
&\lesssim \sum_{q\leqslant 0} 2^q \sum_{j\geqslant q-2} \|\Delta_j B\|_{L^2_T L^2}\|\Delta_j E\|_{L^2_T L^2} \\
&\lesssim \sum_j \sum_{q\leqslant \inf(0,j+2)} 2^q \|\Delta_j B\|_{L^2_T L^2}\|\Delta_j E\|_{L^2_T L^2} \\
&\lesssim \sum_{j\leqslant 0} 2^j \|\Delta_j B\|_{L^2_T L^2}\|\Delta_j E\|_{L^2_T L^2} \\
&\qquad + \sum_{j\geqslant 0} \|\Delta_j B\|_{L^2_T L^2}\|\Delta_j E\|_{L^2_T L^2} \\
&\lesssim \|E\|_{L^2_T L^2}\|B\|_{L^2_T \dot{H}^{1,0}}.
\end{aligned}$$

To estimate (A 8), Hölder's inequality (in t, x) and the Cauchy–Schwarz inequality (in j) give that

$$\begin{aligned}\|(I-S_2)R(E,B)\|_{\tilde{L}^2_T\dot{\mathcal{B}}^{-1}_{2,1}} &\lesssim \sum_{q\geqslant 0}\sum_{j\geqslant q-2}\|\Delta_j E\|_{L^2_T L^2}\|\Delta_j B\|_{L^\infty_T L^2}\\ &\lesssim \sum_{j\geqslant -2}\sum_{0\leqslant q\leqslant j+2}\|\Delta_j E\|_{L^2_T L^2}\|\Delta_j B\|_{L^\infty_T L^2}\\ &\lesssim \sum_{j\geqslant -2}\max(j,1)\|\Delta_j E\|_{L^2_T L^2}\|\Delta_j B\|_{L^\infty_T L^2}\\ &\lesssim \|E\|_{L^2_T L^2_{\log}}\|B\|_{L^\infty_T L^2_{\log}}.\end{aligned}$$

□

Proof of (3.7). As for the proof of (3.6), we split uB following the paraproduct decomposition

$$uB = T_B u + T_u B + R(u,B).$$

We only estimate $T_B u$ here, the estimate of $R(u,B)$ being similar, and that of $T_u B$ being easier. By Hölder's inequality,

$$\begin{aligned}\|T_B u\|^2_{L^2_T L^2_{\log}} &= \sum_q \max(1,q)\|S_q B\Delta_q u\|^2_{L^2_T L^2}\\ &\lesssim \sum_q \max(1,q)\|\Delta_q u\|^2_{L^2_T L^2}\|S_q B\|^2_{L^\infty_T L^\infty}.\end{aligned}$$

Now observe that Bernstein's lemma and the Cauchy–Schwarz inequality (in j) give that

$$\begin{aligned}\|S_q B\|_{L^\infty_T L^\infty} &\lesssim \sum_{j<q} 2^j\|\Delta_j B\|_{L^\infty_T L^2}\\ &\lesssim \biggl(\sum_{j<q}\frac{2^{2j}}{\max(1,j)}\biggr)^{1/2}\biggl(\sum_{j<q}\max(1,j)\|\Delta_j B\|^2_{L^\infty_T L^2}\biggr)^{1/2}\\ &\lesssim \frac{2^q}{\sqrt{\max(1,q)}}\|B\|_{\tilde{L}^\infty_T L^2_{\log}}.\end{aligned}$$

Coming back to the bound for $\|T_B u\|_{L^2_T L^2_{\log}}$, this gives that

$$\|T_B u\|^2_{L^2_T L^2_{\log}} \lesssim \sum 2^{2q}\|\Delta_q u\|^2_{L^2_T L^2}\|B\|^2_{\tilde{L}^\infty_T L^2_{\log}}.$$

□

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