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ON A RESULT OF SMITH AND SUBBARAO CONCERNING A DIVISOR PROBLEM

BY

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ABSTRACT. Let $d(n; l, k)$ denote the number of divisors of the positive integer n which are congruent to l modulo k. The objective of the present paper is to prove that (for some exponent $\theta < \frac{1}{3}$)

$$\sum_{1 \le n \le x} d(n; l, k) = \frac{x}{k} \log x + c(l, k)x + O((x/k)^\theta)$$

holds uniformly in l, k and x satisfying $1 \le l \le k \le x$. This improves a recent result due to R. A. Smith and M. V. Subbarao [3].

Let k and l be positive integers, $1 \le l \le k$, and denote by $d(n; l, k)$ the number of (positive) divisors of the positive integer n which are congruent to l modulo k. To study the average behaviour of $d(n; l, k)$ one considers the Dirichlet's summatory function

$$D(x; l, k) = \sum_{1 \le n \le x} d(n; l, k). \tag{1}$$

R. A. Smith and M. V. Subbarao have proved recently in this journal [3] that

$$D(x; l, k) = \frac{x}{k} \log x + \left(\gamma(l, k) + \frac{1}{k}(\gamma - 1)\right)x + O((kx)^{1/3} d(k) \log x) \tag{2}$$

(provided that l and k are coprime and $k \le x$) where

$$\gamma(l, k) = \lim_{x \to \infty} \left(\sum_{\substack{1 \le n \le x \\ n \equiv l (\mathrm{mod}\, k)}} \frac{1}{n} - \frac{1}{k} \log x \right),$$

$\gamma = \gamma(1, 1)$ is Euler's constant and $d(k)$ is the ordinary divisor function.

The purpose of the present note is to show that the above result can be improved by an argument similar to that which is customary from the classical divisor problem.

THEOREM. *There exists a constant $\theta < \frac{1}{3}$ such that*

$$D(x; l, k) = \frac{x}{k} \log x + \left(\gamma(l, k) + \frac{1}{k}(\gamma - 1)\right)x + O\left(\left(\frac{x}{k}\right)^\theta\right) \tag{3}$$

holds uniformly in l, k and x, provided that $l \le k \le x$.

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REMARK. Of course, the main significance of this result lies in the uniformity in k (as noted also in [3], p. 39, for the estimate (2)). For fixed k, our improvement is only a marginal one, but our O-term becomes the better the larger k is in comparison with x, whereas that in (2) increases with k. Thus the main terms in (3) always dominate the error term (for $x\to\infty$), which is true in (2) only for $k \ll x^{1/2-\varepsilon}$ (cf. the remark in [3], p. 39).

The following relation is an immediate consequence of our theorem (as Corollary 3 in [3] followed from (2)):

COROLLARY.

$$\sum_{\substack{1\le n\le x\\ n\equiv l \pmod k}} \left\{\frac{x}{n}\right\} = \frac{x}{k}(1-\gamma) + O\left(\left(\frac{x}{k}\right)^{\theta}\right) \quad (\theta<\tfrac{1}{3})$$

holds uniformly in l, k and x, if $l\le k\le x$ ($\{\cdot\}$ denotes the fractional part). (Throughout the paper, the O- and $\ll$-constants are absolute ones.) Proof of the theorem. Denoting by $|\mathcal{M}|$ the number of elements of a finite set M, we get

$$D(x;l,k) = |\{(u,t)\in\mathbb{N}^2 : ut\le x,\ t\equiv l \pmod k\}| =$$

$$= |\{(u,v)\in\mathbb{N}\times\mathbb{N}_0 : u(kv+l)\le x\}|.$$

Denote this last set by $M_0 = M_0(x;l,k)$ and put $P=(x/k)^{1/2}$, $Q=P-l/k$, then we conclude that

$$D(x;l,k) = |\mathcal{M}_1| + |\mathcal{M}_2| - |\mathcal{M}_3|$$

where

$$\mathcal{M}_1 = \mathcal{M}_0 \cap (]0,P]\times\mathbb{Z}),\ \mathcal{M}_2 = \mathcal{M}_0\cap(\mathbb{Z}\times[0,Q]),\ \mathcal{M}_3 = \mathcal{M}_1\cap\mathcal{M}_2$$

and therefore

$$(4)\qquad D(x;l,k) = \sum_{1\le u\le P}\left[\frac{1}{k}\left(\frac{x}{u}-l\right)+1\right] + \sum_{0\le v\le Q}[x(kv+l)^{-1}] - [P][Q+1].$$

Applying Euler's summation formula (in the form given e.g. by Titchmarsh [4], p. 13) we obtain (with $\psi(y)=y-[y]-\frac{1}{2}$)

$$\sum_{\frac{1}{2}<u\le P}\left(\frac{1}{k}\left(\frac{x}{u}-l\right)+1\right) = \frac{x}{k}(\log P+\log 2) - \psi(P)Q - \int_{1/2}^{\infty}\psi(u)u^{-2}\,du\,\frac{x}{k} + (P-\tfrac{1}{2})\times\left(1-\frac{l}{k}\right)$$

$$(5)\qquad = \frac{x}{k}(\log P+\gamma) - \psi(P)Q + P\left(1-\frac{l}{k}\right) + O(1)$$

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and

$$\sum_{0\le v\le Q} x(kv+l)^{-1} = \frac{x}{l}+\frac{x}{k}(\log(kQ+l)-\log l)-\psi(Q)P-\frac{x}{2l}-$$

$$(6)\quad -kx\int_0^\infty \psi(v)(kv+l)^{-2}\,dv+O(1)=\frac{x}{2k}\log(kx)+\gamma(l,k)x-\psi(Q)P+O(1).$$

Substituting (5) and (6) into (4) and writing $[y]=y-\psi(y)-\frac{1}{2}$ we arrive at

$$(7)\qquad D(x;l,k)=\frac{x}{k}\log x+\left(\gamma(l,k)+\frac{1}{k}(\gamma-1)\right)x-S_1-S_2+O(1)$$

where

$$S_1=\sum_{1\le u\le P}\psi\left(\frac{1}{k}\left(\frac{x}{u}-l\right)\right),\qquad S_2=\sum_{1\le v\le P}\psi(x(kv+l)^{-1}).$$

Thus our theorem will be proved if we can show that (for some $\theta<\frac{1}{3}$)

$$(8)\qquad S=\sum_{1\le u\le P}\psi(f(u))=O(P^{2\theta})$$

where $f(u)=P^2((u+\alpha)^{-1}-\beta)$ and either (α,β) or (β,α) equals $(k/l,0)$. To establish (8) we make use of a classical result due to Van der Corput [1]:

LEMMA. *Let $f(u)$ be a real-valued function on the interval $I=[a,b]$ with continuous derivatives up to the fifth order and suppose that there exists $\delta>0$ such that, for each triple (p,q,r) of nonnegative integers with $p+q+r=3$, one has (for $u\in I$)*

$$\text{(a)}\qquad |f^{(p+2)}(u)f^{(q+2)}(u)f^{(r+2)}(u)|\le|f''(u)|^{(17/3)+\delta}.$$

Suppose further that $f''(u)$ is monotone and vanishes nowhere on I and that, for suitable $\delta'>0$,

$$\text{(b)}\qquad |f'''(u)|\le|f''(u)|^{(4/3)+\delta'}$$

for $u\in I$. Then there exist real numbers C and $\omega>0$ (depending at most on δ and δ') such that

$$\left|\sum_{a\le u\le b}\psi(f(u))\right|<C\left(\int_a^b|f''(u)|^{(1/3)+\omega}\,du+\max_{[a,b]}|f''(u)|^{-1/2}\right).$$

In order to verify condition (a) for our case we note that

$$P^2u^{-1-j}\ll|f^{(j)}(u)|\ll P^2u^{-1-j}$$

for $j=1,\ldots,5$ and infer (writing L_a and R_a for the left and right side of (a), respectively) that

$$L_aR_a^{-1}\ll P^{-16/3-\delta}u^{5+3\delta}\ll P^{-1/3+\delta}.$$

Choosing e.g. $\delta=\frac{1}{6}$, this is less than 1 if P exceeds some absolute constant (otherwise (8) is trivial). Similarly,

$$L_b R_b^{-1} \ll P^{-2/3-2\delta'} u^{3\delta'} \ll P^{-2/3+\delta'} \leq 1$$

for $\delta'=\frac{1}{3}$ (say) and P sufficiently large.

We now apply the lemma to the interval $T \leq u \leq P$, where $T=P^{2/3+\varepsilon}$ and $\varepsilon>0$ is a (sufficiently small) constant at our disposal. We obtain

$$\text{(9)} \qquad \sum_{T\leq u\leq P} \psi(f(u)) \ll P^{2/3+2\omega} \int_T^P u^{-1-3\omega}\, du + |f''(P)|^{-1/2} \ll P^{2/3-3\omega\varepsilon}.$$

For the interval $W \leq u < T$ (where $W=P^{2/3-\varepsilon}$) we use another classical result of Van der Corput (see e.g. Titchmarsh [4], p. 92) to estimate the corresponding Weyl sums; this yields (for positive integers h)

$$S(h) := \sum_{W\leq u<T} e^{2\pi i h f(u)} \ll h^{1/6} P^{(5/9)+(13\varepsilon/3)}.$$

Combining the inequalities of Erdös-Turán and of Koksma (see e.g. Hlawka [2], p. 104 and 107) we thus obtain (with $H=[P^{2/21}]$)

$$\text{(10)} \qquad \sum_{W\leq u<T} \psi(f(u)) \ll TH^{-1} + \sum_{h=1}^{H} h^{-1}\, |S(h)| \ll P^{(4/7)+(13\varepsilon/3)}.$$

Since the sum over $1 \leq u < W$ can be estimated trivially by $O(P^{2/3-\varepsilon})$, we immediately infer (8) from (9) and (10) (if $\varepsilon>0$ is chosen sufficiently small). This completes the proof of our theorem.

REMARK. Using more elaborate variants of Van der Corput's method one could establish slight refinements of our result, i.e. one could prove (3) for some explicitly given exponent $\theta<\frac{1}{3}$ as it is well-known for the classical Dirichlet's divisor problem.

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