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FUNCTIONS OF BOUNDED MEAN SQUARE, AND GENERALIZED FOURIER-STIELTJES TRANSFORMS

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1. Introduction. A complex function on the real line is said to be bounded in mean square if it is locally in L^2 (i.e. on each finite interval) and satisfies

$$(1.1) \qquad ||f||_B^2 = \sup_{T>0} \frac{1}{1+2T} \int_{-T}^{T} |f(t)|^2 \, dt < \infty.$$

The set of all such functions clearly forms a linear space over the complex numbers and is a Banach space B under the norm $||\cdot||_B$ defined by (1.1). This space, among others, has been discussed by Beurling in [**1**], where it was shown to be the dual, in the Banach space sense, of a certain Banach (convolution) algebra of functions. We have used Beurling's characterization of B and others of his results throughout this paper, and indeed the essence of one or two of the proofs has been derived from his theorems.

Our aim in this paper is to provide the basic theory for the harmonic analysis of functions in B. One would therefore expect some similarity between the present attempt and Wiener's [**6**], and indeed we follow Wiener in using the integrated Fourier transform $s(u)$ of functions in B [**6**, §§ 5, 6; **7**, § 20]. But in this part of his theory, Wiener's interest was confined mainly to the behaviour of the functions in a neighbourhood of infinity, and to functions for which the limit of the expression in (1.1) exists as $T \to \infty$, whereas we shall deal with the full space B, and with various topologies on B including that given by the norm of B.

The Banach algebra A of which B is the dual is a subset of L^1 so that one can take Fourier transforms in A to form a new Banach algebra $\hat{A}$ with pointwise multiplication as product. As is done in the theory of distributions, we can define the Fourier transform $\hat{f}$ of $f \in B$ to be the functional on $\hat{A}$ defined by $\hat{f}(\hat{\phi}) = f(\phi)$, $\phi \in A$. It turns out that $\hat{f}$ can be expressed as a (generalized) measure on $\hat{A}$,

$$\hat{f}(\hat{\phi}) = \int \hat{\phi}(u) \, ds(u) = \lim_{\epsilon \to 0} \int \hat{\phi}(u) \frac{s(u+\epsilon) - s(u)}{\epsilon} \, du,$$

where the function $s(u)$ generating the measure ds is the integrated Fourier transform of f mentioned above. The Fourier transformation can be reversed to yield f as a generalized Fourier-Stieltjes transform,

$$f(x) = \frac{1}{2\pi} \int e^{ixu} \, ds(u) = \lim_{\epsilon \to 0} \frac{1}{2\pi} \int e^{ixu} \frac{s(u+\epsilon) - s(u)}{\epsilon} \, du,$$

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where the limit can be taken to be in mean square over each finite interval (cf. Wiener's integration by parts and Cesàro summation procedure [**6**, § 6]). The functions $s(u)$ generating measures ds in $\hat{B}$ are characterized in Theorem 3.1.

The Poisson integral can be used to extend any $f \in B$ to a harmonic function in the half-plane $\mathrm{Re}(z) > 0$, and this fact leads to the definition of a space B_a of analytic functions bearing the same relationship to the Hardy class H^2 of the half-plane as B does to L^2. The main theorems concerning H^2 (see e.g. [**4**]) can be extended to B_a, culminating in a Paley-Wiener type theorem for B_a (Theorem 4.4) to the effect that $f(z) \in B_a$ if and only if

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-zu}\, ds(u)$$

for some $ds \in \hat{B}$ such that $ds = 0$ for $u < 0$.

In § 5 the general theory developed so far is applied to Dirichlet series $\sum a_n e^{-\lambda_n z}$, where $a_n = a(n)$, $\lambda_n = \lambda(n)$ for suitable functions $a(x)$, $\lambda(x)$. A method is given enabling one to show, under certain conditions, that the Dirichlet series represents the sum $f(z) + c(z)$ of a function f, analytic in $\mathrm{Re}(z) > 0$ and belonging to B on each line $\mathrm{Re}(z) = \sigma > 0$, and a function $c(z)$ having singularities perhaps, but expressible as the limit of a sequence of Fourier transforms of known functions. Furthermore, f is almost periodic in the sense of Besicovitch on each line $\mathrm{Re}(z) = \sigma > 0$. These results are given in Theorem 5.1 and its corollary. An application of these results to the Riemann ζ-function yields information on it generalizing a result due to Hardy and Littlewood.

This paper provides the background theory for certain results concerning the closure in B of trigonometric polynomials, which I intend to publish at a later date. These results lead to new characterizations of Besicovitch almost-periodic functions, and to questions concerning ergodic properties of functions in B. The theory so developed should also provide a natural setting for a theory of harmonic analysis of weighted sequences, as introduced by Guinand in [**3**].

2. Some properties of the Banach spaces A and B. We require the following from Beurling's paper [**1**]. Let Ω denote the set of strictly positive summable functions $\omega(x)$ which are non-increasing functions of $|x|$ and for which

$$\omega(0) = \lim_{x \to 0} \omega(x) < \infty.$$

Let $N(\omega) = \omega(0) + \int \omega\, dx$. If ω_1 and ω_2 belong to Ω, then the sum $\omega_1 + \omega_2$ and the convolution $\omega_1 * \omega_2$ also belong to Ω, and $N(\omega_1 + \omega_2) \leqq N(\omega_1) + N(\omega_2)$ and $N(\omega_1 * \omega_2) \leqq N(\omega_1)N(\omega_2)$ hold true. (Ω is the family Ω_1 defined in [**1**, p. 9].) We denote by Ω_0 the subset of Ω consisting of those ω such that $N(\omega) = 1$.

With Ω we associate the set A of functions ϕ satisfying the condition that

$$\phi \in L^2\left(\frac{1}{\omega}\,dx\right)$$

for at least one $\omega \in \Omega$, and we set

$$(2.1) \qquad ||\phi|| = ||\phi||_A = \inf_{\omega\in\Omega_0} \left\{ \int \frac{|\phi(x)|^2}{\omega(x)}\,dx \right\}^{1/2}.$$

A is a Banach algebra under ordinary addition and convolution with the norm (2.1), and $||\phi_1 * \phi_2|| \leqq ||\phi_1||\,||\phi_2||$ [**1**, Theorem I].

We let A^* denote the set of functions f such that for all $\omega \in \Omega$ we have $f \in L^2(\omega dx)$, and we set

$$(2.2) \qquad ||f|| = ||f||_{A^*} = \sup_{\omega\in\Omega_0} \left\{ \int |f(x)|^2 \omega(x)\,dx \right\}^{1/2}.$$

Under this norm A^* is a Banach space and is the dual of A in the sense that each linear functional $F(\phi)$ on A has the form

$$F(\phi) = \int f(x)\phi(x)\,dx$$

for a unique element $f \in A^*$, and conversely each $f \in A^*$ yields in this manner a functional on A, and

$$\sup_{||\phi||=1} \int f(x)\phi(x)\,dx = ||f||_{A^*}.$$

Then Beurling proved that the space A^* is identical with the space B defined in the introduction, and $||\ ||_{A^*} = ||\ ||_B$ [**1**, Theorem II].

A is a subset of L^1 and thus the Fourier transform

$$\hat{\phi}(x) = \int_{-\infty}^{\infty} \phi(t)e^{-itx}\,dt$$

exists for all $\phi \in A$, so that A is mapped thereby onto a subset of the continuous functions vanishing at infinity, which becomes a Banach space when equipped with the norm $||\hat{\phi}||_{\hat{A}} = ||\phi||_A$. The members of $\hat{A}$ belong to L^2 and satisfy a sort of smoothness condition [**1**, Theorems VIII and III] which yields a quantity equivalent to the norm of $\hat{A}$, but which will not be used in the remainder of the paper.

It should be noted that any $\omega \in \Omega$ also belongs to A and that it is an immediate consequence of the definition of the norm of A that $||\omega||_A \leqq \int \omega\,dx < N(\omega)$ if $\omega \neq 0$.

The following theorem shows the relationship between certain types of convergence which will be considered in B, all of which are weaker than convergence in norm.

THEOREM 2.1. *Let f_n be a sequence in B. Consider the statements that f_n be a Cauchy sequence in* (a) *the weak-star topology* (A *topology*) *of B;* (b) $L^2(\omega dt)$

for each $\omega \in \Omega$; (c) $L^2(xdt/(x^2+t^2))$ *for* $x>0$; (d) $L^2(-T, T)$ *for each* $T>0$. *Then* (a) $\Leftarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d). *If* $\{f_n\}$ *is a bounded set in* B, *then* (b), (c), *and* (d) *are equivalent. Furthermore, if* f_n *is bounded, and is a Cauchy sequence in any of* (a)–(d), *then there exists an* $f \in B$ *toward which* f_n *converges in the respective topology.*

Proof. If $\phi \in A$ and $f \in B$ and ω is an element of Ω such that $\phi \in L^2((1/\omega)\,dx)$, then Hölder's inequality can be written as

$$|\textstyle\int f(x)\phi(x)\,dx\,|^2 \leqq \{\int |f|^2\omega\,dx\}\{\int |\phi|^2/\omega\,dx\}. \tag{2.3}$$

Thus if $f_n - f_m \to 0$ as $n, m \to \infty$ in $L^2(\omega\,dx)$, it follows that $|\int f_n\phi\,dx - \int f_m\phi\,dx| \to 0$ also, which proves that (b) $\Rightarrow$ (a). (b) $\Rightarrow$ (c) since $x/(x^2+t^2) \in \Omega$ for $x>0$. That (c) $\Rightarrow$ (d) is evident.

Now suppose that f_n is bounded in B. We shall prove that then (d) $\Rightarrow$ (b). If (d) is true, there exists a function f such that $f_n \to f$ in L^2 on each interval $(-T, T)$. Furthermore, $f \in B$ since

$$\frac{1}{1+2T}\int_{-T}^{T} |f_n|^2\,dt \to \frac{1}{1+2T}\int_{-T}^{T} |f(t)|^2\,dt$$

and so

$$||f|| \leqq \limsup_{n\to\infty} ||f_n||.$$

We can assume without restriction that $f=0$, so that we must prove that $f_n \to 0$ in $L^2(\omega\,dt)$ for each $\omega \in \Omega$. Suppose on the contrary that there is an $\eta > 0$ such that $\int |f_n|^2\omega\,dt > 2\eta$ for an infinite number of values of n, where ω is a fixed member of Ω. This means, in view of our assumption (d), that, given any $T>0$, we can find an $n(T)$ such that

$$\int_{|t|\geqq T} |f_{n(T)}(t)|^2\omega(t)\,dt > \eta. \tag{2.4}$$

We shall show that this implies the unboundedness of $\{f_n\}$ in B.

For each T we construct an $\omega_T \in \Omega_0$ by putting $\omega_T(t) = C_1$ for $|t| \leqq T$ and $\omega_T(t) = C_2\omega(t)$ for $t \geqq T$. The constants C_1 and C_2 are chosen so that $C_1 = C_2\omega(T)$, which ensures that ω_T is non-increasing, and then C_2 is chosen so that $N(\omega_T) = 1$. This implies that

$$C_2 = C_2(T) = [(1+2T)\omega(T) + \textstyle\int_{|t|\geqq T}\omega(t)\,dt]^{-1}.$$

We also have

$$\begin{aligned}\textstyle\int |f_{n(T)}|^2\omega_T\,dt &> C_2\textstyle\int_{|t|\geqq T}|f_{n(T)}|^2\omega\,dt\\ &> C_2(T)\eta \quad \text{by (2.4)}.\end{aligned}$$

Since $\omega_T \in \Omega_0$, this means that $||f_{n(T)}|| > C_2(T)\eta$. But since $\omega(t)$ is summable, it must be true that $\omega(t) = o(1/t)$ for at least a sequence of values $t_j \to \infty$. But by the formula for $C_2(T)$ this means that $C_2(t_j) \to \infty$, which shows that $f_{n(t_j)}$ is unbounded as $t_j \to \infty$.

This contradiction then shows that if f_n is bounded and is a Cauchy sequence under any of the topologies in (b), (c) or (d), then there exists an $f \in B$ toward which f_n converges in any of the topologies (a)–(d). However, by a theorem of Alaoglu [**2**, Theorem V.4.2] the closed unit sphere in B is compact in the A topology of B. Thus if f_n is a bounded set under the norm of B and a Cauchy sequence in the weak-star topology, there exists an $f \in B$ such that $f_n \to f$ in the weak-star topology of B.

The following two theorems concern the existence of approximate identities in B, and will be useful at various stages in the theory. Theorem 2.3 will be central in the discussion of the extension of B to the half-plane.

THEOREM 2.2. *Let Ω^1 denote the subset of Ω such that $N^1(\omega) = \int \omega\, dx \leqq 1$ if $\omega \in \Omega^1$. If $f \in B$, $\phi \in A$, then $f * \phi(x) = \int f(x+t)\phi(t)\, dt$ belongs to B, and*

$$(2.5) \qquad \begin{aligned} \|f * \phi\|_B &\leqq \|f\| \inf_{\omega\in\Omega^1} \int |\phi|^2/\omega\, dx \\ &\leqq \|f\|_B \|\phi\|_A. \end{aligned}$$

Proof. If ω is an element of Ω such that $\phi \in L^2((1/\omega)\, dt)$, then

$$(2.6) \qquad \begin{aligned} \int |f * \phi(x)|^2 \omega_1(x)\, dx &\leqq \int dx\, \omega_1(x) \{\int |f(x+t)|^2 \omega(t)\, dt\} \{\int |\phi|^2/\omega\, dt\} \\ &= \{\int dt\, |f(t)|^2 \int \omega_1(x)\omega(t-x)\, dx\} \{\int |\phi|^2/\omega\, dt\}, \end{aligned}$$

the last step being provided by the Fubini theorem. Now if $\omega_1 \in \Omega$, and $\omega_2 = \omega_1 * \omega$, then $N(\omega_2) < \omega_1(0) \int \omega\, dx + \int \omega_1\, dx \int \omega\, dx = N(\omega_1)N^1(\omega)$. Thus if $\omega_1 \in \Omega_0$ and $N^1(\omega) = 1$, then $N(\omega_1 * \omega) \leqq 1$. If on the right side of (2.6) we take the infimum over $\omega \in \Omega^1$, and then the supremum on both sides over $\omega_1 \in \Omega_0$, we obtain

$$\|f * \phi\|_B \leqq \|f\|_B \inf_{\omega\in\Omega^1} \{\int |\phi|^2/\omega\, dt\}.$$

The second inequality of (2.5) follows from the fact that $\Omega_0 \subset \Omega^1$.

THEOREM 2.3.

$$P_x(t) = \frac{1}{\pi} \frac{x}{x^2 + t^2}$$

is an approximate identity for B as $x \to 0$ in each of the topologies listed in Theorem 2.1, in the sense that if $f_x = f * P_x$, then $f_x \to f$ as $x \to 0$. Furthermore, $\|f_x\| \leqq \|f\|$.

Proof. In view of Theorem 2.1 we need only prove that $\{f_x\}$ is a bounded set in B and $f_x \to f$ in $L^2(-T, T)$ for each $T > 0$. (It is clear that the integer n can be replaced by a real continuous parameter x.) Since $\int P_x(t)\, dt = 1$ and $P_x \in \Omega$, we have by Theorem 2.2 that $f_x \in B$ and $\|f_x\| \leqq \|f\|$. Thus f_x is bounded in B.

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Since $f_x - f = \int [f(y + t) - f(t)]P_x(y)\, dy$, an application of the Hölder inequality and then the Fubini theorem yields

$$\int_{-T}^{T} |f_x - f|^2\, dt \leqq \int dy\, P_x(y) \int_{-T}^{T} |f(y + t) - f(t)|^2\, dt. \tag{2.7}$$

We now divide the range of integration over y into the two ranges $|y| \leqq \delta$ and $|y| \geqq \delta$. Given any $\epsilon > 0$ we can find a $\delta > 0$ such that

$$\int_{-T}^{T} |f(y + t) - f(t)|^2\, dt < \epsilon$$

if $|y| < \delta$ since functions in $L^2(-T, T)$ are continuous under translation. Hence for all x,

$$\int_{|y|<\delta} dy P_x(y) \int |f(y + t) - f(t)|^2\, dt < \epsilon. \tag{2.8}$$

For the range $|y| \geqq \delta$, we note that $|f(y + t) - f(t)|^2 \leqq 2[|f(y + t)|^2 + |f(t)|^2]$ and that $\int_{|y|>\delta} P_x(y)\, dy \to 0$ as $x \to 0$. Thus it only remains to show that

$$\int_{|y|\geqq\delta} dy P_x(y) \int_{-T}^{T} |f(y + t)|^2\, dt \to 0 \quad \text{as } x \to 0.$$

In order to deal with this double integral let us put $R_x(y)$ equal to $P_x(y)$ for $|y| \geqq \delta$ and equal to $P_x(\delta)$ for $|y| < \delta$. Then $R_x \in \Omega$ and

$$N(R_x) = (1 + 2\delta)P_x(\delta) + \int_{|y|\geqq\delta} P_x(y)\, dy,$$

which tends to zero as $x \to 0$. It follows then that the double integral is less than

$$\int_{-T}^{T} dt \int R_x(y)|f(y + t)|^2\, dt \leqslant N(R_x) \int_{-T}^{T} ||f_t||\, dt$$
$$\to 0 \quad \text{as } x \to 0.$$

This fact, taken together with (2.8), shows that the right side of (2.7) is less than ϵ for sufficiently large x. Thus

$$\limsup_{x\to 0} \int_{-T}^{T} |f_x - f|^2\, dt < \epsilon,$$

and since ϵ was arbitrarily chosen, the theorem follows.

3. Basic properties of $\hat{B}$. Let $\mathscr{F}(\phi) = \hat{\phi}$ denote the Fourier transform of a function $\phi \in A$ as defined in § 2. If $\hat{A} = \mathscr{F}(A)$ and $\hat{A}$ is given the norm $||\hat{\phi}|| = ||\phi||$, then $\mathscr{F}$ is an isometric isomorphism from A onto the Banach space $\hat{A}$. Then the equation $\hat{f}(\hat{\phi}) = f(\phi)$ defines an isometric isomorphism $\mathscr{F}$ between the dual B of A and the dual $\hat{B}$ of $\hat{A}$, and $\hat{f}$ will be called the Fourier transform of f (we are denoting both the linear functional on A and the function in B representing it by the same symbol f).

We can now proceed to characterize the elements of $\hat{B}$ as measures.

If $f \in B$, then

$$f \in L^2\left(\frac{dt}{1 + t^2}\right) \quad \text{since } \frac{1}{1 + t^2} \in \Omega.$$

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Thus by the Plancherel theorem,

$$(3.1)\qquad s(u) = \lim_{X\to\infty}\left(\int_{-X}^{-1} + \int_{1}^{X}\right)\frac{f(t)}{it}e^{iut}\,dt + \int_{-1}^{1} f(t)\,\frac{e^{iut}-1}{it}\,dt$$

exists for almost all u, and s is locally in L^2. Furthermore,

$$(3.2)\qquad \frac{1}{\epsilon}[s(u+\epsilon) - s(u)] = \underset{X\to\infty}{\text{l.i.m.}}\int_{-X}^{X} f(t)\,\frac{e^{i\epsilon t}-1}{i\epsilon t}\,e^{iut}\,dt$$

if $\epsilon \neq 0$, where l.i.m. denotes convergence in $L^2(-\infty,\infty)$, since $f_\epsilon(x) = f(x)[e^{i\epsilon x} - 1]/i\epsilon x$ belongs to $L^2(-\infty,\infty)$. Since any ϕ in A belongs to L^2, the Parseval equation yields

$$(3.3)\qquad \int f_\epsilon(x)\,\phi(x)\,dx = \frac{1}{2\pi}\int \hat{\phi}(u)\,\frac{s(u+\epsilon)-s(u)}{\epsilon}\,du.$$

Clearly $||f_\epsilon|| \leqq ||f||$ so that f_ϵ is a bounded set in B, and furthermore $f_\epsilon \to f$ in $L^2(-T, T)$ as $\epsilon \to 0$. Thus by Theorem 2.1, the left side of (3.3) tends to $f(\phi)$, and

$$(3.4)\qquad \hat{f}(\hat{\phi}) = \lim_{\epsilon\to 0}\frac{1}{2\pi}\int \hat{\phi}(u)\,\frac{s(u+\epsilon)-s(u)}{\epsilon}\,du.$$

The relation (3.4) justifies our saying that $\hat{f}$ is represented by the measure ds on $\hat{A}$ generated by the function $s(u)$. For (3.4) we also write

$$\hat{f}(\hat{\phi}) = \frac{1}{2\pi}\int \hat{\phi}\,ds.$$

A similar expression can be given for the inverse Fourier transform from $\hat{B}$ to B. By the Plancherel theorem, for each $\epsilon \neq 0$,

$$(3.5)\qquad f_\epsilon(x) = \underset{X\to\infty}{\text{l.i.m.}}\frac{1}{2\pi}\int_{-X}^{X}\frac{s(u+\epsilon)-s(u)}{\epsilon}\,e^{-iux}\,du.$$

$f_\epsilon \to f$ in $L^2(-T, T)$ for each $T > 0$ and $\{f_\epsilon\}$ is bounded in B, so that we have

$$(3.6)\qquad f(x) = \underset{\epsilon\to 0}{\text{Lim}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{s(u+\epsilon)-s(u)}{\epsilon}\,e^{-iux}\,du$$

where Lim means convergence in any one of the four types of convergence given in Theorem 2.1. For (3.6) we shall write

$$f(x) = \frac{1}{2\pi}\int e^{-ixu}\,ds(u).$$

In order to characterize those functions $s(u)$ generating measures in $\hat{B}$ we require the following result.

LEMMA 3.1. *If f is a function locally in L^2, then f belongs to B if and only if*

$$(3.7)\qquad S(f) = \sup_{0<\mu<\frac{1}{2}}\frac{1}{\mu}\int |f(t)|^2\,\frac{\sin^2\mu t}{t^2}\,dt < \infty.$$

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Furthermore, there exist constants k_1 and k_2, independent of f such that

$$k_1||f|| \leqq S(f) \leqq k_2||f||. \tag{3.8}$$

Proof. A straightforward integration by parts yields the identity

$$\frac{1}{\mu}\int\left|f(x)\frac{\sin \mu x}{x}\right|^2 dx = \frac{1}{\mu}\int_{-1}^{1}\left|f(x)\frac{\sin \mu x}{x}\right|^2 dx \tag{3.9}$$
$$+\frac{4}{\mu}\int_1^\infty dx\,\frac{1+2x}{x^3}\cdot\frac{1}{1+2x}\int_{-x}^{x}|f(t)\sin \mu t|^2\,dt.$$

Letting $||f|| = M$, the first expression on the right side is at most $(3/2)M$ if $0 < \mu < \frac{1}{2}$ since $\sin^2 \mu x \leqq \mu^2x^2$. In order to estimate the second expression we divide the range of integration into the ranges from 1 to $1/\mu$ and from $1/\mu$ to ∞. For the first range we use the approximation $\sin^2 \mu t \leqq \mu^2x^2$, valid for $|t| \leqq |x|$, and obtain

$$\frac{1}{\mu}\int_1^{1/\mu} dx\,\frac{1+2x}{x^3}\cdot\frac{1}{1+2x}\int_{-x}^{x}|f(t)\sin \mu t|^2\,dt \leqq M\mu\int_1^{1/\mu}(2+1/x)\,dx$$
$$< 3M.$$

For the range of integration from $1/\mu$ to ∞ we simply use the approximation $|\sin \mu t| \leqq 1$, and thereby obtain a similar result with upper bound $3M$. Thus we can take k_2 to be 8.

In order to prove the converse, suppose that for some T^1,

$$\int_{-T^1}^{T^1}|f(t)|^2\,dt > (1+2T^1)M,$$

where $M > 0$. We can assume that $T^1 \geqq 2$, since if $\int_{-T}^{T}|f(t)|^2\,dt > M(1+2T)$ for $T < 2$ then $\int_{-2}^{2}|f(t)|^2\,dt > M$ so that we could take $M/5$ instead of M. If $\mu = \pi/4T^1$, then $0 < \mu < \frac{1}{2}$ and $\sin^2 \mu t \geqq \frac{1}{2}$ for $T^1 \leqq t \leqq 2T^1$. Also because of the definition of T^1,

$$\frac{1}{1+2T}\int_{-T}^{T}|f(t)|^2\,dt > M/2 \qquad \text{for } T^1 \leqq T \leqq 2T^1.$$

Thus the second expression on the right side of (3.9) is greater than

$$\frac{T^1}{\pi}\int_{T^1}^{2T^1}\left(\frac{1}{x^3}+\frac{2}{x^2}\right)\cdot\frac{M}{2}\,dx \geqq \frac{M}{2\pi}.$$

As a result, if the mean square of f is unbounded, then $S(f) = \infty$, and furthermore we can take k_1 to be $1/10\pi$.

THEOREM 3.1. *A function $s(u)$ generates a measure ds in $\hat{B}$ if and only if it is locally in L^2 and satisfies*

$$\mathcal{N}(s) = \sup_{0<\epsilon<1}\frac{1}{\epsilon}\int_{-\infty}^{\infty}|s(u+\epsilon)-s(u)|^2\,du < \infty. \tag{3.10}$$

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Under these conditions s defines a member $\hat{f}$ of $\hat{B}$ through equation (3.4), *corresponding to a function f in B, say, defined by* (3.6). *Furthermore, there exist constants c_1 and c_2, independent of s, such that*

$$c_1||ds|| \leqq \mathcal{N}^{\frac{1}{2}}(s) \leqq c_2||ds||, \tag{3.11}$$

so that $\mathcal{N}^{\frac{1}{2}}$ is equivalent to the norm of $\hat{B}$.

Proof. We have just seen that any $\hat{f} \in \hat{B}$ can be represented by equation (3.4) for some function $s(u)$ which is locally in L^2. $[s(-u + \epsilon) - s(-u)]/\epsilon$ is the Fourier transform of f_ϵ, and by the Parseval equation,

$$\int |f_\epsilon|^2\, dx = \frac{1}{2\pi\epsilon^2} \int |s(u - \epsilon) - s(u)|^2\, du.$$

But $|e^{i\epsilon x} - 1|^2 = 4 \sin^2 \frac{1}{2}\epsilon x$ so that we can rewrite this equation as

$$\frac{1}{\epsilon} \int |f(x)|^2 \frac{\sin^2 \frac{1}{2}\epsilon x}{x^2}\, dx = \frac{1}{8\pi\epsilon} \int |s(u + \epsilon) - s(u)|^2\, du. \tag{3.12}$$

By Lemma 3.1, the left side of this equation is bounded for $0 < \epsilon < 1$ (take $\mu = \epsilon/2$), and we obtain (3.10) as a result. The double inequality (3.11) follows directly from (3.8) and (3.12) by taking $c_j = 4\pi k_j$.

Conversely, suppose that s is a function locally in L^2 satisfying (3.10). Then by the Plancherel theorem we can define a function $f(x, \epsilon)$, such that $f(x, \epsilon)$ is locally in L^2 for each $\epsilon > 0$ and $f(x, \epsilon)[(e^{i\epsilon x} - 1)/ix]$ is the inverse Fourier transform of $[s(-u + \epsilon) - s(-u)]$. Using the identity

$$\begin{aligned}[s(t + \epsilon + \epsilon^1) - s(t + \epsilon)] - [s(t + \epsilon^1) - s(t)]&\\ = [s(t + \epsilon + \epsilon^1) - s(t + \epsilon^1)] &- [s(t + \epsilon) - s(t)]\end{aligned}$$

valid for all ϵ, ϵ^1, t, we obtain on taking transforms that

$$(e^{i\epsilon x} - 1)(e^{i\epsilon^1 x} - 1)f(x, \epsilon^1) = (e^{i\epsilon^1 x} - 1)(e^{i\epsilon x} - 1)f(x, \epsilon)$$

almost everywhere. But this implies that $f(x, \epsilon^1) = f(x, \epsilon) = f(x)$ for almost all x. By the Parseval theorem,

$$\frac{1}{2\pi} \int |s(u + \epsilon) - s(u)|^2\, du = \int |f(x)|^2 \frac{\sin^2 \frac{1}{2}\epsilon x}{x^2}\, dx.$$

Since s satisfies (3.10), Lemma 3.1 then implies that $f \in B$. It follows as before that s generates a measure in $\hat{B}$ defined by (3.4), and that f can be represented by (3.6).

It is perhaps of interest to state (without proof) the corresponding results for Beurling's Banach space B^2 consisting of functions f satisfying

$$\sup_{T>0} \frac{1}{2T} \int_{-T}^{T} |f|^2\, dt < \infty \tag{3.13}$$

with a corresponding change in the norm. B^2 is the dual of the Banach algebra

A^2 which differs from A ($\mathscr{A}^2$ in Beurling's notation) in that Ω is replaced by the set of positive non-increasing functions in $|x|$ satisfying $N(\omega) = \int \omega\, dx < \infty$. With a bit more care one can prove a lemma corresponding to Lemma 3.1 showing that (3.13) is true if and only if (3.7) is true with the supremum over $0 < \mu < \frac{1}{2}$ replaced by the supremum over $0 < \mu < \infty$. The theory of this section can be modified to characterize $\hat{B}^2$ as a set of measures generated by functions s satisfying (3.10) with the range $0 < \epsilon < 1$ replaced by $0 < \epsilon < \infty$. Relation (3.3), for example, has to be modified slightly since functions in A^2 are not necessarily in L^2. However, A is dense in A^2 and one can replace (3.3) by a similar relation involving a limiting procedure over ϕ.

For later reference we should note the condition under which two functions s and s_1 generate the same measure in B, or equivalently, when s generates the null measure. By Theorem 3.1 this is true if and only if $\mathscr{N}(s) = 0$, and this means that $s(u + \epsilon) - s(u) = 0$ a.e. for each ϵ, $0 < \epsilon < 1$. If $S(u) = \int_0^u s(t)\, dt$, this condition implies that $\int_a^u [s(t + \epsilon) - s(t)]dt = 0$, or

$$S(u + \epsilon) - S(u) = S(a + \epsilon) - S(a). \tag{3.14}$$

By a well-known theorem for the differentiation of the indefinite integral of a Lebesgue-integrable function, $S'(u)$ exists and equals $s(u)$ almost everywhere. Taking a in (3.14) to be a point such that $S'(a) = s(a)$, we conclude that $S'(u) = s(a)$ a.e., that is, $s(u) = s(a)$ a.e. It follows that if $ds = 0$, then $s(u)$ is a constant almost everywhere, and the converse is obviously true. Thus $ds = ds_1$ if and only if s and s_1 differ by a constant almost everywhere.

4. Extension of B to the half-plane. The basic theory for the extension of B to the right half-plane has been furnished by Theorem 2.3. As has already been observed, any $F \in B$ belongs to $L^2(dt/(1 + t^2))$, and the standard theory of the Poisson integral for the half-plane (see Hoffman [**4**, p. 123]) ensures that

$$f(x + iy) = F * P_x = \frac{1}{\pi} \int F(t + y) \frac{x}{x^2 + t^2}\, dt$$

is a harmonic function of $z = x + iy$ for $x > 0$. Our first theorem, then, is really a restatement of Theorem 2.3.

THEOREM 4.1. *Let F be any member of B and put $f(x + iy) = F * P_x(y)$. Then $f(x + iy)$ is a harmonic function of $z = x + iy$ for $x > 0$, and $f_x(y) = f(x + iy) \in B$ for $x > 0$. Furthermore, $||f_x||_B \leqq ||F||$ and $f_x \to F$ in any one of the weak topologies considered in Theorem* 2.1, *and in particular, in the weak-star topology of B.*

In general, f_x does not converge to F in the norm topology of B. For this a further condition is required, namely that F should be continuous under translation, or more precisely, that if $F_y(t) = f(t + y)$, then $||F_y - F|| \to 0$

as $y \to 0$. In the general case, of course, we always have $||F_y|| \to ||F||$. This follows from the estimate

$$(4.1) \qquad ||F_y|| \leqq (1 + 2|y|)^{\frac{1}{2}}||F||.$$

To prove this one can use (1.1) for defining the norm of B:

$$\frac{1}{1+2T}\int_{-T}^{T}|F(y+t)|^2\,dt = \frac{1}{1+2T}\int_{-T+y}^{T+y}|F(t)|^2\,dt$$
$$\leqq \left(1+\frac{2y}{1+2T}\right)\frac{1}{1+2(T+y)}\int_{-T-y}^{T+y}|F(t)|^2\,dt$$
$$\text{if } y > 0.$$

From this, (4.1) follows immediately. That $||F_y|| \to ||F||$ results from (4.1) and a reverse inequality obtained by substituting F_{-y} for F in (4.1). We shall now use (4.1) to prove the following theorem.

THEOREM 4.2. *If $F \in B$ is continuous under translation in the norm topology of B, and f is the Poisson integral of F, then $f(x + iy) \to F(y)$ as $x \to 0$ in the norm topology of B.*

Proof. Since $F - f_x = \int[F(y) - F(t + y)]P_x(t)\,dt$, the Fubini theorem gives, for any $\phi \in A$,

$$(4.2) \qquad \int[F(y) - f(x + iy)]\phi(y)\,dy = \int dt\,P_x(t)\int[F(y) - F(t + y)]\phi(y)\,dy.$$

In functional notation the second integral in the expression on the right side is $(F - F_t)(\phi)$, the left side is $(F - f_x)(\phi)$, and if we take the absolute values we obtain

$$|(F - f_x)(\phi)| \leqq \int dt\,P_x(t)|(F - F_t)(\phi)|$$
$$\leqq ||\phi||\int||F - F_t||P_x(t)\,dt,$$

where we have used the fact that $P_x \geqq 0$. On taking the supremum over $||\phi|| = 1$ we obtain

$$(4.3) \qquad ||F - f_x|| \leqq \int||F - F_t||P_x(t)\,dt.$$

Given any $\epsilon > 0$, we can find a $\delta > 0$ such that $||F - F_t|| < \epsilon$ if $|t| < \delta$. Since $\int P_x\,dt = 1$, we have then that $\int_{-\delta}^{\delta}||F - F_t||P_x(t)\,dt < \epsilon$. For the range $|t| > \delta$ we note that $\int_{|t|>\delta}(1 + 2|t|)^{\frac{1}{2}}P_x(t)\,dt \to 0$ as $x \to 0$. Combining these with (4.1) in (4.3) results in

$$\limsup_{x\to 0}||F - f_x|| \leqq \epsilon.$$

Since ϵ was arbitrarily chosen, the theorem follows.

We shall now introduce a class of functions analytic in the half-plane which bear the same relationship to B as the Hardy class H^2 in the half-plane does to L^2.

Definition 4.1. *A function* $f(z)$ *is said to belong to* B_a *if it is analytic for* $\operatorname{Re}(z) > 0$, $f_x = f(x + iy)$ *belongs to* B *for each* $x > 0$, *and* $\sup_{x>0} \|f_x\| < \infty$.

We shall characterize B_a in much the same way that H^2 is characterized, by reducing B_a to H^2 as follows. If $f(z) \in B_a$ then $f_x \in L^2(dy/(1 + y^2))$ and f_x is bounded in the $L^2(dy/(1 + y^2))$ norm for $x > 0$. As a result, $F(z) = f(x)/(1 + z)$ belongs to H^2. Then by one of the theorems for H^2 (see [**4**, p. 128]):

$F(z)$ has non-tangential limits at almost every point of the imaginary axis, say $F(iy)$ and this function belongs to L^2. Furthermore, $F(x + iy) \to F(iy)$ in L^2 as $x \to 0$.

But this implies that $\lim_{x\to 0} f(x + iy) = f(iy)$ exists for almost all y and that $f_x \to f(iy)$ in $L^2(dy/(1 + y^2))$. Since f_x is bounded in the norm of B for $x > 0$, we know by Theorem 2.1 that $f(iy) \in B$ and that $f_x \to f$ in any one of the topologies of that theorem. We have proved the following result.

THEOREM 4.3. *Let $f \in B_a$. Then f has non-tangential limits almost everywhere on the imaginary axis, defining a function $f(iy)$ belonging to B. Furthermore, $f_x \to f(iy)$ in any one of the topologies considered in Theorem* 2.1.

It is also true that $f(z)$ is the Poisson integral of $f(iy)$, but we shall prove this later as a corollary of the next theorem.

THEOREM 4.4 (*The Paley-Wiener theorem for* B_a). *$f(z)$ belongs to B_a if and only if*

$$f(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-zu}\, ds(u) \tag{4.4}$$

for some $ds \in \hat{B}$ such that $ds = 0$ for $u < 0$. This representation is unique.

Proof. First suppose that $f(z) \in B_a$. If $\epsilon > 0$, then $F_\epsilon(z) = f(z)[e^{-\epsilon z} - 1]/z$ belongs to H^2 since f_x is bounded in $L^2(dy/(1 + y^2))$ for $x > 0$. By the Paley-Wiener theorem for the half-plane,

$$F_\epsilon(iy) = \lim_{x\to 0} F_\epsilon(x + iy)$$

exists almost everywhere, $F_\epsilon(iy) \in L^2$ and

$$\hat{F}_\epsilon(u) = \int F_\epsilon(iy) e^{iuy}\, dy = 0 \quad \text{a.e. for } u < 0. \tag{4.5}$$

If $ds \in \hat{B}$ corresponds to

$$f(iy) = \lim_{x\to 0} f(x + iy) \in B,$$

then (4.5) implies that $s(u - \epsilon) - s(u) = 0$ a.e. for $u < 0$ (cf. (3.2)). But this means that $ds = 0$ for $u < 0$.

Furthermore, by the Paley-Wiener theorem,

$$F_\epsilon(z) = \frac{1}{2\pi} \int_0^{\infty} e^{-zu} \hat{F}_\epsilon(u)\, du,$$

so that

$$f(z) = -\lim_{\epsilon\to 0+} \frac{1}{\epsilon} F_\epsilon(z)$$

$$= \lim_{\epsilon\to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-zu} \frac{s(u) - s(u-\epsilon)}{\epsilon} du$$

which is (4.4). The uniqueness of ds follows from the uniqueness of $f(iy)$.

Conversely, let $ds = 0$ for $u < 0$, where ds belongs to $\hat{B}$ and corresponds to f_0 in B. For $x > 0$ put $\phi_z(u) = e^{-x|u|}e^{-iyu}$. Then $\mathscr{F}^{-1}(\phi_z) = P_x(t-y) \in A$, so that $\phi_z \in \hat{A}$ and

$$f(z) = \frac{1}{2\pi} \int \phi_z(u)\, ds(u) \tag{4.6}$$

$$= \frac{1}{2\pi} \int e^{-zu}\, ds$$

exists for each z, $\mathrm{Re}(z) > 0$. But (4.6) can also be written as

$$f(z) = \int f_0(t) P_x(t-y)\, dt = f_0 * P_x,$$

the Poisson integral of f_0 so that by Theorem 4.1, $||f_x||$ is bounded for $x > 0$ and $||f_x|| \leqq ||f_0||$. Thus to prove that $f \in B_a$, we need only prove that f is analytic.

By the Paley-Weiner theorem for H^2, each

$$F_\epsilon(z) = \frac{1}{2\pi\epsilon} \int e^{-zu}[s(u) - s(u-\epsilon)]\, du$$

is analytic for $\mathrm{Re}(z) > 0$, since $s(u) - s(u-\epsilon) = 0$ a.e. for $u < 0$. But $F_\epsilon(z) \to f(z)$ uniformly in z on each compact set in $\mathrm{Re}(z) > 0$. (This is most easily seen by noting that $f^\epsilon = f_0(t)[e^{-i\epsilon t} - 1]/i\epsilon t \to -f_0$ boundedly and in each $L^2(-T, T)$, hence in $L^2(dt/(1+t^2))$ as $\epsilon \to 0$, and that F_ϵ and f are the Poisson integrals of f^ϵ and f_0, respectively). Thus $f(z)$ is analytic in $\mathrm{Re}(z) > 0$, which completes the proof.

COROLLARY. *If $f \in B_a$, then f is the Poisson integral of its boundary values $f(iy)$.*

For by the necessary part of the above theorem the Fourier transform ds of $f(iy)$ is zero for $u < 0$, and

$$f(z) = \frac{1}{2\pi} \int e^{-zu}\, ds.$$

But by the proof of sufficiency part of the theorem, the Poisson integral of $f(iy)$ has the same expression.

5. An application to Dirichlet series. By a Dirichlet series is meant a formal series $\sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ involving the complex variable $z = \sigma + iy$, where

$$0 \leqq \lambda_1 < \lambda_2 < \ldots; \qquad \lambda_n \to \infty. \tag{5.1}$$

We pose the following question for such a series:

Under what conditions does the Dirichlet series represent a function F in B (with respect to a sequence $C_N(t)$) for $\sigma = 0$, in the sense that each C_N belongs to B, F belongs to B, and

$$F_N(t) = \sum_{\lambda_n < N} a_n e^{-i\lambda_n t} - C_N(t) \to F(t) \tag{5.2}$$

as $N \to \infty$ under some topology on B?

If each C_N can be chosen so that its Fourier transform vanishes for $u < 0$, then by Theorem 4.4, $F(t)$ can be extended to an analytic function $f(z)$ for $\mathrm{Re}(z) > 0$, and furthermore, f belongs to B_a. In this case each C_N defines an analytic function c_N in the right-hand half-plane, but the sequence c_N need not converge in any sense. In some cases, however, the sequence $\{c_N\}$ can be interpreted as representing the singularities of the Dirichlet series. For example, it can happen that c_N converges to an analytic function $c(z)$ in some half-plane $\mathrm{Re}(z) > \sigma_0 > 0$, which has an analytic continuation to all of $\mathrm{Re}(z) > 0$ with the exception of certain singularities, and the Dirichlet series can be thought of as representing $f(z) + c(z)$ in $\mathrm{Re}(z) > 0$. This behaviour will be illustrated by the example of the Riemann ζ-function for $\mathrm{Re}(z) > \frac{1}{2}$.

We shall consider in this paper only the existence of such an F as defined in (5.2); we shall not concern ourselves with the various topologies under which convergence in (5.2) might take place. Thus only convergence in norm will be considered.

The discussion will now be confined to the following special case of a Dirichlet series. The coefficients will be assumed to have the form $a_n = a(\lambda_n)$, where $a(x)$ is a complex function of the real variable x, continuous everywhere, and differentiable except perhaps at isolated points. This ensures that for all x and y we have

$$|a(x) - a(y)| < \sup_{x < \zeta \leqq y} |a'(\zeta)|\,|x - y|.$$

We suppose also that $\lambda_n = \lambda(n)$ where λ is a real non-decreasing function of x. Its inverse function $l(x)$ is assumed to be strictly increasing with a bounded derivative $l'(x)$ such that l is the integral of l'.

The Dirichlet condition (5.1) will take on significance in the following way:

(*) *For each $\epsilon > 0$ we can find an $N_\epsilon > 0$ such that $\lambda_{n+1} - \lambda_n > \epsilon$ for $\lambda_n < N_\epsilon$, and $N_\epsilon \to \infty$ as $\epsilon \to 0$.*

With these conventions for a_n and λ_n we can now state the following result.

THEOREM 5.1. *Let $a(x) \in L^2(l'(x)\,dx)$ and suppose that $a^* \in L^2(0, \infty)$ where*

$$a^*(x) = \sup_{\lambda_n \leqq y < \lambda_{n+1}} |a'(y)|$$

for $\lambda_n \leqq x < \lambda_{n+1}$. Then $ds \in \hat{B}$ where

$$s(u) = \sum_{\lambda_n < u} a(\lambda_n) - \int_0^u a(x) l'(x)\,dx,$$

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and

$$(5.3) \qquad L(s) \equiv \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int |s(u+\epsilon) - s(u)|^2 \, du = \sum |a_n|^2,$$

where the existence of the limit defining $L(s)$ is part of the conclusion, as is the convergence of $\sum |a(\lambda_n)|^2$.

Furthermore, ds can be approximated under the norm of $\hat{B}$ by its truncations ds_N, where $s_N(u) = s(u)$ for $0 \leqq u \leqq N$, and $s_N(u) = s(N)$ for $u > N$.

Proof. In order to find an upper bound for $\mathscr{N}(s)$ (see (3.10)) we shall be required to estimate sums of the form

$$(5.4) \qquad s(q) - s(p) = \sum_{p \leqq \lambda_n < q} a(\lambda_n) - \int_p^q a(x) l'(x) \, dx.$$

Let λ' be the first λ_n greater than or equal to p, and let λ'' be the greatest λ_n which is less than q. Noting that

$$\int_{\lambda_n}^{\lambda_{n+1}} l'(x) \, dx = l(\lambda(n+1)) - l(\lambda(n)) = 1,$$

we can then write

$$(5.5) \quad s(q) - s(p) = \sum_{\lambda' \leqq \lambda_n < \lambda''} \int_{\lambda_n}^{\lambda_{n+1}} \{a(\lambda_n) - a(x)\} l'(x) \, dx - \left(\int_p^{\lambda'} + \int_{\lambda''}^q \right) a(x) l'(x) \, dx.$$

Now

$$|a(\lambda_n) - a(x)| \leqq a^*(\lambda_n)(\lambda_{n+1} - \lambda_n) = \int_{\lambda_n}^{\lambda_{n+1}} a^*(x) \, dx,$$

so that the absolute value of the summation term on the right side of (5.5) is less than

$$(5.6) \qquad \sum \int_{\lambda_n}^{\lambda_{n+1}} a^*(x) \, dx \leqq \int_p^q a^*(x) \, dx \leqq (q-p) \int_p^q |a^*(x)|^2 \, dx,$$

the last step being a result of the Cauchy-Schwarz inequality. The Cauchy-Schwarz inequality applied to the two integrals on the extreme right side of (5.5) yields

$$(5.7) \qquad \left| \int_p^{\lambda'} a(x) l'(x) \, dx \right| \leqq \left\{ \int_p^{\lambda'} |a(x)|^2 l'(x) \, dx \right\}^{\frac{1}{2}} \left\{ \int_p^{\lambda'} l'(x) \, dx \right\}^{\frac{1}{2}} \leqq \left\{ \int_p^q |a(x)|^2 l'(x) \, dx \right\}^{\frac{1}{2}},$$

and the identical estimate for the integral from λ'' to q. If we now set $p = u$, $q = u + \epsilon$, apply the Minkowski inequality to (5.5), and use the upper bounds given by (5.6) and (5.7), we obtain

$$(5.8)\quad \left\{\frac{1}{\epsilon}\int_N^\infty |s(u+\epsilon)-s(u)|^2\,du\right\}^{\frac{1}{2}} \leqq \left\{\frac{1}{\epsilon}\int_N^\infty du\cdot\int_u^{u+\epsilon}|a^*(x)|^2\,dx\right\}^{\frac{1}{2}} + 2\left\{\frac{1}{\epsilon}\int_N^\infty du\int_u^{u+\epsilon}|a(x)|^2 l'(x)\,dx\right\}^{\frac{1}{2}} \leqq \left\{\int_N^\infty |a^*(x)|^2\,dx\right\}^{\frac{1}{2}} + 2\left\{\int_N^\infty |a(x)|^2 l'(x)\,dx\right\}^{\frac{1}{2}}.$$

This last step results from the fact that

$$\int_a^b dx\,\frac{1}{\delta}\int_0^\delta g(x+t)\,dt \leqq \int_a^{b+\delta} g(x)\,dx$$

for any $g(x) \geqq 0$ integrable in $(a, b+\delta)$. That $ds \in \hat{B}$ follows by taking the supremum of (5.8) over $0 < \epsilon < 1$ for $N = 0$.

We now show that $\mathcal{N}(s - s_N) \to 0$ as $N \to \infty$. We have

$$\{s(u+\epsilon) - s_N(u+\epsilon)\} - \{s(u) - s_N(u)\} = \begin{cases} 0 & \text{for } u < N-\epsilon, \\ s(u+\epsilon) - s(u) & \text{for } u \geqq N, \\ s(u+\epsilon) - s(N) & \text{for } N-\epsilon \leqq u < N. \end{cases}$$

Thus

$$(5.9)\quad \mathcal{N}(s - s_N) = \sup_{0<\epsilon<1}\left\{\frac{1}{\epsilon}\int_{N-\epsilon}^N |s(u+\epsilon) - s(N)|^2\,du + \frac{1}{\epsilon}\int_N^\infty |s(u+\epsilon)-s(u)|^2\,du\right\}.$$

The integral over $u \geqq N$ has been dealt with in (5.8). For the integral over $N - \epsilon \leqq u < N$ we can use the methods used in (5.4)–(5.7) (with $p = N$, $q = u + \epsilon$) to obtain

$$(5.10)\quad \left\{\frac{1}{\epsilon}\int_{N-\epsilon}^N |s(u+\epsilon) - s(N)|^2\,du\right\}^{\frac{1}{2}} \leqq \left\{\int_{N-\epsilon}^{N+\epsilon} |a^*(x)|^2\,dx\right\}^{\frac{1}{2}} + 2\left\{\int_{N-\epsilon}^{N+\epsilon} |a(x)|^2 l'(x)\,dx\right\}^{\frac{1}{2}}.$$

That $\mathcal{N}(s - s_N) \to 0$ now follows from the integrability conditions on $a(x)$ and $a^*(x)$ on substituting (5.8) and (5.10) into (5.9).

In order to prove (5.3) we show first that

$$(5.11)\quad L(s_N) = \sum_{\lambda_n < N} |a(\lambda_n)|^2.$$

To this end we use the Dirichlet condition (*), and for any $N > 0$ choose ϵ so small that $N_\epsilon > N$. Then the discontinuous part of s contributes to $s_N(u + \epsilon) - s_N(u)$ precisely $a(\lambda_n)$ for $\lambda_n - \epsilon \leqq u < \lambda_n$, and 0 in the comple-

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mentary intervals, for $u < N - \epsilon$. Separating the continuous and discontinuous parts of s_N, and using the Minkowski inequality, we obtain

$$(5.12)\quad \left|\left\{\frac{1}{\epsilon}\int |s_N(u+\epsilon) - s_N(u)|^2\, du\right\}^{\frac{1}{2}} - \left\{\sum_{\lambda_n < N-\epsilon} |a(\lambda_n)|^2\right\}^{\frac{1}{2}}\right| \leqq \left\{\frac{1}{\epsilon}\int_0^{N-\epsilon} du \left|\int_u^{u+\epsilon} a(x)l'(x)\, dx\right|^2\right\}^{\frac{1}{2}} + R_N,$$

where R_N takes care of the expression for $N - \epsilon \leqq u < N$. But

$$R_N = \left\{\frac{1}{\epsilon}\int_{N-\epsilon}^{N} |s(N) - s(u)|^2\, du\right\}^{\frac{1}{2}},$$

and we can deal with this expression precisely as we treated the expression in the left side of (5.10) to show that $R_N \to 0$ as $\epsilon \to 0$. The integral expression on the right side of (5.12) is less than

$$\int_0^{N-\epsilon} du\, \frac{1}{\epsilon}\int_u^{u+\epsilon} |a(x)l'(x)|^2\, dx \int_u^{u+\epsilon} dx \leqq \epsilon \int_0^N |a(x)l'(x)|^2\, dx.$$

This last integral exists since $a(x)$ is continuous and l' is integrable and bounded. Thus the right side of (5.12) tends to zero as $\epsilon \to 0$, which proves (5.11).

Since $\mathscr{N}(s) < \infty$, and $\mathscr{N}(s - s_N)$ is bounded in N, we know that $\mathscr{N}(s_N) \leqq K < \infty$ for all N, and hence the same upper bound holds for $L(s_N)$. Thus $\sum |a_n|^2 < \infty$. Given any $\eta > 0$, choose N so large that

$$\sum_{\lambda_n \geqq N-1} |a_n|^2 < (\eta/3)^2 \qquad \text{and} \qquad \mathscr{N}(s - s_N) < (\eta/3)^2.$$

Then choose ϵ' so small that the right side of (5.12) is less than $\eta/3$ for all $\epsilon < \epsilon'$. Then for all $\epsilon < \epsilon'$ we have

$$\left|\left\{\frac{1}{\epsilon}\int |s(u+\epsilon) - s(u)|^2\, du\right\}^{\frac{1}{2}} - \left\{\sum |a_n|^2\right\}^{\frac{1}{2}}\right| \leqq \mathscr{N}^{\frac{1}{2}}(s - s_N) + (\text{l.s. of } (5.12)) + \left(\sum_{\lambda_n \geqq N-1} |a_n|^2\right)^{\frac{1}{2}} < \eta/3 + \eta/3 + \eta/3 = \eta.$$

Since η was arbitrarily chosen, this proves the existence of $L(s)$ and equation (5.3).

COROLLARY. *Under the conditions of the theorem,*

$$F_N(t) = \sum_{\lambda_n < N} a_n e^{-i\lambda_n t} - \int_0^N a(x)l'(x)\, e^{-itx}\, dx$$

converges to a function $F(t)$ in B, which can be extended to an analytic function $f(z)$ in $\mathrm{Re}(z) > 0$ *belonging to B_a. Furthermore, F is almost periodic in the sense of Besicovitch* (B^2a.p.).

In this case, then, the function $C_N(t)$ in (5.2) can be taken to be

$$C_N(t) = \int_0^N a(x)l'(x)e^{-itx}\,dx.$$

Proof. We need to show that $F_N(t)$ is the Fourier transform of $2\pi ds_N$ (in the sense of (3.6)). The discontinuous part of s_N clearly yields the trigonometric polynomial in F_N and it remains to show that the continuous part of s_N yields the integral in F_N. In other words we must prove that

$$\lim_{\epsilon\to 0}\int_0^N e^{-itu}\frac{1}{\epsilon}\int_u^{u+\epsilon} a(x)l'(x)\,dx = \int_0^N a(x)l'(x)e^{-itx}\,dx. \tag{5.13}$$

But if $g \in L(0,\infty)$, then by the Fubini theorem,

$$\left|\int due^{-itu}\frac{1}{\epsilon}\int_u^{u+\epsilon} g(x)\,dx - \int e^{-ixu}g(u)\,du\right|$$
$$\leqq \frac{1}{\epsilon}\int_0^\epsilon dx\int_0^\infty |g(x+u)-g(u)|\,du,$$

which tends to zero as $\epsilon\to 0$ since g is continuous in the norm of $L^2(0,\infty)$. This proves (5.13).

By the Riemann-Lebesgue lemma, $C_N(t)\to 0$ as $t\to\pm\infty$. Thus

$$\mathscr{M}\{|C_N|^2\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|C_N(t)|^2\,dt = 0,$$

and it follows that $F_N(t)$ is B^{2}a.p. But $\mathscr{M}\{|F_N - F|^2\}\to 0$ as $N\to\infty$, since $\mathscr{M}\{|\cdot|^2\}\leqq \mathscr{N}(\cdot)$, so that F itself is B^{2}a.p.

This corollary can be applied to the Riemann ζ-function. Let α be greater than $\frac{1}{2}$ and put $a(x) = e^{-\alpha x}$, $\lambda_n = \log n$ so that $l(x) = e^x$. Then it is not difficult to see that $a^*(x)\leqq |a'(x-1)| = \alpha e^\alpha e^{-\alpha x}$. Clearly $a^*\in L^2(0,\infty)$, and also $|a(x)|^2 l'(x) = e^{-(2\alpha-1)x}$, so that $a(x)\in L^2(l'dx)$. Thus the conditions of the corollary are satisfied, and $F_N\to F$ in B where

$$F_N(t) = \sum_{\log n<N}\frac{1}{n}e^{-it\log n} - \int_0^N e^{(1-\alpha)x}\,e^{-itx}\,dx$$
$$= \sum_{\log n<N}\frac{1}{n^{\alpha+it}} - \frac{e^{N(1-\alpha-it)}-1}{1-\alpha-it}.$$

Moreover,

$$f_N(z) = \sum_{n<M}\frac{1}{n^z} - \frac{M^{(1-z)}-1}{1-z}$$

converges to an analytic function $f(z)$ for $\mathrm{Re}(z)>\alpha>\frac{1}{2}$; i.e. for $\mathrm{Re}(z)>\frac{1}{2}$. In the region $\mathrm{Re}(z)>1$, $f(z)$ is clearly $\zeta(z) - 1/(1-z)$, and so we have:

$$\zeta_N(z) = \sum_{u<M}\frac{1}{n^z} - \frac{M^{(1-z)}}{1-z}\to\zeta(z)$$

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in the norm of B on each line $\text{Re}(z) = \sigma > \frac{1}{2}$, $\sigma \neq 1$. (That $\zeta_N \rightarrow \zeta$ uniformly on finite intervals of each line $\text{Re}(z) = \sigma > 0$ follows from a result due to Hardy and Littlewood [**5**, p. 67].)

Since the function $M(z) = M^{(1-z)}/(1 - z)$ is trivially B^2a.p., since

$$\mathscr{M}_t\{|M(\sigma + it)|^2\} = 0,$$

we can conclude also that $\zeta(\sigma + it)$ is a B^2a.p. function of t for $\sigma > \frac{1}{2}$, $\sigma \neq 1$. This result, of course, can be proved by other methods (see [**5**, pp. 132–133]).

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