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# Characterization of isochronous foci for planar analytic differential systems

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We consider the two-dimensional autonomous systems of differential equations of the form

$$\dot{x} = \lambda x - y + P(x, y), \qquad \dot{y} = x + \lambda y + Q(x, y),$$

where $P(x, y)$ and $Q(x, y)$ are analytic functions of order greater than or equal to 2. These systems have a focus at the origin if $\lambda \neq 0$, and have either a centre or a weak focus if $\lambda = 0$. In this work we study the necessary and sufficient conditions for the existence of an isochronous critical point at the origin. Our result is, to the best of our knowledge, original when applied to weak foci and gives known results when applied to strong foci or to centres.

## 1. Introduction

Let us consider an autonomous differential system

$$\dot{x} = \lambda x - y + P(x, y), \qquad \dot{y} = x + \lambda y + Q(x, y), \tag{1.1}$$

where $P(x, y)$ and $Q(x, y)$ are analytic functions in a neighbourhood $\mathcal{U}$ of the origin O and of order greater or equal than two. We assume that O is an isolated singular point of (1.1). We denote by $\mathcal{X}$ the equivalent vector field:

$$\mathcal{X} = (\lambda x - y + P(x, y))\frac{\partial}{\partial x} + (x + \lambda y + Q(x, y))\frac{\partial}{\partial y}.$$

An isolated singular point of (1.1) is said to be a *focus* if it has a neighbourhood where all the orbits spiral forward or backward in time. An isolated singular point of (1.1) is said to be a *centre* if it has a punctured neighbourhood filled with periodic orbits. System (1.1) has a strong focus at the origin if $\lambda \neq 0$ and it has a weak focus or a centre at the origin if $\lambda = 0$.

A major problem is that of studying the existence and properties of periodic solutions in a neighbourhood of the origin of (1.1). In this field, different methods have been used to study isolated periodic solutions, i.e. limit cycles, or non-isolated solutions, i.e. period annuli. The stability of the singular point O does not imply the stability of the cycles close to the singular point. In fact, a non-isolated cycle is Lyapunov stable if and only if every neighbouring cycle has the same period. This fact motivates the definition of isochronicity. We give a precise definition of





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isochronicity in a forthcoming paragraph. Isochronicity has been widely studied not only for its physical meaning and its role in stability theory, but also for its relationship with bifurcation problems and to boundary-value problems.

An essential tool with which to study the stability of the origin of system (1.1) is the *Poincaré map* [11, 12]. Let us consider a neighbourhood $\mathcal{U}$ of the origin and let $\Sigma$ be a section of system (1.1) through the origin, i.e. a transversal curve through the origin for the flow of system (1.1).

More precisely, we define a *section through the origin* as a simple arc without contact with the origin O as an endpoint (see [2, p. 55] for a precise definition of simple arc without contact). We also need some assumptions on its regularity for technical reasons. Given a section through the origin $\Sigma \subset \mathbb{R}^2$, we consider a parametrization $c : \mathbb{R} \to \mathbb{R}^2$ such that $\Sigma = \{c(\sigma) \mid \sigma \in \mathbb{R}\}$ and $\lim_{\sigma \to -\infty} c(\sigma) = \mathrm{O}$. We assume that $c(\sigma)$ is analytic for all $\sigma \in \mathbb{R}$.

For each point $p \in \Sigma$, the flow of system (1.1) through $p$ will again cross $\Sigma$ at a point $\mathcal{P}(p) \in \Sigma$ near $p$. The map $p \mapsto \mathcal{P}(p)$ is called the Poincaré map. If we denote by $\Phi_t(p)$ the flow of system (1.1) with the initial condition $\Phi_0(p) = p$, we can define the Poincaré map in the following way. Given $p \in \Sigma$, there is a unique analytic function $\tau(p)$ such that $\Phi_{\tau(p)}(p) \in \Sigma$ and $\Phi_t(p) \notin \Sigma$ for any $0 < t < \tau(p)$ (see [11]). In these terms, we have $\mathcal{P}(p) = \Phi_{\tau(p)}(p)$. We remark that the functions $\mathcal{P}$ and $\tau$ both depend on the chosen section $\Sigma$. The function $\tau : \Sigma \to \mathbb{R}^+$ is called the *period function*. As usual, $\mathbb{R}^+$ denotes the set of positive real numbers. In this paper we study the existence of a section $\Sigma$ such that $\tau : \Sigma \to \mathbb{R}^+$ is constant. When such a $\Sigma$ exists, we say that the origin O of (1.1) is *isochronous* and that $\Sigma$ is an *isochronous section*.

We will call a 'centre' an analytic system of the form (1.1) with $\lambda = 0$ and where there exists a neighbourhood of the origin O filled up with a period annulus. Isochronicity has been widely studied for centres (see, for example, [6, 8] and the references therein). We remark that the period function of a centre does not depend on the chosen section $\Sigma$. The main methods used in order to study isochronicity of centres can be roughly classified in two categories: linearization and commutation.

Finding a linearization for a centre $\mathcal{X}$ means finding a transformation $\phi : \mathcal{U} \to \mathcal{U}$ that is analytic in a neighbourhood of the origin such that $D\phi(O) = I$, where $I$ denotes the $2 \times 2$ identity matrix, such that the transformed system is a linear centre, i.e. $\phi_*(\mathcal{X}) = -y\partial/\partial x + x\partial/\partial y$. If such a transformation exists, then all the orbits have the same period, coinciding with the period of the linear centre. So a centre is isochronous if and only if a linearization can be found.

Finding a commutator for a centre $\mathcal{X}$ means finding a second vector field $\mathcal{Y}$ analytic in a neighbourhood of the origin and of the form

$$\mathcal{Y} = (x + A(x, y))\frac{\partial}{\partial x} + (y + B(x, y))\frac{\partial}{\partial y}, \tag{1.2}$$

where $A$ and $B$ are analytic functions of order greater than or equal to 2, such that the Lie bracket $[\mathcal{X}, \mathcal{Y}]$ of the centre $\mathcal{X}$ and $\mathcal{Y}$ identically vanishes.

An isolated singular point of a real planar analytic autonomous system is called a *star node* if the linear part of the vector field at the singular point has equal non-zero eigenvalues and it is diagonalizable. Clearly, the origin is a star node for (1.2).

By an affine change of coordinates any vector field with a star node can be brought to the form (1.2).

Given two analytic vector fields defined in an open set $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$, we say that they are *transversal* at noncritical points when $\mathcal{X}$ and $\mathcal{Y}$ have isolated singular points, they both have the same critical points in $\mathcal{U}$, and if $p \in \mathcal{U}$ is such that $\mathcal{X}(p) \neq 0$ then the function given by the wedge product of $\mathcal{X}$ and $\mathcal{Y}$ is not zero at $p$. From now on, we always assume that $\mathcal{X}$ and $\mathcal{Y}$ are analytic vector fields defined in a neighbourhood $\mathcal{U}$ of the origin and transversal at non-critical points.

We will always consider analytic vector fields although many of the stated results apply also for vector fields with weaker differentiability restrictions. The results of Sabatini [16] go on this direction. We define a *smooth* function as a function of class $\mathcal{C}^\infty$ in a neighbourhood $\mathcal{U}$ of the origin O. Analogously, a smooth vector field is defined by smooth functions.

The following result, proved in [1, theorem 2.4, p. 140], characterizes centres in terms of Lie brackets.

Theorem 1.1 (Algaba *et al*. [1]). *System (1.1) with* $\lambda = 0$ *has a centre at the origin if, and only if, there exists a smooth vector field* $U$ *of the form (1.2) and a smooth scalar function* $\nu(x, y)$ *with* $\nu(0, 0) = 0$ *such that* $[\mathcal{X}, U] = \nu\mathcal{X}$.

The most important result on characterization of isochronous centres appears in [14, 17]. A further study can be found in, for example, [1, 7] and the references therein. See [4] for a constructive method to determine $U$ and $\nu$ in special cases for polynomial vector fields.

The following theorem, which is stated and proved in [14, theorem 2, p. 98], gives the equivalence between commutation and isochronicity for centres.

Theorem 1.2 (Sabatini [14]). *Let O be a centre of system (1.1), with* $\lambda = 0$. *Then O is isochronous for system (1.1) if, and only if, there exists an analytic vector field* $\mathcal{Y}$ *of the form (1.2), transversal to* $\mathcal{X}$ *and such that* $[\mathcal{X}, \mathcal{Y}] \equiv 0$.

Another work on commuting systems is [15], where Sabatini discusses the local and global behaviour of the orbits of a pair of commuting systems and gives several illustrative examples. A wide collection of commutators and linearizations can be found in [3].

When a centre is isochronous, it is possible to construct an isochronous section $\Sigma$ (see, for example, [16]). However, the existence of an isochronous section is not strictly dependent on the existence of a centre. A system can have a singular focus point with an isochronous section. This implies the existence of a neighbourhood covered with solutions spiralling towards the singular point, all meeting $\Sigma$ at equal time intervals. Such a behaviour may occur, for example, in a pendulum with friction, or in an electric circuit with dissipation (see also [16]). Our main result, theorem 3.1, characterizes the situation when the origin of system (1.1) is isochronous, even when the origin is a centre, a weak focus or a strong focus. In this paper, we adapt the two different techniques usually used for isochronous centres, in order to study isochronous foci.

In § 2 we summarize the known results on isochronicity for foci. It is shown that a strong focus of an analytic system is always isochronous. All the results described in § 2 apply only for systems of the form (1.1) with $\lambda \neq 0$ or for centres.

Section 3 contains the main theorem of this work, characterizing isochronicity for the origin of a system (1.1). Our result is original when applied to weak foci and gives known results when applied to strong foci or to centres. We modify the commutators' method to study isochronous critical points. We prove that system (1.1) has a transversal vector field $\mathcal{Y}$ such that the vector field $[\mathcal{X}, \mathcal{Y}]$ is proportional to $\mathcal{Y}$ if, and only if, system (1.1) has an isochronous critical point at the origin.

We give two examples of weak isochronous foci and we give an example of a family of quadratic systems depending on a parameter $w \in \mathbb{R}$ which never has an isochronous critical point at the origin. When $w = 0$, the system is a centre and, when $w \neq 0$, the system is a weak focus (stable if $w < 0$ and unstable if $w > 0$). Hence, we show that there is no isochronous section for any system of this family.

## 2. Summary of known results

We denote by $\mathcal{U}$ any open neighbourhood of the origin and by $\rho : \mathcal{U} \to \mathbb{R}^+ \times \mathbb{R}$ the change to polar coordinates, that is $\rho(x, y) = (r, \theta)$ with $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$. As usual, $\rho_*$ is the push-forward defined by $\rho$ and $\rho^*$ is the corresponding pull-back.

In order to give the definition of isochronous critical point, we consider the form of (1.1) in polar coordinates, that is

$$\rho_*(\mathcal{X}) = rf(r, \theta)\frac{\partial}{\partial r} + g(r, \theta)\frac{\partial}{\partial \theta},$$

where $f$ and $g$ are analytic functions in a neighbourhood of $\rho(O)$.

Definition 2.1. The critical point O of (1.1) is said to be *isochronous* if there exists a local analytic change of variables $\phi$ with $D\phi(O) = I$ and such that

$$\rho_*\phi_*(\mathcal{X}) = rf(r, \theta)\frac{\partial}{\partial r} + g(\theta)\frac{\partial}{\partial \theta}.$$

A system (1.1) with an isochronous critical point at the origin is more easily written using the *arc length* $\varphi$, defined by

$$\varphi = \int_0^\theta \mathrm{d}\theta \, \frac{1}{g(\theta)},$$

as new angular variable. In this formulation we end up with the following definition.

Definition 2.2. The critical point O of (1.1) is said to be *isochronous* if there exists a local analytic change of variables $\phi$ with $D\phi(O) = I$ and such that

$$\rho_*\phi_*(\mathcal{X}) = rf(r, \theta)\frac{\partial}{\partial r} + k\frac{\partial}{\partial \theta}, \quad k \in \mathbb{R}, \ k \neq 0.$$

The existence of an isochronous section is equivalent to the existence of the local analytic change of variables $\phi$, as we will show in theorem 3.1. We state the definition of isochronous critical point by means of $\phi$, since this is its classical definition which allows us to give the summary of known results.

Linear foci,

$$(\lambda x - y)\frac{\partial}{\partial x} + (x + \lambda y)\frac{\partial}{\partial y},$$

are isochronous, since their angular speed is constant along rays through the origin. For a linear focus, every ray through the origin is an isochronous section. We say that a vector field $\mathcal{X}$ of the form (1.1) is *linearizable* when there exists a local change of variables $\phi$ with $D\phi(O) = I$ such that $\phi_*(\mathcal{X})$ is a linear focus.

By the above definition 2.2, every analytic linearizable focus is isochronous. If $\phi$ is the linearizing transformation and $\Sigma$ is a ray, then $\phi^{-1}(\Sigma)$ is an isochronous section of the analytic linearizable focus. The next theorem, which is a special case of the classical Poincaré theorem, shows that every strong focus of an analytic system is linearizable and therefore isochronous. For a proof, see [5, 12].

Theorem 2.3 (Poincaré [12, 13]). *Let us consider the planar real analytic system*

$$\dot{x} = \alpha x - \beta y + g_1(x, y), \qquad \dot{y} = \beta x + \alpha y + g_2(x, y), \tag{2.1}$$

*with $\alpha\beta \neq 0$, and $g_1$ and $g_2$ are of second order in $x$ and $y$. Then there exists a real local analytic change of variables $\phi(x, y) = (u, v)$ with $D\phi(O) = I$, which transforms system (2.1) into $\dot{u} = \alpha u - \beta v$, $\dot{v} = \beta u + \alpha v$.*

This result can also be stated for a system of the form (2.1) satisfying weaker differentiability restrictions. Since we are concerned only with analytic vector fields, we state the result for the analytic case only.

We have seen that every analytic linearizable focus is isochronous, but finding the linearization, and hence the isochronous sections, is usually too difficult. The next theorem, proved in [16], shows that it is not necessary to find the explicit form of the linearization, since the orbits of a suitable commutator are isochronous sections of $\mathcal{X}$.

Theorem 2.4 (Sabatini [16]). *If the vector field $\mathcal{X}$ given by (1.1) has a focus $O$ and a non-trivial commutator $\mathcal{Y}$ with a star node at $O$, then every orbit of $\mathcal{Y}$ contained in a neighbourhood of $O$ is an isochronous section of $\mathcal{X}$.*

This result applies only when the vector field $\mathcal{X}$ has a strong focus at the origin or has a centre, because if the vector field $\mathcal{X}$ has a weak focus at the origin with a non-trivial commutator $\mathcal{Y}$ with a star node at O, then, by theorem 1.1, the vector field has a centre at the origin. The next corollary, proved in [16], shows that every system with a strong focus and a non-trivial commutator has a commutator with a star node.

Corollary 2.5 (Sabatini [16]). *If the vector field $\mathcal{X}$ has eigenvalues with non-zero real part at a focus $O$ and a non-trivial commutator $\mathcal{Y}$, then it has infinitely many isochronous sections.*

Different sufficient conditions for an analytic vector field to have an isochronous weak focus at O are given in [9] and [16]. In [16], the particular case of a differential system equivalent to a Liénard equation is taken into account.

## 3. Characterization of isochronous critical points

The following theorem characterizes the situation when the origin O of a system (1.1) has a section $\Sigma$ such that the period function $\tau : \Sigma \to \mathbb{R}^+$ is constant,

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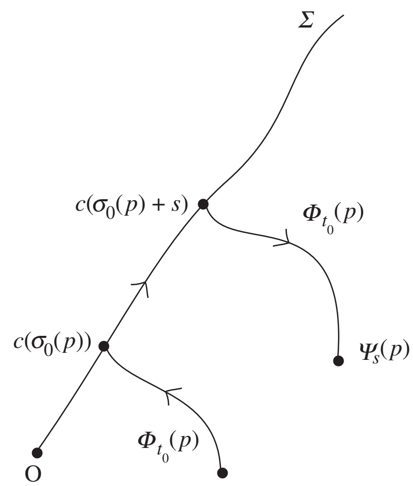


Figure 1. Definition of $\Psi_s(p)$.

that is, it does not depend on the point $p \in \Sigma$ considered. We will see that if such a section exists, then there are an infinite number of sections. In particular, the next theorem characterizes the existence of isochronous critical points.

THEOREM 3.1. *Let us consider an analytic system (1.1). The following statements are equivalent.*

(i) *There exists an analytic change of variables $\phi : \mathcal{U} \to \mathcal{U}$, where $\mathcal{U}$ is a neighbourhood of the origin, with $D\phi(O) = I$, such that the transformed system is given by*

$$\rho_* \phi_*(\mathcal{X}) = r f(r,\theta) \frac{\partial}{\partial r} + g(\theta) \frac{\partial}{\partial \theta}.$$

(ii) *There exists an analytic vector field $\mathcal{Y}$, defined in a neighbourhood of the origin, of the form*

$$\mathcal{Y} = (x + A(x,y)) \frac{\partial}{\partial x} + (y + B(x,y)) \frac{\partial}{\partial y}, \tag{3.1}$$

*with $A$ and $B$ analytic functions of order greater than or equal to 2, such that $[\mathcal{X}, \mathcal{Y}] = \mu(x,y)\mathcal{Y}$, where $\mu(x,y)$ is a scalar function with $\mu(0,0) = 0$.*

(iii) *There exists a section $\Sigma$ such that the period function $\tau : \Sigma \to \mathbb{R}^+$ is constant.*

*Proof of Theorem 3.1.* In order to prove the equivalence of the three statements, it suffices to show that (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i). We also include the proof of (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (ii), for completeness.

In the following, we will denote a partial derivative by a subscript; for example, if $f(r,\theta)$ is a function of $(r,\theta)$, then $\partial f/\partial r$ is replaced by $f_r$.

(i) $\Rightarrow$ (ii). We define

$$\mathcal{Y} = \phi^* \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right),$$

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and we find that $\mathcal{Y}$ has the form described, since $\phi$ is an analytic change such that $D\phi(O) = I$. Moreover,

$$\begin{aligned}[\mathcal{X}, \mathcal{Y}] &= \left[\phi^*\phi_*\mathcal{X}, \phi^*\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\right] \\ &= \phi^*\rho^*\left(\left[rf(r,\theta)\frac{\partial}{\partial r} + g(\theta)\frac{\partial}{\partial \theta}, r\frac{\partial}{\partial r}\right]\right) \\ &= \phi^*\rho^*\left(-r^2 f_r(r,\theta)\frac{\partial}{\partial r}\right).\end{aligned}$$

We define $\mu(x,y) = \phi^*\rho^*(-rf_r(r,\theta))$. It is obvious that is an analytic scalar function with $\mu(0,0) = 0$. We then have

$$[\mathcal{X}, \mathcal{Y}] = \mu(x,y)\phi^*\rho^*\left(r\frac{\partial}{\partial r}\right) = \mu(x,y)\phi^*\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = \mu(x,y)\mathcal{Y}.$$

(ii) $\Rightarrow$ (i). From normal form theory [5, 12, 13], we find that there exists an analytic change of variables $\phi$, defined in a neighbourhood $\mathcal{U}$ of the origin and with $D\phi(O) = I$, such that

$$\phi_*(\mathcal{Y}) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Since $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{Y}$, we obtain $[\phi_*(\mathcal{X}), \phi_*(\mathcal{Y})] = \phi_*(\mu)\phi_*(\mathcal{Y})$. We introduce the following notation:

$$\tilde{\mu}(r,\theta) := \rho_*\phi_*(\mu(x,y)) \quad \text{and} \quad \rho_*\phi_*(\mathcal{X}) := rf(r,\theta)\frac{\partial}{\partial r} + g(r,\theta)\frac{\partial}{\partial \theta}.$$

Hence,

$$\left[rf(r,\theta)\frac{\partial}{\partial r} + g(r,\theta)\frac{\partial}{\partial \theta}, r\frac{\partial}{\partial r}\right] = \tilde{\mu}(r,\theta)r\frac{\partial}{\partial r}.$$

We compute the Lie bracket and then we have the equality

$$-r^2 f_r(r,\theta)\frac{\partial}{\partial r} - rg_r(r,\theta)\frac{\partial}{\partial \theta} = \tilde{\mu}(r,\theta)r\frac{\partial}{\partial r},$$

which implies $g_r(r,\theta) \equiv 0$ and, therefore, $g(r,\theta) = g(\theta)$. We remark that, since the origin of the system defined by $\mathcal{X}$ is a monodromic critical point, we find that $g(\theta) > 0$ or $g(\theta) < 0$ for all $\theta \in \mathbb{R}$. Moreover, as before, we may consider the arc length

$$\varphi = \int_0^\theta \mathrm{d}\theta \frac{1}{g(\theta)}.$$

This integral is well defined and it gives a change of variable, since $g(\theta)$ has a definite sign for all $\theta \in \mathbb{R}$. After this change, the angular speed of the corresponding system is constant.

(ii) $\Rightarrow$ (iii). This statement is a clear corollary of theorem 2.4. However, a geometric outline of its proof is easy enough to be given here.

For any $p \in \mathcal{U}$, let $\Phi_t(p)$ be the flow of $\mathcal{X}$ and $\Psi_s(p)$ that of $\mathcal{Y}$, with the initial condition $\Phi_0(p) = \Psi_0(p) = p$. Without loss of generality, we can assume that O is the unique singular point for $\mathcal{X}$ and $\mathcal{Y}$ in $\mathcal{U}$. Let $p, q \in \mathcal{U}$, $p, q \neq \mathrm{O}$. By classical Lie theory, the relation $[\mathcal{X}, \mathcal{Y}] = \mu(x, y)\mathcal{Y}$ implies that if $\Sigma = \{\Psi_s(p) \mid s \in \mathbb{R}\}$ is a solution of $\mathcal{Y}$, then, for any $t \in \mathbb{R}$, $\Phi_t(\Sigma)$ is another solution for $\mathcal{Y}$.

It is clear that $\Sigma$ is a transversal section for $\mathcal{X}$. Let $\tau$ and $\mathcal{P}$ be the corresponding period function and Poincaré map defined on it. We will show that any two points $p, q \in \Sigma$ have the same period function. We find that $\mathcal{P}(p) = \Phi_{\tau(p)}(p)$. The time $\tau(p)$ leaves $\Sigma$ invariant: $\Phi_{\tau(p)}(\Sigma) \subseteq \Sigma$. Let $q \in \Sigma$. Then there exists $s \in \mathbb{R}$ such that $q = \Psi_s(p)$. The minimal time taken to meet $\Sigma$ again, that is $\tau(q)$, must coincide with $\tau(p)$, since the time $\tau(p)$ brings the solution $\Sigma$ into itself. Then $\tau(p) = \tau(q)$.

(iii) $\Rightarrow$ (ii). We consider $\Phi_t(p)$, the flow of system $\mathcal{X}$ defined in the neighbourhood $\mathcal{U}$ of the origin and with the initial condition $\Phi_0(p) = p$.

Given a section through the origin, $\Sigma \subset \mathbb{R}^2$, we consider its parametrization by its arc parameter $\sigma$, that is, there exists a map $c : \mathbb{R} \to \Sigma$ such that $\Sigma = \{c(\sigma) \mid \sigma \in \mathbb{R}\}$. We can assume without loss of generality that $\lim_{\sigma \to -\infty} c(\sigma) = \mathrm{O}$ and that $\lim_{\sigma \to -\infty} c'(\sigma) \neq (0, 0)$. As usual, $c'(\sigma)$ denotes the derivative of the parametrization of the curve $c : \sigma \mapsto c(\sigma)$ at the value $\sigma$. We define the set of transformations $\Psi : \mathbb{R} \times \mathcal{U} \to \mathcal{U}$ in the following way (see figure 1).

If $p \in \Sigma$, that is $p = c(\sigma_0)$ for a certain $\sigma_0 \in \mathbb{R}$, and $s \in \mathbb{R}$, then $\Psi_s(p) := c(\sigma_0 + s)$.

If $p \notin \Sigma$, there exists $t_0(p) \in \mathbb{R}$ such that $\Phi_{t_0(p)}(p) \in \Sigma$, that is, there exists $\sigma_0 \in \mathbb{R}$ such that $c(\sigma_0) = \Phi_{t_0(p)}(p)$. Assume that $t_0(p) > 0$ is the lowest positive real number with this property. For any $s \in \mathbb{R}$ we define $\Psi_s(p) = \Phi_{-t_0(p)}(c(\sigma_0 + s))$.

In the following, for any $p \in \mathcal{U}$, we denote by $t_0(p)$ the lowest positive real number such that $\Phi_{t_0(p)}(p) \in \Sigma$. It is clear that $t_0 : \mathcal{U} \to [0, T)$, where $T > 0$ is the period defined by the section $\Sigma$. We denote by $\sigma_0(p) \in \mathbb{R}$ the value of the parameter such that $\Phi_{t_0(p)}(p) = c(\sigma_0(p))$.

We are going to prove that the set of transformations defined by $\Psi_s$ is a one-parameter Lie group of point transformations. We need to show the following statements:

(a) for all $s \in \mathbb{R}$, $\Psi_s : \mathcal{U} \to \mathcal{U}$ is bijective;

(b) $\Psi_0$ is the identity map;

(c) for any $s_1, s_2 \in \mathbb{R}$, $\Psi_{s_1} \circ \Psi_{s_2} = \Psi_{s_1 + s_2}$;

(d) $\Psi \in \mathcal{C}^\omega(\mathbb{R}) \times \mathcal{C}^\omega(\mathcal{U})$.

(a) Fixed $s \in \mathbb{R}$, let us consider any $p \in \mathcal{U}$ and we have $\Psi_s(p) = \Phi_{-t_0(p)}(c(\sigma_0(p) + s))$. Let $p_1, p_2 \in \mathcal{U}$. If $\Psi_s(p_1) = \Psi_s(p_2)$, let $q$ be this point $q = \Psi_s(p_i)$. Then, the points $\Phi_{t_0(p_1)}(q) = c(\sigma_0(p_1) + s)$ and $\Phi_{t_0(p_2)}(q) = c(\sigma(p_2) + s)$ belong to $\Sigma$. Both $t_0(p_1)$ and $t_0(p_2)$ are defined as the minimum positive time with this property, so $t_0(p_1) = t_0(p_2)$. Therefore, $c(\sigma_0(p_2) + s) = c(\sigma_0(p_1) + s)$ and this gives $\sigma_0(p_1) = \sigma_0(p_2)$, which implies $p_1 = p_2$. Then $\Psi_s$ is injective.

Let us see that it is exhaustive. Given $q \in \mathcal{U}$, let $p = \Phi_{-t_0(q)}(c(\sigma_0(q) - s))$. Then $t_0(p) = t_0(q)$, $\sigma_0(p) = \sigma_0(q) - s$ and $\Psi_s(p) = \Phi_{-t_0(q)}(c(\sigma_0(q))) = q$. The fact that the section $\Sigma$ is isochronous ensures the well-definedness of this $p$.

(b) Given $p \in \mathcal{U}$, we find that $\Psi_0(p) = \Phi_{-t_0(p)}(c(\sigma_0(p)))$, where $c(\sigma_0(p)) = \Phi_{t_0(p)}(p)$. Then, clearly, $\Psi_0(p) = p$.

(c) Given $p \in \mathcal{U}$, it is clear that

$$t_0(\Phi_{-t_0(p)}(c(\sigma_0 + s_1))) = t_0(\Phi_{-t_0(p)}(c(\sigma_0 + s_1 + s_2))) = t_0(p).$$

We have

$$\begin{aligned}\Psi_{s_1} \circ \Psi_{s_2}(p) &= \Psi_{s_1}(\Phi_{-t_0(p)}(c(\sigma_0 + s_1))) \\ &= \Phi_{-t_0(p)}(c(\sigma_0 + s_1 + s_2)) \\ &= \Psi_{s_1+s_2}(p).\end{aligned}$$

(d) The regularity of $\Psi$ is clear due to the regularity of $\Phi$ and $c$.

Once we have that $\Psi$ is a one-parameter Lie group of point transformations, we apply the first fundamental theorem of Lie [10], and we find that there exists an analytic vector field $\mathcal{Y}$ whose flow coincides with $\Psi_s(p)$. Moreover, $\mathcal{Y}$ is given by

$$\frac{\partial \Psi_s}{\partial s}(p)_{|s=0}.$$

By the definition $\Psi_s(p) = \Phi_{t_0(p)}(c(\sigma_0(p) + s))$, we see that

$$\mathcal{Y}(p) = D\Phi_{t_0(p)}(c(\sigma_0(p) + s)) \cdot c'(\sigma_0(p) + s)_{|s=0} = D\Phi_{t_0(p)}(c(\sigma_0(p))) \cdot c'(\sigma_0(p)),$$

where $D\Phi_t(q)$ denotes the Jacobian matrix of the analytic change of variables $\Phi_t$ at the point $q$ and, as before, $c'(\sigma)$ denotes the derivative of the parametrization of the curve $c : \sigma \mapsto c(\sigma)$ at the value $\sigma$.

Moreover, by construction, $\mathcal{Y}$ has a star node at the origin. This is clear by the fact that each of its orbits $\Phi_t(\Sigma)$, $t \in [0, T)$, has a different tangent at the origin. Let

$$\mathcal{Y} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}.$$

Since $\mathcal{Y}$ has a star node at the origin, by a classical result stated in [18, p. 63] we find that $\xi(x, y) = xh(x, y) + \text{h.o.t.}$ and $\eta(x, y) = yh(x, y) + \text{h.o.t.}$, where $h(x, y)$ is a homogeneous polynomial and 'h.o.t.' denotes higher-order terms. Therefore, in order to see that $\mathcal{Y}$ is of the form (1.2), we need only to show that the divergence of the vector field $\mathcal{Y}$, that is $\operatorname{div}\mathcal{Y}$, is different from zero at the origin, where

$$\operatorname{div}\mathcal{Y}(x, y) = \frac{\partial \xi}{\partial x}(x, y) + \frac{\partial \eta}{\partial y}(x, y).$$

The divergence of the vector field $\mathcal{Y}$ is related to the inverse-integrating factor of $\mathcal{Y}$. The inverse-integrating factor of $\mathcal{Y}$ is given by $V(x, y) = (\lambda x - y + P(x, y))\eta(x, y) - (x + \lambda y + Q(x, y))\xi(x, y)$, which is defined in the neighbourhood $\mathcal{U}$ of the origin. An easy computation shows that

$$V(\Psi_s(x_0, y_0)) = V(x_0, y_0)\exp\left\{\int_0^s \operatorname{div}\mathcal{Y}(\Psi_u(x_0, y_0))\,\mathrm{d}u\right\} \tag{3.2}$$

for any $(x_0, y_0) \in \mathcal{U}$. It is clear that $V(0, 0) = 0$. Let $p_0 := (x_0, y_0) \in \mathcal{U} - \{(0, 0)\}$ and assume that $V(p_0) = 0$. This implies that the vectors $\mathcal{Y}(p_0)$ and $\mathcal{X}(p_0)$ are parallel. By the definition of $\mathcal{Y}$, we obtain

$$\mathcal{Y}(p_0) = D\Phi_{-t_0(p_0)}(c(\sigma_0(p_0))) \cdot c'(\sigma_0(p_0)) = D\Phi_{-t_0(p_0)}(\Phi_{t_0(p_0)}(p_0)) \cdot \mathcal{Y}(\Phi_{t_0(p_0)}(p_0)).$$

We define $q_0 = \Phi_{t_0(p_0)}(p_0)$ and we obtain $D\Phi_{t_0(p_0)}(q_0) \cdot \mathcal{Y}(p_0) = \mathcal{Y}(q_0)$. Since $\Phi$ is the flow of $\mathcal{X}$, we have $D\Phi_{t_0(p_0)}(q_0) \cdot \mathcal{X}(p_0) = \mathcal{X}(q_0)$. Therefore, if $\mathcal{Y}(p_0)$ and $\mathcal{X}(p_0)$ are parallel, then $\mathcal{Y}(q_0)$ and $\mathcal{X}(q_0)$ are parallel. However, $q_0 \in \Sigma$ and the vector $\mathcal{Y}(q_0)$ is tangential to $\Sigma$ at $q_0$, so the parallelism between $\mathcal{Y}(q_0)$ and $\mathcal{X}(q_0)$ is a contradiction with $\Sigma$ being a transversal section for $\mathcal{X}$. Therefore, we conclude that $V(x_0, y_0) \neq 0$ for any $(x_0, y_0) \in \mathcal{U} - \{(0,0)\}$.

By using this fact, we prove that $\operatorname{div} \mathcal{Y}(0,0) \neq 0$. Let us consider $(x_0, y_0) \in \mathcal{U} - \{(0,0)\}$, and that $\lim_{s \to -\infty} \Psi_s(x_0, y_0) = (0,0)$. By continuity and the identity (3.2), we find that the integral

$$I(x_0, y_0) := \int_{-\infty}^{0} \operatorname{div} \mathcal{Y}(\Psi_u(x_0, y_0)) \, \mathrm{d}u$$

diverges. $I(x_0, y_0)$ is continuous, so $I(0,0)$ also diverges. Hence, if $\operatorname{div} \mathcal{Y}(0,0) = 0$, then

$$I(0,0) = \int_{-\infty}^{0} \operatorname{div} \mathcal{Y}(\Psi_u(0,0)) \, \mathrm{d}u = \int_{-\infty}^{0} \operatorname{div} \mathcal{Y}(0,0) \, \mathrm{d}u = 0,$$

in contradiction with being divergent. Therefore, $\operatorname{div} \mathcal{Y}(0,0) \neq 0$.

Moreover, by definition, it is clear that the flow of $\mathcal{X}$ takes solutions of $\mathcal{Y}$ to solutions of $\mathcal{Y}$. Another classical result on Lie symmetries gives that $\mathcal{X}$ is a Lie symmetry for $\mathcal{Y}$ and therefore, there exists an analytic scalar function $\mu : \mathcal{U} \to \mathbb{R}$ such that $[\mathcal{X}, \mathcal{Y}] = \mu \mathcal{Y}$. Moreover, $\mu(0,0) = 0$, since both functions defining the vector field $[\mathcal{X}, \mathcal{Y}]$ have order two at the origin, and the vector field $\mathcal{Y}$ has order one at the origin.

(i) $\Rightarrow$ (iii). The ray $\tilde{\Sigma} = \{(x,0) \mid x > 0\}$ is an isochronous section for the system $\phi_*(\mathcal{X})$, where

$$\rho_* \phi_*(\mathcal{X}) = r f(r,\theta) \frac{\partial}{\partial r} + g(\theta) \frac{\partial}{\partial \theta},$$

since $\tilde{\tau} : \tilde{\Sigma} \to \mathbb{R}^+$ is given by

$$\tilde{\tau}(x) = \int_0^{2\pi} \mathrm{d}\theta \, \frac{1}{g(\theta)},$$

which is constant for every $x \in \tilde{\Sigma}$. Then $\Sigma := \phi^{-1}(\tilde{\Sigma})$ is an isochronous section for system (1.1) and the period function is given by $\tau := \phi^*(\tilde{\tau})$. □

Using theorems 1.1 and 3.1, we reencounter the following result which characterizes isochronous centres and which is stated and proved in [1, theorem 2.3, p. 140].

Theorem 3.2. *System (1.1) with $\lambda = 0$ has an isochronous centre at the origin if, and only if, there exists a smooth vector field $\mathcal{Z}$ of the form (1.2) such that $[\mathcal{X}, \mathcal{Z}] = 0$.*

*Proof.* Assume that system (1.1), with $\lambda = 0$, has an isochronous centre at the origin. Then, by theorem 1.1, there exists a smooth vector field $\mathcal{U}$ of the form (1.2), and a smooth function $\nu$, with $\nu(0,0) = 0$ such that $[\mathcal{X}, \mathcal{U}] = \nu \mathcal{X}$. Moreover, by theorem 3.1 there exists an analytic vector field $\mathcal{Y}$ of the form (1.2) and an analytic function $\mu$, with $\mu(0,0) = 0$, satisfying $[\mathcal{X}, \mathcal{Y}] = \mu \mathcal{Y}$. Since $\mathcal{X}$ and $\mathcal{Y}$ are transversal in the neighbourhood of the origin, they define a basis in this neighbourhood and therefore there exist two smooth functions $\alpha$, $\beta$ such that $\mathcal{U} = \alpha\mathcal{X} + \beta\mathcal{Y}$. Since both $\mathcal{U}$ and $\mathcal{Y}$ have the form (1.2), we find that $\beta = 1 + \beta_1$, where $\beta_1$ is a smooth function of order greater than or equal to 1. We compute

$$\begin{aligned}[\mathcal{X},\mathcal{U}] &= [\mathcal{X}, \alpha\mathcal{X} + \beta\mathcal{Y}] \\ &= \mathcal{X}(\alpha)\mathcal{X} + \alpha[\mathcal{X},\mathcal{X}] + \mathcal{X}(\beta)\mathcal{Y} + \beta[\mathcal{X},\mathcal{Y}] \\ &= \mathcal{X}(\alpha)\mathcal{X} + (\mathcal{X}(\beta) + \beta\mu)\mathcal{Y}.\end{aligned}$$

Since $[\mathcal{X},\mathcal{Y}] = \nu\mathcal{X}$, we deduce that $\mathcal{X}(\beta) = -\mu\beta$.

We define $\mathcal{Z} = \beta\mathcal{Y}$, which is a smooth vector field with the form (1.2) since $\beta = 1+\beta_1$, where $\beta_1$ is a smooth function of order greater than or equal to 1. Then

$$[\mathcal{X},\mathcal{Z}] = [\mathcal{X},\beta\mathcal{Y}] = \beta[\mathcal{X},\mathcal{Y}] + \mathcal{X}(\beta)\mathcal{Y} = \beta\mu\mathcal{Y} - \mu\beta\mathcal{Y} = 0.$$

□

We remark on a difference between theorems 3.2 and 1.2. In theorem 1.2, the origin is required to be a centre and the result characterizes its isochronicity. In theorem 3.2, only the origin is a linear centre (the condition that the equilibrium is a centre is not required), due to theorem 1.1.

The methods developed in this work can be used to classify isochronous critical points for polynomial systems. We give some examples of systems of the form (1.1) with an isochronous critical point at the origin. The determination of the origin as a focus is straightforward by computing Lyapunov constants (see, for example, [9]). When the origin is a centre, a first integral defined on a neighbourhood is provided. We also give an example of a family of quadratic systems depending on a real parameter $w \neq 0$, which never has an isochronous critical point at the origin. When $w = 0$, the system has a centre, and when $w \neq 0$ the system has a weak focus at the origin.

Example 3.3. The following system has an isochronous critical point at the origin:

$$\left.\begin{aligned}\dot{x} &= -y + \lambda_2 x^3 + \lambda_3 x^2 y + \lambda_4 x y^2, \\ \dot{y} &= x + \lambda_2 x^2 y + \lambda_3 x y^2 + \lambda_4 y^3,\end{aligned}\right\} \tag{3.3}$$

with $\lambda_2, \lambda_3, \lambda_4 \in \mathbb{R}$. In polar coordinates, this system is given by

$$\dot{r} = \tfrac{1}{2} r^3(\lambda_2 + \lambda_4 + (\lambda_2 - \lambda_4)\cos(2\theta) + \lambda_3 \sin(2\theta)), \quad \dot{\theta} = 1.$$

Then, by definition, the origin is an isochronous critical point. A first integral for system (3.3) is given by

$$H(x,y) = \frac{x^2 + y^2}{1 - \lambda_3 x^2 + (\lambda_2 - \lambda_4)xy + (\lambda_2 + \lambda_4)(x^2 + y^2)\arctan(y/x)}.$$

When $\lambda_2 + \lambda_4 \neq 0$, the origin is a focus, and when $\lambda_2 + \lambda_4 = 0$, the origin is a centre. Let us consider $\mathcal{X}$, the corresponding vector field, and

$$\mathcal{Y} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

We have $[\mathcal{X},\mathcal{Y}] = -2(\lambda_2 x^2 + \lambda_3 xy + \lambda_4 y^2)\mathcal{Y}$.

Example 3.4. The following system has an isochronous focus at the origin:

$$\left.\begin{aligned}\dot{x} &= -y - 2xy + xy^2 - 2y^3 + \mu_2(x^3 - xy^2) + \mu_3 x^2 y - y^4 \\ &\quad + \mu_2(x^2y^2 + y^4) - \mu_2 xy^4 - \mu_3 y^5 - \mu_2 y^6, \\ \dot{y} &= x + y^2 + y^3 + \mu_2(x^2 y - y^3) + \mu_3 xy^2 + 2\mu_2 xy^3 + \mu_3 y^4 + \mu_2 y^5,\end{aligned}\right\} \tag{3.4}$$

where $\mu_i$ are arbitrary real constants for $i = 2, 3$. This system has no constant angular speed. An easy computation shows that the first Lyapunov constant equals $-\frac{1}{2}$, so the origin of (3.4) is a stable weak focus. We use theorem 3.1 to ensure the property of isochronicity.

Let us consider $\mathcal{X}$, the corresponding vector field, and $\mathcal{Y}$, the following analytic vector field:

$$\mathcal{Y} = (x - y^2)\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

The Lie bracket $[\mathcal{X}, \mathcal{Y}]$ gives

$$[\mathcal{X}, \mathcal{Y}] = -2(y^2 + \mu_2(x^2 - y^2 + 2xy^2 + y^4) + \mu_3(xy + y^3))\mathcal{Y}.$$

Therefore, the hypothesis of theorem 3.1 are satisfied and the origin of system (3.4) is an isochronous focus.

Example 3.5. The family of quadratic systems

$$\dot{x} = -y, \qquad \dot{y} = x - 4wxy + 2y^2, \tag{3.5}$$

depending on the parameter $w \in \mathbb{R}$, never has an isochronous critical point at the origin.

It can be shown that $w$ is the first Lyapunov constant for this family of quadratic systems. Hence, when $w > 0$ the origin is an unstable weak focus, and when $w < 0$ the origin is a stable weak focus. When $w = 0$, we find that

$$H(x, y) = (4x + 8y^2 - 1)\,\mathrm{e}^{4x}$$

defines a first integral which is analytic in a neighbourhood of the origin. So, the origin is a centre for $w = 0$.

We will try to construct a vector field $\mathcal{Y}$ and a function $\mu$ satisfying theorem 3.1, and we will obtain a contradiction. Assume that there exists a vector field $\mathcal{Y}$ with a star node at the origin such that the Lie bracket between the vector field $\mathcal{X}_w$, defined by (3.5), and $\mathcal{Y}$ is equal to $\mu(x, y)\mathcal{Y}$ for a certain scalar analytic function $\mu(x, y)$ with $\mu(0, 0) = 0$. We can write

$$\mathcal{Y} = \left(x + \sum_{i>1} A_i(x, y)\right)\frac{\partial}{\partial x} + \left(y + \sum_{i>1} B_i(x, y)\right)\frac{\partial}{\partial y},$$

where $A_i(x, y)$, $B_i(x, y)$ are homogeneous polynomials of degree $i$ and $\mu(x, y) = \sum_{i>0} m_i(x, y)$, where $m_i(x, y)$ is a homogeneous polynomial of degree $i$.

Equating the terms of order 2 in the equation $[\mathcal{X}_w, \mathcal{Y}] = \mu\mathcal{Y}$, we obtain the following two equations:

$$-y\frac{\partial A_2}{\partial x} + x\frac{\partial A_2}{\partial y} + B_2 = m_1 x,$$

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$$-y\frac{\partial B_2}{\partial x}+x\frac{\partial B_2}{\partial y}+4xwy-2y^2-A_2=m_1y.$$

The solutions of these two equations are $A_2(x,y)=ax^2+bxy-\frac{2}{3}y^2$, $B_2(x,y)=\frac{4}{3}wx^2+axy+by^2$ and $m_1(x,y)=(b+\frac{4}{3}w)x-(\frac{4}{3}+a)y$, where $a$, $b$ are any two real numbers.

Equating the terms of order 3 in the equation $[\mathcal{X}_w,\mathcal{Y}]=\mu\mathcal{Y}$, we obtain the following two equations:

$$-y\frac{\partial A_3}{\partial x}+x\frac{\partial A_3}{\partial y}+(2y^2-4wxy)\frac{\partial A_2}{\partial y}+B_3=m_2x+m_1A_2,$$

$$-y\frac{\partial B_3}{\partial x}+x\frac{\partial B_3}{\partial y}+(2y^2-4wxy)\frac{\partial B_2}{\partial y}+4wyA_2-A_3-4(y-wx)B_2=m_2y+m_1B_2.$$

Let us write

$$m_2(x,y)=\sum_{i+j=2}m_{ij}x^iy^j,$$
$$A_3(x,y)=\sum_{i+j=3}a_{ij}x^iy^j,$$
$$B_3(x,y)=\sum_{i+j=3}b_{ij}x^iy^j.$$

We consider the vector of unknowns

$$\boldsymbol{v}=\{m_{20},m_{11},m_{02},a_{30},a_{21},a_{12},a_{03},b_{30},b_{21},b_{12},b_{03}\}$$

and we can write the previous two equations as a linear system of eight equations in these eleven unknowns: $\boldsymbol{M}\boldsymbol{v}=\boldsymbol{k}$. The matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M}=\begin{pmatrix}-1&0&0&0&1&0&0&1&0&0&0\\0&-1&0&-3&0&2&0&0&1&0&0\\0&0&-1&0&-2&0&3&0&0&1&0\\0&0&0&0&0&-1&0&0&0&0&1\\0&0&0&-1&0&0&0&0&1&0&0\\-1&0&0&0&-1&0&0&-3&0&2&0\\0&-1&0&0&0&-1&0&0&-2&0&3\\0&0&-1&0&0&0&-1&0&0&-1&0\end{pmatrix},$$

which can be seen to be of rank 7. The vector $\boldsymbol{k}$ is given by

$$\boldsymbol{k}=\{\tfrac{1}{3}a(3b+4w),\tfrac{1}{3}(-4a-3a^2+3b^2+16bw),\tfrac{1}{9}(-36b-9ab-56w),\tfrac{2}{9}(16+3a),\tfrac{4}{9}(3b-8w)w,\tfrac{1}{9}(9ab+32w-36aw),\tfrac{1}{3}(2a-3a^2+3b^2+4bw),\tfrac{1}{3}(-4b-3ab+8w)\}.$$

The matrix $(\boldsymbol{M}\mid\boldsymbol{k})$ has rank 8, as the determinant of one of its $8\times 8$ minors is equal to $1+w^2$. Therefore, the linear system does not satisfy the compatibility condition and, hence, no such $\mathcal{Y}$ or $\mu$ can exist.

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