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DECOMPOSITION VARIETIES IN SEMISIMPLE LIE ALGEBRAS

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ABSTRACT. The notion of decompositon class in a semisimple Lie algebra is a common generalization of nilpotent orbits and the set of regular semisimple elements. We prove that the closure of a decomposition class has many properties in common with nilpotent varieties, *e.g.*, its normalization has rational singularities.

The famous Grothendieck simultaneous resolution is related to the decomposition class of regular semisimple elements. We study the properties of the analogous commutative diagrams associated to an arbitrary decomposition class.

1**. Introduction.** Let $\mathfrak{g}$ be a semisimple Lie algebra over an algebraically closed field k of characteristic zero with adjoint group G. We shall say that two elements x and x' are in the same *decomposition class* D if x and x' have a similar Jordan decomposition. There are only finitely many different decomposition classes; they are all smooth; and the closure of any one of them, called *decomposition variety*, is a union of decomposition classes. Decomposition classes were first defined and studied by Borho and Kraft [8], and their analogs in the group G first by Lusztig and Spaltenstein [38]. Their properties have important applications in representation theory.

To be more precise, let $x = x_s + x_n$ and $x' = x'_s + x'_n$ be the Jordan decompositions of x and x'. Then x and x' are in the same decomposition class if and only if there exists a $g \in G$ such that $x_n = gx'_n$ and $G_{x_s} = G_{gx'_s}$. For $\mathfrak{sl}_n$ this means that in a decomposition class we vary the continuous parameters (eigenvalues) but fix all discrete parameters (sizes of Jordan blocks).

In this article we study the algebraic geometric properties of decomposition varieties. As motivation we give first some examples. First of all the collection of regular semisimple elements is a decomposition class. At the other extreme, all adjoint orbits consisting of nilpotent elements, called *nilpotent orbits*, are decomposition classes. The determinantal varieties of $n \times n$-matrices occur as normalizations of decomposition varieties in $\mathfrak{sl}_n$ (see Section 9.6). The determinantal varieties of symmetric $2n \times 2n$-matrices are isomorphic to decomposition varieties in $\mathfrak{sp}_{2n}$ (see Section 9.5). Varieties defined by Pfaffians of fixed order of a generic anti-symmetric $n \times n$ matrix are decomposition varieties in $\mathfrak{so}_n$ (see Section 9.7). As a last example we mention the symplectic rank variety first appearing in a study of the subregular nilpotent variety [10], and thoroughly studied by Klimek *et al* [28]; it is a decomposition variety in $\mathfrak{sp}_{2n}$ (see Section 9.8).

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We have tried to generalize some known properties for determinantal varieties and for the closures of nilpotent orbits, called *nilpotent varieties*. For example, the properties that there are only finitely many nilpotent orbits, that they are all smooth and that each nilpotent variety is a union of nilpotent orbits all extend to decomposition classes (see Proposition 2.3). It is known that determinantal varieties and the normalization of any nilpotent variety (by Hinich and Panyushev) have rational singularities. We show that this generalizes: the *normalization* $\tilde{D}$ of any decomposition variety $\bar{D}$ has rational singularities (see Theorem 7.1). In general, decomposition varieties are not normal, not even for $\mathfrak{sl}_n$.

Fix an element x in a decomposition class D with Jordan decomposition $x = x_s + x_n$. Let L be the stabilizer of the semisimple part x_s, and $\mathfrak{c}$ the center of the Lie algebra $\mathfrak{l}$ of L. The subset $\mathfrak{c}^\circ$ of $\mathfrak{c}$ consisting of semisimple elements with stabilizer L is an affine open subset. The decomposition class containing $x = x_s + x_n$ is then

$$D := G \cdot (\mathfrak{c}^\circ + x_n).$$

The finite group $\Gamma_s := N_G L / L$ acts on $\mathfrak{c}$, stabilizing $\mathfrak{c}^\circ$. It also acts on the set of irreducible components of $N_G L \cdot x_n$; we call Γ the stabilizer in Γ_s of the component Lx_n. In general Γ_s is not equal to Γ. When it is equal we shall call the decomposition class *stable*. By general results due to Luna, the inclusion of $\mathfrak{c}$ into the closure $\bar{D}$ of D induces a map

$$\bar{\nu}\colon \mathfrak{c}/\Gamma_s \longrightarrow \bar{D}/\!/G$$

between quotient spaces which is just the *normalization* map. Here $\bar{D}/\!/G$ is the affine variety with coordinate ring $\mathrm{k}[\bar{D}]^G$. We classify the decomposition varieties such that $\bar{D}/\!/G$ is normal (see Theorem 3.1), completing work begun by Richardson [43].

In general all the fibres of the G-quotient map $\bar{\pi}\colon \bar{D} \longrightarrow \bar{D}/\!/G$ are of the same dimension, but they need neither be irreducible nor reduced. The G-quotient space of the normalization $\tilde{D}$ can be identified with $\mathfrak{c}/\Gamma$ and the fibres of the quotient map $\tilde{\pi}\colon \tilde{D} \longrightarrow \mathfrak{c}/\Gamma$ are all irreducible containing a dense orbit. We give criteria for when the fibres of $\tilde{\pi}$ are all reduced (see Theorem 7.5); if that is the case, then the quotient map is flat with Gorenstein fibres having rational singularities, and then $\tilde{D}$ is also Gorenstein.

Borho and Kraft showed that we can choose, for any decomposition variety $\bar{D}$, a parabolic subgroup P and a solvable ideal $\mathfrak{r}$ of $\mathfrak{p}$ such that $G\mathfrak{r} = \bar{D}$. We can assume that $\mathfrak{c}$ is contained in $\mathfrak{r}$, but L is not necessarily the Levi factor of P. We can write $\mathfrak{r} = \mathfrak{c} \oplus \mathfrak{n}$, for some nilpotent ideal $\mathfrak{n}$. The collapsing map

$$\gamma\colon G \times^P \mathfrak{r} \longrightarrow \bar{D}\colon g * y \longmapsto g \cdot y$$

of the homogeneous vector bundle on G/P with fibre $\mathfrak{r}$ is a projective morphism, but it is not birational in general. Since $G \times^P \mathfrak{r}$ is smooth, γ factors over the normalization of $\bar{D}$. We define the normal affine variety Y by its ring of global regular functions: $\mathrm{k}[Y] := \mathrm{k}[G \times^P \mathfrak{r}]$. We can choose P and $\mathfrak{r}$ in such a way that additionally the following properties hold. The finite group Γ acts on this remarkable variety Y (although it does

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not act on $G \times^P \mathfrak{r}$) having $\tilde{D}$ as orbit space Y/Γ (see Theorem 4.9). We show that Y is Gorenstein and has rational singularities (see Corollary 6.7), and that Y does not depend on the various possible good choices of P and $\mathfrak{r}$.

There are other interesting varieties and morphisms involved; their study is necessary for the understanding of various geometric, combinatorial or algebraic aspects of decomposition varieties. Using the smooth map

$$\theta: G \times^P \mathfrak{r} \longrightarrow \mathfrak{r}/\mathfrak{n} \simeq \mathfrak{c}: g * y \longmapsto y + \mathfrak{n}$$

and the collapsing $\gamma: G \times^P \mathfrak{r} \longrightarrow \bar{D}$, factorizing over the normalization $\tilde{D}$ by

$$\tilde{\gamma}: G \times^P \mathfrak{r} \longrightarrow \tilde{D}$$

and the normalization map $\nu: \tilde{D} \longrightarrow \bar{D}$, we get the following commutative diagram

$$\begin{array}{ccccc} G \times^P \mathfrak{r} & \xrightarrow{\tilde{\gamma}} & \tilde{D} & \xrightarrow{\nu} & \bar{D} \\ {\scriptstyle\theta}\downarrow & & \downarrow{\scriptstyle\tilde{\pi}} & & \downarrow{\scriptstyle\bar{\pi}} \\ \mathfrak{c} & \longrightarrow & \mathfrak{c}/\Gamma & \longrightarrow & \bar{D}/\!/G. \end{array}$$

If we write $\tilde{X}$ for the image of the product map $(\theta, \tilde{\gamma})$, then there is an induced map $G \times^P \mathfrak{r} \longrightarrow \tilde{X}$. This map is a resolution of singularities. We study the various properties of this diagram. Although various aspects of the diagram in special cases have been extensively studied by various authors, much is still unknown about the algebraic geometric properties.

In the special case of the decomposition class consisting of the regular semisimple elements we obtain the famous so-called *Grothendieck simultaneous resolution*. There $\mathfrak{c}$ is the Lie algebra of a maximal torus T contained in a Borel subgroup B and $\Gamma_s = \Gamma$ is the Weyl group W. Chevalley proved that the inclusion $\mathfrak{t} \subset \mathfrak{g}$ induces an isomorphism $\mathfrak{t}/W \simeq \mathfrak{g}/\!/G$ of quotient spaces. Now $P = B$, $\mathfrak{r} = \mathfrak{b}$ and the diagram simplifies to

$$\begin{array}{ccc} G \times^B \mathfrak{b} & \longrightarrow & \mathfrak{g} \\ \downarrow & & \downarrow \\ \mathfrak{t} & \longrightarrow & \mathfrak{g}/\!/G. \end{array}$$

This diagram has many good properties we sought to generalize. The vertical maps are flat, having irreducible and reduced fibres containing a dense orbit. The diagram gives rise to a birational proper morphism of $G \times^B \mathfrak{b}$ to the Cartesian product $\mathfrak{t} \times_{\mathfrak{g}/\!/G} \mathfrak{g}$ and an isomorphism of algebras of global regular functions:

$$\mathrm{k}[G \times^B \mathfrak{b}] \simeq \mathrm{k}[\mathfrak{t}] \otimes_{\mathrm{k}[\mathfrak{g}]^G} \otimes \mathrm{k}[\mathfrak{g}],$$

and indeed W acts on $\mathrm{k}[G \times^B \mathfrak{b}]$ with ring of W-invariants $\mathrm{k}[\mathfrak{g}]$.

Unfortunately many of the good properties of the Grothendieck simultaneous resolution are not present in our generalized set-up. One of the aims of this research is to understand what is actually happening. This study was already begun by Borho-Brylinski [7], Soergel [49] and Knop [30]. We analyze each of the occurring maps and morphisms. For example, we classify the parabolic subgroups that induce diagrams similar to the Grothendieck simultaneous resolution (the classification is up to normality results for nilpotent varieties in exceptional Lie algebras of types E_6, E_7 and E_8). More precisely, if $\mathfrak{r}$ is the solvable radical of the Lie algebra of $\mathfrak{p}$, we require in this classification that the decomposition variety $\bar{D} := G\mathfrak{r}$ is normal and the pull-back $\mathrm{k}[\bar{D}] \otimes_{\mathrm{k}[\bar{D}]^G} \mathrm{k}[\mathfrak{c}]$ identifies with the ring of global regular functions on $G \times^P \mathfrak{r}$ (see Theorem 7.9).

There are applications to the theory of primitive ideals of enveloping algebras and rings of differential operators. In the special case where decomposition classes contain semisimple elements, decomposition varieties and all the other varieties defined above have non-commutative analogs (see Section 8) involving quotients of the universal enveloping algebra of $\mathfrak{g}$ and rings of global differential operators on homogeneous spaces. See Section 8 and [7], [49].

This is one way where representation theory enters the picture. The same decomposition classes also appear in representation theory in connection to the orbit method as first studied by Dixmier. The geometry and topology of some special cases of the analog of decomposition classes in the group G also play an important role in Lusztig's theories of the generalized Springer correspondence and of the character sheafs, see [36] and [37]. The examples of decomposition varieties arising in Lusztig's study having beautiful properties, see Section 9.10.

In the last section we collect several examples and counter-examples. We show that most of the results above remain true for $\mathfrak{gl}_n$ if we allow the algebraically closed field k to be of positive characteristic, see Section 9.1.

2**. Decomposition classes.** We shall fix a reductive group G of rank r defined over an algebraically closed field k of characteristic zero. We denote the Lie algebra of an algebraic group by its corresponding gothic character, *e.g.*, the Lie algebra of G is $\mathfrak{g}$.

If K is a reductive group acting on an affine variety X, we denote the K-quotient space by $X//K$ with coordinate ring $\mathrm{k}[X]^K$. If K acts on any variety X with only closed orbits (for example if K is a finite group) then we write X/K for the K-orbit space $X//K$. If P is any closed subgroup of K acting on a variety Y, then it acts freely on $K \times Y$ by $p(k, y) := (kp^{-1}, py)$ and we let $K \times^P Y$ denote its orbit space. We denote the class containing (k, y) by $k * y$.

Whenever we fix an element $x \in \mathfrak{g}$ we shall adopt the following notation. When we write $x = x_s + x_n$, we mean the Jordan decomposition, *i.e.*, x_s is the semisimple part of x, x_n its nilpotent part and $[x_s, x_n] = 0$. We write $L := G_{x_s}$ for the stabilizer in G of the semisimple part of x, $\mathfrak{c}$ for the center of $\mathfrak{l}$ and $\mathfrak{c}^\circ$ for the open subset of $\mathfrak{c}$ of the elements in $\mathfrak{g}$ with stabilizer L. The finite group $\Gamma_s := N_G L/L$ acts on the collection of irreducible components of $(N_G L)x_n$. The stabilizer in Γ_s of the irreducible component Lx_n is denoted by Γ.

We say that x and y are in the same decomposition class if and only if there is a $g \in G$ such that $G_{x_s} = G_{gy_s}$ and $x_n = gy_n$. We can describe the decomposition class D containing x as $D = G(\mathfrak{c}^\circ + x_n)$. Obviously $\mathfrak{g}$ is a the disjoint union of its decomposition classes. We refer to [8], [5] and [13] for an introduction into the theory of decomposition classes. See [17, Ch. 7] for Lusztig-Spaltenstein's related notion of induction.

Our first aim is to show that decomposition classes are smooth, first shown in the literature by Lusztig [36, p. 216]. We shall then classify their orbits. To prove these result we shall apply Luna's fundamental lemma [2] at several places; for a somewhat different proof see our lecture notes [13]. We start with a useful lemma.

LEMMA 2.1. *Let M be a Levi subgroup of G. Define $\mathfrak{m}^\circ$ to be the collection of points $y \in \mathfrak{g}$ such that the stabilizer of its semisimple part y_s is contained in M.*

(i) Then $\mathfrak{m}^\circ$ is an M-stable affine open subset of $\mathfrak{m}$.

(ii) The morphism $G \times^M \mathfrak{m}^\circ \longrightarrow \mathfrak{g}\colon g * y \longmapsto gy$ induces a Cartesian diagram

$$\begin{array}{ccc} G \times^M \mathfrak{m}^\circ & \longrightarrow & \mathfrak{g} \\ \downarrow & \square & \downarrow{\scriptstyle \pi} \\ \mathfrak{m}^\circ /\!/ M & \longrightarrow & \mathfrak{g} /\!/ G \end{array}$$

with étale horizontal maps and vertical G-quotient maps.

PROOF. (i) Let f be the product of all the weights of $\mathfrak{g}/\mathfrak{m}$, considered as a homogeneous W_M-invariant polynomial function on $\mathfrak{t}$, where W_M is the Weyl group of M. By Chevalley's isomorphism $\mathfrak{t}/W_M \simeq \mathfrak{m}/\!/M$, we can extend f uniquely to an M-invariant homogeneous polynomial function F on $\mathfrak{m}$. We claim that $\mathfrak{m}^\circ$ is the M-stable principal affine open subset of $\mathfrak{m}$ defined by F. Let $y \in \mathfrak{m}^\circ$, we have to show that $F(y) \neq 0$. Since $\mathfrak{g}_{y_s} \subset \mathfrak{m}$ and y_s, y_n and y all centralize y_s, it follows that $y \in \mathfrak{m}$. Since F is M-invariant, it suffices to show that $F(y_s) \neq 0$ for $y_s \in \mathfrak{t}$. By assumption $[x_\alpha, y_s] = \alpha(y_s) \neq 0$ if α is a weight of $\mathfrak{g}/\mathfrak{m}$, where x_α is the corresponding root vector. So $f(y_s) = F(y_s) \neq 0$. The argument can be reversed, so this proves (i).

(ii) Let $y = y_s + y_n$, then it is easy to show that

$$\mathfrak{g} = [\mathfrak{g}, y_s] \oplus \mathfrak{g}_{y_s},$$

and $[\mathfrak{g}, y_s] \subset [\mathfrak{g}, y]$. The tangent map of $G \times \mathfrak{m} \longrightarrow \mathfrak{g}\colon (g, y) \longmapsto gy$ at $(1, y)$ is $(X, Y) \longmapsto [X, y] + Y$, where $X \in \mathfrak{g}$ and $Y \in \mathfrak{m}$. It follows that this tangent map is surjective at $(1, y)$ (and $(1, y_s)$) when $y \in \mathfrak{m}^\circ$. Hence the tangent map at $1 * y$ (and at $1 * y_s$) of $G \times^M \mathfrak{m}^\circ \longrightarrow \mathfrak{g}$ is also surjective if $y \in \mathfrak{m}^\circ$, hence is étale at $1 * y$ and $1 * y_s$ (by [22, Proposition 10.4]). In this situation we can apply Luna's fundamental lemma (see [2]), so that the diagram is indeed a pull-back diagram, with étale horizontal maps. And indeed $k[G \times^M \mathfrak{m}^\circ] \simeq k[\mathfrak{m}^\circ]^M$. ∎

In general, the intersection of a levi subalgebra $\mathfrak{m}$ with a decomposition variety for G is not necessarily a union of decomposition classes for M. But we shall use the previous lemma to show that the intersection of a G-decomposition class with the open affine subset $\mathfrak{m}^\circ$ is a union of M-decomposition classes.

LEMMA 2.2. *Let M be a Levi subgroup of G, and* $\mathfrak{m}^\circ$ *the principal open subset in* $\mathfrak{m}$ *as defined in the previous lemma. Let* D *be any decomposition class such that* $D \cap \mathfrak{m}^\circ$ *is not empty, or, equivalently, such that* $D \subset G\mathfrak{m}^\circ$.

Then the intersection $D \cap \mathfrak{m}^\circ$ *is a union of decomposition classes for M in* $\mathfrak{m}$*, all of equal dimension* $\dim D - \dim G/M$*, and the intersection is reduced.*

More precisely, take any element $x = x_s + x_n$ *in* $D \cap \mathfrak{m}^\circ$*. Write* $L := G_{x_s}$ *and* $\mathfrak{c}^\circ$ *for the collection of elements in* $\mathfrak{g}$ *having stabilizer L. Let* $\{L_1, \ldots, L_m\}$ *be the collection of different conjugates of L contained in M and containing a fixed maximal torus. Fix a* $g_i \in G$ *such that* $g_i L g_i^{-1} = L_i$ *and write* $\mathfrak{c}_i^\circ := g_i \mathfrak{c}^\circ$ *and* $y_i := g_i x_n$*. Then*

$$D \cap \mathfrak{m}^\circ = \bigcup_{i=1}^{m} M(\mathfrak{c}_i^\circ + N_G L_i y_i \cap \mathfrak{m}).$$

In particular, in the special case where $L = M$,

$$D \cap \mathfrak{m}^\circ = \mathfrak{c}^\circ + N_G L x_n.$$

PROOF. Both x_s and x_n are in $\mathfrak{l}$ by the theory of Jordan decompositions. By assumption $\mathfrak{l} \subset \mathfrak{m}$, hence $\mathfrak{c}^\circ$ is also equal to the collection of points in $\mathfrak{m}$ with centralizer exactly $\mathfrak{l}$. So the whole decomposition class $M(\mathfrak{c}^\circ + x_n)$ is contained in $\mathfrak{m}^\circ$. If $m(a_1 + x_n) = a_2 + x_n$, for $a_i \in \mathfrak{c}^\circ$ and $m \in M$, then $ma_1 = a_2$ and $mx_n = x_n$, by unicity of Jordan decomposition. Thus $m \in N_M L$, and the orbit $N_M L a_1$ is finite. We conclude that the decomposition class is a union of orbits, each of dimension $\dim G - \dim G_x$, and each orbit intersects $\mathfrak{c}^\circ + x_n$ in finitely many points. Hence its dimension is $\dim \mathfrak{c} + \dim G - \dim G_x$.

By the lemma above the natural map $G \times^M (D \cap \mathfrak{m}^\circ) \longrightarrow D$ is a pull-back of an étale map, hence is itself étale. It follows that the intersection $D \cap \mathfrak{m}^\circ$ is reduced, and each component has dimension $\dim D - \dim G/M$. In particular, $M(\mathfrak{c}^\circ + x_n)$ is one of those components, and all components are of this form.

Let $g(a + x_n) \in \mathfrak{m}^\circ \cap D$ with $a \in \mathfrak{c}^\circ$ and $g \in G$. Then $G_{ga} = gLg^{-1} \subset M$. Now take an $m \in M$ such that $mgL(mg)^{-1}$ contains the fixed maximal torus. Then $G_{mga} = L_i = G_{g_i a}$ for some i, and so $g_i^{-1} mg \in N_G L$ and $g_i^{-1} mgx_n \in N_G L x_n$. Thus $mga \in \mathfrak{c}_i^\circ$, $mgx_n \in N_G L_i y_i \cap \mathfrak{m}$ and $ga + gx_n \in M(\mathfrak{c}_i^\circ + N_G L_i y_i \cap \mathfrak{m})$. ∎

2.3. We shall now use the two lemmas to describe the basic orbit structure of decomposition classes and show that they are all smooth. The precise statement is longer than its proof.

PROPOSITION 2.3. *Let* D *be a decomposition class containing* $x \in \mathfrak{g}$*. Let* $x = x_s + x_n$*, L,* $\mathfrak{c}$*,* $\mathfrak{c}^\circ$ *and* Γ_s *be as defined before. Then:*

(i) The decomposition class D *is smooth.*

(ii) The natural action of Γ_s *on*

$$V^\circ := G \times^L (\mathfrak{c}^\circ + N_G L \cdot x_n)$$

is free, and its orbit space is isomorphic to D*, i.e.,*

$$D \simeq V^\circ/\Gamma_s = G \times^{N_G L} (\mathfrak{c}^\circ + N_G L \cdot x_n).$$

(iii) The morphism

$$\theta^\circ \colon V^\circ \longrightarrow \mathfrak{c}^\circ \colon g * (s+e) \longmapsto s,$$

where $g \in G$, $s \in \mathfrak{c}^\circ$, and $e \in N_G L \cdot x_n$, induces a morphism between orbit spaces

$$\pi^\circ \colon \mathcal{D} \simeq V^\circ / \Gamma_s \longrightarrow \mathfrak{c}^\circ / \Gamma_s,$$

and a Cartesian diagram

$$\begin{array}{ccc} V^\circ & \longrightarrow & \mathcal{D} \\ {\scriptstyle \theta^\circ}\downarrow & \square & \downarrow{\scriptstyle \pi^\circ} \\ \mathfrak{c}^\circ & \longrightarrow & \mathfrak{c}^\circ / \Gamma_s. \end{array}$$

Moreover, the horizontal maps are finite, étale Γ_s-quotient maps, and the vertical maps are constant on G-orbits.

(iv) Let $\bar{\mathcal{D}}$ be the closure of $\mathcal{D}$. Write the G-quotient map as

$$\bar{\pi} \colon \bar{\mathcal{D}} \longrightarrow \bar{\mathcal{D}} /\!/ G;$$

then $\bar{\pi}$ is the restriction to $\bar{\mathcal{D}}$ of the quotient map $\pi \colon \mathfrak{g} \longrightarrow \mathfrak{g} /\!/ G$, and we get Cartesian squares

$$\begin{array}{ccc} G \times^L (\mathfrak{c}^\circ + \overline{N_G L x_n}) & \longrightarrow & \bar{\mathcal{D}} \\ \downarrow & \square & \downarrow{\scriptstyle \bar{\pi}} \\ \mathfrak{c}^\circ & \longrightarrow & \bar{\mathcal{D}} /\!/ G, \end{array}$$

and

$$\begin{array}{ccc} G \times^{N_G L} (\mathfrak{c}^\circ + \overline{N_G L x_n}) & \hookrightarrow & \bar{\mathcal{D}} \\ \downarrow & \square & \downarrow{\scriptstyle \bar{\pi}} \\ \mathfrak{c}^\circ / \Gamma_s & \hookrightarrow & \bar{\mathcal{D}} /\!/ G, \end{array}$$

with étale horizontal maps and vertical G-quotient maps, where the horizontal maps are open immersions. It follows that the natural map $\mathfrak{c}/\Gamma_s \longrightarrow \bar{\mathcal{D}} /\!/ G$ is the normalization map.

PROOF. (i) follows from (ii). Since $\mathcal{D} \cap \mathfrak{l}^\circ = \mathfrak{c}^\circ + N_G L x_n$, by the previous lemma, we get that the morphism

$$G \times^L (\mathfrak{c}^\circ + N_G L x_n) \longrightarrow \mathcal{D}$$

is étale, as it is the pull-back of an étale map (by Lemma 2.1). Hence $\mathcal{D}$ is smooth. Since Γ_s acts freely on the left-hand side, we get another étale morphism

$$G \times^{N_G L} (\mathfrak{c}^\circ + N_G L x_n) \longrightarrow \mathcal{D}.$$

Since the latter is bijective, it is an isomorphism (see [1, p. 122]), hence (ii).

As Γ_s also acts freely on $\mathfrak{c}^\circ$, the quotient map $\mathfrak{c}^\circ \longrightarrow \mathfrak{c}^\circ / \Gamma_s$ and its pull-back $\mathfrak{c}^\circ \times_{\mathfrak{c}^\circ/\Gamma_s} \mathcal{D} \longrightarrow \mathcal{D}$ are étale. Since Γ_s acts freely on both V° and $\mathfrak{c}^\circ \times_{\mathfrak{c}^\circ/\Gamma_s} \mathcal{D}$ with isomorphic Γ_s-quotients, the induced map $V^\circ \longrightarrow \mathfrak{c}^\circ \times_{\mathfrak{c}^\circ/\Gamma_s} \mathcal{D}$ is a bijective étale morphism, hence is an isomorphism, and (iii) is shown.

For (iv), the previous lemma that $\bar{D} \cap \mathfrak{l}^\circ = (\mathfrak{c}^\circ + \overline{N_G L x_n})$ gives us étale morphisms

$$G \times^L (\mathfrak{c}^\circ + \overline{N_G L x_n}) \longrightarrow \bar{D}$$
$$G \times^{N_G L} (\mathfrak{c}^\circ + \overline{N_G L x_n}) \longrightarrow \bar{D}.$$

The latter is injective and therefore an open immersion (see [1, p. 122]). The second Cartesian diagram and subsequently the first diagram follow as in (iii) (or by using Luna's fundamental lemma). ∎

By restricting to a connected component we obtain a twin version of the proposition with an analogous proof.

PROPOSITION 2.4. *Assume the same notation as before. Recall that* Γ *is the stabilizer in* Γ_s *of the irreducible component* Lx_n *of* $N_G L \cdot x_n$. *Then:*

(i) The natural action of Γ *on*

$$\tilde{V}^\circ := G \times^L (\mathfrak{c}^\circ + Lx_n)$$

is free. Its orbit space is isomorphic to D, *i.e.,*

$$D \simeq \tilde{V}^\circ / \Gamma = G \times^{N_G L^\circ} (\mathfrak{c}^\circ + Lx_n),$$

where $N_G L^\circ$ *is the normalizer in* G *of the component* Lx_n *in* $N_G L \cdot x_n$.

(ii) The morphism

$$\tilde{\theta}^\circ : \tilde{V}^\circ \longrightarrow \mathfrak{c}^\circ : g * (s + e) \longmapsto s,$$

where $g \in G$, $s \in \mathfrak{c}^\circ$, *and* $e \in Lx_n$, *induces a morphism between orbit spaces*

$$\tilde{\pi}^\circ : D \simeq \tilde{V}^\circ / \Gamma \longrightarrow \mathfrak{c}^\circ / \Gamma,$$

and a Cartesian diagram

$$\begin{array}{ccc} \tilde{V}^\circ & \longrightarrow & D \\ {\scriptstyle \tilde{\theta}^\circ} \downarrow & \square & \downarrow {\scriptstyle \tilde{\pi}^\circ} \\ \mathfrak{c}^\circ & \longrightarrow & \mathfrak{c}^\circ / \Gamma \end{array}$$

such that the horizontal maps are finite, étale Γ*-quotient maps, and any fibre of the vertical maps is a* G*-orbit. Hence* $\tilde{\pi}^\circ$ *and* $\tilde{\theta}^\circ$ *are* G*-quotient maps, so* $\mathfrak{c}^\circ / \Gamma$ *classifies the orbits in* D.

(iii) Let $\tilde{D}$ *be the normalization of* $\bar{D}$. *The* G*-quotient variety* $\tilde{D} /\!/ G$ *can be identified with the* Γ*-quotient variety* $\mathfrak{c} / \Gamma$.

2.5. *Positive characteristics.* The definition of decomposition classes and the elementary properties given in the two propositions and their proofs remain valid over more general algebraically closed fields. We only need that for all $x \in \mathfrak{g}$ it is true that the Lie algebra of G_x is the centralizer of x and that the stabilizer of any semisimple element in $\mathfrak{g}$ is the Levi-component of some parabolic subgroup of G. This guarantees every adjoint orbit map is separable. These conditions are known to be satisfied for $G = GL_n$. Let p be the characteristic of k, and suppose $\mathfrak{g}$ is simple. Then these conditions are satisfied if and only if p does not divide n when $\mathfrak{g} = \mathfrak{sl}_n$, $p \neq 2$ if $\mathfrak{g} \neq A_r$, $p \neq 3$ if $\mathfrak{g}$ is exceptional and $p \neq 5$ if $\mathfrak{g} = E_8$.

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2.6. *The finite groups* Γ_s *and* Γ. From the two previous propositions, it follows that $\mathfrak{c}/\Gamma_s$ and $\mathfrak{c}/\Gamma$ play important roles in the theory of decomposition varieties. The first quotient can be identified with the normalization of $\bar{D}/\!/G$ and the latter with $\tilde{D}/\!/G$. These G-quotient spaces classify closed orbits, and under the normalization map $\nu\colon \tilde{D} \longrightarrow \bar{D}$ closed orbits can be covered by several closed orbits. This happens generically if and only if $\Gamma_s \neq \Gamma$.

Recall from the introduction that the decomposition class containing $x = x_s + x_n$ is called *stable* if $Lx_n = N_G L \cdot x_n$. This desirable property does not depend on the choice of x. For example the SL_4-decomposition class containing the matrix

$$x = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \in \mathfrak{sl}_4$$

is unstable. We give some characterizations of stableness.

COROLLARY 2.7. *The following are equivalent for a decomposition variety* D *containing* x*:*

(i) D *is stable;*
(ii) $\Gamma_s = \Gamma$*;*
(iii) V° *is irreducible;*
(iv) $\tilde{D}/\!/G$ *is the normalization of* $\bar{D}/\!/G$*;*
(v) The intersection of Gx *with* $\mathfrak{c} + x_n$ *is just* $\Gamma_s x_s + x_n$.

PROOF. This all follows from the two propositions. ∎

In almost, but not all, cases, Γ_s acts as a reflection group on $\mathfrak{c}$, so $\mathfrak{c}/\Gamma_s$ is smooth. In general Γ_s contains a normal subgroup Γ_s' acting as a reflection group on $\mathfrak{c}$ with an elementary abelian group $(\mathbb{Z}/2\mathbb{Z})^\ell$ as factor group Γ_s/Γ_s'. For example, if $\mathfrak{g}$ is simple of type A_r, B_r, C_r, F_4 or G_2 then Γ_s is a reflection group, and for the exceptional simple Lie algebras, the order of Γ_s/Γ_s' is either one or two. For a precise description of Γ_s and Γ_s' see Howlett [25]. If $G = GL_n$, then Γ is always a reflection group (see Section 9.1).

Write $\kappa_s\colon \mathfrak{c} \longrightarrow \mathfrak{c}/\Gamma_s$ and $\kappa\colon \mathfrak{c} \longrightarrow \mathfrak{c}/\Gamma$ for the quotient maps. Then the restriction to $\mathfrak{c}$ of $\bar{\pi}$ is surjective and factors as

$$\mathfrak{c} \xrightarrow{\kappa_s} \mathfrak{c}/\Gamma_s \xrightarrow{\bar{\nu}} \bar{D}/\!/G.$$

Moreover, we already showed Luna's result that $\bar{\nu}$ identifies with the normalization map.

LEMMA 2.8. *The following three conditions are equivalent.*

(refl.*1)* $\kappa\colon \mathfrak{c} \longrightarrow \mathfrak{c}/\Gamma$ *is flat;*
(refl.*2)* Γ *acts as a reflection group on* $\mathfrak{c}$*;*
(refl.*3)* $\mathfrak{c}/\Gamma$ *is smooth.*

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PROOF. The Killing form on $\mathfrak{g}$ restricts to a non-degenerate bilinear form on $\mathfrak{l}$ such that $\mathfrak{l} = \mathfrak{c} \oplus [\mathfrak{l},\mathfrak{l}]$ is an orthogonal direct sum. In particular $\mathfrak{c}$ carries a non-degenerate form which is Γ_s and Γ invariant. So if Γ (or Γ_s) acts as a pseudo-reflection group, it even acts as a reflection group. Now apply Chevalley's theorem. ∎

We shall say that (refl) is satisfied, if one of the conditions in the lemma is satisfied.

2.10. The most important decomposition classes are stable. For any integer, d the union of all adjoint orbits of dimension d is a locally closed subvariety of $\mathfrak{g}$ whose irreducible components are called *sheets*. They are unions of decomposition classes. The class $D(x)$ is dense in some sheet if and only if the nilpotent orbit Lx_n is in $[\mathfrak{l},\mathfrak{l}]$ (see [5, 4.3 Satz]), *i.e.*, if Lx_n itself is a sheet in $[\mathfrak{l},\mathfrak{l}]$. Sheets are classified by Spaltenstein and Elashvili. If a sheet contains a semi-simple element it is called a *Dixmier sheet*. Borho [5, 4.5 Lemma] showed that the dense decomposition class in a sheet is stable. Also the decomposition class $D(x)$ of a regular or a nilpotent element x is stable.

3. **The normality question for $\bar{D}//G$.** The closure of a decomposition class, *i.e.*, a decomposition variety, is usually not normal; for example, when the decomposition class is not stable, or when it is stable but $\overline{Lx_n}$ is not normal (by Proposition 2.3(iv)). Richardson [43] discussed the question whether the quotient space $\bar{D}//G$ is normal, or equivalently whether $\mathfrak{c}/\Gamma_s$ equals $\bar{D}//G$. He answered this question for the classical simple Lie algebras using Howlett's calculations [25]. His method extends to the exceptional Lie algebras. We present the classification. The closure $\bar{D}(x_s)$ of the decomposition class containing x_s is contained in the closure $\bar{D}(x)$ of the decomposition class containing x and both have the same quotient spaces $\bar{D}(x_s)//G \simeq \bar{D}(x)//G$, so the question of normality of $\bar{D}(x)//G$ only depends on the semisimple part of x. But the normality of $\bar{D}(x)$ itself depends on both x_s and x_n.

Recall that $\bar{\pi}\colon \bar{D} \to \bar{D}//G$ is the quotient map, and write

$$\tilde{\pi}\colon \tilde{D} \to \tilde{D}//G \simeq \mathfrak{c}/\Gamma$$

for the quotient map of the normalization. We shall show later that both quotient maps are equidimensional.

In the following we shall always use Bourbaki's enumeration of simple roots, say $\alpha_1, \ldots, \alpha_r$.

THEOREM 3.1. *Let $\mathfrak{g}$ be simple and $x \in D$. As usual $L := G_{x_s}$.*

(i) If $\bar{D}//G$ is normal then $\bar{D}//G$ is also smooth.

(ii) The quotient $\bar{D}//G$ is normal if and only if L is either a maximal torus or the full group G or

- *(a) of type pA_q with $p(q+1) = r+1$ and $q \geq 0$, when $\mathfrak{g} = A_r$;*
- *(b) its type is $pA_q + B_j$ with $j, p, q \geq 0$ and $r = j + p(q+1)$, when $\mathfrak{g} = B_r$ (here B_1 is distinguished from A_1 by root length);*
- *(c) its type is $pA_q + C_j$ with $j, p, q \geq 0$ and $r = j + p(q+1)$, when $\mathfrak{g} = C_r$ (here C_1 is distinguished from A_1 by root length);*

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(d) *its type is* $pA_q + D_j$ *with* $j \geq 2$ *and* $p(q+1)+j = r$*, or its type is* pA_q *with* $p(q+1) = r$ *and* q *is odd, when* $\mathfrak{g} = D_r$ *with* $r \geq 4$ *(here* L *has component* D_j *for* $j \geq 2$ *if and only if* α_{r-1} *and* α_r *are the roots of* L*);*

(e6) *its type is* $A_2 + A_2$, A_5 *or* $A_2 + A_2 + A_1$*, when* $\mathfrak{g} = E_6$*;*

(e7) *its type is* A_5 *(with simple roots* α_2, α_4, α_5, α_6 *and* α_7*),* $3A_1 + A_2$ *(with simple roots* α_1, α_2, α_3, α_5, α_7*) or when its semisimple rank is* 6*, when* $\mathfrak{g} = E_7$*;*

(e8) *its semisimple rank is* 7*, when* $\mathfrak{g} = E_8$*;*

(f) *its type is* A_2 *(both possible cases) or if the semisimple rank is* 3*, when* $\mathfrak{g} = F_4$*;*

(g) *any other type, when* $\mathfrak{g} = G_2$.

PROOF. By the remarks before the theorem it follows that we can assume that x is semisimple, *i.e.*, $\mathcal{D} = G(\mathfrak{c}^\circ)$. Richardson [43] showed that $\bar{\mathcal{D}}//G$ is normal if and only if the map

$$\mathrm{k}[\mathfrak{t}]^W \longrightarrow \mathrm{k}[\mathfrak{c}]^{\Gamma_s}$$

induced by restriction is surjective. He also showed (a), (b), (c) and (d). His method of proof extends to the exceptional Lie algebras as well, using the tables of Γ_s compiled by Howlett [25].

If L is a maximal torus, then $\bar{\mathcal{D}} = \mathfrak{g}$. If $L = G$ then $\bar{\mathcal{D}} = \{0\}$, so in both cases normality is clear. Suppose that $0 < \dim \mathfrak{c} < r$.

If $\dim \mathfrak{c} = 1$ then Γ_s is either trivial or $\{1, -1\}$. In the first case there is a linear Γ_s-invariant on $\mathfrak{c}$ but no linear W-invariant on $\mathfrak{t}$, so the map between invariant rings is not surjective and $\bar{\mathcal{D}}//G$ is not normal. If $\Gamma_s = \{-1, 1\}$, then $\mathrm{k}[\mathfrak{c}]^{\Gamma_s}$ is generated by a quadratic invariant. On the other hand, since the quadratic form associated to the Killing form does not vanish on $\bar{\mathcal{D}}$, we get that its restriction to $\mathfrak{c}$ is non-zero. So the restriction map is surjective and hence $\bar{\mathcal{D}}//G$ is normal. Summarizing, if $\dim \mathfrak{c} = 1$, then $\bar{\mathcal{D}}//G$ is normal if and only if Γ_s is non-trivial. For example, if $-1 \in W$ then Γ_s is non-trivial. For the remaining cases we check Howlett's tables.

Next we restrict to the case where $\mathfrak{g}$ is exceptional and $1 < \dim \mathfrak{c} < r$. If Γ_s is a reflection group but not irreducible, then $\mathrm{k}[\mathfrak{c}]^{\Gamma_s}$ has either a linear invariant or two independent quadratic invariants. Since $\mathrm{k}[\mathfrak{t}]^W$ has no linear W-invariant and no pair of independent quadratic invariants, it follows that the map between invariant rings is not surjective and so $\bar{\mathcal{D}}//G$ is not normal. Suppose next that Γ_s is an irreducible reflection group. If it is of type A, B, C or D, then Γ_s has at least a degree 3 or a degree 4 generating invariant. But W does not (since $\mathfrak{g}$ is exceptional), so $\bar{\mathcal{D}}//G$ is not normal in that case. So suppose that Γ_s is of exceptional type. Then it has a degree 6 generating invariant (since it is not E_8), hence two linearly independent degree six invariants. But if $\mathfrak{g}$ is of type E_8, then W has only one independent degree 6 invariant, hence $\bar{\mathcal{D}}//G$ is not normal. So we assume now that $\mathfrak{g}$ is not of type E_8. Then the type of Γ_s can only be F_4 or G_2. From Howlett's tables it follows that F_4 does not occur as a Γ_s, and G_2 occurs 5 times: For $\mathfrak{g} = E_6$ and L of type $2A_2$, for $\mathfrak{g} = E_7$ and L is of type A_5 or of type $3A_1 \times A_2$ and for $\mathfrak{g} = F_4$ and both cases where L is of type A_2. We'll show that in each of these five cases $\bar{\mathcal{D}}//G$ is normal.

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Using the Killing form we can define the gradient of a G-invariant function f on $\mathfrak{g}$, which will be a G-equivariant vector field $\operatorname{grad} f\colon \mathfrak{g} \longrightarrow \mathfrak{g}$. Analogously, we can use the restriction of the Killing form to $\mathfrak{c}$ to define the gradient $\operatorname{grad} g\colon \mathfrak{c} \longrightarrow \mathfrak{c}$ of a Γ_s-invariant g on $\mathfrak{c}$. If $\bar{f}$ and $\overline{\operatorname{grad} f}$ are the restrictions to $\mathfrak{c}$, we have that

$$\operatorname{grad} \bar{f} = \overline{\operatorname{grad} f}.$$

We now consider the five cases, where the type of Γ_s is G_2. First of all the Killing quadratic form f_2 restricts to a quadratic generator g_2 of $k[\mathfrak{c}]^{\Gamma_s}$. Let f_6 be any degree six generating G-invariant on $\mathfrak{g}$. The so-called Richardson nilpotent orbit corresponding to L is contained in $\bar{D}$. Richardson [44] showed that in each of the five cases $\operatorname{grad} f_6$ does not vanish on the Richardson orbit. In particular $\operatorname{grad} f_6$ is non-zero on $\bar{D} = \overline{G\mathfrak{c}^\circ}$, so necessarily non-zero on $\mathfrak{c}$. This does not depend on the choice of the generator f_6. So $\operatorname{grad} \overline{f_6} \neq 0$, and therefore $\overline{f_6} \neq 0$. Suppose the restriction $\overline{f_6}$ is not a Γ_s-generator. Then $\overline{f_6} = cg_2^3$, for some constant c. But then $f_6 - cf_2^3$ is also a degree 6 generating G-invariant, but now with $\operatorname{grad} \overline{f_6 - cf_2^3} = 0$, which contradicts Richardson's non-vanishing result. So $\overline{f_6}$ is a Γ_s-generator, and so the map $k[\mathfrak{g}]^G \simeq k[\mathfrak{t}]^W \longrightarrow k[\mathfrak{c}]^{\Gamma_s}$ is surjective, and $\bar{D}//G$ is normal in all five cases.

Next suppose that $\mathfrak{g}$ is exceptional, $1 < \dim \mathfrak{c} < r$ and that Γ_s is not a reflection group. That can not happen when $\mathfrak{g} = F_4$. According to Howlett's tables Γ_s contains an index two normal subgroup Γ_s' that acts as a reflection group on $\mathfrak{c}$ and an order two subgroup V such that $V\Gamma_s' = \Gamma_s$. Suppose that Γ_s' is not irreducible, then there is either a linear and a quadratic generating Γ_s' invariant, or two generating quadratic Γ_s' invariants. It is no harm to assume that both are eigenvectors for the $\Gamma_s/\Gamma_s' = \{-1, 1\}$ action. One of the two quadratic Γ_s' invariants is the restriction of the Killing quadratic form, hence is already invariant by Γ_s. In either case, there are two independent Γ_s-invariants of degree 4. But there are no two W-invariants of degree 4 on $\mathfrak{t}$, hence the map between invariant rings is not surjective and so $\bar{D}//G$ is not normal.

Consider next the special case where $\mathfrak{g} = E_8$ and Γ_s' of type A_s, then there are at least two independent degree six Γ_s-invariants (by the same argument as before), but only one degree six W-invariant. Hence non-normality. Apart from this special case, it follows from Howlett's tables that there are only two remaining cases to be considered.

The first is where $\mathfrak{g} = E_8$ and L of type A_2. Then Γ_s' is of type E_6. There are two generating invariants for Γ_s' of degree five and six. If Γ_s/Γ_s' does not act trivially on either of them, then it acts trivially on its product. Since $k[\mathfrak{t}]^W$ contains no two independent elements of degree six nor invariants of degree five or eleven it follows that $k[\mathfrak{t}]^W \longrightarrow k[\mathfrak{c}]^{\Gamma_s}$ cannot be surjective, and so $\bar{D}//G$ is not normal.

The last remaining case to be considered is where $\mathfrak{g} = E_7$ and L of type A_2 (roots α_1 and α_3). Here Γ_s' is of type A_5. There are two generating invariants for Γ_s' of degree three and four. If Γ_s/Γ_s' does not act trivially on either of them, then it acts trivially on its product. Since $k[\mathfrak{t}]^W$ contains no two independent elements of degree four nor elements of degree three or seven it follows that $k[\mathfrak{t}]^W \longrightarrow k[\mathfrak{c}]^{\Gamma_s}$ cannot be surjective, and so

$\bar{D}//G$ is not normal. We have now considered all possible cases, hence the proof of the classification in (ii) is complete.

We saw that whenever Γ_s is not a reflection group, then $\bar{D}//G$ is not normal, whence (i). ∎

REMARK. We give the type of the reflection groups Γ_s coming up in the classification. If L is a maximal torus, then $\Gamma_s = W$. If $L = G$ then $\Gamma_s = \{1\}$ and if the semisimple rank of L is $r-1$, then $\Gamma_s = \{1, -1\}$. For the exceptional Lie algebras there remain five cases, and in each of these cases, the type of Γ_s is G_2. For the classical Lie algebras we get that in case (a) Γ_s is of type A_{p-1} and, in case (b), (c) and (d), Γ_s is of type $B_p(= C_p)$.

Even when $\bar{D}//G$ is not normal it can happen that the normalization map is still a bijection, so that $\mathfrak{c}/\Gamma_s$ still parametrizes the closed G-orbits in $\bar{D}$. This situation is described in the next lemma.

LEMMA 3.2. *Let $x \in D$ and $L := G_{x_s}$. Let $z \in \mathfrak{c}$ with stabilizer $M := G_z$. The following three statements are equivalent.*

(i) The variety $\bar{D}//G$ is unibranch at $\bar{\pi}(z)$, i.e., the fibre of $\bar{\pi}(z) \in \bar{D}//G$ under the normalization map $\bar{\nu}\colon \mathfrak{c}/\Gamma_s \longrightarrow \bar{D}//G$ consists of only one element;

(ii) We have $Gz \cap \mathfrak{c} = \Gamma_s z$;

(iii) For any $g \in G$ such that $gLg^{-1} \subset M$ there exists an $m \in M$ such that $mgL(mg)^{-1} = L$, i.e., such that $mg \in N_G L$.

PROOF. It is clear that the fibre of $\bar{\pi}(z)$ consists exactly of the Γ_s-orbits in $Gz\cap\mathfrak{c}$, so (i) and (ii) are equivalent. Suppose (ii) holds and that $gLg^{-1} \subset M$. Then $L \subset g^{-1}Mg = G_{g^{-1}z}$, *i.e.*, $g^{-1}z \in \mathfrak{c}$. By assumption there is an $n \in N_G L$ such that $g^{-1}z = nz$, or $gn \in M$. Take $m := (gn)^{-1}$, then $mgL(mg)^{-1} = n^{-1}Ln = L$. So (iii) follows.

Suppose (iii) holds and $z' := gz \in \mathfrak{c}$. Then $L \subset G_{z'} = gMg^{-1}$, or $g^{-1}Lg \subset M$. By assumption there exists an $m \in M$ such that $n := mg^{-1} \in N_G L$, so $z' = gz = gm^{-1}z = n^{-1}z \in N_G Lz$. Hence (ii) holds. ∎

REMARK. The condition (iii) is often easy to check. It holds when M contains only one conjugacy class of Levi subgroups isomorphic to L. For example, if L is of semisimple type $A_1 \times A_2 \times A_3$ in G of type A_8, then the normalization map $\mathfrak{c} \longrightarrow \bar{D}//G$ is bijective (here Γ_s is trivial), but not an isomorphism.

4. Collapsing of a vector bundle.

4.1. *Indicators and notation.* A decomposition class is completely determined by giving a Levi subgroup L and a nilpotent $N_G L$-orbit in $[\mathfrak{l}, \mathfrak{l}]$, where L is only determined up to conjugacy. Nilpotent conjugacy classes Ge in $\mathfrak{g}$ were classified by Dynkin and Kostant, using certain weighted Dynkin diagrams constructed from $\mathfrak{sl}_2$-triples. The weighted Dynkin diagram determines a parabolic group P and a nilpotent ideal $\mathfrak{n} \subset \mathfrak{p}$ together with a collapsing map $G \times^P \mathfrak{n} \longrightarrow \overline{Ge}$ which happens to be a resolution of singularities.

We shall call an *indicator* any sequence $[n_1, n_2, \ldots, n_r]$ of r numbers in $\{0, 1, 2\}$, where additionally some of the n_i's are underlined, and where the non-underlined n_i have value 2.

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Any indicator determines a decomposition class in the following way. The underlined indices correspond to the simple roots of a Levi subgroup L. Let $\mathfrak{c}^\circ$ be the collection of points in $\mathfrak{g}$ with stabilizer L. The values n_i at underlined positions define a graded Lie algebra structure $\mathfrak{l} = \bigoplus_i \mathfrak{l}_i$, by imposing that a simple root vector of $\mathfrak{l}$ corresponding to the simple root α_i has degree n_i. There is a nilpotent e in $\mathfrak{n}_1 := \bigoplus_{i\geq 2} \mathfrak{l}_i$ with the property that the intersection $\mathfrak{n}_1^\circ$ of Le with $\mathfrak{n}_1$ is dense in $\mathfrak{n}_1$. Then we define $D := G(\mathfrak{c}^\circ + e)$ as the corresponding decomposition class.

For a given decomposition class there may be many different indicators. For example, take $\mathfrak{g} = \mathfrak{sl}_3$. The indicator $[2, 2]$ defines the decomposition class of regular semisimple elements, $[\underline{2}, \underline{2}]$ that of the regular nilpotent elements and $[\underline{2}, 2]$, $[2, \underline{2}]$ both define the class of non-semisimple, non-nilpotent, regular elements. The indicators $[\underline{2}, \underline{1}]$, $[\underline{1}, \underline{2}]$, $[\underline{2}, \underline{0}]$, $[\underline{0}, \underline{2}]$, $[\underline{1}, \underline{1}]$, $[\underline{1}, \underline{0}]$, and $[\underline{0}, \underline{1}]$ all define the non-zero, non-regular, nilpotent orbit, and $[\underline{0}, \underline{0}]$ defines the zero-orbit. The indicators $[2, \underline{1}]$, $[\underline{1}, 2]$, $[2, \underline{0}]$ and $[\underline{0}, 2]$ all define the decomposition class of non-zero, non-regular, semi-simple elements.

An indicator determines much more than just a decomposition class. We introduce more *notation*. $\mathfrak{p}_1 := \bigoplus_{i\geq 0} \mathfrak{l}_i$ is a parabolic subalgebra in $\mathfrak{l}$ corresponding to a parabolic subgroup $P_1 \subset L$. Define analogously a grading on $\mathfrak{g}$ by using all n_i's. Write

$$\mathfrak{n} := \bigoplus_{i\geq 2} \mathfrak{g}_i \subset \mathfrak{p} := \bigoplus_{i\geq 0} \mathfrak{g}_i$$

with corresponding parabolic subgroup $P \subset G$. Next write $\mathfrak{r} := \mathfrak{c} + \mathfrak{n}$, where $\mathfrak{c}$ is the center of $\mathfrak{l}$. Generally the Levi factor of P is not L.

If we change the indicator by putting all underlined n_i equal to zero, we get a new attached parabolic subgroup $P_2 \subset G$ this time having L as Levi factor. Let $\mathfrak{n}_2$ be the nilradical of $\mathfrak{p}_2$, so

$$\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2.$$

We get several collapsing maps of homogeneous vector bundles:

$$\gamma_1 \colon L \times^{P_1} \mathfrak{n}_1 \longrightarrow \mathfrak{l} \colon g * y \longmapsto gy,$$

with image $\overline{Le}$, and

$$\gamma \colon G \times^P \mathfrak{r} \longrightarrow \mathfrak{g} \colon g * y \longmapsto gy,$$

with image $\bar{D}$, *i.e.*, $\bar{D} = G\mathfrak{r}$, where the decomposition class $D = G(\mathfrak{c}^\circ + e)$ is as defined before. Both collapsings are proper morphisms. We write

$$Y := G \times^P \mathfrak{r},$$

and define a map

$$\theta \colon Y \longrightarrow \mathfrak{c} \colon g * (s + u) \longmapsto s,$$

for $g \in G$, $s \in \mathfrak{c}$ and $u \in \mathfrak{n}$.

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We write X for the image of $\mathcal{Y}$ under the product map $(\theta, \gamma)\colon \mathcal{Y} \longrightarrow \mathfrak{c} \times \bar{D}$; it is closed and an irreducible component of the Cartesian product $V := \mathfrak{c} \times_{\bar{D}//G} \bar{D}$:

$$\begin{array}{ccccccc} \mathcal{Y} & \longrightarrow & X & \subset & V & \longrightarrow & \bar{D} \\ & & & & \big\downarrow & \square & \big\downarrow \\ & & & & \mathfrak{c} & \longrightarrow & \bar{D}//G. \end{array}$$

Then V contains V° (as defined in Proposition 2.3) as an open, smooth (but not necessarily dense) subvariety.

Recall $\nu_A\colon \tilde{A} \longrightarrow A$ is the normalization map of any variety A. Since $\mathcal{Y}$ is smooth, any morphism $\phi\colon \mathcal{Y} \longrightarrow A$ factorizes over $\tilde{A}$ as

$$\mathcal{Y} \xrightarrow{\tilde{\phi}} \tilde{A} \xrightarrow{\nu_A} A,$$

for a unique map $\tilde{\phi}\colon \mathcal{Y} \longrightarrow \tilde{A}$.

Let $\tilde{X}$ be the image of $\mathcal{Y}$ under the product map $(\theta, \tilde{\gamma})\colon \mathcal{Y} \longrightarrow \mathfrak{c} \times \tilde{D}$; it is closed and is an irreducible component of the Cartesian product $\tilde{V} := \mathfrak{c} \times_{\tilde{D}//G} \tilde{D}$ from the following diagram:

$$\begin{array}{ccccccc} \mathcal{Y} & \longrightarrow & \tilde{X} & \subset & \tilde{V} & \longrightarrow & \tilde{D} \\ & & & & \big\downarrow & \square & \big\downarrow \\ & & & & \mathfrak{c} & \longrightarrow & \tilde{D}//G. \end{array}$$

Then $\tilde{V}$ contains $\tilde{V}^\circ$, as defined in Proposition 2.4, as a dense smooth subvariety.

The algebra $\mathrm{k}[\mathcal{Y}] \simeq \mathrm{k}[G \times \mathfrak{r}]^P$ of global regular functions on $\mathcal{Y}$ is finitely generated and defines a normal affine variety Y, the *affinization* of $\mathcal{Y}$. Let

$$\alpha\colon \mathcal{Y} \longrightarrow Y := \operatorname{Spec} \mathrm{k}[\mathcal{Y}]$$

be the canonical map, which is proper and birational. We have the property that any morphism $\phi\colon \mathcal{Y} \longrightarrow A$ of $\mathcal{Y}$ to an affine variety A factors through $\alpha\colon \mathcal{Y} \longrightarrow Y$ as

$$\mathcal{Y} \xrightarrow{\alpha} Y \xrightarrow{\phi_a} A,$$

for a unique morphism $\phi_a\colon Y \longrightarrow A$. In particular we get surjective maps $\theta_a\colon Y \longrightarrow \mathfrak{c}$, $\gamma_a\colon Y \longrightarrow \bar{D}$, $\tilde{\gamma}_a\colon Y \longrightarrow \tilde{D}$, $(\theta, \gamma)_a\colon Y \longrightarrow X$ and $(\theta, \tilde{\gamma})_a\colon Y \longrightarrow \tilde{X}$.

All these varieties and morphisms are determined by a single indicator. We shall consider how much only depends on the decomposition class.

4.2. *Good indicators.* We shall single out among all indicators the most useful ones. We shall call a indicator *good* if the collapsing $\gamma_1\colon L \times^{P_1} \mathfrak{n}_1 \longrightarrow \overline{Le}$ is birational, or, equivalently, if $L_e \subset P_1$ or, equivalently, if γ_1 restricts to an isomorphism

$$L \times^{P_1} \mathfrak{n}_1^\circ \simeq Le.$$

The indicator corresponding to the weighted Dynkin diagram of a nilpotent orbit (all indices underlined) is a good diagram for this orbit, but there might be more good indicators.

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LEMMA 4.3. *Any decomposition class $\mathcal{D}$ is associated to at least one good indicator.*

PROOF. To define one, pick an $x = x_s + x_n \in \mathcal{D}$, and define $L = G_{x_s}$ and $\mathfrak{c}^\circ$ as before. Then n_i will be underlined iff α_i is a root of $\mathfrak{l}$, and we'll put $n_i := 2$ whenever α_i is not a root of L. The nilpotent orbit Lx_n has an ordinary weighted Dynkin diagram, as defined by Dynkin and Kostant. These give the underlined n_i, corresponding to the simple roots of $\mathfrak{l}$. Then $\mathcal{D}$ corresponds to this indicator. It follows from the general theory of nilpotent orbits, that such an indicator is good (see [16]). Such an indicator for a decomposition class is not uniquely determined, but it only depends on the choice of $L \supset T$ and on the choice of a component of $N_G L \cdot e$. ∎

4.4. *Description of an open set of $\mathcal{Y}$. From now on we shall fix a good indicator and all the notation that comes with it.* We show that an open subset of the vector bundle $\mathcal{Y}$ can be identified with the variety $\tilde{V}^\circ$ defined in Proposition 2.4 and that the map $\tilde{\theta}^\circ$ (also defined there) identifies with the restriction of $\tilde{\theta}$ (defined in Section 4.1).

PROPOSITION 4.5. *We fix a good indicator and corresponding notation as in Section 4.1. Then:*

(i) The open dense subset $\mathcal{Y}^\circ := G \times^P (\mathfrak{c}^\circ + \mathfrak{n}_1^\circ + \mathfrak{n}_2)$ of $\mathcal{Y} := G \times^P \mathfrak{r}$ is isomorphic to the Γ-stable irreducible component $\tilde{V}^\circ = G \times^L (\mathfrak{c}^\circ + Le)$ of $V^\circ = G \times^L (\mathfrak{c}^\circ + N_G L \cdot e)$ (see Proposition 2.4), where e is any element of $\mathfrak{n}_1^\circ$.

(ii) $\mathcal{Y}^\circ$ identifies with the preimage of $\mathcal{D}$ under the collapsing $\gamma \colon \mathcal{Y} \to \bar{\mathcal{D}}$,

$$\begin{array}{ccc} \mathcal{Y}^\circ & \longrightarrow & \mathcal{D} \\ \cap & \square & \cap \\ \mathcal{Y} & \longrightarrow & \bar{\mathcal{D}}, \end{array}$$

and the restriction to $\mathcal{Y}^\circ$ of the collapsing γ is a Galois covering of $\mathcal{D}$ with Galois group Γ.

(iii) The commutative diagram

$$\begin{array}{ccc} \mathcal{Y}^\circ & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathfrak{c} & \longrightarrow & \mathfrak{c}/\Gamma \end{array}$$

identifies with the Cartesian square in Proposition 2.4(ii).

PROOF. With N_2 the unipotent subgroup of P_2 with Lie algebra $\mathfrak{n}_2$, we claim that the following map is an isomorphism

$$N_2 \times \mathfrak{c}^\circ \times \mathfrak{n}_1^\circ \xrightarrow{\sim} \mathfrak{c}^\circ \times \mathfrak{n}_1^\circ \times \mathfrak{n}_2 \colon (n, z) \mapsto n \cdot z.$$

We shall prove the claim first. Let $(s, y) \in \mathfrak{c}^\circ \times \mathfrak{n}_1^\circ$. Since N_2 is unipotent, the orbit $N_2 \cdot (s, y) \subset (s, y) \times \mathfrak{n}_2$ is closed by the Kostant-Rosenlicht lemma. Let $n \in N_2$ be in the stabilizer of (s, y). By the unicity of Jordan decompositions it follows that n fixes s

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and y separately. So $n \in N_2 \cap L$ and hence n is trivial. It follows that the orbit equals $(s, y) \times \mathfrak{n}_2$ and that the map in the lemma is bijective. Since both varieties are smooth the claim follows.

We have the following sequence of isomorphisms:

$$\begin{aligned} G \times^L (\mathfrak{c}^\circ + Ly) &\simeq G \times^L (\mathfrak{c}^\circ \times L \times^{P_1} \mathfrak{n}_1^\circ) \\ &\simeq G \times^{P_1} (\mathfrak{c}^\circ + \mathfrak{n}_1^\circ) \\ &\simeq G \times^P (\mathfrak{c}^\circ + \mathfrak{n}_1^\circ + \mathfrak{n}_2). \end{aligned}$$

Here the first isomorphism comes from the goodness of the indicator, and the last isomorphism follows from the claim. This shows (i). The remaining statements follow from Propositions 2.3 and 2.4. ∎

COROLLARY 4.6. *The maps $(\theta, \gamma)_a : Y \longrightarrow X$ and $(\theta, \tilde{\gamma})_a : Y \longrightarrow \tilde{X}$ are normalization maps.*

PROOF. From the proposition it follows that these maps are birational, and Y is normal. ∎

The following corollary is very important in applications.

COROLLARY 4.7. *The open subset $\mathcal{Y}^\circ$ of $\mathcal{Y}$ and the varieties $\tilde{X}$ and Y do not depend on the choice of a good indicator, but only on the associated decomposition class.*

PROOF. That $\mathcal{Y}^\circ$ does not depend on the choice of a good indicator follows from the proposition. The preimage of $\mathcal{D}$ under the projection $\tilde{V} \longrightarrow \tilde{\mathcal{D}}$ is independent of the good indicator, and identifies with $\mathcal{Y}^\circ$. The image of $\mathcal{Y}$ in $\tilde{V}$ is $\tilde{X}$, which identifies with the closure of the preimage of $\mathcal{D}$, and hence does not depend on the good indicator. Since Y is the normalization of $\tilde{X}$, it does not depend on the good indicator either. ∎

4.8. *The normalization of $\bar{\mathcal{D}}$ as the Γ-quotient of Y.* In particular, it follows from the last proposition that Γ acts on the open set $\mathcal{Y}^\circ$. This action does not extend to the whole variety $\mathcal{Y}$, but Γ does act on Y, commuting with the $G \times \mathrm{k}^*$-action. This fact is very useful for obtaining information on $\bar{\mathcal{D}}$ from information on Y, and vice versa.

THEOREM 4.9. *(i) Y admits a $G \times \Gamma \times \mathrm{k}^*$-action such that the G-quotient map $\theta_a : Y \longrightarrow \mathfrak{c}$ is $\Gamma \times \mathrm{k}^*$-equivariant.*

(ii) The map $\tilde{\gamma}_a : Y \longrightarrow \tilde{\mathcal{D}}$, induced by the collapsing $\gamma : \mathcal{Y} \longrightarrow \bar{\mathcal{D}}$, can be identified with the Γ-quotient map of the Γ-action on Y; in particular $Y/\Gamma \simeq \tilde{\mathcal{D}}$.

(iii) The G-quotient map $\theta_a : Y \longrightarrow \mathfrak{c}$ induces (by taking Γ-quotients) the G-quotient map

$$\tilde{\pi} : \tilde{\mathcal{D}} \longrightarrow \mathfrak{c}/\Gamma.$$

PROOF. We consider the commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{\tilde{\gamma}_a} & \tilde{\mathcal{D}} \\ {\scriptstyle \theta_a}\downarrow & & \downarrow{\scriptstyle \tilde{\pi}} \\ \mathfrak{c} & \xrightarrow[\kappa]{} & \mathfrak{c}/\Gamma. \end{array}$$

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Γ_s acts on the Cartesian product V by pulling back the Γ_s-action on $\mathfrak{c}$. By the foregoing proposition the subgroup Γ acts on an open set of X isomorphic to Y°, and therefore on its closure. The surjection of Y onto X is the normalization map, and so the Γ-action extends to Y as well. Since the surjection $V \longrightarrow \mathfrak{c}$ is Γ_s-equivariant, it follows that θ_a is Γ-equivariant. By the previous proposition, $\tilde{\gamma}_a: Y \longrightarrow \tilde{D}$ is generically a Galois cover with Galois group Γ. Since $\tilde{D}$ is normal it follows from Zariski's main theorem that $\tilde{\gamma}_a$ is the Γ-quotient map. It follows that the natural map $Y/\Gamma = \tilde{D} \longrightarrow \mathfrak{c}/\Gamma$ induced by the G-quotient map θ_a is the G-quotient map of $\tilde{D}$. ■

5. **Regular elements in decomposition varieties.** Although the Γ-action on Y° does not extend to Y, it does extend to the union Y^r of all orbits of maximal dimension, and this subvariety is also independent of the good indicator.

For an irreducible G-variety W we denote the open set consisting of orbits of maximal dimension by W^r, and call it the set of *regular* elements in W. If V is a subset of W, we denote the open (possibly empty) set $V \cap W^r$ by V^r; it should be clear from the context what the corresponding variety W is. This open set is of interest in the theory of sheets. If $\bar{D}$ is the closure of some sheet, then this sheet is just $\bar{D}^r$. A useful property is that the regular parts of Y and Y coincide, hence Y^r is smooth. Next we get information on the normalization of $\bar{D}^r$, since it is the Γ-orbit space of Y^r.

The decomposition class D is contained in the subvariety $\bar{D}^r$ of regular elements in $\bar{D}$, but in general the inclusion is strict. We shall need the following lemma.

LEMMA 5.1. *Let $\mu: Y^r \longrightarrow X^r \subset \mathfrak{c} \times \bar{D}^r$ (resp. $\tilde{\mu}: Y^r \longrightarrow \tilde{X}^r \subset \mathfrak{c} \times \tilde{D}^r$) be the restriction of (θ, γ) (resp. $(\theta, \tilde{\gamma})$). For any $y \in Y^r$ the tangent map $d\mu_y$ (resp. $d\tilde{\mu}_y$) is injective on $T_y Y^r$.*

PROOF. We can assume $y = 1 * a$, with $a \in \mathfrak{r}$. Write $X_1 := \{1*(s+a) \in Y^r; s \in \mathfrak{c}\}$; this is an open subset of an affine space in the fibre above $P \in G/P$. Put $X_2 := Gy$; then X_1 and X_2 are smooth varieties. Restricted to X_1, the map θ is an embedding, but X_2 maps onto a single point. Since $\dim X_1 + \dim X_2 = \dim Y^r$, this implies that $T_y Y^r = T_y X_1 \oplus T_y X_2$. Since $\operatorname{Ker} d\mu_y = \operatorname{Ker} d\theta_y \cap \operatorname{Ker} d\gamma_y$ it follows also that $\operatorname{Ker} d\mu_y$ is equal to the kernel of the restriction of $d\gamma_y$ to $T_y X_2$. But since γ restricted to X_2 is just a finite covering of one G-orbit by another of the same dimension, it follows that this kernel is trivial. The statements involving normalizations are proved the same way. ■

5.2. *The (1-1)-condition.* The following corollary and its proof are inspired by unpublished work of Brylinski-Kostant [14].

COROLLARY 5.3. *(i) The affinization map $\alpha: Y \longrightarrow Y$ restricts to an isomorphism $\alpha^r: Y^r \simeq Y^r$ on the open subsets of regular elements. It follows that Y^r is independent of the good indicator and the Γ-action on Y° extends to Y^r.*

(ii) The following statements are equivalent.

(1-1.1) $\mu: Y^r \longrightarrow X^r$ is injective;

(1-1.2) $\mu: Y^r \longrightarrow X^r$ is an isomorphism;

(1-1.3) *The fibre* $\gamma^{-1}(e)$ *of* $\gamma\colon Y \longrightarrow \bar{D}$ *consists of a single point for any nilpotent element* e *in* $\bar{D}^r$.

(iii) *The following statements are equivalent.*

$(\widetilde{1\text{-}1}.1)$ $\tilde{\mu}\colon Y^r \longrightarrow \tilde{X}^r$ *is injective;*

$(\widetilde{1\text{-}1}.2)$ $\tilde{\mu}\colon Y^r \longrightarrow \tilde{X}^r$ *is an isomorphism;*

$(\widetilde{1\text{-}1}.3)$ *The fibre* $\tilde{\gamma}^{-1}(e)$ *of* $\tilde{\gamma}\colon Y \longrightarrow \tilde{D}$ *consists of a single point for any nilpotent element* e *in* $\tilde{D}^r$.

PROOF. Since α^r is the pull-back of the proper map α, it is proper; therefore it is a finite map by Chevalley's theorem since it has finite fibres. Since Y^r is normal and α is birational, Zariski's main theorem implies that α^r is an isomorphism. This proves (i). By the same arguments μ and $\tilde{\mu}$ are normalization maps (in the notation of the previous lemma), hence finite. Since both tangent maps are injective at all points, by Lemma 5.1 it follows that (1-1.1) and (1-1.2) (resp. $(\widetilde{1\text{-}1}.1)$ and $(\widetilde{1\text{-}1}.2)$) are equivalent to each other (see [46, Lemma, p. 136]). Assuming (1-1.3) it follows by the same arguments that X^r is normal in $(0, e)$. If X is not normal, then the locus of non-normal points X^{nn} is a $k^* \times G \times \Gamma$ stable closed subvariety of X. The G-quotient map $X^{nn} \longrightarrow X^{nn}/\!/G$ attains its maximal fibre dimension in the fibre containing $(0, 0)$ which has dimension strictly smaller than $\dim G(0, e) = \dim Ge$ since $(0, e)$ is a normal point by (ii). So $X^{nn} \cap X^r$ is empty and hence $Y^r \simeq X^r$. This proves (ii), and (iii) is proved similarly. ∎

REMARKS. (i) We shall write (1-1) for the equivalent conditions on D in (ii) of the corollary, and $(\widetilde{1\text{-}1})$ for the conditions in (iii). Usually it is not difficult to check (1-1.3) in contrast to $(\widetilde{1\text{-}1})$, using for example [23]. For $\mathfrak{g} = \mathfrak{sl}_n$ (1-1) is always satisfied; this is not the case for the other simple Lie algebras.

(ii) Let $\chi = [\underline{n_1}, \ldots, \underline{n_r}]$ be the weighted Dynkin diagram of any nilpotent variety N. As remarked by Collingwood-McGovern [17, p. 110], the n_i that have values 0 or 1 form the weighted Dynkin diagram of a nilpotent variety for the Levi L defined by the corresponding α_i. This is an empirical fact, and an a priori proof would be interesting. If we define a decomposition variety $\bar{D}$ by the indicator χ where we only underline the $\underline{n_i}$'s with value unequal to 2, then $\bar{D}$ satisfies (1-1) and N is the (reduced) intersection of $\bar{D}$ with the full nilpotent variety. In this way one can put any nilpotent variety with 2's in its weighted Dynkin characteristic in a strictly larger decomposition variety, and try to study it using information on the generic orbit closures in the decomposition variety (just like polarizable orbits are studied by comparing them to semisimple orbits).

6**. Cohomological results.** Motivated by the result that $Y/\Gamma = \tilde{D}$, we wish to know more properties of Y that might descend to $\tilde{D}$. We show in this section that Y has rational singularities using results of Hinich and Panyushev. By a theorem due to Boutot, this property descends to $\tilde{D}$. Rational singularities implies Cohen-Macaulayness. It is also true that Y is Gorenstein, but this property does not always descend to $\tilde{D}$. We shall also give some information on the minimal resolution of $k[Y]$ as a module over the polynomial ring R of regular functions on $\mathfrak{g}$, and that information also descends in principle.

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6.1. *Coverings of nilpotent varieties.* Let N be any nilpotent variety. Panyushev and Hinich showed that the normalization of N has rational singularities. This is a very useful result, which we will show also holds for decomposition varieties. To prove this generalization we need a small extension of the Panyushev-Hinich result, saying that finite coverings of N have the same property. We need the following general result, proved using a result of Flenner [19]. Recall that our base field has characteristic zero.

THEOREM 6.2. *Let $f\colon Z_1 \longrightarrow Z_2$ be a finite morphism between two normal varieties, and let $Z_2^\circ \subset Z_2$ be an open subset of smooth points with complement of codimension at least two. Suppose the restriction of f to $Z_1^\circ := f^{-1}(Z_2^\circ)$ is étale. Then if Z_2 is Gorenstein with rational singularities, Z_1 is also Gorenstein with rational singularities.*

PROOF. Suppose Z_2 is Gorenstein with rational singularities. Since the statements in the theorem are local we can assume that Z_1 and Z_2 are affine and that the Grothendieck dualizing sheaf ω_{Z_2} is trivial (by the Gorenstein property), with generating global section s. So the restriction $\omega_{Z_2^\circ}$ is trivial too, and also $\omega_{Z_1^\circ}$ by étaleness. By the normality of Z_1 and the codimension condition it follows that $\omega_{Z_1} = i_*\omega_{Z_1^\circ}$ is also trivial, where $i\colon Z_1^\circ \longrightarrow Z_1$ is the inclusion map. Put another way, $f^*\omega_{Z_2} = \omega_{Z_1}$ is isomorphic to the structure sheaf.

Let
$$\rho_2\colon \widetilde{Z_2} \longrightarrow Z_2$$
be a resolution of singularities. Since Z_2 has rational singularities we have $\rho_{2,*}\omega_{\widetilde{Z_2}} = \omega_{Z_2}$. Identifying Z_2° with an open subset of $\widetilde{Z_2}$, we see that the restriction of s to Z_2° extends to a global regular section $\tilde{s}$ of $\omega_{\widetilde{Z_2}}$, *i.e.*, to a global regular differential form on the smooth variety $\widetilde{Z_2}$.

We can identify Z_1° with an open subset of the cartesian product $\widetilde{Z_2} \times_{Z_2} Z_1$; write $\widehat{Z_1}$ for its closure. Let $\hat{f}\colon \widehat{Z_1} \longrightarrow \widetilde{Z_2}$ and $\hat{\rho}_2\colon \widehat{Z_1} \longrightarrow Z_1$ be the corresponding projections. Taking a resolution of singularities
$$\tilde{\rho}_1\colon \widetilde{Z_1} \longrightarrow \widehat{Z_1},$$
we then obtain the following diagram:

$$\begin{array}{ccccc} \widetilde{Z_1} & \xrightarrow{\tilde{\rho}_1} & \widehat{Z_1} & \xrightarrow{\hat{\rho}_2} & Z_1 \\ & & \downarrow \hat{f} & & \downarrow f \\ & & \widetilde{Z_2} & \xrightarrow{\rho_2} & Z_2 \end{array}$$

In the above, the composition $\rho_1 := \hat{\rho}_2 \circ \tilde{\rho}_1$ is a resolution of singularities for Z_1, and the composed map $\tilde{f} := \hat{f} \circ \tilde{\rho}_1$ is a morphism between smooth varieties.

The pull-back $\tilde{f}^*(\tilde{s})$ is a global regular differential form on $\widetilde{Z_1}$, extending the global section $f^*(s)$ of $\omega_{Z_1^\circ}$, where Z_1° is identified with an open subset of $\widetilde{Z_1}$.

In general we have an inclusion $\rho_{1,*}\omega_{\widetilde{Z_1}} \subset \omega_{Z_1}$. But we just saw that the generator of ω_{Z_1} extends to a global regular differential form on $\widetilde{Z_1}$, so this inclusion is an isomorphism.

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To sum up, ω_{Z_1} is invertible and $\rho_{1,*}\omega_{\widetilde{Z_1}} = \omega_{Z_1}$, where $\rho_1 \colon \widetilde{Z_1} \longrightarrow Z_1$ is a resolution of singularities. These are exactly the conditions of a theorem of Flenner [19], implying that Z_1 is Gorenstein with rational singularities. ∎

The slight extension of the result of Hinich [24] and Panyushev [41] is then as follows.

COROLLARY 6.3. *Let $N \subset \mathfrak{g}$ be the closure of a nilpotent orbit. Suppose $\nu \colon \hat{N} \longrightarrow N$ is a finite G-equivariant covering. Then the normalisation of $\hat{N}$ is Gorenstein with rational singularities.*

PROOF. Let Z_1 be the normalization of $\hat{N}$, Z_2 the normalization of N and $f \colon Z_1 \longrightarrow Z_2$ the associated finite G-equivariant map. All orbits on N and therefore on Z_2 and Z_1 are even dimensional, and both Z_1 and Z_2 are the closure of one orbit, so we can take Z_2 to be that orbit. By the result of Hinich and Panyushev, Z_2 is Gorenstein with rational singularities. By the above theorem it follows that Z_2 also is Gorenstein with rational singularities. ∎

REMARK. If I remember correctly, the idea of this proof came up in a conversation with F. Knop around 1991 in Basel after discussing Hinich and Panyushev's work.

6.4. *Syzygies.* G acts on the graded coordinate ring $R := \mathrm{k}[\mathfrak{g}]$. This we extend to a $G \times \Gamma$ action where Γ acts trivially. Consider $\mathrm{k}[Y]$ as a graded R-module with compatible $G \times \Gamma$ action. Since $G \times \Gamma$ is linearly reductive, $\mathrm{k}[Y]$ has a $G \times \Gamma$-equivariant minimal resolution by finitely generated, free, graded R-modules of the form

$$\cdots \longrightarrow R \otimes_{\mathrm{k}} M_2 \longrightarrow R \otimes_{\mathrm{k}} M_1 \longrightarrow R \otimes_{\mathrm{k}} M_0 \longrightarrow \mathrm{k}[Y] \longrightarrow 0,$$

where each M_i is a finite dimensional graded $G \times \Gamma$-module. Here minimality means that all maps become zero after tensoring with k−), where k_0 is the quotient of R by its maximal graded ideal. The $G \times \Gamma_s$-modules M_i are uniquely determined and isomorphic to the finite dimensional doubly-graded associative Tor-algebra $\mathrm{Tor}^R_\bullet(\mathrm{k}_0, \mathrm{k}[Y])$. In fact, for all i,

$$M_i \simeq \mathrm{Tor}^R_i(\mathrm{k}_0, \mathrm{k}[Y])$$

as a graded $G \times \Gamma$-module.

By taking Γ-invariants of a minimal resolution of $\mathrm{k}[Y]$,

$$\cdots R \otimes_{\mathrm{k}} M_2^\Gamma \longrightarrow R \otimes_{\mathrm{k}} M_1^\Gamma \longrightarrow R \otimes_{\mathrm{k}} M_0^\Gamma \longrightarrow \mathrm{k}[Y]^\Gamma \longrightarrow 0,$$

we get a minimal free G-equivariant resolution of $\mathrm{k}[\tilde{D}] \simeq \mathrm{k}[Y]^\Gamma$ (see Theorem 4.9) by graded R-modules.

Remarkably enough, these Tor-modules can be calculated as sheaf cohomology groups of certain homogeneous vector bundles on G/P. If M is a P-module, we write $L_{G/P}(M)$ for the locally free sheaf of sections of the homogeneous vector bundle $G \times^P M \longrightarrow G/P$.

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THEOREM 6.5. *Notations as before.*

(i) We have G-module isomorphisms

$$\left(\mathrm{Tor}_j^R(\mathrm{k}_0, \mathrm{k}[Y])\right)_{i+j} \simeq H^i\left(G/P, \bigwedge^{i+j} L_{G/P}(\mathfrak{g}/\mathfrak{r})^*\right),$$

and

$$\mathrm{Tor}_\bullet^R(\mathrm{k}_0, \mathrm{k}[\tilde{D}]) \simeq \left(\mathrm{Tor}_\bullet^R(\mathrm{k}_0, \mathrm{k}[Y])\right)^\Gamma.$$

(ii) We have the vanishing results

(1) $$H^i\left(G/P, \bigwedge^{j} L_{G/P}(\mathfrak{g}/\mathfrak{r})^*\right) = 0 \quad \text{for } i > j, \text{ and}$$

(2) $$H^k(Y, \mathcal{O}_Y) = 0 \quad \text{for } k \geq 1.$$

PROOF. Given the vanishing results of (ii), the same proof as [10, Lemma 3.9] gives the first statement of (i); the second follows from the remarks made before the statement of the theorem.

Since the indicator is assumed to be good, it follows that the dimensions of $G\mathfrak{n}$ and $G \times^P \mathfrak{n}$ are the same, so $\mathrm{Spec}\,\mathrm{k}[G \times^P \mathfrak{n}]$ is a normal finite covering of $G\mathfrak{n}$. From Corollary 6.3 it follows that $\mathrm{Spec}\,\mathrm{k}[G \times^P \mathfrak{n}]$ is Gorenstein with rational singularities. Since the affinization map

$$G \times^P \mathfrak{n} \longrightarrow \mathrm{Spec}\,\mathrm{k}[G \times^P \mathfrak{n}]$$

is a resolution of singularities, it follows that the higher cohomology groups of the structure sheaf of $G \times^P \mathfrak{n}$ vanish.

Consider the graded Koszul complex associated to the global section $s\colon g*x \longmapsto g*(x, \bar{x})$ of the vector bundle

$$G \times^P (\mathfrak{g} \times \mathfrak{g}/\mathfrak{n}) \longrightarrow G \times^P \mathfrak{g} \simeq G/P \times \mathfrak{g}.$$

The scheme of zeros of the section s is just $G \times^P \mathfrak{n}$. Using the two spectral sequences of hypercohomology of this complex, an argument as in [10, Section 2.12] shows that the higher vanishing of the structure sheaf of $G \times^P \mathfrak{n}$ implies that

$$H^i\left(G/P, \bigwedge^{j} L_{G/P}(\mathfrak{g}/\mathfrak{n})^*\right) = 0 \quad \text{for } i > j.$$

The short exact sequence

$$0 \longrightarrow \mathfrak{r}/\mathfrak{n} \longrightarrow \mathfrak{g}/\mathfrak{n} \longrightarrow \mathfrak{g}/\mathfrak{r} \longrightarrow 0$$

induces the long exact sequence (using the triviality of the bundle $L_{G/P}(\mathfrak{r}/\mathfrak{n})^* \simeq \mathcal{O}_{G/P} \otimes \mathfrak{c}^*$, since P acts trivially on $\mathfrak{r}/\mathfrak{n} \simeq \mathfrak{c}$)

$$\begin{aligned} 0 \longrightarrow \bigwedge^{i} L_{G/P}(\mathfrak{g}/\mathfrak{r})^* \longrightarrow \bigwedge^{i} L_{G/P}(\mathfrak{g}/\mathfrak{n})^* \longrightarrow \bigwedge^{i-1} L_{G/P}(\mathfrak{g}/\mathfrak{n})^* \otimes S^1\mathfrak{c}^* \longrightarrow \\ \cdots \longrightarrow \bigwedge^{i-2} L_{G/P}(\mathfrak{g}/\mathfrak{n})^* \otimes S^2\mathfrak{c}^* \longrightarrow L_{G/P}(\mathfrak{g}/\mathfrak{n})^* \otimes S^{i-1}\mathfrak{c}^* \longrightarrow \mathcal{O}_{G/P} \otimes S^i\mathfrak{c}^* \longrightarrow 0. \end{aligned}$$

By breaking this up into short exact sequences and using the vanishing results for $\bigwedge^j L_{G/P}(\mathfrak{g}/\mathfrak{n})^*$, the proof of (ii) follows straightforwardly. ∎

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REMARKS. (i) The theorem implies that the cohomology of certain homogeneous vector bundles carries a remarkable $G \times \Gamma$ action. Let $\mathrm{k} = \mathbb{C}$, the field of complex numbers, K be a compact form of G and $H \subset K$ be such that $K/H \simeq G/P$ as manifolds. Then Γ_s acts naturally on the manifold K/H; can we lift this action (or restricted to Γ) to the complex vector bundle $G \times^P (\mathfrak{g}/\mathfrak{r})^* \longrightarrow K/H$ and to its cohomology groups? This is possible for $P = B$ and $\mathfrak{r} = \mathfrak{b}$. In that case, this bundle is just the complex cotangent bundle.

(ii) One can get an explicit upper bound for the dominant weights occurring in $\mathrm{Tor}^R_\bullet(\mathrm{k}_0, \mathrm{k}[Y])$ as in [10, Lemma 2.10].

6.6. *Y has rational singularities.* As a first corollary, we find the key result of this article, namely that Y is Gorenstein and has rational singularities.

COROLLARY 6.7. *The affine variety Y is a rational Gorenstein variety with rational singularities and with resolution of singularities $\alpha: \mathcal{Y} \longrightarrow Y$.*

PROOF. From Bruhat's lemma it follows that G/P is rational, and therefore also $\mathcal{Y}$ is rational, since it is the total space of a vector bundle on G/P. Since $Y^r \simeq \mathcal{Y}^r$, it follows that Y is rational. Since α is a resolution of singularities of Y, the corollary follows from the theorem and the definition of rational singularities. ∎

REMARKS. (i) In the case of decomposition classes of semisimple elements this result together with its non-commutative analog was already known, using an observation of Elkik (see [49, Proposition 10]). For the decomposition classes of nilpotent elements it is due to Hinich and Panyushev.

(ii) Suppose we have an indicator with the property that $L \times^{P_1} \mathfrak{n}_1^\circ \longrightarrow Le$ is only a finite covering. Then the corresponding affine variety Y is still a Gorenstein variety with rational singularities, and the full proof extends to this case.

6.8. *Identification with X.* As a second corollary (of the proof), we derive that Y identifies with its image X in $\mathfrak{c} \times \mathfrak{g}$ if and only if (1-1) is satisfied and the nilpotent variety $G\mathfrak{n}$ is normal. These hypotheses are always satisfied in case $\mathfrak{g} = \mathfrak{sl}_n$.

COROLLARY 6.9. *(a) The following five statements are equivalent.*

(i) The variety X is normal, i.e., $Y \simeq X$;

(ii) The variety X is Cohen-Macaulay and (1-1) is satisfied;

(iii) The nilpotent variety $G\mathfrak{n}$ is normal and (1-1) is satisfied;

(iv) The following holds:

$$H^i\Big(G/P, \bigwedge^i L_{G/P}(\mathfrak{g}/\mathfrak{n})^*\Big) = \begin{cases} \mathrm{k}, & \text{if } i = 0; \\ 0, & \text{otherwise;} \end{cases}$$

(v) G acts trivially on $\mathrm{Tor}^R_0(\mathrm{k}_0, \mathrm{k}[Y])$, i.e., G acts trivially on $H^i\big(G/P, \bigwedge^i L_{G/P}(\mathfrak{g}/\mathfrak{r})^*\big)$, for all i.

(b) Suppose the stable class $\mathcal{D}$ satisfies the conditions in (a) and that $\bar{\mathcal{D}}//G$ is unibranch (see Lemma 3.2). Then $\bar{\mathcal{D}}$ is unibranch, and $\tilde{\mathcal{D}}$ is the underlying variety of the pull-back $\mathfrak{c}/\Gamma_s \times_{\bar{\mathcal{D}}//G} \bar{\mathcal{D}}$.

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PROOF. Since Y is Cohen-Macaulay, (i) implies (ii). If we assume (ii), then the non-smooth locus of X is of codimension at least two. Serre's normality criterion shows that X is normal, hence (i).

By a variation of [10, Lemma 3.9] the conditions in (iii) and (iv) are equivalent. It follows from Theorem 6.5(i) that $k[Y]$ is a quotient of $k[\mathfrak{c}] \otimes_k k[\mathfrak{g}] = k[Y]^G \otimes_k k[\mathfrak{g}]$ if and only if $H^i\big(G/P, \bigwedge^i L_{G/P}(\mathfrak{g}/\mathfrak{r})^*\big)$ consists only of G-invariants for all i. It follows easily that (i) and (iv) are equivalent. The equivalence of (i) and (v) follows from the theorem. Finally, (b) follows directly from (a). ∎

LEMMA 6.10. *The pull-back variety $\tilde{V}$ is irreducible with underlying variety $\tilde{X}$, i.e., $\tilde{X} = \tilde{V}_{\mathrm{red}}$. Moreover $\tilde{X}$ is normal, i.e. $Y \simeq \tilde{X}$, if and only if $\tilde{X}$ is Cohen-Macaulay and $\widetilde{(1\text{-}1)}$ holds.*

PROOF. Consider the following commutative square of quotient maps:

$$\begin{array}{ccc} Y & \longrightarrow & \tilde{D} \\ \downarrow & & \downarrow \\ \mathfrak{c} & \longrightarrow & \mathfrak{c}/\Gamma \end{array}$$

Let $s \in \mathfrak{c}$ have image $\bar{s} := \kappa(s)$. Then Γ acts transitively on the (closed) points of the fibre $\kappa^{-1}(\bar{s})$. Since all fibres of θ_a are irreducible by Theorem 6.12(iv), Γ_s acts transitively on the set of irreducible components of the fibre $\theta_a^{-1}\big(\kappa^{-1}(\bar{s})\big)$. This implies that the fibre $\tilde{\pi}^{-1}(\bar{s})$ is irreducible. So all fibres of $\tilde{V} \longrightarrow \mathfrak{c}$ are irreducible of the same dimension and $\tilde{V}_{\mathrm{red}} = \tilde{X}$. The proof of the second statement uses the same argument as the one used in the previous proof. ∎

6.11. *Fibres of the G-quotient map of Y.* In the next proposition we assemble some more results on Y and its G-quotient map.

PROPOSITION 6.12. *(i) $\mathcal{Y} \xrightarrow{\alpha} Y \xrightarrow{\gamma_a} \bar{D}$ is the Stein factorization of the collapsing γ.*

(ii) The map $\theta\colon \mathcal{Y} \longrightarrow \mathfrak{c}$ is smooth.

(iii) The G-quotient map $\theta_a\colon Y \longrightarrow \mathfrak{c}$ is flat. Its fibres are irreducible, reduced, Gorenstein, have rational singularities and contain a dense open G-orbit.

(iv) For $s \in \mathfrak{c}$, the dense orbits in $\theta^{-1}(s)$ and $\theta_a^{-1}(s)$ are isomorphic and the induced map $\theta^{-1}(s) \longrightarrow \theta_a^{-1}(s)$ is a resolution of singularities.

PROOF. (i) is standard (see [22]). The map θ factors through the smooth maps of vector bundles

$$\mathcal{Y} \longrightarrow G \times^P \mathfrak{r}/\mathfrak{n} \simeq G/P \times \mathfrak{r}/\mathfrak{n}$$

and the smooth projection $G/P \times \mathfrak{r}/\mathfrak{n} \longrightarrow \mathfrak{r}/\mathfrak{n} \simeq \mathfrak{c}$. Thus θ is smooth.

Since $k[G \times^P \mathfrak{n}]$ does not contain non-constant G-invariants, it follows that $k[Y]^G \simeq k[\mathfrak{c}]$. It follows that the isotypical components of $k[Y]$ are finitely generated maximal dimensional Cohen-Macaulay graded modules over the ring of invariants $k[\mathfrak{c}]$, which is a polynomial ring. From the Auslander-Buchsbaum equality (see [39, Theorem 19.1]) it follows that the isotypical components are projective, hence free. So $k[Y]$ is $k[\mathfrak{c}]$-free and θ_a is flat. (One can also use [21, Proposition 15.4.2] directly.) Then from general theorems (see [39, Theorem 23.4]), flatness of θ_a implies that Y is Gorenstein if and only if the special fibre Y_0 is Gorenstein.

Since $f: \theta^{-1}(s) \longrightarrow \theta_a^{-1}(s)$ is a pull-back of α, it is proper, surjective with connected fibres (by (i)). Each fibre of θ and θ_a is a complete intersection by global G-invariant regular functions, and all have the same dimension. It follows that f is generically one-to-one. By the vanishing of higher cohomology of $\mathcal{O}_Y$, it follows that $k[\theta^{-1}(s)] \simeq k[\theta_a^{-1}(s)]$, in particular $\theta_a^{-1}(s)$ is normal, and the higher cohomology of $\mathcal{O}_{\theta^{-1}(s)}$ vanishes. Hence f is a generically finite map with connected fibres between normal varieties, so by Zariski's main theorem, f is birational. Since $\theta^{-1}(s) \simeq G \times^P (s + \mathfrak{n})$ is smooth, f is a resolution of singularities and $\theta_a^{-1}(s)$ has rational singularities. To show that these fibres contain a dense open G-orbit, it is enough to show that $\theta^{-1}(0) = G \times^P \mathfrak{n}$ contains a dense orbit (compare [32, p. 130]). Since $G \times^P \mathfrak{n}$ and $G\mathfrak{n}$ have the same dimension and $G\mathfrak{n}$ contains an open, dense orbit of nilpotent elements, it follows that $G \times^P \mathfrak{n}$ has a dense open orbit. This finishes the proof of (iv). ∎

7. Fibres of the G-quotient map of $\tilde{D}$.

In the last section we derived various very good properties for Y and its G-quotient map. Now we shall try to induce those properties on $\tilde{D}$ and on its quotient map. Fibres of the quotient map remain irreducible, but no longer need to be reduced. We shall give several characterizations of when all fibres are reduced.

We first show that $\tilde{D}$ has rational singularities and study some properties of the fibres of the G-quotient map $\tilde{\pi}: \tilde{D} \longrightarrow \mathfrak{c}/\Gamma$.

THEOREM 7.1. *(i) The normalization of any decomposition variety has rational singularities. Furthermore* $\tilde{D} \simeq Y/\Gamma$ *and* $\tilde{D}^r \simeq Y^r/\Gamma$.

(ii) The fibres of the G-quotient map $\tilde{\pi}: \tilde{D} \longrightarrow \mathfrak{c}/\Gamma$ *are all irreducible containing a dense G-orbit.*

(iii) Recall that $\kappa: \mathfrak{c} \longrightarrow \mathfrak{c}/\Gamma$ *is the quotient map. Then G acts trivially on* $\mathrm{Tor}_0^R(k_0, k[Y])^\Gamma$ *if and only if* $\tilde{D}$ *is the image of* $(\kappa \circ \theta, \gamma): Y \longrightarrow \mathfrak{c}/\Gamma \times \bar{D}$*. In particular, this is the case if (1-1) is satisfied and* $G\mathfrak{n}$ *is normal.*

(iv) $\bar{D}$ *is normal if and only if* $\mathrm{Tor}_0^R(k_0, k[Y])^\Gamma = k$.

PROOF. Since Y has rational singularities by Corollary 6.7, (i) follows from Boutot's theorem [9] and Theorem 4.9. In the proof of Lemma 6.10 we already showed (ii). And (iii) follows from Corollary 6.9 and (i). If $\bar{D} = \tilde{D}$ then $k[\tilde{D}]$ is a quotient of R and vice versa, this proves (iv). ∎

REMARK. It follows that $\mathfrak{c}/\Gamma$ parametrizes the regular orbits of $\tilde{D}$, by associating to $c \in \mathfrak{c}/\Gamma$ the dense orbit in the fibre of $\tilde{\pi}$, and we at least get a surjection of $\mathfrak{c}/\Gamma$ on the orbit space $\bar{D}^r/G$. If $\bar{D}^r$ is a sheet (and so $\Gamma = \Gamma_s$), then the main result of Borho [5] gives that $\mathfrak{c}/\Gamma$ also parametrizes the orbits of $\bar{D}^r$. If the normalization map $\tilde{D}^r \longrightarrow \bar{D}^r$ is bijective, this is obvious, but I don't know whether an analogous result is true in general.

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Katsylo [27] proved that the orbit space $\bar{D}^r/G$ has the structure of an algebraic variety for any decomposition class D.

7.2. *Conditions for flatness of* $\tilde{\pi}$. If A is an affine G-variety, the G-multiplicities are the multiplicities of simple G-modules in $\mathrm{k}[A]$, ranging over all simple G-modules.

LEMMA 7.3. *(i) The following conditions are equivalent.*

(1) The G-quotient map $\tilde{\pi}\colon \tilde{D} \longrightarrow \mathfrak{c}/\Gamma$ *is flat;*

(2) The G-multiplicities are constant along the fibres of $\tilde{\pi}\colon \tilde{D} \longrightarrow \mathfrak{c}/\Gamma$*;*

(3) The G-quotient map $\tilde{V} \longrightarrow \mathfrak{c}$ *is flat;*

(4) The G-multiplicities are constant along the fibres of $\tilde{V} \longrightarrow \mathfrak{c}$*;*

(5) $\tilde{V}$ *is Cohen-Macaulay and reduced (i.e.,* $\tilde{X} = \tilde{V}$*).*

(ii) If (refl) *is satisfied, then the conditions in (i) are also satisfied.*

(iii) If $\tilde{D}^r$ *is smooth, then the conditions in (i) are satisfied if and only if* (refl) *holds.*

PROOF. The coordinate ring $\mathrm{k}[\tilde{D}]$ considered as a graded module for the invariant ring $\mathrm{k}[\tilde{D}]^G \simeq \mathrm{k}[\mathfrak{c}/\Gamma]$ is a direct sum of its G-isotypical components. By Nakayama's lemma for graded modules, it follows that the rank of any isotypical component as a graded $\mathrm{k}[\tilde{D}]^G$-module is equal to the minimal number of homogeneous generators if and only if the isotypical components are free. So (1) $\iff$ (2). Analogously (3) $\iff$ (4). Since $\tilde{V} \longrightarrow \mathfrak{c}$ is a pull-back of $\tilde{\pi}$, it follows that the corresponding fibres are isomorphic. Hence (2) $\iff$ (4). If $\tilde{V}$ is Cohen-Macaulay, then $\tilde{V} \longrightarrow \mathfrak{c}$ is flat, so (5) implies (3).

For any irreducible G-character λ, there is a $\mathrm{k}[\mathfrak{c}]$-linear map between isotypical components

$$f_\lambda\colon \mathrm{k}[\tilde{V}]_\lambda \twoheadrightarrow \mathrm{k}[\tilde{X}]_\lambda \hookrightarrow \mathrm{k}[Y]_\lambda, \tag{3}$$

where $\mathrm{k}[Y]_\lambda$ is free by the flatness of θ_a (see Theorem 6.12(iii)). Let $s \in \mathfrak{c}^\circ$ and let k_s be the corresponding $\mathrm{k}[\mathfrak{c}]$-module defined by $f \cdot t := f(s)t\ (f \in \mathrm{k}[\mathfrak{c}])$. The open orbits in $\theta_a^{-1}(s)$ and the fibre of s under $\tilde{V} \longrightarrow \mathfrak{c}$ are isomorphic, by Proposition 4.5, so the singularity locus of the latter fibre is at least of codimension two (all orbits have even dimension). Assuming flatness of $\tilde{\pi}$, then both fibres are Cohen-Macaulay, and, by Serre's criterion, both are normal and hence isomorphic. So there is an isomorphism of vector spaces

$$f_\lambda \otimes \mathrm{k}_s\colon \mathrm{k}[\tilde{V}]_\lambda \otimes \mathrm{k}_s \simeq \mathrm{k}[Y]_\lambda \otimes \mathrm{k}_s.$$

This shows that the three isotypical components in (3) all have the same rank as $\mathrm{k}[\mathfrak{c}]$-modules. Again, by the flatness assumption on $\tilde{\pi}$, we have that $\mathrm{k}[\tilde{V}]_\lambda$ is free for all λ. Since free modules are torsion free, it follows that f_λ is injective for all λ. Since $\mathrm{k}[\tilde{V}] \longrightarrow \mathrm{k}[\tilde{X}]$ is surjective it follows by Theorem 7.1 that it is an isomorphism, *i.e.*, $\tilde{X} = \tilde{V}$. Since $\mathfrak{c}/\Gamma$ and $\tilde{D}$ are Cohen-Macaulay, it follows from the flatness of $\tilde{\pi}$ that all its fibres are Cohen-Macaulay. This also holds for the fibres of the pull-back $\tilde{V} \longrightarrow \mathfrak{c}$, so $\tilde{V}$ is Cohen-Macaulay. Hence (1) implies (5) and we have finished the proof of (i).

Assuming (refl.3), it follows from the Cohen-Macaulayness of $\tilde{D}$ that $\tilde{\pi}$ is flat (apply [21, Proposition 15.4.2] or [39, Theorem 23.1]). Hence (ii).

Suppose $\tilde{D}^r$ is smooth, then flatness of $\tilde{\pi}$ implies that $\mathfrak{c}/\Gamma$ is smooth. Hence (iv). ∎

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7.4. *Grothendieck simultaneous resolutions for* $\tilde{\pi}$. The usefulness of the following theorem lies in the fact that very often we can check condition (cart.3). But once this is satisfied, many other very desirable properties follow.

THEOREM 7.5. *(i) The following statements are equivalent.*

(cart.*1)* *The zero fibre* $\tilde{\pi}^{-1}(\bar{0})$ *of* $\tilde{\pi}\colon \tilde{D} \longrightarrow \mathfrak{c}/\Gamma$ *is reduced;*

(cart.*2)* *All the fibres of* $\tilde{\pi}$ *are reduced and irreducible, are Gorenstein and have rational singularities;*

(cart.*3)* $\tilde{\pi}$ *is flat and* $(\widetilde{1\text{-}1})$ *is satisfied;*

(cart.*4)* $Y \simeq \tilde{X} \simeq \tilde{V}$, *i.e., the following diagram of quotient maps is Cartesian*

$$\begin{array}{ccccc} Y & \longrightarrow & Y & \longrightarrow & \tilde{D} \\ & & \big\downarrow & \square & \big\downarrow \\ & & \mathfrak{c} & \longrightarrow & \mathfrak{c}/\Gamma. \end{array}$$

(ii) Suppose the conditions in (i) are satisfied. Then $\tilde{D}$ *is Gorenstein and the smooth locus of* $\tilde{D}^r$ *is the preimage under* $\tilde{\pi}$ *of the smooth locus of* $\mathfrak{c}/\Gamma$. *In particular* $\tilde{D}^r$ *is smooth if and only if* (refl) *is satisfied.*

PROOF. Obviously, using Proposition 6.12, both (cart.2) and (cart.4) imply (cart.1). Assume (cart.1), so the special fibre of $\tilde{\pi}$ is reduced (and irreducible). The map that surjects the normal variety $\theta_a^{-1}(0)$ onto $\tilde{\pi}^{-1}\big(\tilde{\pi}(0)\big)$ is in fact the Γ-quotient map, from which it follows that $\Big(\tilde{\pi}^{-1}\big(\tilde{\pi}(0)\big)\Big)_{\text{red}} = \Big(\tilde{\pi}^{-1}\big(\tilde{\pi}(0)\big)\Big)$ is normal. Then it follows from Borho-Kraft's associated cone construction (see [32, II 4.2]) that all the (closed) fibres are reduced, irreducible and normal and that the multiplicities along the fibres are constant. Hence by (ii) $\tilde{\pi}$ is flat. By applying [21, Corollaire 12.1.7], it follows that the collection of (not necessarily closed) points z in $\tilde{D}$ such that z is normal and reduced in the (not necessarily closed) fibre $\tilde{\pi}^{-1}\big(\tilde{\pi}(z)\big)$ is open and $G \times \mathrm{k}^*$-stable. Its complement is a closed G-stable cone. Suppose it is non-empty. Then it contains a point in the zero-fibre $\tilde{\pi}^{-1}\big(\tilde{\pi}(0)\big)$, which is normal and reduced. This is a contradiction, hence all fibres of $\tilde{\pi}$ are reduced and normal.

By pulling back (see [21, Proposition 6.8.2]), we see that the quotient map $\pi_{\tilde{V}}\colon \tilde{V} \longrightarrow \mathfrak{c}$ has the same properties. It follows that $\tilde{V}$ is reduced and normal (see [21, Corollaires 6.4.2 and 6.5.4]). Because $Y \longrightarrow \tilde{V}$ is the normalization map it follows that $Y \simeq \tilde{X} \simeq \tilde{V}$, *i.e.* (cart.1) implies (cart.4). Using the equivalence of (cart.1) and (cart.4), it also follows that $(\widetilde{1\text{-}1})$ is satisfied. So (cart.1) also implies (cart.3).

Assume (cart.3), then by the lemma before we have that $\tilde{X} = \tilde{V}$ is reduced and Cohen-Macaulay, and by Corollary 5.3 that $Y^r \simeq Y^r \simeq \tilde{X}^r$. Let $\tilde{X}^s$ be the singular locus of $\tilde{X}$, then $\tilde{X}^s$ has codimension at least three, since $\dim \tilde{X}^s//G < \dim \tilde{X}//G = \dim \mathfrak{c}$ and $\dim \theta_a^{-1}(s) \cap \tilde{X}^s < \dim \theta_a^{-1}(s) \cap (\tilde{X} \setminus \tilde{X}^r) \leq \dim \theta_a^{-1}(s) - 2$, all orbits having even dimension in $\tilde{X}$. So $\tilde{V} = \tilde{X}$ is Cohen-Macaulay and smooth up to a subset of codimension at least three. Then it follows from Serre's criterion for normality that $\tilde{X}$ is normal. Thus $Y \simeq \tilde{X}$ and (cart.4) follows. Hence (i).

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If the conditions are satisfied, then $\tilde{D}$ is Gorenstein since all the fibres of $\tilde{\pi}$ are Gorenstein. In addition, all the fibres of the flat map $\tilde{\pi}^r\colon \tilde{D}^r \longrightarrow \mathfrak{c}/\Gamma$ are smooth. Now apply [39, Theorem 27.7] to get (iii). ∎

REMARKS. (i) We shall say that (cart) is satisfied if one of the equivalent conditions in (i) of the theorem is satisfied. If that is the case, the commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \tilde{D} \\ \big\downarrow & & \big\downarrow \\ \mathfrak{c} & \longrightarrow & \mathfrak{c}/\Gamma \end{array}$$

is a *simultaneous resolution* of the flat quotient map $\tilde{D} \longrightarrow \mathfrak{c}/\Gamma$ (see [48]). (cart) is always satisfied if $\mathfrak{g}$ is of type A_r.

(ii) As remarked in the proof, the Killing form restricts to a non-degenerate form on $\mathfrak{c}$. It follows that the induced quadratic invariant on $\mathfrak{g}$ vanishes on $\bar{D}$ if and only if $\mathfrak{c} = 0$.

(iii) In general $\tilde{D}$ need not be Gorenstein, even if $\bar{D} = \tilde{D}$ (see example Section 9.5).

(iv) Let Γ' be the (normal) subgroup of Γ generated by the reflections in Γ. Suppose $(\widetilde{1\text{-}1})$ holds; then the same arguments show that Y is isomorphic to the pull-back $\mathfrak{c} \times_{\mathfrak{c}/\Gamma'} Y/\Gamma'$ and $Y/\Gamma' = (\mathfrak{c}/\Gamma' \times_{\mathfrak{c}/\Gamma} \tilde{D})_{\mathrm{red}}$.

7.6. *Grothendieck simultaneous resolutions for* $\bar{\pi}$. We describe next the situation where $\bar{D}$ itself is part of a Grothendieck simultaneous resolution. The conditions in (b) are in general easy to check, using tables already published in the literature.

THEOREM 7.7. *Let D be a decomposition class.*

(i) The following two statements are equivalent.

(1) The quotient $\bar{D}//G$ is normal and $V = Y$, i.e., we have a Cartesian diagram

$$\begin{array}{ccccc} Y & \longrightarrow & Y & \longrightarrow & \bar{D} \\ & & \big\downarrow & \square & \big\downarrow \\ & & \mathfrak{c} & \longrightarrow & \mathfrak{c}/\Gamma. \end{array}$$

(2) D is stable, satisfies (1-1), and both $\bar{D}//G$ and $G\mathfrak{n}$ are normal.

(ii) Suppose the conditions in (i) are satisfied. Then $\bar{D}$ has rational singularities, is Gorenstein; its quotient $\bar{D}//G \simeq \mathfrak{c}/\Gamma$ is smooth; its quotient map $\bar{\pi}$ is flat with irreducible, reduced, Gorenstein fibres having rational singularities, and $\bar{D}^r$ is smooth.

PROOF. Suppose (1) is satisfied. Then $Y = X$, hence (1-1) and normality of $G\mathfrak{n}$ (see Corollary 6.9). Since $V = Y$ is irreducible, V° is also irreducible, and D is stable (see Corollary 2.7). Hence $\bar{D} = Y/\Gamma = \tilde{D}$ is normal and $\mathfrak{c}/\Gamma$ is smooth by Theorem 3.1(i). Hence (2).

Conversely, suppose (2) is satisfied. (1-1) and normality of $G\mathfrak{n}$ imply that $Y = X$ is an irreducible component of V, and $\Gamma_s = \Gamma$ acts on Y with quotient $\tilde{D}$. From $\bar{D}//G \simeq \mathfrak{c}/\Gamma_s$ it follows again that (refl) holds, and the group Γ_s also acts on V with quotient $\bar{D}$. So $\tilde{D}$ is a component of $\bar{D}$, hence $\bar{D}$ is normal. So $V = \tilde{V} = Y$ by Theorem 7.5.

Hence (i), and (ii) follows from Theorem 7.5. ∎

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REMARKS. (i) Assuming only the condition $Y = V$ does not imply the normality of $\bar{D}//G$ (see example Section 9.3). But then $\bar{D}$ is normal if and only if $\bar{D}^r$ is normal if and only if $\bar{D}//G$ is normal.

(ii) The conditions in the theorem are satisfied for any stable decomposition class in $A_r = \mathfrak{sl}_{r+1}$ such that the type of $\mathfrak{l}$ is pA_q with $p(q+1) = r+1$, (see [34]).

(iii) Another equivalent condition is as follows. $Y = V$ and $\bar{D}//G$ is normal if and only if $\dim \mathrm{Tor}_0^R(\mathrm{k}_0, \mathrm{k}[Y]) = \#\Gamma$ if and only if the representation of Γ_s on $\mathrm{Tor}_0^R(\mathrm{k}_0, \mathrm{k}[Y])^\Gamma$ is the regular representation. The ordinary Grothendieck simultaneous resolution was used by Slodowy [48] to define a W-action on the cohomology of the fibres of γ using monodromy. These actions turned out to be equivalent to Springer's famous W-representations. One might ask whether Slodowy's method generalizes to obtain Γ-actions on fibres of γ, at least in the situation of the last theorem.

7.8. *Classification of Dixmier sheets where $Y = V$ and the closure is normal.* The following theorem classifies the Dixmier sheets that resemble most the regular sheet in the sense that its closure is normal and that there is a natural simultaneous resolution for the quotient map of $\bar{D}$. The classification is complete only up to normality results of nilpotent varieties in Lie algebras of type E_r.

THEOREM 7.9. *Let $\mathfrak{g}$ be simple. Suppose the decomposition class D contains a semisimple element x with stabilizer L so that $\bar{D}^r$ is a Dixmier sheet.*

Then $\bar{D}//G$ is normal and $Y = V$ if and only if L is either a maximal torus or $L = G$ or

(a) its type is pA_q, with $q \geq 0$ and $r+1 = p(q+1)$, if $\mathfrak{g} = A_r$;
(b) its type is $pA_q + B_j$, with $q, j \geq 0$, $r = j + p(q+1)$ and $2j \geq q$, if $\mathfrak{g} = B_r$;
(c) its type is $pA_q + C_j$, with $q, j \geq 0$, $r = j + p(q+1)$ and $2j \leq q+1$, if $\mathfrak{g} = C_r$;
(d) its type is $pA_q + D_j$, with $q \geq 0$, $j \geq 2$, $r = j + p(q+1)$ and $2j \geq q+1$ or its type is pA_q, with q odd and $r = p(q+1)$, if $\mathfrak{g} = D_r$;
(e) the types in Theorem 3.1(e6) (resp. (e7), (e8)) where the nilpotent variety $G\mathfrak{n}$ is normal (we conjecture that these are all normal), if $\mathfrak{g} = E_r$.
(f) its type is A_2 (short roots), $A_1 + A_2$ (short roots and one long simple root) or C_3, if $\mathfrak{g} = F_4$;
(g) its type is A_1 (corresponding to the short simple root) if $\mathfrak{g} = G_2$.

PROOF. Since we require $\bar{D}//G$ to be normal, we only have to consider the types given in Theorem 3.1. Among those types we have to consider which satisfy both (1-1) and that $G\mathfrak{n}$ is normal. For $\mathfrak{g} = A_r$ both conditions are satisfied for the given types, by Kraft and Procesi [34], which gives (a).

Let $\mathfrak{g}$ be classical. Then we can check (1-1) using Hesselink [23] and normality using Kraft-Procesi [35]. Let $\mathfrak{g} = \mathfrak{so}_{2r+1}$ and L of type $pA_q + B_j$, where $r = p(q+1)+j$. Order the sequence $\big((q+1)^{2j}, 2j+1\big)$ to obtain a partition of $2r+1$ and then take the dual partition and obtain λ. From a result of Hesselink [23, Theorem 7.1] it follows that (1-1) is satisfied if and only if λ is the partition of a nilpotent in $\mathfrak{so}_{2r+1}$. If $2j \geq q$ then $\lambda = \big((2p+1)^{q+1}, 1^{2j-q}\big)$ is the partition of a nilpotent (*i.e.*, all even parts occur an even number of times), hence

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(1-1) is satisfied. To check normality, we have to find the minimal ϵ-degenerations of λ. Then normality fails if and only if a minimal ϵ degeneration of type (e) occurs (see Kraft-Procesi[35, Theorem 16.2]). In this case, type (e) does not occur, hence normality holds for $G\mathfrak{n}$. If $q > 2j$, then $\lambda = \big((2p+1)^{2j+1}, 2p^{q-2j}\big)$. If q is odd this is not a partition of a nilpotent, hence (1-1) is not satisfied. But if q is even, it is a partition of a nilpotent, hence (1-1) is satisfied. But then there is a minimal degeneration of type (e), hence non-normality. This handles type B_r.

Type C_r is similar. Here L is of type $pA_q + C_j$, where $r = p(q+1) + j$. If $2j \geq q$, then we get (1-1) and normality. If $q > 2j$, then (1-1) is satisfied if and only if q is even. But if q is even, normality fails. This handles type C_r.

Let $\mathfrak{g} = \mathfrak{so}_{2r}$. Here Procesi-Kraft's normality criterion only holds for partitions that are not very even. First consider the case where the type of L is $pA_q + D_j$, where $j \geq 2$ and $p(q+1) + j = r$. If $2j \geq q+1$, we obtain that $\lambda = \big((2p+1)^{q+1}, 1^{2j-(q+1)}\big)$ is a partition of a nilpotent, hence (1-1). This is not very even and there is no minimal degeneration of type (e), hence normality. If $2j < q+1$ then $\lambda = \big((2p+1)^{2j}, 2p^{q+1-2j}\big)$. This is a partition of a nilpotent (hence (1-1) holds) if and only if q is odd. But if q is odd, then the partition is not very even but a degeneration of type (e) occurs, hence non-normality. Now consider the second case where L is of type pA_q, where $p(q+1) = r$ and q is odd. Now the partition $\lambda = \big((2p)^{q+1}\big)$ is the partition of a nilpotent, hence (1-1) holds. This time the partition is very even and we can use Kraft-Procesi [35, 17.3 Theorem (b)] to conclude the normality of $G\mathfrak{n}$. This handles type D_r.

For F_4 the following cases given in Theorem 3.1 are ruled out: A_2 (long roots) since $G\mathfrak{n}$ is not normal; $A_2 + A_1$ (long roots and one short simple root) since (1-1) is not satisfied; B_3 since $G\mathfrak{n}$ is not normal. The remaining cases satisfy (1-1) and the normality condition: see [12].

For G_2 we only have to check both rank one cases. In any case the closure of the nilpotent orbit in $\bar{D}^r$ is normal (Kraft), and (1-1) is satisfied if the simple root in the Levi factor of P is short.

For E_r, the cases given in Theorem 3.1 all satisfy (1-1). ∎

REMARKS. (i) Many of the cases of sheets studied by Rubenthaler [45] occur in the list above. A special case was studied by Brylinski-Kostant [14], where the decomposition class contains an element h that is part of an even $\mathfrak{sl}_2$-triple $\{e, f, h\}$, such that G_h is of semi-simple rank $r-1$. In general Rubenthaler's $\overline{D}$ and $\overline{Ge}$ need not be normal. For example, in type C_4 with Levi group L of semisimple type $A_1 \times C_2$, or in type F_4 with Levi of semisimple type B_3. But we conjecture that they are always normal in the simply-laced case.

(ii) If $\mathfrak{g} = A_r$, then all sheets are smooth (due to Peterson and Kraft-Luna, see [4]). Also for the other classical types it is conjectured that all sheets are smooth, so we could forget about the normality condition on $\bar{D}//G$. So far only one non-smooth sheet has been found (the subregular Dixmier sheet in G_2 corresponding to a long root, found by Borho and Kraft), but in that case (1-1) is not satisfied. One might ask whether there are non-smooth Dixmier sheets where (1-1) is satisfied.

(iii) Here is a remark on the classification of normal nilpotent varieties. In 1989 Kraft [33] gave a summary of what was known about normality of nilpotent varieties. For the classical simple Lie algebras and for G_2 this remains up to date, but for the exceptional Lie algebras some new results appeared. In [11], we showed that certain nilpotent varieties are normal, and in [12] we completely handled the case of type F_4. We obtained the normality of the following nilpotent varieties.

In E_6 those with Bala-Carter labels: E_6, $E_6(a_1)$, D_5, $E_6(a_3)$, $3A_1$, $2A_1$, A_1, A_0. In E_7 those with labels E_7, $E_7(a_1)$, $E_7(a_2)$, $E_7(a_3)$, E_6, $E_6(a_1)$, $4A_1$, $(3A_1)'$, $(3A_1)''$, $2A_1$, A_1, A_0. In E_8 those with labels E_8, $E_8(a_1)$, $E_8(a_2)$, $E_8(a_3)$, $E_8(a_4)$, $4A_1$, $3A_1$, $2A_1$, A_1, A_0.

Richardson [44] calculated the multiplicity of the adjoint representation in the coordinate ring of any nilpotent variety. For some nilpotent varieties he also calculated the multiplicity of the adjoint representation in the coordinate ring of the normalization. If these multiplicities are not equal, then obviously the nilpotent variety is not normal. We calculated the multiplicity of the adjoint representation in the normalization of any nilpotent variety in the exceptional Lie algebras. We used the methods exposed in [12], and calculated most cases by hand and some with the help of a computer (and found that Richardson made some errors). Eric Sommers independently programmed a computer and obtained the same multiplicity results. In this way the non-normality is detected for the following nilpotent varieties in type E_r.

In E_6 those with labels: A_4, A_3+A_1, A_3, $2A_2$ and A_2+A_1. In E_7 those with labels $D_6(a_1)$, $D_6(a_2)$, $(A_5)''$, A_4, A_3+A_2, $D_4(a_1)+A_1$, A_3+2A_1, $(A_3+A_1)'$, $(A_3+A_1)''$, A_3. In E_8 those with labels $E_7(a_1)$, $E_7(a_2)$, $D_7(a_1)$, $E_7(a_3)$, E_6, D_6, $E_6(a_1)$, $E_7(a_4)$, $D_6(a_1)$, A_6, D_5+A_1, $E_7(a_5)$, $E_6(a_3)+A_1$, $D_6(a_2)$, $D_5(a_1)+A_2$, A_5+A_1, D_5, $E_6(a_3)$, D_4+A_2, $D_5(a_1)+A_1$, A_5, $D_5(a_1)$, D_4+A_1, A_4, A_3+A_2, A_3+2A_1, D_4, A_3+A_1 and A_3.

In general the normalization map of a nilpotent variety N is not bijective. For type E_r Beynon-Spaltenstein [3] provided a table saying exactly how many points lie over a given point in N. In this way the non-normality of two more non-normal nilpotent varieties can be shown, namely in E_8 those with labels $2A_2+A_1$ and $D_4(a_1)$.

We expect that the remaining nilpotent varieties in E_r are all normal. At least they are all unibranch, *i.e.*, the normalization map is bijective (Beynon-Spaltenstein), and the multiplicities of the adjoint representation in $\mathbb{C}[N]$ and $\mathbb{C}[\tilde{N}]$ are the same.

For any semisimple Lie algebra we conjecture (together with Panyushev and Sommers) that all distinguished nilpotent varieties (in the sense of Bala-Carter) are normal. This is correct at least for the classical Lie algebras and for E_6, F_4 and G_2. For E_7 two cases remain to be settled and for E_8 six cases.

8. **Non-commutative analogs.** The decomposition varieties allow non-commutative analogs, at least when they allow a good, even indicator. In that case $\mathfrak{r} = \mathfrak{c} + \mathfrak{n}$ with $\mathfrak{n}$ the nilradical of $\mathfrak{p}$. We shall assume also that D is stable. There is a connected subgroup $F < G$ with the properties that $(P, P) \leq F \leq P$, $P/F = A$, a torus with Lie algebra $\mathfrak{c}$, and $(\mathfrak{g}/\mathfrak{f})^*$ can be identified with $\mathfrak{r}$ using the Killing form. Let I be the annihilator in $U(\mathfrak{g})$ of the left-module

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{f})} \mathbb{C}_{-2\rho_P},$$

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where $\mathbb{C}_{-2\rho_P} = \bigwedge^{\mathrm{top}}(\mathfrak{g}/\tilde{\mathfrak{f}}) = \bigwedge^{\mathrm{top}}(\mathfrak{g}/\mathfrak{p})$, hence $2\rho_P$ is the sum of all roots in $\mathfrak{p}$ (or $\mathfrak{n}$). Then $\mathrm{k}[\bar{D}]$ is the associated graded ring of the Poincaré-Birkhoff-Witt filtration on $U(\mathfrak{g})/I$. Here $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$.

Also, the other varieties associated to such a good indicator, such as $Y, \tilde{D}, V, \tilde{V}, \bar{D}//G$, *etc.*, have non-commutative analogs as we shall indicate. If Z is any smooth variety we denote its ring of global algebraic differential operators by D_Z. If G acts on Z, the Lie algebra $\mathfrak{g}$ acts by global vector fields on Z, and therefore we get an operator representation $\omega\colon U(\mathfrak{g}) \longrightarrow D_Z$, where the elements in the enveloping algebra are interpreted as differential operators on Z.

The following results are due to or follow easily from work of Borho-Brylinski and Soergel.

THEOREM 8.1. *Fix a good, even indicator. The completely prime ideal I defined above is the kernel of the operator representation $\omega\colon U(\mathfrak{g}) \longrightarrow D_{G/F}$ induced by the G action on G/F. The action of A on the right of G/F induces an inclusion $U(\mathfrak{c}) \longrightarrow D_{G/F}$. Write $U_{\mathfrak{c}}$ for the image of $U(\mathfrak{g} \times \mathfrak{c})$ in $D_{G/F}$, it is contained in the ring of A-invariant differential operators $D := D^A_{G/F}$. The centers of D and $U_{\mathfrak{c}}$ can both be identified with $U(\mathfrak{c})$. D is flat as a module over its center.*

Let K be the fraction field of $U(\mathfrak{c})$. Then the ring of A-invariant differential operators D is the integral closure of $U_{\mathfrak{c}}$ in $D_K = (U_{\mathfrak{c}})_K$, i.e., D is equal to

$$\{x \in D_K \mid \text{the subalgebra of } D_K \text{ generated by } U_{\mathfrak{c}} \text{ and } x \text{ is a finitely generated } U_{\mathfrak{c}}\text{-bimodule}\}.$$

The associated graded ring of D with respect to the operator filtration is $\mathrm{k}[Y]$. *The associated graded ring of $\bar{U} := U(\mathfrak{g})/I$ with respect to the* PBW*-filtration is* $\mathrm{k}[\bar{D}]$ *and the associated graded ring of $\bar{Z} := Z(\mathfrak{g})/\big(I \cap Z(\mathfrak{g})\big)$ is* $\mathrm{k}[\bar{D}//G]$. *The associated graded ring of the image $U_{\mathfrak{c}}$ of $U(\mathfrak{g} \times \mathfrak{c})$ in D with respect to the* PBW*-filtration is* $\mathrm{k}[X]$. *The associated graded ring of $U(\mathfrak{c}) \otimes_{\bar{Z}} \bar{U}$ is* $\mathrm{k}[V]$.

PROOF. See Borho-Brylinski [6] and Soergel [49]. There the algebra $D_{G/S}$ of global differential operators on any homogeneous space G/S is studied. It allows a faithful representation on the local cohomology $M = H^n_x(G/S, \mathcal{O}_{G/H})$, where $n = \dim G/S$ and $x = S \in G/S$. Its restriction to $U(\mathfrak{g})$ can be identified with the induced module

$$M = U(\mathfrak{g}) \otimes_{U(\mathfrak{s})} \bigwedge^n(\mathfrak{g}/\mathfrak{s}).$$

Apply this to the homogeneous space G/F. Let $Q \subset G \times A$ be the image of $P \longrightarrow G \times A\colon p \longrightarrow (p, pF)$. Then

$$G/F = (G \times A)/Q. \qquad \blacksquare$$

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8.2. *An action of* Γ *on D.* To give a non-commutative analog of $\tilde{D}$, we need to define a twisted action of Γ_s on $\mathfrak{c}$ (or on $\mathfrak{c}^*$ and using the Killing form). The extension of this action to D hasn't been explicitly used in the literature before.

The $\bullet$-action of W on $\mathfrak{t}^*$ is defined by

$$w \bullet \mu := w(\mu + \rho) - \rho,$$

where $\rho = \rho_B$ is half the sum of positive roots. The projection $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{t} \oplus \mathfrak{u} \longrightarrow \mathfrak{t}$ induces a linear map from $U(\mathfrak{g}) \longrightarrow U(\mathfrak{t})$ and an algebra homomorphism

$$\phi\colon Z(\mathfrak{g}) \longrightarrow U(\mathfrak{t}).$$

For every $z \in Z(\mathfrak{g})$ there is a $z' \in U(\mathfrak{g})\mathfrak{u}$ such that $z = \phi(z) + z'$. Harish-Chandra showed that ϕ induces an isomorphism of the center of $U(\mathfrak{g})$ onto the invariant ring by the $\bullet$-action of W on $U(\mathfrak{t}) \simeq \mathrm{k}[\mathfrak{t}^*]$.

Let ρ_L be half the sum of positive roots in L and $\rho_P = \rho - \rho_L$ half the sum of roots in $\mathfrak{n}$. The surjection $j^\#\colon \mathfrak{t} \subset \mathfrak{p} \longrightarrow \mathfrak{p}/\tilde{\mathfrak{k}} \simeq \mathfrak{c}$ induces an injection $j\colon \mathfrak{c}^* \longrightarrow \mathfrak{t}^*$. Write

$$j_L\colon \mathfrak{c}^* \longrightarrow \mathfrak{t}^*, \quad j_L(\lambda) := j(\lambda) - 2\rho_L,$$

with comorphism

$$j_L^\#\colon U(\mathfrak{t}) \longrightarrow U(\mathfrak{c}).$$

Let $\tilde{\Gamma}$ be the normalizer of this affine subspace $j_L(\mathfrak{c}^*) \subset \mathfrak{t}^*$ for the $\bullet$-action of W. Let $N_W(\mathfrak{c}^*)$ be the normalizer of $\mathfrak{c}^*$ and $C_W(\mathfrak{c}^*)$ its centralizer for the ordinary W-action. Then the finite group Γ_s identifies with the quotient $N_W(\mathfrak{c}^*)/C_W(\mathfrak{c}^*)$ (by identifying $\mathfrak{t}$ and $\mathfrak{t}^*$ using the restriction of the Killing form). Howlett [25, Corollary 3] showed that $\tilde{\Gamma}$ is just the subgroup of $N_W(\mathfrak{c}^*)$ permuting the positive roots of L, and $N_W(\mathfrak{c}) = C_W(\mathfrak{c}) \cdot \tilde{\Gamma}$.

There is an involution commuting with the $\bullet$-action of W defined by

$$\iota\colon \mathfrak{t}^* \longrightarrow \mathfrak{t}^*, \quad \iota(\lambda) := -\lambda - 2\rho.$$

Write $\iota^\#\colon U(\mathfrak{t}) \longrightarrow U(\mathfrak{t})$ for the comorphism.

The operator representation $\omega\colon U(\mathfrak{g}) \longrightarrow U_{\mathfrak{c}}$ restricts to a homomorphism of centers, *i.e.*, a morphism $\omega_{Z(\mathfrak{g})}\colon U(\mathfrak{g}) \longrightarrow U(\mathfrak{c})$.

LEMMA 8.3. *The restriction $\omega_{Z(\mathfrak{g})}$ of the operator representation to $Z(\mathfrak{g})$ can be identified with $j_L^\# \circ \iota^\# \circ \phi$. Its image is contained in the ring of invariants $U(\mathfrak{c})^{\tilde{\Gamma},\bullet}$ for the $\bullet$-action of $\tilde{\Gamma}$.*

PROOF. See Soergel [49, Proposition 16]. By Borho-Brylinsky, there is a faithful action of $U(\mathfrak{g} \times \mathfrak{c})$ on $M = U(\mathfrak{g} \times \mathfrak{c}) \otimes_{U(\mathfrak{q})} \bigwedge^n(\mathfrak{g} \times \mathfrak{c})/\mathfrak{q}$, with notation as in the previous proof. Let v be a generator for $\bigwedge^n(\mathfrak{g} \times \mathfrak{c})/\mathfrak{q}$, where $n = \dim G/S$.

For $t \in \mathfrak{t}$ we have

$$t \otimes v = \Big[\big(j^\#(t) + t\big) - j^\#(t)\Big] \otimes v = [-2\rho_P(t) - j^\#(t)] \otimes v = j_L^\#\big(\iota^\#(t)\big) \otimes v.$$

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So, for every $t \in U(\mathfrak{t})$, $t \otimes v = j_L^{\#}\big(\iota^{\#}(t)\big) \otimes v$.

For $c \in Z(\mathfrak{g})$ and $u \in U(\mathfrak{g} \times \mathfrak{c})$, we then get that

$$cu \otimes v = uc \otimes v = u\phi(c) \otimes v = uj_L^{\#}\Big(\iota^{\#}\big(\phi(c)\big)\Big) \otimes v = j_L^{\#}\Big(\iota^{\#}\big(\phi(c)\big)\Big)u \otimes v.$$

So c acts in the same way as $j_L^{\#}\Big(\iota^{\#}\big(\phi(c)\big)\Big)$ on the faithful module M. This proves the lemma. ∎

8.4. Now we can extend the $\bullet$-action of $\tilde{\Gamma}$ on $U(\mathfrak{c})$ to $D = D^A_{G/F}$. Its ring of invariants is a non-commutative analog of $\tilde{D}$.

THEOREM 8.4. *There is a natural $\tilde{\Gamma}$-action on $D = D^A_{G/F}$ preserving the filtration. The associated graded Γ_s-action coincides with the action on* $\mathrm{k}[Y]$ *defined earlier in the paper. The associated graded of the ring of invariants $D^{\tilde{\Gamma}}$ is* $\mathrm{k}[\tilde{D}]$. *The center of $D^{\tilde{\Gamma}}$ can be identified with the invariant ring $U(\mathfrak{c})^{\tilde{\Gamma},\bullet}$. $D^{\tilde{\Gamma}}$ is the integral closure of $\bar{U} \otimes_{\bar{Z}} U(\mathfrak{c})^{\tilde{\Gamma},\bullet}$ modulo $U(\mathfrak{c})^{\tilde{\Gamma},\bullet}$-torsion in the ring obtained by localizing in the quotient field of $U(\mathfrak{c})^{\tilde{\Gamma},\bullet} = K^{\tilde{\Gamma},\bullet}$.*

PROOF. The $\bullet$-action of $\tilde{\Gamma}$ extends naturally to an action on the tensor product $\bar{U} \otimes_{\bar{Z}} U(\mathfrak{c})$. We get an action on the quotient $R_{\mathfrak{c}}$ obtained by dividing out the $U(\mathfrak{c})$-torsion (see Soergel [49, Corollar 20]). If K is the quotient field of $U(\mathfrak{c})$, $\tilde{\Gamma}$ also acts on the localization $(R_{\mathfrak{c}})_K$ and therefore on the integral closure $\widetilde{R_{\mathfrak{c}}}$ of $R_{\mathfrak{c}}$ in $(R_{\mathfrak{c}})_K$ defined in the following sense:

$$\widetilde{R_{\mathfrak{c}}} := \{x \in R_{\mathfrak{c}} \mid \text{the subring of } (R_{\mathfrak{c}})_K \text{ generated by } R_{\mathfrak{c}} \text{ and } x \text{ is finitely generated as } R_{\mathfrak{c}}\text{-bimodule}\}.$$

According to Soergel [49, Theorem 13] this is just D, hence we have extended the $\tilde{\Gamma}$-action.

The remaining statements follow from the results in this article by considering associated graded rings. ∎

REMARK. It would be interesting to get a direct definition of the $\tilde{\Gamma}$-action on D; for the classical case of the decomposition class of regular semisimple elements over the complex numbers this was done by Gel'fand-Kirillov (see [20, Remark 10.3]).

We give a sample of immediate applications.

COROLLARY 8.5. *(i) Assume the conditions in Corollary 6.9 are satisfied. Then the operator representation*

$$\omega\colon U(\mathfrak{g} \times \mathfrak{c}) \longrightarrow D$$

is surjective.

(ii) Assume the conditions (cart) *in Theorem 7.5(i) are satisfied, then $D^{\tilde{\Gamma}}$ is flat over its center and $D \simeq D^{\tilde{\Gamma}} \otimes_{U(\mathfrak{c})^{\tilde{\Gamma},\bullet}} U(\mathfrak{c})$. For each maximal ideal m of $U(\mathfrak{c})^{\tilde{\Gamma},\bullet}$, the ring $D^{\tilde{\Gamma}}/mD^{\tilde{\Gamma}}$ is an integral domain.*

(iii) If the conditions in Theorem 7.7 are satisfied then

$$D \simeq \bar{U} \otimes_{\bar{Z}} U(\mathfrak{c}).$$

In particular, those conditions are satisfied in the situation of Theorem 7.9.

PROOF. (i) is due to Soergel [49, Theorem 30]. The other assertions follow from the results of this article by considering associated graded rings. ∎

9**. Examples.** In this section we give various examples and counter-examples.

9.1. *Type A_r over fields in any characteristic.* We shall sum up now the results for $\mathfrak{gl}_{r+1}$. In fact, most of the results for this type remain true when we allow k to be an algebraically closed field of any characteristic. We shall indicate briefly the changes we have to make in the proofs.

Suppose $a_1, \ldots, a_s$ are the different eigenvalues of $x \in \mathfrak{gl}_{r+1}$ acting on k^{r+1}, with generalized eigenspaces $E_1, \ldots, E_s$ of dimensions $e_1, \ldots, e_s$. We suppose that $e_1 \geq e_2 \geq \cdots \geq e_s$. The restriction of $x - a_i 1$ to E_i is nilpotent with partition $\lambda_i = \lambda_{i1} \geq \lambda_{i2} \geq \cdots$ of e_i. Let Γ_s (resp. Γ) be the subgroup of the symmetric group on $\{1, 2, \ldots, s\}$ of permutations τ such that $e_{\tau(i)} = e_i$ (resp. $\lambda_{\tau(i)} = \lambda_i$) for all i.

PROPOSITION 9.2. (GL_{r+1}, CHARACTERISTIC k ARBITRARY) *Let $\mathcal{D}$ be the decomposition class containing $x \in \mathfrak{gl}_{r+1}$. Then $\mathcal{D}$ is stable if and only if $\lambda_i = \lambda_j$ whenever $e_i = e_j$. $\bar{\mathcal{D}}//G$ is normal if and only all e_i are equal. Both $\mathrm{k}[\mathfrak{c}]^{\Gamma_s}$ and $\mathrm{k}[\mathfrak{c}]^{\Gamma}$ are polynomial rings.*

Every decomposition class $\mathcal{D}$ has a good, even indicator, i.e., where all labels are even. It gives a decomposition $\mathfrak{r} = \mathfrak{c} + \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$. $G\mathfrak{n}$ is Gorenstein with rational singularities and (1-1) is satisfied.

The normalization $\tilde{\mathcal{D}}$ is a normal Gorenstein variety with flat quotient map $\tilde{\pi}: \tilde{\mathcal{D}} \longrightarrow \mathfrak{c}/\Gamma$ (and has rational singularities if the characteristic of k is zero). $G\mathfrak{n}$ is isomorphic to the zero fibre of $\tilde{\pi}$, hence is isomorphic to a complete intersection in $\tilde{\mathcal{D}}$.

The affinization $\alpha: \mathcal{Y} \longrightarrow Y$ is a resolution of singularities, the canonical bundle of $\mathcal{Y}$ is trivial, and Y is Gorenstein with rational singularities. Y allows a Γ action with quotient $\tilde{\mathcal{D}}$, the quotient map is flat. $\mathrm{k}[Y]$ is a free graded $\mathrm{k}[\tilde{\mathcal{D}}]$-module of rank $\#\Gamma$ with $\mathrm{k}[\tilde{\mathcal{D}}]$ as a direct summand.

There is a simultaneous resolution of singularities of $\tilde{\pi}$

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & Y & \longrightarrow & \tilde{\mathcal{D}} \\ & \searrow & \downarrow & \square & \downarrow \\ & & \mathfrak{c} & \longrightarrow & \mathfrak{c}/\Gamma. \end{array}$$

We have additionally that $Y = \tilde{\mathcal{D}}$ if and only if $\lambda_i \neq \lambda_j$ whenever $e_i = e_j$. And $\bar{\mathcal{D}}$ is normal if and only if all e_i are equal and all λ_i are equal.

Y is an irreducible component of V, and $\bar{\mathcal{D}}$ is an irreducible component of V/Γ.

$\tilde{\mathcal{D}}^r$ is always smooth. If $\mathcal{D}$ is semisimple, then the sheet $\bar{\mathcal{D}}^r$ is smooth with orbit space $\mathfrak{c}/\Gamma_s$.

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PROOF. Write

$$p_i := \#\{j; e_j = i\} \quad \text{and} \quad p_{i,\lambda} := \#\{j; e_j = i, \lambda_j = \lambda\},$$

for an integer j and a partition λ of j. Write $\tilde{\lambda}$ for the dual partition of λ. Write S_j for the symmetric group on j letters, then

$$\Gamma_s \simeq \prod_i S_{p_i}; \quad \Gamma \simeq \prod_{i,\lambda} S_{p_{i,\lambda}}$$

acting as a group generated by reflections on $\mathfrak{c}$. The fundamental theorem on symmetric polynomials holds true in any characteristic, hence $k[\mathfrak{c}]^{\Gamma_s}$ and $k[\mathfrak{c}]^{\Gamma}$ are both polynomial rings and $k[\mathfrak{c}]^{\Gamma}$ is a direct summand of $k[\mathfrak{c}]$ as a graded $k[\mathfrak{c}]^{\Gamma}$-module.

We construct a good indicator as follows. We underline n_i if and only if i is one of the integers $e_1, e_1 + e_2, \ldots, e_1 + e_2 + \cdots + e_{s-1}$. We put $n_i := 2$ if and only if i is of the form

$$e_1 + \cdots + e_j + \tilde{\lambda}_{j1} + \tilde{\lambda}_{j2} + \cdots + \tilde{\lambda}_{jk},$$

for some j and k. All the other n_i are put 0. That this indicator is good and that (1-1) holds follows from a result of Spaltenstein (see [40, Theorem 4.8]). That $G\mathfrak{n}$ is Gorenstein with rational singularities is a result of Mehta-van der Kallen [40, Theorem 4.6]. It follows that Y is Gorenstein with rational singularities.

We show next that $k[\tilde{V}]$ is reduced. It is a free $k[\tilde{D}]$ module of rank $\#\Gamma$ and it allows a surjection to $k[Y]$ which becomes an isomorphism after localizing at a generic maximal ideal of $k[\tilde{D}]$, by the description of $\tilde{V}^{\circ}$. By Nakayama's lemma it follows that $k[\tilde{V}]$ and $k[Y]$ are isomorphic.

So $k[Y]$ is a free $k[\tilde{D}]$-module of finite rank, so $k[\tilde{D}]$ is Cohen-Macaulay if and only if $k[Y]$ is Cohen-Macaulay. And hence by commutative algebra that $\tilde{\pi}$ is flat with Gorenstein fibres, hence $\tilde{D}$ is Gorenstein.

That the sheet $\bar{D}^r$ is smooth in any characteristic is a result of Bongartz [4]. ∎

9.3. *Complement of the regular semisimple elements in A_2.* Peterson's theorem says that $\bar{D}^r$ is always smooth in type A_r, if D is dense in a sheet. This does not generalize to general stable decomposition classes. Even if $Y = V$ and $\bar{D}$ is a hypersurface $\bar{D}^r$ can be non-normal.

This is so for example for the decomposition class of

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \in \mathfrak{sl}_3$$

of indicator $[\underline{2}, 2]$, consisting of regular elements. Its closure is the complement of the affine open set of regular semisimple elements. This hypersurface is defined by the homogeneous invariant $F = f_2^3 - 6f_3^2$, where $f_2(X) = \mathrm{tr}(X^2)$ and $f_3 = \mathrm{tr}(X^3)$. The quotient $\bar{D}//G$ with coordinate ring $k[\bar{D}]^G = k[f_2, f_3]/(f_2^3 - 6f_3^2)$ is a cusp, so it is not normal and neither is $\bar{D}$. But $\bar{D}$ is a hypersurface, and so it follows from Serre's criterion of normality that its singularities form a subvariety of codimension one; in fact it is the cone of nilpotent elements in $\bar{D}$. In this case Y equals the normalization $\tilde{D}$, which can be identified with a subvariety in $\mathfrak{gl}_3$ defined by the equations $\operatorname{tr}(X^2) = (\operatorname{tr} X)^2$ and $\operatorname{tr}(X^3) = (\operatorname{tr} X)$ and the normalization map $\tilde{D} \longrightarrow \bar{D}$ identifies with $X \longrightarrow X - (\operatorname{tr} X/3)I$, where I is the identity matrix. And $Y = V = \mathfrak{c} \times_{\bar{D}//G} \bar{D}$, even though $\bar{D}//G$ is not normal.

$$\begin{array}{ccccc} Y & \longrightarrow & Y = \tilde{D} & \longrightarrow & \bar{D} \\ & & \downarrow & \square & \downarrow \\ & & \mathfrak{c} & \longrightarrow & \bar{D}//G. \end{array}$$

This example generalizes easily to the complement of the set of regular semisimple elements in any $\mathfrak{sl}_n$. Compare with Richardson [43, Proposition 9.3].

9.4. *Various counter-examples.* For a stable example where V is not irreducible, or equivalently where V° is not dense in V, take $[2, 2, \underline{0}]$ in A_3. In this example

$$x = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad x' = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

generate two different regular orbits in the same fibre of $\bar{\pi}$.

The class containing x, with indicator $[\underline{2}, 2, \underline{0}]$ is unstable. Here Γ is trivial, Γ_s has order two, and so $Y = \tilde{D}$, and $\tilde{D}$ can be identified with the hypersurface of the determinantal variety of 4×4-matrices of rank ≤ 2 defined by the invariant $2\operatorname{tr}(x^2) - (\operatorname{tr} x)^2$. The normalization map is induced by the natural projection $\mathfrak{gl}_4 \longrightarrow \mathfrak{sl}_4$.

For a stable example where (1-1) is not satisfied but $\bar{D}$ is normal still (see example 9.5); for a stable example where (1-1) is not satisfied, where $\bar{D}//G$ is normal but where $\bar{D}$ is not normal and $\bar{D}^r$ is not smooth take $[2, \underline{0}]$ in G_2.

For a stable example where (1-1) is satisfied, but where (refl) is not satisfied, take $\mathfrak{g} = D_4$ and indicator $[\underline{2}, \underline{0}, 2, 2]$. Here the generator of Γ_s acts as multiplication by -1 on $\mathfrak{c}$ and $\mathfrak{c}/\Gamma_s$ has only one singular point.

I don't know of an example where $(\widetilde{1\text{-}1})$ is satisfied but where (1-1) is not.

9.5. *Determinantal varieties and* $\mathfrak{sp}_{2r}$. In this subsection we give examples of stable decomposition varieties where (cart) and $(\widetilde{1\text{-}1})$ are not satisfied, but where still $\bar{D}$ is normal but not Gorenstein and where $\bar{D}^r$ is smooth.

Let $\mathfrak{g}$ be of type C_r, realized as the Lie algebra of $2r \times 2r$-matrices X such that $XJ + JX^t = 0$, with $J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, where I (resp. O) is the $r \times r$ identity (resp. zero) matrix. Let L be the Levi subgroup of G of type C_l, and $\bar{D}$ the closure of the corresponding Dixmier sheet. In this case (1-1) is not satisfied (see [23]), but still Γ_s acts like the reflection group $W(C_{r-l})$ (see [25]).

Multiplication $X \longmapsto JX$ induces an isomorphism between $\mathfrak{g}$ and the space $\operatorname{Sym}(2r)$ of symmetric $2r \times 2r$-matrices. Both $\mathfrak{c}^\circ$ and its image in $\operatorname{Sym}(2r)$ consist of matrices of rank $2(r-l)$, so $J\bar{D} \subset \mathrm{Sym}(2r)$ is contained in the determinantal variety D of symmetric $2r \times 2r$-matrices of rank at most $2(r-l)$. Since both varieties are irreducible of the same codimension $l(2l+1)$, it follows that $\bar{D}$ is isomorphic to D. This determinantal variety is well-studied, *e.g.*, it is normal with rational singularities and its ideal is minimally generated by the determinants of the $2(r-l)+1$-dimensional minors of the generic symmetric $2r \times 2r$-matrix of coordinate functions, also the higher syzygies are known (see [26]). Its coordinate ring is isomorphic to the invariant ring associated to the representation of the orthogonal group $O_{2(r-l)}$ on $2r$ copies of its natural representation, but it is not Gorenstein. It follows that (cart) is not satisfied, that $\bar{D}$ is normal and that it is not Gorenstein.

In the special case where $l = r-1$, hence where $\bar{D}$ has dimension $4r-1$, we claim that Y is isomorphic to the determinantal subvariety D_2 of $\mathfrak{gl}_{2r}$ of matrices of rank ≤ 1. This is also a $(4r-1)$-dimensional affine cone; it is normal Gorenstein with rational singularities and its ideal in the coordinate ring of $\mathfrak{gl}_{2r}$ is minimally generated by the determinants of the 2×2-minors of the generic matrix of coordinate functions. The endomorphism $\tau\colon x \longmapsto Jx^tJ$ of $\mathfrak{gl}_{2r}$ is an involution with fixed points space $\mathfrak{g} = \mathfrak{sp}_{2r}$, D_2 is stable under τ; Γ_s acts on D_2 as the group generated by τ. Let $\beta\colon \mathfrak{gl}_{2r} \longrightarrow \mathfrak{g}$ be the projection defined by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \frac{1}{2}\begin{pmatrix} A-D^t & B+B^t \\ C+C^t & -A^t+D \end{pmatrix},$$

with A, B, C and D $r \times r$-matrices. Then β is Sp_{2r}-equivariant and constant on Γ_s-orbits. The Γ_s-quotient map $\gamma_a\colon Y \longrightarrow \bar{D}$ can be identified with the restriction of β to D_2. Indeed the image of D_2 contains $\beta\big(\mathrm{diag}(1, 0 \ldots, 0)\big) \in \mathfrak{c}^\circ$, which implies that this image is contained in $\bar{D}$. For the non-commutative analog of this example see [49, 8.2].

9.6. *Determinantal varieties as normalization of decomposition varieties.* Let D_{r-l} be the determinantal variety of matrices in $\mathfrak{gl}_{r+1}$ of rank at most $r-l$; it is a Gorenstein variety with rational singularities. The image of D_{r-l} under the projection

$$\beta\colon \mathfrak{gl}_{r+1} \longrightarrow \mathfrak{sl}_{r+1}\colon X \longmapsto X - \frac{\mathrm{tr}(X)}{r+1}I$$

is the closure $\bar{D}$ of a sheet corresponding to a Levi factor of type A_l. The restriction of β to D_{r-l} is the normalization map. For the non-commutative analog of this example in case $l = r-1$ see [49, 8.1].

9.7. *Pfaffian varieties as decomposition varieties.* Let $\mathfrak{g}$ be of type B_r and L a Levi subgroup of type B_l for $l < r$. The corresponding Dixmier sheet is dense in the stable decomposition variety $\bar{D}$ of any semisimple element with stabilizer L, it has Dynkin indicator $[2, 2, \ldots, 2, \underline{0}, \underline{0}, \ldots, \underline{0}]$. The nilpotent variety $G\mathfrak{n} \subset \bar{D}$ has the same indicator (see [23]), so (1-1) is satisfied. By [25], Γ_s acts like $W(B_{r-l})$ on $\mathfrak{c}$, so (refl) is satisfied. We consider $\mathfrak{g}$ as the Lie algebra of anti-symmetric $2r+1 \times 2r+1$-matrices. Any $x \in \mathfrak{c}^\circ$ has rank precisely $2(r-l)$, so $\bar{D}$ is contained in the variety Pf of anti-symmetric

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$2r+1 \times 2r+1$-matrices of rank at most $2(r-l)$. Since $\bar{D}$ and Pf both are irreducible of the same dimension $2(r^2-l^2)+(r-l)$ it follows that $\bar{D} = \mathrm{Pf}$.

The variety Pf has been well-studied, its coordinate ring is isomorphic to the invariant ring associated to the representation of the symplectic group $\mathrm{Sp}_{2(r-l)}$ acting on $2r+1$ copies of its natural representation. The ideal of Pf is generated by $2(r-l)+2$-order Pfaffians and much is known on the higher syzygies (see [26]). Pf is normal, Gorenstein with rational singularities. It follows that $\bar{D}$ has the same properties, (cart) is satisfied, and the sheet $\bar{D}^r$ is smooth. It also follows that the minimal resolution of the coordinate ring $\mathrm{k}[G\mathfrak{n}]$ (which is normal) is the tensor product of the minimal resolution of $\mathrm{k}[\mathrm{Pf}]$ (see [26]) and the Koszul complex on fundamental invariants $f_2, f_4, \ldots, f_{2l}$.

Analogous results hold for the closure of the sheet in $\mathfrak{g} = D_r$ corresponding to L of type D_l for $2 \leq l < r$.

9.8. *Subregular decomposition varieties.* An element $x \in \mathfrak{g}$ with a stabilizer of dimension $r+2$ is called *subregular*. Let $\mathfrak{g}$ be simple then there is a unique class of nilpotent subregular elements, its closure is called the subregular nilpotent variety N_{sr}. We studied its algebraic properties in [10]; for example, we showed it is normal and we described a minimal set of generators for its ideal.

PROPOSITION 9.9. *Let x be a subregular element in the simple Lie algebra $\mathfrak{g}$ with decomposition class $D = D(x)$. Suppose $\bar{D}//G$ is normal and (1-1) is satisfied. Then $\bar{D}$ is a normal Gorenstein variety with rational singularities, the quotient map $\bar{\pi}\colon \bar{D} \longrightarrow \bar{D}//G$ is flat and $Y = V$.*

Furthermore, let $\delta_1, \ldots, \delta_a$ be the fundamental degrees of the action $(\Gamma_s, \mathfrak{c})$ and $d_1, \ldots, d_r$ the fundamental degrees of $(G, \mathfrak{g})$, arranged in such a way that $\delta_i = d_i$, for $i \leq a$, and d_r is the largest degree. Let δ be the height of the short dominant root ϕ. Then the ideal of $\bar{D}$ is generated by fundamental invariants of degree $d_{a+1}, \ldots, d_r$ and by homogeneous functions of degree δ forming a basis for a G-module of highest weight ϕ.

The subregular nilpotent variety N_{sr} is the complete intersection of $\bar{D}$ by fundamental invariants of degrees $d_1, \ldots, d_a$. So $\mathrm{Tor}^R_\bullet(\mathrm{k}_0, \mathrm{k}[N_{sr}])$ is the product of $\mathrm{Tor}^R_\bullet(\mathrm{k}_0, \mathrm{k}[\bar{D}])$ with an exterior algebra on generators of degree $2, 4, \ldots, 2r-6$ in $\mathrm{Tor}^R_1(\mathrm{k}_0, \mathrm{k}[N_{sr}])$.

The only weights occurring in $\mathrm{Tor}^R_\bullet(\mathrm{k}_0, \mathrm{k}[\tilde{D}])$ are the zero-weight and short dominant roots.

PROOF. The proof follows by combining the results of [10] with the results obtained in this article, as Theorem 6.5. ∎

The following cases were studied in [10]. If $\mathfrak{g} = B_r$ and x of type $[2, 2, \ldots, 2, 2, \underline{0}]$; we get the variety defined by the $2r$-Pfaffians studied by Buchsbaum and Eisenbud. If $\mathfrak{g} = C_r$ and x of type $[2, 2, \ldots, 2, \underline{0}, \underline{2}]$ we get a variety also (thoroughly) studied by Klimek, Kraśkiewicz and Weyman in [28]. For D_r and x of type $[2, 2, \ldots, 2, \underline{2}, \underline{2}, \underline{0}]$ we find the variety, whose existence we conjectured in [l.c.]. For F_4 we can take x of type $[2, 2, \underline{0}, \underline{2}]$ and for G_2 we take x of type $[\underline{0}, 2]$.

9.10. *Lusztig's special cases.* In representation theory Lusztig needed to study some special decomposition varieties with extremely good properties.

Let $\mathfrak{g}$ be simple. Suppose L is a Levi-subgroup and Lx_n a nilpotent orbit in $[\mathfrak{l}, \mathfrak{l}]$ with an irreducible L-equivariant cuspidal local system (see Lusztig [37] for the definition). Let $\mathfrak{c}$ be the center of $\mathfrak{l}$ and $\bar{D}$ the decomposition variety $\bar{D} := \overline{G(\mathfrak{c} + x_n)}$. Lusztig determined the decomposition varieties arising in this way. See [36] and [37, p. 160] (when $L \neq G$). He showed normality of $\bar{D}//G$, that $\Gamma = \Gamma_s$ is a reflection group, that (1-1) holds, that there is an even, good indicator obtained in the following way. Take the weighted Dynkin characteristic of the nilpotent class in $\bar{D}^r$, and underline the indices corresponding to L. Furthermore, he proved that $Y^r \simeq \mathcal{Y}^r$ is isomorphic to the pull-back $\mathfrak{c} \times_{\mathfrak{c}/\Gamma} \bar{D}^r$ and hence that $\bar{D}^r$ is smooth.

Using Kraft-Procesi's normality results we checked case by case as in the proof of Theorem 7.9 that whenever $G \neq L$ that the closure of the nilpotent class in $\bar{D}^r$ is normal. When $G = L$ (so when the decomposition variety is a nilpotent variety) this is also the case except maybe for $E_8(a_7)$ in E_8 or for $E_7(a_5)$ in E_7, where the normality property is conjectured (see Remark 7.8(iii)) but not yet shown.

So we get additional properties (with possibly at most two exceptions) for Lusztig's special cases of decomposition varieties arising from cuspidal local systems. They are normal, Gorenstein with rational singularities and $Y = \mathfrak{c} \times_{\mathfrak{c}/\Gamma} \bar{D}$, *i.e.*, there is a Grothendieck simultaneous resolution diagram (using Theorem 7.7)

$$\begin{array}{ccccc} \mathcal{Y} & \longrightarrow & Y & \longrightarrow & \bar{D} \\ & & \downarrow & \square & \downarrow \\ & & \mathfrak{c} & \longrightarrow & \mathfrak{c}/\Gamma. \end{array}$$

Finally, there are non-commutative analogs of all varieties in the diagram, with corresponding properties.

9.11. *Sheets in F_4.* In general, if $\bar{D}^r$ is a sheet it is known what the Dynkin diagram is of its nilpotent orbit. For exceptional Lie algebras these results are due to Elashvili, see Spaltenstein's book [51]. If $\bar{D}^r$ is a Dixmier sheet it is even known in all cases what $\#G_e/P_e$ is, *i.e.*, whether (1-1) is satisfied. Let $\mathfrak{g}$ be of type F_4. Then according to Howlett [25] the (refl)-condition is always satisfied. Suppose x is semisimple then (1-1) is *not* satisfied if and only if the semisimple type is A_1 (long root), B_2, or $A_1 \times A_2$ (one short simple root and the long simple roots). This is checked using Elashvili's tables reproduced in [51, p. 174] together with the knowledge of the component group G_e/G_e^0, where e is an element in the dense P-orbit of $\mathfrak{n}$ (see [16, p. 401]).

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We collect some information in the following table. The columns with headings $G\mathfrak{n}$, $\bar{D}//G$ and $\bar{D}$ indicate whether these varieties are normal "+", non-normal "$-$"; the column with heading $\bar{D}^r$ indicates whether the sheet is smooth. We use the normality results obtained in [12].

type sheet	type $G\mathfrak{n}$	Γ_s	(1-1)	$G\mathfrak{n}$	$\bar{D}//G$	$\bar{D}$	$Y = V$	$\bar{D}^r$
$[2,2,2,2]$	$[2,2,2,2]$	F_4	+	+	+	+	+	+
$[2,2,2,\underline{0}]$	$[2,2,0,2]$	B_3	+	+	–	–	–	?
$[\underline{0},2,2,2]$	$[2,2,0,2]$	B_3	–	+	–	–	–	?
$[\underline{0},2,2,\underline{0}]$	$[0,2,0,2]$	$2A_1$	+	+	–	–	–	?
$[2,\underline{0},\underline{1},2]$	$[0,2,0,2]$	B_2	–	+	–	–	–	?
$[2,2,\underline{0},\underline{0}]$	$[2,2,0,0]$	G_2	+	+	+	+	+	+
$[\underline{0},\underline{0},2,2]$	$[1,0,1,2]$	G_2	+	–	+	–	–	+
$[\underline{0},2,\underline{0},\underline{0}]$	$[0,2,0,0]$	A_1	+	+	+	+	+	+
$[\underline{0},\underline{0},2,\underline{0}]$	$[0,2,0,0]$	A_1	–	+	+	?	–	?
$[2,\underline{0},\underline{0},2]$	$[0,2,0,0]$	B_2	–	+	–	–	–	?
$[\underline{0},\underline{1},\underline{0},2]$	$[1,0,1,0]$	A_1	+	–	+	–	–	+
$[2,\underline{0},\underline{0},\underline{1}]$	$[2,0,0,1]$	A_1	+	–	+	–	–	+
$[\underline{0},\underline{1},\underline{0},\underline{1}]$	$[1,0,1,0]$	A_0	+	–	+	–	–	+
$[\underline{0},\underline{0},\underline{1},\underline{0}]$	$[0,0,1,0]$	A_0	+	+	+	+	+	+
$[2,\underline{0},\underline{0},\underline{0}]$	$[2,0,0,0]$	A_1	+	+	+	+	+	+
$[\underline{0},\underline{0},\underline{0},2]$	$[0,0,0,2]$	A_1	+	–	+	–	–	+
$[\underline{0},\underline{1},\underline{0},\underline{0}]$	$[0,1,0,0]$	A_0	+	+	+	+	+	+
$[\underline{0},\underline{0},\underline{0},\underline{1}]$	$[0,0,0,1]$	A_0	+	+	+	+	+	+
$[\underline{1},\underline{0},\underline{0},\underline{0}]$	$[1,0,0,0]$	A_0	+	+	+	+	+	+
$[\underline{0},\underline{0},\underline{0},\underline{0}]$	$[0,0,0,0]$	A_0	+	+	+	+	+	+

REFERENCES

1. A. Altman and S. Kleiman, *Introduction to Grothendieck duality theory*. Lecture Notes in Math. **146**, Springer-Verlag, New York, 1970.

2. P. Bardsley and R. W. Richardson, *Etale slices for algebraic transformation groups in characteristic p*. Proc. London. Math. Soc. (3)**51**(1985), 295–317.

3. W. M. Beynon and N. Spaltenstein, *Green functions of finite Chevalley groups of type E_n ($n = 6, 7, 8$)*. J. Algebra **88**(1984), 584–614.

4. K. Bongartz, *Schichten von Matrizen sind rationale Varietäten*. Math. Ann. **283**(1989), 53–64.

5. W. Borho, *Über Schichten halbeinfacher Lie-Algebren*. Invent. Math. **65**(1981), 283–317.

6. W. Borho and J.-L. Brylinski, *Differential operators on homogeneous spaces. I. Irreducibility of the associated variety for annihilators of induced modules*. Invent. Math. **69**(1982), 437–476.

7. ———, *Differential operators on homogeneous spaces. II. Relative enveloping algebras*. Bull. Soc. Math. France **117**(1989), 167–210.

8. W. Borho and H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen*. Comment. Math. Helv. **54**(1979), 61–104.

9. J.-F. Boutot, *Singularités rationnelles et quotients par les groupes réductifs*. Invent. Math. **88**(1987), 65–68.

10. A. Broer, *Line bundles on the cotangent bundle of the flag variety*. Invent. Math. **113**(1993), 1–20.

11. ———, *Normality of some nilpotent varieties and cohomology of line bundles on the cotangent bundle of the flag variety*. In: J.-L. Brylinski *et al*, Lie theory and geometry, In honor of Bertrant Kostant, Progress in Math. **123**(1994), 1–19.

12. ———, *Normal nilpotent varieties in F_4*, J. Algebra, to appear.

13. ______, *Lectures on decomposition classes*. Representation theories and algebraic geometry, (ed. A. Broer), NATO ASI Series C, Kluwer Academic Publishers, Dordrecht, 1998, to appear.
14. R. Brylinski and B. Kostant, *On the variety associated to a* TDS. December 1992, preprint.
15. R. W. Carter, *Conjugacy classes in the Weyl group*. In: Borel *et al*, Seminar on algebraic groups and related finite groups, Lecture Notes in Math. **131**, Springer-Verlag, New York, 1970.
16. ______, *Finite groups of Lie type. Conjugacy classes and complex characters*. Wiley, Chichester, 1985.
17. D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold, New York, 1993.
18. R. Elkik, *Singularités rationnelles et déformations*. Invent. Math. **47**(1978), 139–147.
19. H. Flenner, *Rationale quasihomogene Singularitäten*. Arch. Math. **36**(1981), 35–44.
20. I. M. Gel'fand and A. A. Kirillov, *The structure of the Lie field connected with a split semisimple Lie algebra*. Functional Anal. Appl. **3**(1969), 6–21.
21. A. Grothendieck, *Éléments de géométrie algébrique, IV*. Publ. Math. I.H.E.S. **20, 24, 28, 32**(1964–67).
22. R. Hartshorne, *Algebraic Geometry*. Graduate Texts in Math. **52**, Springer-Verlag, New York, 1977.
23. W. H. Hesselink, *Polarizations in the classical groups*. Math. Z. **160**(1978), 217–234.
24. V. Hinich, *On the singularities of nilpotent orbits.* Israel J. Math. **73**(1991), 297–308.
25. R. B. Howlett, *Normalizers of parabolic subgroups of reflection groups*. J. London Math. Soc. **21**(1980), 62–80.
26. T. Jósefiak, P. Pragacz and J. Weyman, *Resolutions of determinantal varieties*. In: Tableaux de Young et foncteurs de Schur en algèbre et géométrie, Toruń, Astérisque **87–88**, Société Mathématique de France, (1981), 109–189.
27. P. I. Katsylo, *Sections of sheets in a reductive Lie algebra*. Math. USSR-Izv. **20**(1983), 449–457.
28. J. Klimek, W. Kraśkiewicz and J. Weyman, *A free resolution of a symplectic rank variety*. J. Algebra, to appear.
29. F. Knop, *Weylgruppe und Momentabbildung*. Invent. Math. **99**(1990), 1–23.
30. ______, *A Harish-Chandra homomorphism for reductive group actions*. Ann. of Math. **140**(1994), 253–288.
31. H. Kraft, *Paramatrisierung von Konjugationsklassen in* $\mathfrak{sl}_n$. Math. Ann. **234**(1978), 209–220.
32. ______, *Geometrische Methoden in der Invariantentheorie*. Vieweg Verlag, Braunschweig, 1984.
33. H. Kraft, *Closures of conjugacy classes in* G_2. J. Algebra **126**(1989), 454–465.
34. H. Kraft and C. Procesi, *Closures of conjugacy classes of matrices are normal*. Invent. Math. **53**(1979), 227–247.
35. ______, *On the geometry of conjugacy classes in classical groups*. Comment. Math. Helv. **57**(1982), 539–602.
36. G. Lusztig, *Intersection cohomology complexes on a reductive group*. Invent. Math. **75**(1984), 205–272.
37. ______, *Cuspidal local systems and graded Hecke algebras, I*. Publ. Math. I.H.E.S. **67**(1988), 145–202.
38. G. Lusztig and N. Spaltenstein, *Induced unipotent classes*. J. London Math. Soc. **19**(1979), 41–52.
39. H. Matsumura, *Commutative ring theory*. Cambridge Univ. Press, Cambridge, 1986.
40. V. B. Mehta and W. van der Kallen, *A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices*. Compositio Math. **84**(1992), 169–178.
41. D. I. Panyushev, *Rationality of singularities and the Gorenstein property for nilpotent orbits.* Functional Anal. Appl. **25**(1991), 225–226.
42. D. Peterson, *Geometry of the adjoint representation of a complex semisimple Lie algebra*. thesis, Harvard University, 1978.
43. R. W. Richardson, *Normality of G-stable subvarieties of a semisimple Lie algebra*. In: A. Cohen *et al*, Algebraic groups, Utrecht 1986, Lecture Notes in Math. **1271**, Springer-Verlag, New York, 1987.
44. ______, *Derivatives of invariant polynomials on a semisimple Lie algebra*. In: Harmonic analysis and operator theory, Proc. Cent. Math. Anal. Austral. Nat. Univ. **15**(1987).
45. H. Rubenthaler, *Paramétrisation d'orbites dans les nappes de Dixmier admissibles*. In: Duflo *et al*, Analyse harmonique sur les groupes de Lie et les espaces symétriques, Mém. Soc. Math. France (N.S.) **15**(1984), 255–275.
46. I. R. Shafarevich, *Basic algebraic geometry* 1, *Varieties in projective space, Second, revised and expanded version*. Springer-Verlag, New York, 1994.
47. P. Slodowy, *Simple singularities and simple algebraic groups*. Lecture Notes in Math. **815**, Springer-Verlag, New York, 1980.
48. ______, *Four lectures on simple groups and singularities*. Communications of the Mathematical Institute **11-1980**, Rijksuniversiteit Utrecht, 1980.

49. W. Soergel, *Universelle versus relative Einhüllende: Eine geometrische Untersuchung von Quotienten von universellen Einhüllenden halbeinfacher Lie-Algebren*. Math. Ann. **284**(1989), 177–198.

50. N. Spaltenstein, *Nilpotent classes and sheets of Lie algebras in bad characteristic*. Math. Z. **181**(1982), 31–48.

51. ———, *Classes unipotents et sous-groupes de Borel*. Lecture Notes in Math. **946**, Springer-Verlag, New York, 1982.

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