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On the existence of multiple positive solutions to some superlinear systems

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We use the method of upper and lower solutions combined with degree-theoretic techniques to prove the existence of multiple positive solutions to some superlinear elliptic systems of the form

$$-\Delta u = g_1(x,u,v), \qquad -\Delta v = g_2(x,u,v),$$

on a smooth, bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions, under suitable conditions on g_1 and g_2. Our techniques apply generally to subcritical, superlinear problems with a certain concave–convex shape to their nonlinearity.

1. Introduction

In this paper we study the multiplicity of solutions to an elliptic system of the form

$$\left.\begin{aligned} -\Delta u &= g_1(x,u,v) && \text{for all } x \in \Omega,\\ -\Delta v &= g_2(x,u,v) && \text{for all } x \in \Omega,\\ u, v &> 0 && \text{for all } x \in \Omega,\\ u = v &= 0 && \text{for all } x \in \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, and $g_i(x,u,v)\colon \bar{\Omega}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $i = 1,2$, are differentiable functions subject to further restrictions to be named below. We assume that $g_i(x,u,v)$, $i = 1,2$, satisfy suitable conditions on a bounded rectangle so that a positive strict lower solution pair and a positive strict upper solution pair can be constructed. These assumptions suffice to prove the existence of at least one positive solution, provided the lower solution pair and upper solution pair are ordered componentwise. For example, in [4], Chhetri and Robinson prove the existence of a positive solution for a single equation analogous to (1.1) by constructing an ordered pair of lower and upper solutions. There, the nonlinearity is negative at the origin and satisfies additional conditions. (Further related references may be found therein.)

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To obtain the second solution, we assume that for $i, j = 1, 2$ there exist constants q_{ij} with $0 \leqslant q_{ij} < 2^* - 1 = (n+2)/(n-2)$ for $n > 2$ or $0 \leqslant q_{ij}$ for $n = 2$ so that the following holds: there exist continuous functions $h_{ij}(x)$ on $\bar{\Omega}$ which are strictly positive in Ω such that

(H1) $g_i(x, u, v) = h_{i1}(x)u^{q_{i1}} + h_{i2}(x)v^{q_{i2}} + r_i(x, u, v)$

with $|r_i(x, u, v)| \leqslant C(1 + |u|^{\beta_{i1}} + |v|^{\beta_{i2}})$ and $\beta_{ij} < q_{ij}$ for $i, j = 1, 2$. (If $q_{ij} = 0$, β_{ij} may also be 0.)

The purpose of this subcriticality condition is to enable us to obtain *a priori* bounds for our problem. Using a blow-up technique as first introduced in [13], in the scalar equation case, we can reduce the question of finding *a priori* bounds to the question of an appropriate Liouville theorem. If the *a priori* bound holds, then we can do a degree calculation to obtain a second solution with a larger L^∞ norm than the first solution obtained via upper and lower solutions. Such Liouville theorems depend on the form of the interaction between u and v in the nonlinearities. If the system is weakly coupled, as defined in § 2, then the Liouville theorem follows from the scalar case as in the work of Gidas and Spruck [13]. In the present paper, we use the method introduced in [8] to apply the blow-up procedure to weakly coupled systems. On the other hand, if the system is strongly coupled, then we obtain a Hamiltonian system after applying the blow-up procedure. Although a complete Liouville theory does not yet exist in this case, a great deal of work has been done. For example, in [9] de Figueiredo and Felmer obtain a Liouville theorem for some subcritical exponents; in [10] these authors prove a Liouville theorem in which they actually allow for interaction terms in the blown-up system; in [16] Guo and Liu prove a generalized result which unifies the above cases and includes some non-pure-power nonlinearities. Additionally, some work has been done to obtain *a priori* bounds for strongly coupled systems via non-blow-up methods [19]. In this paper, we will not attempt to prove a Liouville theorem, but will instead assume, as a condition on the system (if strongly coupled), that either it admits a Liouville theorem or an *a priori* bound holds for positive solutions via some other technique.

Sun *et al*. [22] obtained multiple positive solutions for the single equation case when the nonlinearity is of the form $\lambda u^\beta + p(x)u^{-\alpha}$ with $0 < \alpha < 1 < \beta < 2^* - 1$. The interesting feature about such a nonlinearity is that it exhibits concave–convex type behaviour, thus making it possible to obtain the Ambrosetti–Brézis–Cerami-type result initiated by by Ambrosetti *et al*. [2]. Our nonlinearities roughly exhibit this concave–convex behaviour in the sense that there is a finite rectangle on which they are bounded above by an upper shelf, whereas for large values of u and v they are superlinear. Hence, the two-solution conclusion that we obtain, with one solution in the concave region and one solution reaching into the convex region, is not unexpected.

The novelty of this paper lies in the fact that we are able to get not just one positive solution to a superlinear problem, but also a second positive solution. Our motivation in this work was to study the semi-positone case, where the nonlinearity is negative at origin; however, our results do not depend on the sign of the nonlinearity at the origin and can be applied in great generality. To the best of our knowledge, this paper is the first to deal with multiple positive solutions of general superlinear systems. In the scalar case, Crandall and Rabinowitz proved the existence of two positive solutions by combining the variational method and bifurcation theory for $\lambda > 0$ small, in [6, theorem 2.1]. In [11, corollary 2.2], de Figueiredo *et al.* proved the existence of two positive solutions for λ small using degree theory and an *a priori* bound on positive solutions. In fact, the proof shows that the pair of positive solutions lies on a continuum of positive solutions. In [17, theorem 2.1(iii)], the existence of two positive solutions was also established by combining degree theory with an upper–lower solution method. Thus, the abstract result in our paper can be considered as an extension of those established in these three papers: specifically [17] because the approach is similar. We note that in these three papers, the nonlinearity f was assumed to be positive at the origin (the so-called *positone* case) and the parameter λ was considered as part of the nonlinearity. However, in this paper we do not assume any sign condition of the nonlinearities g_i at the origin, and g_i are parameter free. The only multiplicity result we know for the corresponding *semi-positone* case (when the nonlinearity is negative at the origin) is [7, theorem 1.1(C)]. Castro and Shivaji prove that, for $n = 1$, there is a finite range of $\lambda > 0$ for which the problem has two positive solutions. Thus, our result for the general bounded domain is new even for a scalar equation in the semi-positone case. It is also important to note that our methods allow a large class of differential operators and nonlinear forcing terms. In particular, our proofs find both positive solutions using degree-theoretic arguments, rather than a mountain-pass argument. Thus, our results generalize to non-variational elliptic operators. They also generalize easily to larger systems.

This paper is organized as follows: in § 2, we state and prove a general theorem about the existence of two solutions. In § 3, we prove the existence of positive solutions to an auxiliary problem. These solutions will become the lower solutions for our main theorem. In § 4, we state and prove an existence theorem in the weakly coupled quasi-monotone non-decreasing case, and provide two examples of nonlinearities that satisfy the hypotheses of the theorem. In § 5, we state and prove a theorem that deals with a quasi-monotone non-decreasing Hamiltonian (and thus strongly coupled) system and provide an example satisfying the hypotheses of the theorem.

2. A general two-solution existence theorem

Using the method of upper and lower solutions, an *a priori* bound and some degree theory, we formulate very general conditions under which two solutions will be guaranteed to exist for an elliptic superlinear problem. This is phrased in terms of a 2×2 system of equations for simplicity of exposition, but the technique also applies to single equations or larger systems. Throughout this paper we will use the vector notation $\boldsymbol{u} = (u, v)$ to indicate a pair of functions being considered in the first and second equations of the system, respectively.

First, we must define some terms for the general types of systems that we may encounter. The following are taken from [18, § 8.4, p. 402].

Definition 2.1. The nonlinearities $g_i(x, u, v)$ are called *quasi-monotone non-decreasing* if, for all x, u, v,

$$\frac{\partial g_1}{\partial v} \geqslant 0 \quad \text{and} \quad \frac{\partial g_2}{\partial u} \geqslant 0.$$

In this case, two pairs of functions $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are called an ordered lower–upper solution pair of (1.1) if $\underline{u}(x) \leqslant \bar{u}(x)$ and $\underline{v}(x) \leqslant \bar{v}(x)$ for all $x \in \bar{\Omega}$, $\underline{u}(x) = 0$ and $\underline{v}(x) = 0$ on $\partial\Omega$, $\bar{u}(x) \geqslant 0$ and $\bar{v}(x) \geqslant 0$ on $\partial\Omega$ and

$$\left.\begin{aligned} -\Delta\underline{u} &\leqslant g_1(x, \underline{u}, \underline{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\underline{v} &\leqslant g_2(x, \underline{u}, \underline{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\bar{u} &\geqslant g_1(x, \bar{u}, \bar{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\bar{v} &\geqslant g_2(x, \bar{u}, \bar{v}) \quad \text{for all } x \in \Omega. \end{aligned}\right\} \tag{2.1}$$

Definition 2.2. The nonlinearities $g_i(x, u, v)$ are called *quasi-monotone non-increasing* if, for all x, u, v,

$$\frac{\partial g_1}{\partial v} \leqslant 0 \quad \text{and} \quad \frac{\partial g_2}{\partial u} \leqslant 0.$$

In this case, two pairs of functions $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are called an ordered lower–upper solution pair of (1.1) if $\underline{u}(x) \leqslant \bar{u}(x)$ and $\underline{v}(x) \leqslant \bar{v}(x)$ for all $x \in \bar{\Omega}$, $\underline{u}(x) = 0$ and $\underline{v}(x) = 0$ on $\partial\Omega$, $\bar{u}(x) \geqslant 0$ and $\bar{v}(x) \geqslant 0$ on $\partial\Omega$ and

$$\left.\begin{aligned} -\Delta\underline{u} &\leqslant g_1(x, \underline{u}, \bar{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\underline{v} &\leqslant g_2(x, \bar{u}, \underline{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\bar{u} &\geqslant g_1(x, \bar{u}, \underline{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\bar{v} &\geqslant g_2(x, \underline{u}, \bar{v}) \quad \text{for all } x \in \Omega. \end{aligned}\right\} \tag{2.2}$$

Definition 2.3. The nonlinearities $g_i(x, u, v)$ are called *quasi-monotone mixed* if, for all x, u, v,

$$\frac{\partial g_1}{\partial v} \leqslant 0 \quad \text{and} \quad \frac{\partial g_2}{\partial u} \geqslant 0.$$

(If the opposite holds, we switch u and v.) In this case, two pairs of functions $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are called an ordered lower–upper solution pair of (1.1) if $\underline{u}(x) \leqslant \bar{u}(x)$ and $\underline{v}(x) \leqslant \bar{v}(x)$ for all $x \in \bar{\Omega}$, $\underline{u}(x) = 0$ and $\underline{v}(x) = 0$ on $\partial\Omega$, $\bar{u}(x) \geqslant 0$ and $\bar{v}(x) \geqslant 0$ on $\partial\Omega$ and

$$\left.\begin{aligned} -\Delta\underline{u} &\leqslant g_1(x, \underline{u}, \bar{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\underline{v} &\leqslant g_2(x, \underline{u}, \underline{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\bar{u} &\geqslant g_1(x, \bar{u}, \underline{v}) \quad \text{for all } x \in \Omega, \\ -\Delta\bar{v} &\geqslant g_2(x, \bar{u}, \bar{v}) \quad \text{for all } x \in \Omega. \end{aligned}\right\} \tag{2.3}$$

We say that the nonlinearities are *quasi-monotone* if they are either quasi-monotone non-decreasing, non-increasing or mixed. In each case, an upper–lower solution pair is called *strict* if each of the differential inequalities is strict for all $x \in \Omega$.

The following definitions are taken from [8].

Definition 2.4. The system (1.1) under condition (H1) is *weakly coupled* if there are positive numbers c_1 and c_2 such that

$$\begin{aligned} c_1 + 2 - c_1 q_{11} = 0, &\qquad c_1 + 2 - c_2 q_{12} > 0, \\ c_2 + 2 - c_1 q_{21} > 0, &\qquad c_2 + 2 - c_2 q_{22} = 0. \end{aligned}$$

DEFINITION 2.5. The system (1.1) under condition (H1) is *strongly coupled* if there are positive numbers c_1 and c_2 such that

$$\begin{aligned} c_1 + 2 - c_1 q_{11} > 0, \qquad c_1 + 2 - c_2 q_{12} = 0, \\ c_2 + 2 - c_1 q_{21} = 0, \qquad c_2 + 2 - c_2 q_{22} > 0. \end{aligned}$$

DEFINITION 2.6. Suppose the system (1.1) is strongly coupled as in definition 2.5. Then we say that the system satisfies the *Liouville condition* if there are no non-trivial non-negative solutions to the system

$$\left.\begin{aligned} -\Delta u = h_{12}(x_0) v^{q_{12}} \quad \text{for all } x \in \mathbb{R}^n, \\ -\Delta v = h_{21}(x_0) u^{q_{21}} \quad \text{for all } x \in \mathbb{R}^n, \end{aligned}\right\} \tag{2.4}$$

for some $x_0 \in \bar{\Omega}$, and the same holds on the half-space $\mathbb{R}^n_+$.

If the system (1.1) is strongly coupled, as defined above, then there exists some $R_0 > 0$ such that, for all $x \in \bar{\Omega}$,

$$\begin{aligned} g_1(x, u, v) > \mu_1 v + 1 \quad \text{for all } u \geqslant 0 \text{ and } v > R_0, \\ g_2(x, u, v) > \mu_1 u + 1 \quad \text{for all } v \geqslant 0 \text{ and } u > R_0, \end{aligned}$$

where μ_1 is the first eigenvalue of $(-\Delta)$ on Ω. Let

$$m_1(x, u, v) = \begin{cases} g_1(x, u, v) & \text{for all } u \geqslant 0 \text{ and } v \geqslant R_0, \\ \max\{\mu_1 v + 1, g_1(x, u, v)\} & \text{for all } u \geqslant 0 \text{ and } 0 \leqslant v < R_0, \end{cases}$$

and similarly

$$m_2(x, u, v) = \begin{cases} g_2(x, u, v) & \text{for all } v \geqslant 0 \text{ and } u \geqslant R_0, \\ \max\{\mu_1 u + 1, g_2(x, u, v)\} & \text{for all } u \geqslant 0 \text{ and } 0 \leqslant u < R_0, \end{cases}$$

and $\boldsymbol{m}(x, \boldsymbol{u}) = (m_1(x, \boldsymbol{u}), m_2(x, \boldsymbol{u}))$. Let

$$\boldsymbol{p}_t(x, \boldsymbol{u}) = (1 - t)\boldsymbol{g}(x, \boldsymbol{u}) + t\boldsymbol{m}(x, \boldsymbol{u}),$$

where $\boldsymbol{g} = (g_1, g_2)$. Then we have the following definition.

DEFINITION 2.7. Suppose the system (1.1) is strongly coupled as in definition 2.5. Then we say that the system satisfies the *a priori bound condition* if there is a $T > 0$ such that the estimate $\|u\|_{L^\infty} + \|v\|_{L^\infty} < T$ holds for any solution triple (t, u, v) of

$$\begin{aligned} -\Delta \boldsymbol{u} = \boldsymbol{p}_t(x, \boldsymbol{u}) \quad & \text{for all } x \in \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \quad & \text{for all } x \in \partial\Omega. \end{aligned}$$

with $0 \leqslant t \leqslant 1$.

Note that the Liouville condition implies the *a priori* bound for positive solutions through a blow-up procedure, but there are other techniques which can also yield appropriate *a priori* bounds in some cases. See [19] for some of the techniques used in the literature.

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Now we state our general two-solution theorem.

THEOREM 2.8. *Suppose that $g_i(x,u,v)$, $i=1,2$, satisfy (H1) and the following:*

(C1) $g_i(x,u,v)$ *are quasi-monotone;*

(C2) *there exists a strictly positive ordered upper–lower solution pair $(\underline{u},\underline{v})$ and $(\bar{u},\bar{v})$ according to the definition corresponding to the nonlinearities' quasi-monotone type, and either*

(a) *the system is weakly coupled or*

(b) *the system is strongly coupled, and satisfies either the Liouville condition (definition 2.6) or the a priori bound condition (definition 2.7).*

Then (1.1) has at least two solutions.

REMARK 2.9. In the strongly coupled case, an *a priori* bound for the problem is not known in general. We will use the blow-up method first developed by Gidas and Spruck [13] to prove an *a priori* bound in the weakly coupled case. We will also use the same method in the strongly coupled case, but we can only conclude the *a priori* bound if an appropriate Liouville theorem holds in $\mathbb{R}^n$ and $\mathbb{R}^n_+$. This will be discussed further in the proof of the theorem for the strongly coupled case below.

We will prove this theorem via several lemmata. Using the method of upper and lower solutions we will find a first solution and show that the degree of an operator corresponding to the system (1.1) is 1 on an appropriate set. We will then prove (or assume) that positive solutions of the system satisfies an *a priori* bound. This will allow us to show that the degree on a larger set is 0 and conclude that there is a second solution using the excision property of Leray–Schauder degree.

In order to establish the existence of solutions, we shall need to represent the boundary-value problem (1.1) as an operator equation in the proper form and then perform a Leray–Schauder degree computation. Similar arguments for single equations can be found in many references (see [1,21] for details).

In order to work in the appropriate function space setting, we consider the auxiliary problem

$$\left.\begin{aligned} -\Delta z &= 1 \quad \text{for all } x\in\Omega, \\ z &= 0 \quad \text{for all } x\in\partial\Omega. \end{aligned}\right\} \tag{2.5}$$

By the Hopf maximum principle we know that z is strictly positive in Ω and that

$$\left|\frac{\partial z}{\partial\nu}\right| > 0 \quad \text{on } \partial\Omega,$$

where ν represents the unit outward normal on the boundary. Let

$$C_z(\bar{\Omega}) := \{u\in C(\bar{\Omega})\colon -tz\leqslant u\leqslant tz \text{ in } \bar{\Omega} \text{ for some } t>0\},$$

and let $\|u\|_z := \inf\{t>0\colon -tz\leqslant u\leqslant tz\}$. Define $X := C_z(\bar{\Omega})\times C_z(\bar{\Omega})$. Notice that the rectangle $W := \{\boldsymbol{u}\in X\colon \underline{u}(x) < u(x) < \bar{u}(x),\ \underline{v}(x) < v(x) < \bar{v}(x)\} = (\underline{u},\underline{v})\times(\bar{u},\bar{v})$ is open in the X topology.

Now let $F\boldsymbol{u} := I\boldsymbol{u} - L^{-1}N(\boldsymbol{u})$ for $\boldsymbol{u} \in X$, where I is the identity, L^{-1} is the inverse of $L := (-\Delta, -\Delta)$ and $N(\boldsymbol{u}) := (g_1(x, \boldsymbol{u}), g_2(x, \boldsymbol{u}))$. The standard arguments applied to elliptic operators and substitution operators show that F is a compact perturbation of the identity, and so it is valid to discuss Leray–Schauder degree computations for F.

Lemma 2.10. *If g_1 and g_2 are quasi-monotone, then the system (1.1) has a solution in the set W. Moreover,* $\deg(F, \mathbf{0}, W) = 1$.

Proof. For simplicity, we provide full details only for the case when the nonlinearities are quasi-monotone non-decreasing. We indicate below how to modify the proof in the other two cases.

The first goal is to transform the problem (1.1) to one with helpful monotonicity properties. Choose $t > 0$ such that

$$\frac{\partial g_1}{\partial u}(x, \boldsymbol{u}) \geqslant -t \quad \text{and} \quad \frac{\partial g_2}{\partial v}(x, \boldsymbol{u}) \geqslant -t$$

for $\underline{u}(x) \leqslant u(x) \leqslant \bar{u}(x)$ and $\underline{v}(x) \leqslant v(x) \leqslant \bar{v}(x)$. Let $a_t(x, \boldsymbol{u}) := g_1(x, \boldsymbol{u}) + tu$ and $b_t(x, \boldsymbol{u}) := g_2(x, \boldsymbol{u}) + tv$. Let $L_t := -\Delta + t$. The problem (1.1) can now be rewritten as

$$\left.\begin{aligned} L_t u &= a_t(x, \boldsymbol{u}) && \text{for all } x \in \Omega, \\ L_t v &= b_t(x, \boldsymbol{u}) && \text{for all } x \in \Omega, \\ u &= 0 && \text{for all } x \in \partial\Omega, \\ v &= 0 && \text{for all } x \in \partial\Omega, \end{aligned}\right\} \tag{2.6}$$

where L_t satisfies the standard maximum principle for linear elliptic operators and a_t, b_t are monotone in both variables. Moreover, it is easy to check that $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are lower–upper solution pairs for (2.6).

The second goal is to modify the problem to ensure that solutions cannot occur outside of the rectangle W. For a given function $u(x)$, let

$$\tilde{u}(x) := \begin{cases} \underline{u}(x) & \text{if } u(x) \leqslant \underline{u}(x), \\ u(x) & \text{if } \underline{u}(x) < u(x) < \bar{u}(x), \\ \bar{u}(x) & \text{if } \bar{u}(x) \leqslant u(x). \end{cases}$$

Define $\tilde{v}(x)$ similarly. Define the substitution operators

$$\tilde{a}_t(u(x), v(x)) := a_t(x, \tilde{u}(x), \tilde{v}(x)) \quad \text{and} \quad \tilde{b}_t(u(x), v(x)) := b_t(x, \tilde{u}(x), \tilde{v}(x)).$$

We can now state a modified boundary-value problem (BVP) that has useful properties of monotonicity and boundedness:

$$\left.\begin{aligned} L_t u &= \tilde{a}_t(\boldsymbol{u}) && \text{for all } x \in \Omega, \\ L_t v &= \tilde{b}_t(\boldsymbol{u}) && \text{for all } x \in \Omega, \\ u &= 0 && \text{for all } x \in \partial\Omega, \\ v &= 0 && \text{for all } x \in \partial\Omega. \end{aligned}\right\} \tag{2.7}$$

The third goal is to do a degree computation for (2.7), and then relate that computation back to the original BVP. The modified BVP can be represented as

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an operator equation of the form

$$\tilde{F}_t(\boldsymbol{u}) := I\boldsymbol{u} - \boldsymbol{L}_t^{-1}\tilde{N}_t(\boldsymbol{u}) = \boldsymbol{0}$$

on the space X, where $\boldsymbol{L}_t^{-1}$ is the inverse of $\boldsymbol{L}_t := (L_t, L_t)$, and

$$\tilde{N}_t(\boldsymbol{u}) := (\tilde{a}_t(\boldsymbol{u}), \tilde{b}_t(\boldsymbol{u})).$$

Since $\tilde{N}_t$ is bounded, and every solution of $\tilde{F}_t(\boldsymbol{u}) = \boldsymbol{0}$ satisfies

$$\|\boldsymbol{u}\| = \|\boldsymbol{L}_t^{-1}\tilde{N}_t(\boldsymbol{u})\| \leqslant \|\boldsymbol{L}_t^{-1}\|\|\tilde{N}_t(\boldsymbol{u})\|,$$

it is straightforward to obtain an *a priori* bound on solutions. If we then select any $R > 0$ larger than the *a priori* bound and consider the homotopy $h(\lambda, \boldsymbol{u}) := I\boldsymbol{u} - \lambda\boldsymbol{L}_t^{-1}\tilde{N}_t(\boldsymbol{u})$ for $\lambda \in [0, 1]$, we see that $\deg(\tilde{F}_t, \boldsymbol{0}, B_R(\boldsymbol{0})) = \deg(I, \boldsymbol{0}, B_R(\boldsymbol{0})) = 1$.

It follows from the previous argument that (2.7) has at least one solution $\boldsymbol{u} \in B_R(\boldsymbol{0})$. Observe that

$$L_t u(x) = \tilde{a}_t(u(x), v(x)) \leqslant \tilde{a}_t(\bar{u}(x), \bar{v}(x)) < L_t\bar{u}(x). \tag{2.8}$$

By the maximum principle this implies that $u(x) < \bar{u}(x)$ in Ω. Similar arguments show that $\underline{u}(x) < u(x)$ and $\underline{v}(x) < v(x) < \bar{v}(x)$. It follows that all solutions of (2.7) are also solutions of (2.6) and thus of (1.1). Moreover, these solutions must lie strictly between the upper and lower solution pairs, and hence in W. We can now say that

$$\deg(\tilde{F}_t, \boldsymbol{0}, B_R(\boldsymbol{0})) = \deg(\tilde{F}_t, \boldsymbol{0}, W) = \deg(F_t, \boldsymbol{0}, W) = 1,$$

where $N_t(\boldsymbol{u}) := (a_t(\boldsymbol{u}), b_t(\boldsymbol{u}))$ and $F_t := I - \boldsymbol{L}_t^{-1}N_t$.

Finally, we consider t to be a homotopy parameter and let $t \to 0$ so that $F_t \to F$. It is clear that the solutions to (2.6) in $\bar{W}$ do not change as t changes, so there are no solutions on ∂W for any t. Hence, degree is preserved along the homotopy and we get $\deg(F, \boldsymbol{0}, W) = 1$.

The cases where g_1 and g_2 satisfy either definition 2.2 or definition 2.3 can be handled in a similar way. For example, if g_1 and g_2 are quasi-monotone nonincreasing, and if $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are lower and upper solution pairs as described in definition 2.2, then we can modify g_1 exactly as before, and $a_t(x, u, v)$ will then be non-decreasing in u and nonincreasing in v. It is then straightforward to apply this monotonicity and the assumptions in definition 2.2 to get the analogue to (2.8), i.e.

$$L_t u(x) = \tilde{a}_t(u(x), v(x)) \leqslant \tilde{a}_t(\bar{u}(x), \underline{v}(x)) < L_t\bar{u}(x).$$

Other comparisons follow similarly. □

In order to obtain a second solution, we will do a second degree computation on a similar set, $(\underline{u}, T) \times (\underline{v}, T)$, where T is an *a priori* bound on the solutions of (1.1). For $0 \leqslant t \leqslant 1$, define

$$\boldsymbol{p}_t(x, \boldsymbol{u}) = (1 - t)\boldsymbol{g}(x, \boldsymbol{u}) + t\boldsymbol{m}(x, \boldsymbol{u}),$$

where $\boldsymbol{g} = (g_1, g_2)$ and $\boldsymbol{p}_t = (p_{1,t}, p_{2,t})$. Here $\boldsymbol{m}(x, \boldsymbol{u}) := (m_1(x, \boldsymbol{u}), m_2(x, \boldsymbol{u}))$ is defined below for a weakly coupled system and was defined in definition 2.7 for a strongly coupled system. We proceed to study the homotopy class of problems

$$\left.\begin{aligned} -\Delta \boldsymbol{u} &= \boldsymbol{p}_t(x, \boldsymbol{u}) && \text{for all } x \in \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{for all } x \in \partial\Omega. \end{aligned}\right\} \tag{2.9}$$

Observe that $p_{1,t}$ increases as t increases, so

$$-\Delta \underline{u} \leqslant g_1(x, \underline{u}, \underline{v}) \leqslant p_{1,t}(x, \underline{u}, \underline{v})$$

for each t. Thus, combining the facts that $(\underline{u}, \underline{v})$ is a strict lower solution to (2.9) for any $t \in [0, 1]$ and that T is a strict *a priori* bound, it is clear that $(\underline{u}, T) \times (\underline{v}, T) \subset X$ is an open set and that (2.9) has no solutions on its boundary. If we let $F' := I\boldsymbol{u} - L^{-1}\boldsymbol{m}(x, \boldsymbol{u}) = 0$, then it follows from homotopy invariance that

$$\deg(F, \boldsymbol{0}, (\underline{u}, T) \times (\underline{v}, T)) = \deg(F', \boldsymbol{0}, (\underline{u}, T) \times (\underline{v}, T)).$$

We show below that $\deg(F', \boldsymbol{0}, (\underline{u}, T) \times (\underline{v}, T)) = 0$ using lemma 2.12 for the weakly coupled system and using lemma 2.14 for the strongly coupled system. Hence, using homotopy invariance, we have that $\deg(F, \boldsymbol{0}, (\underline{u}, T) \times (\underline{v}, T)) = 0$. By the excision property of Leray–Schauder degree it follows that

$$\deg(F, \boldsymbol{0}, (\underline{u}, T) \times (\underline{v}, T) \setminus (\underline{u}, \bar{u}) \times (\underline{v}, \bar{v})) = -1,$$

and thus (1.1) has a second solution, $(u_2, v_2) \in (\underline{u}, T) \times (\underline{v}, T)$, satisfying

$$(u_2(x_0), v_2(x_0)) > (\bar{u}(x_0), \bar{v}(x_0))$$

at some point $x_0 \in \Omega$. This completes the proof of theorem 1.1.

2.1. Weakly coupled system

Let μ_1 be the first eigenvalue of $-\Delta$ on Ω. Since the system is weakly coupled there exists some $R_0 > 0$ such that, for all $x \in \bar{\Omega}$,

$$\begin{aligned} g_1(x, u, v) &> \mu_1 u + 1 \quad \text{for all } v \geqslant 0 \text{ and } u > R_0, \\ g_2(x, u, v) &> \mu_1 v + 1 \quad \text{for all } u \geqslant 0 \text{ and } v > R_0. \end{aligned}$$

Let

$$m_1(x, u, v) := \begin{cases} g_1(x, u, v) & \text{for all } v \geqslant 0 \text{ and } u \geqslant R_0, \\ \max\{\mu_1 u + 1, g_1(x, u, v)\} & \text{for all } v \geqslant 0 \text{ and } 0 \leqslant u < R_0, \end{cases}$$

and, similarly,

$$m_2(x, u, v) := \begin{cases} g_2(x, u, v) & \text{for all } u \geqslant 0 \text{ and } v \geqslant R_0, \\ \max\{\mu v + 1, g_2(x, u, v)\} & \text{for all } u \geqslant 0 \text{ and } 0 \leqslant v < R_0, \end{cases}$$

and $\boldsymbol{m}(x, \boldsymbol{u}) := (m_1(x, \boldsymbol{u}), m_2(x, \boldsymbol{u}))$.

The argument below is adapted from [13] (for the single equation case) and [8].

LEMMA 2.11. *If the system is weakly coupled, then there is a $T > 0$ such that any solution triple (t, u, v) of (2.9) satisfies $\|u\|_{L^\infty} + \|v\|_{L^\infty} < T$.*

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Proof. Suppose to the contrary that there exists a sequence of solutions (t_j, u_j, v_j) of (2.9) with nonlinearity $\boldsymbol{p}_{t_j}$ such that $\|(u_j, v_j)\|_{L^\infty} \to \infty$. Let c_1 and c_2 be the weak-coupling constants from definition 2.4. Without loss of generality, we may assume that $\|u_j\|_{L^\infty} \to \infty$ and, possibly after passing to a subsequence, that

$$\|u_j\|_{L^\infty}^{1/c_1} \geqslant \|v_j\|_{L^\infty}^{1/c_2} \quad \text{for all } j.$$

Define $M_j := \|u_j\|_{L^\infty}$ and choose x_j such that $u(x_j) = M_j$ (which exists because Ω is a compact domain and u_j is subject to Dirichlet boundary conditions). Without loss of generality, we may assume that the x_j converge to some x_0 in $\bar{\Omega}$. There are two cases, depending on whether $x_0 \in \Omega$ or $x_0 \in \partial\Omega$.

In the first case, define $d := \frac{1}{2}d(x_0, \partial\Omega)$. Define the sequence λ_j so that

$$\lambda_j^{c_1}\|u_j\|_{L^\infty} = 1 \quad \text{for each } j.$$

Note that $\lambda_j \to 0$ as $j \to \infty$. Define

$$\tilde{u}_j(y) := \lambda_j^{c_1} u_j(\lambda_j y + x_j) \quad \text{and} \quad \tilde{v}_j(y) := \lambda_j^{c_2} v_j(\lambda_j y + x_j).$$

Clearly, we have

$$\|\tilde{u}_j\|_{L^\infty} = 1 \quad \text{and} \quad \|\tilde{v}_j\|_{L^\infty} \leqslant 1.$$

Also note that $\tilde{u}_j$ and $\tilde{v}_j$ are well defined on $\tilde{\Omega}_j = \{y \colon \lambda_j y + x_j \in \Omega\}$. Note that $\tilde{\Omega}_j \supseteq B_{R_j}(0)$, where $R_j = d/\lambda_j$, for j sufficiently large.

We have

$$\begin{aligned}
-\Delta_y u_j(y) &= \lambda_j^{c_1}\Delta_y(u_j(\lambda_j y + x_j)) \\
&= \lambda_j^{2+c_1}(\Delta_x u_j(\lambda_j y + x_j)) \\
&= \lambda_j^{2+c_1} p_{1,t_j}(x, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j))
\end{aligned}$$

as $j \to \infty$, where $\boldsymbol{p}_t := (p_{1,t}, p_{2,t})$. Note here that all of the nonlinearities p_{1,t_j} are identical for large values of u. Hence, by condition (H1),

$$p_{1,t_j}(x, u_j, v_j) = h_{11}(x)u_j^{q_{11}} + h_{12}(x)v_j^{q_{12}} + r_1(x, u_j, v_j),$$

so, for j sufficiently large,

$$\begin{aligned}
&\lambda_j^{2+c_1} p_{1,t_j}(x, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j)) \\
&\quad = \lambda_j^{2+c_1}(h_{11}(x)u_j^{q_{11}} + h_{12}(x)v_j^{q_{12}} + r_1(x, u_j, v_j)) \\
&\quad = \lambda_j^{2+c_1-q_{11}c_1}h_{11}(x_j + \lambda_j y)\tilde{u}_j^{q_{11}} + \lambda_j^{2+c_1-q_{12}c_2}h_{22}(x_j + \lambda_j y)\tilde{v}_j + \lambda_j^{2+c_1}r_1(x, u, v),
\end{aligned}$$

which approaches $h_{11}(x_0)\tilde{u}_j^{q_{11}}$ according to the weak coupling condition. That is, the difference between $-\Delta_y u_j(y)$ and $h_{11}(x_0)\tilde{u}_j^{q_{11}}$ approaches 0 uniformly as $j \to \infty$. This holds because the exponent of λ_j in the first term is 0, and in the second term is strictly positive. For the third term, recall that $|r_1(x, u, v)| \leqslant C(1 + |u|^{\beta_{i1}} + |v|^{\beta_{i2}})$ and $\beta_{ij} < q_{ij}$ for $i, j = 1, 2$. Thus,

$$r_1(x, u, v) = o(h_{11}(x)u^{q_{11}} + h_{12}(x)v^{q_{12}})$$

uniformly in x and $\lambda_j^{2+c_1}r_1(x, u, v)$ must go to 0 uniformly.

Note that

$$-\Delta\tilde{u}_j = \lambda_j^{2+c_1-q_{11}c_1} h_{11}(x_j+\lambda_j y)\tilde{u}_j^{q_{11}} + \lambda_j^{2+c_1-q_{12}c_2} h_{12}(x_j+\lambda_j y)\tilde{v}_j + \lambda_j^{2+c_1} r_1(x,u_j,v_j)$$

is uniformly bounded (by the maximum of h_{11} or h_{12}, whichever is larger, because all the other terms are less than or equal to 1). Hence, for any fixed $R > 0$ with $R < d/\lambda_{j_0}$, there exist uniform bounds on $\|\tilde{u}_j\|_{W^{2,\gamma}(B_R(0))}$ for $j \geqslant j_0$, for all $\gamma > 1$ (see [15, lemma 9.17]; without loss of generality, we may assume that the λ_j are monotone decreasing). It follows that there also exist uniform bounds on $\|\tilde{u}_j\|_{C^{1,\beta}(B_R(0))}$ for some $0 < \beta < 1$ by choosing γ sufficiently large. Hence, a subsequence of the $\tilde{u}_j$ converges to some function $\tilde{u}$ in $C^{1,\beta} \cap W^{2,p}$ on $B_R(0)$, with $-\Delta\tilde{u} = h_{11}(x_0)\tilde{u}^{q_{11}}$ and $\tilde{u}(0) = 1$ due to uniform convergence. By a standard argument, $\tilde{u}$ is well defined (and bounded) on all of $\mathbb{R}^n$, which contradicts the Liouville theorem proved in [14].

Note that the second equation is not needed here because u is known to be the larger function. Once the system decouples, we need obtain a contradiction from only one of the equations.

In the second case, when $x_0 \in \partial\Omega$, we must first check that there exists $c > 0$ such that $d(x_j, \partial\Omega)/\lambda_j \geqslant c$ for all j sufficiently large. Consider the set where $\tilde{u}_j \geqslant \frac{1}{2}$. As above, $\Delta\tilde{u}_j$ is uniformly bounded in this set and hence $\|\nabla\tilde{u}_j\|$ is uniformly bounded on $\{y\colon \tilde{u}_j(y) \geqslant \frac{1}{2}\}$, and it immediately follows that the distance from the origin to the boundary of this set is bounded below. Hence, the distance from the origin to $\partial\tilde{\Omega}_j$, being even larger, is uniformly bounded below as well.

We therefore have two final cases to consider. $d(x_j, \partial\Omega)/\lambda_j$ is either bounded or unbounded. In the latter situation, we find that as $j \to \infty$, $\tilde{u}_j$ is defined on $B_R(0)$ for any $R > 0$, and we are back in the situation of case 1. In the former case, without loss of generality we may suppose that $d(x_j, \partial\Omega)/\lambda_j \to \delta > 0$. Moreover, we may suppose (possibly after making a smooth transformation of the domain that will not affect the character of the equations) that $\partial\Omega \subset \{y\colon y_n = -d(x_j, \partial\Omega)\}$ near 0, and, without loss of generality, that $d(x_j, \partial\Omega)/\lambda_j$ is increasing monotonically to δ. For $R > 0$, define

$$D_R = B_R(0) \cap \left\{y\colon y_n > -\delta\left(1 - \frac{1}{R}\right)\right\}.$$

Then for any R, $\tilde{u}_j$ is well defined on D_R for all j sufficiently large. But then, exactly as above, we can conclude that there exists $\tilde{u}$ so that $\tilde{u}_j \to \tilde{u}$ uniformly in D_R and thus, since R was arbitrary, on $\{y\colon y_n > -\delta\}$. Since $\tilde{u}$ satisfies the same elliptic equation and non-triviality condition as above, this contradicts the half-space Liouville theorem proved in [13] and concludes the result. □

LEMMA 2.12. *In the weakly coupled case, the BVP*

$$\left.\begin{aligned} -\Delta u &= m_1(x, \boldsymbol{u}) \quad \text{for all } x \in \Omega, \\ -\Delta v &= m_2(x, \boldsymbol{u}) \quad \text{for all } x \in \Omega, \\ u = v &= 0 \quad \text{for all } x \in \partial\Omega, \end{aligned}\right\} \tag{2.10}$$

has no non-negative solution.

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Proof. The proof is by contradiction. Suppose (u, v) is a non-negative solution of (2.10). We only need to consider one of the two equations, say the equation for $-\Delta u$. Since, for all x, u, v, $m_1(x, u, v) \geqslant \mu_1 u + 1$, we have $-\Delta u = m_1(x, u, v) \geqslant \mu_1 u + 1$. Multiplying both sides by the positive eigenfunction, ϕ_1, of $(-\Delta)$ corresponding to μ_1 and integrating by parts, we get

$$\begin{aligned}\mu_1 \int_\Omega u\phi_1 \, dx = \int_\Omega u(-\Delta\phi_1) \, dx &= -\int_\Omega (\Delta u)\phi_1 \, dx \\ &\geqslant \int_\Omega (\mu_1 u + 1)\phi_1 \, dx \\ &= \mu_1 \int_\Omega u\phi_1 \, dx + \int_\Omega \phi_1 \, dx.\end{aligned}$$

Hence,

$$0 \geqslant \int_\Omega \phi_1 > 0,$$

which is a contradiction. □

2.2. Strongly coupled system

In the strongly coupled case, if an appropriate *a priori* bound holds, then we are done. Alternatively, if the system (1.1) satisfies the Liouville condition as stated in definition 2.7, we may follow the blow-up method as in the weakly coupled case to obtain the *a priori* bound.

LEMMA 2.13. *Suppose that (1.1) satisfies (H1) and is strongly coupled, and suppose that the system (1.1) satisfies the Liouville condition as stated in definition 2.7. Then there is a* $T > 0$ *such that any solution triple* (t, u, v) *of (2.9) satisfies* $\|u\|_{L^\infty} + \|v\|_{L^\infty} < T$.

Proof. We follow the proof of lemma 2.11. Suppose to the contrary that there exists a sequence of solutions (t_j, u_j, v_j) of (1.1) with nonlinearity p_{t_j} such that $\|(u_j, v_j)\|_{L^\infty} \to \infty$. Let c_1 and c_2 be the strong-coupling constants from definition 2.5. Without loss of generality, we may assume that $\|u_j\|_{L^\infty} \to \infty$, and, possibly after passing to a subsequence, that

$$\|u_j\|_{L^\infty}^{1/c_1} \geqslant \|v_j\|_{L^\infty}^{1/c_2} \quad \text{for all } j.$$

Define M_j, x_j, x_0, λ_j, $\tilde{\Omega}_j$, $\tilde{u}_j$ and $\tilde{v}_j$ as in the weakly coupled case.

We compute as before, that

$$\begin{aligned}-\Delta\tilde{u}_j(y) &= \lambda_j^{c_1}\Delta_y(u_j(\lambda_j y + x_j)) \\ &= \lambda_j^{2+c_1}\Delta_x u_j(\lambda_j y + x_j) \\ &= \lambda_j^{2+c_1} p_{1,t_j}(\lambda_j y + x_j, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j))\end{aligned}$$

as $j \to \infty$. At any given point y, either $p_{1,t_j} \leqslant \mu_1 R_0 + 1$ or

$$\begin{aligned}p_{1,t_j} &= g_1(\lambda_j y + x_j, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j)) \\ &= h_{11}(\lambda_j y + x_j)u_j^{q_{11}} + h_{12}(\lambda_j y + x_j)v_j^{q_{12}} + r_1(\lambda_j y + x_j, u_j, v_j).\end{aligned}$$

(The latter case always occurs whenever $v_j(\lambda_j y + x_j) > R_0$.) We find that

$$-\Delta \tilde{u}_j = \lambda_j^{2+c_1} p_{1,t_j}(\lambda_j y + x_j, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j)).$$

As the power of λ_j is positive, if $p_{1,t_j} \leqslant \mu_1 R_0 + 1$, then the right-hand side will go to 0 as $j \to \infty$, uniformly for all such y. On the other hand, if $p_{1,t_j} = g_1$, then

$$\begin{aligned} \lambda_j^{2+c_1} & p_{1,t_j}(\lambda_j y + x_j, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j)) \\ &= \lambda_j^{2+c_1-q_{11}c_1} h_{11}(x_j + \lambda_j y)\tilde{u}_j^{q_{11}} + \lambda_j^{2+c_1-q_{12}c_2} h_{22}(x_j + \lambda_j y)\tilde{v}_j \\ &\quad + \lambda_j^{2+c_1} r_1(\lambda_j y + x_j, u, v), \end{aligned}$$

which is uniformly bounded for all such y. Hence, we conclude that $-\Delta \tilde{u}_j$ is uniformly bounded for all y and as before it follows that a subsequence of the $\tilde{u}_j$ converges to some function $\tilde{u}$ in $C^{1,\beta} \cap W^{2,p}$ on $B_R(0)$, with $\tilde{u}(0) = 1$ due to uniform convergence.

In addition, we compute that

$$\begin{aligned} -\Delta \tilde{v}_j(y) &= \lambda_j^{c_2} \Delta_y (v_j(\lambda_j y + x_j)) \\ &= \lambda_j^{2+c_2} \Delta_x v_j(\lambda_j y + x_j) \\ &= \lambda_j^{2+c_2} p_{2,t_j}(\lambda_j y + x_j, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j)). \end{aligned}$$

Since $u(x_j)$ is eventually greater than R_0, for j sufficiently large, by condition (H1),

$$p_{2,t_j}(x, u, v) = h_{21}(x)u^{q_{21}} + h_{22}(x)v^{q_{22}} + r_2(x, u, v).$$

Thus,

$$\begin{aligned} -\Delta \tilde{v}_j &= \lambda_j^{2+c_2} p_{2,t_j}(\lambda_j y + x_j, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j)) \\ &= \lambda_j^{2+c_2} h_{21}(\lambda_j y + x_j)u^{q_{21}} + h_{22}(\lambda_j y + x_j)v^{q_{22}} + r_2(\lambda_j y + x_j, u, v) \\ &= \lambda_j^{2+c_2-q_{21}c_1} h_{21}(x_j + \lambda_j y)\tilde{u}_j^{q_{21}} + \lambda_j^{2+c_2-q_{22}c_2} h_{22}(x_j + \lambda_j y)\tilde{v}_j \\ &\quad + \lambda_j^{2+c_2} r_2(\lambda_j y + x_j, u, v), \end{aligned}$$

which approaches $h_{21}(x_0)\tilde{u}^{q_{21}}$ according to the strong coupling condition. As above, we may conclude that the $\tilde{v}_j$ converges to some $\tilde{v}$ in $C^{1,\beta} \cap W^{2,\gamma}$, and $\tilde{v}$ is well defined on all of $\mathbb{R}^n$ or an appropriate half-space. Moreover, $\tilde{v}(y) \geqslant 0$ for all y, and $-\Delta \tilde{v} = h_{22}(x_0)\tilde{u}^{q_{21}}$, which is a non-trivial, non-negative function. Hence, by the strong maximum principle, $\tilde{v} > 0$ everywhere. Thus, for j sufficiently large, $v_j(y) > R_0$ for all y. We may conclude that, for j sufficiently large $p_{1,t_j}(x, u, v) = g_1(x, u, v)$. Hence,

$$\begin{aligned} -\Delta u_j &= \lambda_j^{2+c_1} p_{1,t_j}(x, u_j(\lambda_j y + x_j), v_j(\lambda_j y + x_j)) \\ &= \lambda_j^{2+c_1-q_{11}c_1} h_{11}(x_j + \lambda_j y)\tilde{u}_j^{q_{11}} + \lambda_j^{2+c_1-q_{12}c_2} h_{22}(x_j + \lambda_j y)\tilde{v}_j \\ &\quad + \lambda_j^{2+c_1} r_1(x, u, v), \end{aligned}$$

which converges uniformly to $h_{12}(x_0)\tilde{v}^{q_{12}}$ by the strong coupling condition.

Thus, we conclude that $\tilde{u}$ and $\tilde{v}$ satisfy

$$-\Delta\tilde{u} = h_{11}(x_0)\tilde{v}^{q_{12}},$$
$$-\Delta\tilde{v} = h_{22}(x_0)\tilde{u}^{q_{21}},$$

on either $\mathbb{R}^n$ or $\mathbb{R}^n_+$ depending on the location of x_0 as discussed in the proof of lemma 2.11. Additionally, we know that $\tilde{u}(0) = 1$. This contradicts the assumed Liouville theorem. □

We also have the following.

LEMMA 2.14. *In the strongly coupled case, the BVP*

$$\left.\begin{aligned} -\Delta u &= m_1(x,u,v) && \text{for all } x \in \Omega, \\ -\Delta v &= m_2(x,u,v) && \text{for all } x \in \Omega, \\ u = v &= 0 && \text{for all } x \in \partial\Omega, \end{aligned}\right\} \tag{2.11}$$

has no non-negative solution, where $m_i(x, \boldsymbol{u})$ are as defined in definition 2.7.

Proof. Suppose (u, v) is a non-negative solution of (2.11). Since $m_1(x,u,v) \geqslant \mu_1 v + 1$, we have $-\Delta u = m_1(x,u,v) \geqslant \mu_1 v + 1$. Similarly, since $m_2(x,u,v) \geqslant \mu_1 u + 1$, we have $-\Delta v = m_2(x,u,v) \geqslant \mu_1 u + 1$. Multiplying both sides by the positive eigenfunction, ϕ_1, of $(-\Delta)$ corresponding to μ_1 and integrating, we get

$$\begin{aligned} \mu_1^2 \int_\Omega u\phi_1 \,\mathrm{d}x &= \mu_1 \int_\Omega u(-\Delta\phi_1)\,\mathrm{d}x = \mu_1 \int_\Omega (-\Delta u)\phi_1 \,\mathrm{d}x \\ &\geqslant \mu_1 \int_\Omega (\mu_1 v + 1)\phi_1 \,\mathrm{d}x > \int_\Omega (\mu_1 v)(\mu_1\phi_1)\,\mathrm{d}x \\ &= \int_\Omega (\mu_1 v)(-\Delta\phi_1)\,\mathrm{d}x = \mu_1 \int_\Omega (-\Delta v)\phi_1 \,\mathrm{d}x \\ &\geqslant \mu_1 \int_\Omega (\mu_1 u + 1)\phi_1 \,\mathrm{d}x = \mu_1^2 \int_\Omega u\phi_1 \,\mathrm{d}x + \mu_1 \int_\Omega \phi_1 \,\mathrm{d}x, \end{aligned}$$

after integrating by parts repeatedly. (Notice that our boundary conditions are exactly correct to prevent boundary terms in the integrations by parts.) Subtracting

$$\mu_1^2 \int_\Omega u\phi_1 \,\mathrm{d}x$$

from both sides of the resulting inequality, we conclude that

$$0 > \mu_1 \int_\Omega \phi_1 > 0,$$

which is a contradiction. □

3. An auxiliary problem

Our next goal is to discuss some explicit situations in which we know that the conditions of theorem 2.8 are satisfied. In this section we study an important auxiliary problem that has a simplified structure. We prove an existence result that generalizes the positone result in [12] and the radially symmetric positone result in [20]. In the next section we will use the solution of this problem to construct a positive lower solution for (1.1).

Henceforth, we shall denote by z the positive solution of (2.5) and let $M := \|z\|_\infty$. Additionally, ν will always represent the outward unit normal to the boundary of Ω.

Consider the auxiliary problem

$$\left.\begin{aligned} -\Delta\psi &= -k\chi_{\{\psi<1\}} + K\chi_{\{\psi\geqslant 1\}} &&\text{for all } x \in \Omega,\\ \psi &= 0 &&\text{for all } x \in \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

where $\chi_{\{\psi<1\}}$ represents the standard characteristic function on the set $\{x \in \Omega \colon \psi(x) < 1\}$, and $\chi_{\{\psi\geqslant 1\}}$ is defined similarly.

Lemma 3.1. *For each fixed $k > 0$ there exists $K > 0$ such that (3.1) has a positive solution.*

Proof. Let $k > 0$ be fixed and let $B := B_r(x_0) \subset\subset \Omega$. Consider the sub-auxiliary problem

$$\left.\begin{aligned} -\Delta w &= -k\chi_{\bar{B}^c} + K\chi_{\bar{B}} &&\text{for all } x \in \Omega,\\ w &= 0 &&\text{for all } x \in \partial\Omega. \end{aligned}\right\} \tag{3.2}$$

Let w_K represent the unique solution to this problem. Then $v_K := w_K/K$ satisfies

$$\begin{aligned} -\Delta v_K &= -\frac{k}{K}\chi_{\bar{B}^c} + \chi_{\bar{B}} &&\text{for all } x \in \Omega,\\ v_K &= 0 &&\text{for all } x \in \partial\Omega. \end{aligned}$$

Since the right-hand side of (3.2) satisfies a uniform $L^\infty(\Omega)$ bound, we have, without loss of generality, that $v_K \to v$ in $C^1(\bar{\Omega})$ as $K \to \infty$, where v solves

$$\begin{aligned} -\Delta v &= \chi_{\bar{B}} &&\text{for all } x \in \Omega,\\ v &= 0 &&\text{for all } x \in \partial\Omega. \end{aligned}$$

By the maximum principle, $v > 0$ in Ω and $\partial v/\partial\nu < 0$ on $\partial\Omega$. This implies that $w_K > 0$ in Ω for large K. Moreover, it is clear that for all K large enough we have $w_K > 1$ on $\bar{B}$ and $\partial w_K/\partial\nu < 0$ on $\partial\Omega$.

Since $\bar{B}^c \supset \{x \in \Omega \colon w_K < 1\}$, for large K, w_K satisfies

$$-\Delta w_K = -k\chi_{\bar{B}^c} + K\chi_{\bar{B}} \leqslant -k\chi_{\{w_K<1\}} + K\chi_{\{w_K\geqslant 1\}}$$

in Ω. Thus, w_K is a lower solution of (3.1).

Let $K > 0$ be chosen so that w_K is a lower solution of (3.1). Then Kz is an upper solution of (3.1), because $-\Delta(Kz) = K \geqslant -k\chi_{\{Kz<1\}} + K\chi_{\{Kz\geqslant 1\}}$. By the same calculation, $-\Delta(Kz) \geqslant -\Delta w_K$. Hence, by the maximum principle, $Kz \geqslant w_K$, so Kz and w_K are well ordered. Note that the function $h(t) := -k\chi_{\{t<1\}} + K\chi_{\{t\geqslant 1\}}$ is non-decreasing and continuous from the right. It follows that, given well-ordered lower and upper solutions, (3.1) has a solution obtained via monotone iteration from the upper solution. □

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In the arguments that follow we will refer to a solution, u, of (3.1) as *maximal*, if every other solution, v, of (3.1) for the same values of k and K satisfies $v \leqslant u$ in $\bar{\Omega}$. The solution found in the previous lemma will be maximal relative to other solutions lying between the given lower and upper solutions because it was found by monotone iteration from the given upper solution.

LEMMA 3.2. *If (k, K) is a pair such that (3.1) has a positive solution, then (3.1) has a maximal solution.*

Proof. By the maximum principle, $\bar{u} = Kz$ provides an upper bound on any solution, u, to (3.1), because $-\Delta\bar{u} = K \geqslant -k\chi_{\{u<1\}} + K\chi_{\{u\geqslant 1\}} = -\Delta u$. Hence, the solution obtained by monotone iteration from $\bar{u}$ is maximal relative to all solutions. □

The following lemma characterizes the set

$$S_k := \{K > 0\colon (3.1) \text{ has a positive solution}\}.$$

LEMMA 3.3. *The set S_k is a closed ray. That is, for $k > 0$ fixed, let $K_k := \inf\{K > 0 : K \in S_k\}$. Then $S_k = [K_k, \infty)$.*

Proof. First, we already know that every sufficiently large $K \in \mathbb{R}$ is in S_k. We next need to show that S_k is a ray, i.e. if $K_0 \in S_k$ and $K > K_0$, then $K \in S_k$. But this follows immediately because ψ_{K_0} satisfies

$$-\Delta\psi_{K_0} = -k\chi_{\{\psi_{K_0}<1\}} + K_0\chi_{\{\psi_{K_0}\geqslant 1\}} < -k\chi_{\{\psi_{K_0}<1\}} + K\chi_{\{\psi_{K_0}\geqslant 1\}},$$

so ψ_{K_0} is a lower solution for (3.1) with respect to K. Moreover, Kz is a strictly larger upper solution to the problem, as above. Therefore, a solution to (3.1) for K must exist, and $K \in S_k$.

Finally, we must show that this is a closed ray, i.e. $K_k \in S_k$. Let $K_1, K_2 \in S_k$ with $K_1 > K_2$, and let ψ_1, ψ_2 represent the corresponding maximal solutions of (3.1). Then

$$-\Delta\psi_2 = -k\chi_{\{\psi_2<1\}} + K_2\chi_{\{\psi_2\geqslant 1\}} \leqslant -k\chi_{\{\psi_2<1\}} + K_1\chi_{\{\psi_2\geqslant 1\}},$$

so ψ_2 is a positive lower solution for (3.1) with $K = K_1$. Using the maximal property of solutions we get $\psi_1 \geqslant \psi_2$.

Now let $K_n \searrow K_k$ and let ψ_n be the corresponding maximal solutions. Then $\{\psi_n\}$ is monotonically decreasing, by the above argument, and thus the pointwise limit $\psi_k(x) := \lim_{n\to\infty}\psi_n(x)$ exists. Moreover,

$$-k\chi_{\{\psi_n<1\}} + K_n\chi_{\{\psi_n\geqslant 1\}} \searrow -k\chi_{\{\psi_k<1\}} + K_k\chi_{\{\psi_k\geqslant 1\}}$$

pointwise, where we have used the fact that $h(t)$, as defined above, is continuous from the right. Since the right-hand side of (3.1) is uniformly L^∞-bounded, we can apply standard regularity and imbedding theorems (see, for example, [15, theorems 7.22, 9.11 and 9.15]), to derive a subsequence such that $\psi_n \to \psi_k$ in $C^{1,\gamma}(\bar{\Omega})$ for some $\gamma \in (0, 1)$ and ψ_k is a solution of (3.1) with $K = K_k$. □

4. The quasi-monotone non-decreasing case: weakly coupled system

In this section we consider problem (1.1) with the assumption that definition 2.1 is satisfied. We will give explicit conditions for the existence of the required ordered, strictly positive upper and lower solution pair. Our result is complementary to the existence and non-existence results of Chhetri and Girg [3]. Our theorem here also complements the single equation positone results in [12] and generalizes the single equation radially symmetric results in [20].

THEOREM 4.1. *Fix $k_1, k_2 > 0$ and choose $K_i > K_{k_i}$ for $i = 1, 2$. Let*

$$m_{k_i} := \|\psi_{k_i}\|_\infty > 1,$$

where ψ_{k_i} is a solution of (3.1) with $K = K_{k_i}$. Suppose that $g_i(x, s, t)$ are such that (H1) holds. Suppose there exist $C_1 > m_{k_1}/M$ and $C_2 > m_{k_2}/M$ such that the following hold uniformly for $x \in \bar{\Omega}$:

(H2a) *we have*

$$g_1(x, s, t) > -k_1 \quad \text{for } 0 \leqslant s \leqslant 1 \text{ and } t \leqslant m_{k_2},$$
$$g_2(x, s, t) > -k_2 \quad \text{for } 0 \leqslant t \leqslant 1 \text{ and } s \leqslant m_{k_1};$$

(H2b) *we have*

$$g_1(x, s, t) > K_1 \quad \text{for } 1 \leqslant s \leqslant m_{k_1} \text{ and } t \leqslant m_{k_2},$$
$$g_2(x, s, t) > K_2 \quad \text{for } 1 \leqslant t \leqslant m_{k_2} \text{ and } s \leqslant m_{k_1};$$

(H2c) *$g_i(x, s, t) < C_i$ for $0 \leqslant s \leqslant C_1 M$ and $0 \leqslant t \leqslant C_2 M$.*

Then (1.1) has at least two solutions.

The proof of this theorem follows from the series of lemmata established below.

LEMMA 4.2. *$(\underline{u}, \underline{v}) := (\psi_{k_1}, \psi_{k_2})$ is a strict lower solution.*

Proof. This lemma follows from the strict inequalities in (H2a) and (H2b). Indeed, since $0 < \psi_{k_2}(x) \leqslant m_{k_2}$ for all $x \in \Omega$, we have

$$\begin{aligned} -\Delta \underline{u} &= -\Delta \psi_{k_1} \\ &= -k_1 \chi_{\{\psi_{k_1} < 1\}} + K_1 \chi_{\{\psi_{k_1} \geqslant 1\}} \\ &= -k_1 \chi_{\{0 \leqslant \psi_{k_1} < 1\}} + K_1 \chi_{\{1 \leqslant \psi_{k_1} \leqslant m_{k_1}\}} \\ &< g_1(x, \psi_{k_1}, \psi_{k_2}) = g_1(x, \underline{u}, \underline{v}). \end{aligned}$$

The calculation for the $\underline{v}$ equation is similar. □

LEMMA 4.3. *$(\bar{u}, \bar{v}) := (C_1 z, C_2 z)$ is a strict upper solution.*

Proof. The proof follows from the fact that $0 \leqslant C_i z \leqslant C_i M$ for all $x \in \bar{\Omega}$ and so from (H2c) we have the strict inequality

$$-\Delta(C_i z) = C_i > g_i(x, C_1 z, C_2 z)$$

for $i = 1, 2$. □

Observe that $C_i > g_i(x, \underline{u}, \underline{v})$ because $0 \leqslant \underline{u} \leqslant m_{k_1} < C_1 M$ and similarly for $\underline{v}$. It follows that $-\Delta\bar{u} = C_1 > g_1(x, \underline{u}, \underline{v}) > -\Delta\underline{u}$, and so an application of the maximum principle shows that $\underline{u} < \bar{u}$, i.e. the lower and upper solutions are well ordered. An identical calculation can be done to show that $\underline{v} < \bar{v}$. By theorem 2.8, theorem 4.1 immediately follows.

Finally, we provide two examples of reaction terms $g_i(x, s, t)$ that satisfy the conditions of theorem 4.1.

EXAMPLE 4.4 (positone). For $n = 2$ or $n = 3$, let

$$g_1(x, s, t) = \varepsilon s^4 + A \exp\left(\frac{at}{1+t}\right),$$
$$g_2(x, s, t) = \varepsilon t^4 + A \exp\left(\frac{as}{1+s}\right).$$

Condition (H1) is satisfied with $q_{ii} = 4$, which is subcritical for our choice of space dimension, $q_{ij} = 0$ for $i \neq j$, $h_{ij} \equiv 0$ for $i \neq j$; $h_{ii} \equiv \varepsilon$; $r_1(x, s, t) = A \exp(at/(1+t))$ and $r_2(x, s, t) = A \exp(as/(1 + s))$.

(H2a) Clearly, $g_1(x, s, t) = \varepsilon s^4 + A \exp(at/(1 + t))$ is a non-decreasing function for $0 \leqslant s, t \leqslant 1$. Therefore, to satisfy (H2a), it is enough to notice that $-k < 0 < A = g_1(x, \mathbf{0})$.

(H2b) Since $g_1(x, s, t) = \varepsilon s^4 + A \exp(at/(1 + t))$ is a non-decreasing function for $1 \leqslant s, t \leqslant m_k$, to satisfy (H2b), we need to show that $g_1(x, 1, 0) = \varepsilon + A > K$. This holds if A is chosen sufficiently large.

(H2c) Using the fact that $g_1(x, s, t)$ is non-decreasing for $0 \leqslant s, t \leqslant CM$, it suffices to show $Ae^a + \varepsilon(CM)^4 < C$. This condition holds for C large enough relative to Ae^a and ε is sufficiently small.

EXAMPLE 4.5 (semi-positone). For $n = 2, 3$, let

$$g_1(x, s, t) = \varepsilon s^4 + Bs^\eta + At^\theta - \gamma,$$
$$g_2(x, s, t) = \varepsilon t^4 + Bt^\eta + As^\theta - \gamma,$$

where ε, A, B are positive parameters and $0 < \theta, \eta < 1$. Due to the symmetry of this system, we may assume that $k_1 = k_2$ and $C_1 = C_2$ when checking the conditions.

Condition (H1) is satisfied with $q_{ii} = 4$, which is subcritical in 2 or 3 space dimensions, $q_{ij} = \theta$ for $i \neq j$; $h_{ij} \equiv A$ for $i \neq j$; $h_{ii} \equiv \varepsilon$ for $i = j$; $r_1(x, s, t) = Bs^\eta - \gamma$ and $r_2(x, s, t) = Bt^\eta - \gamma$.

(H2) We show below that (H2a)–(H2b) are satisfied for $g_1(x, s, t)$. (The same arguments work for g_2.)

(H2a) Notice that $g_1(x, s, t) = \varepsilon s^4 + At^\theta + Bs^\eta - \gamma$ is a non-decreasing function for $0 \leqslant s, t \leqslant 1$. Therefore, to satisfy (H2a), we need to show $g_1(x, \mathbf{0}) = -\gamma > -k$. Thus, (H2a) is satisfied if $\gamma < k$.

(H2b) Since $g_1(x, s, t) = \varepsilon s^4 + At^\theta - \gamma$ is a non-decreasing function for $1 \leqslant s, t \leqslant m_k$, to satisfy (H2b) it is enough to show $g_1(x, 1, 0) = \varepsilon + B - \gamma > K$. This holds by choosing B sufficiently large.

(H2c) Using the fact that $g_1(x,s,t)$ is non-decreasing for $0 \leqslant s,t \leqslant CM$, it suffices to show $A(CM)^\theta + B(CM)^\eta + \varepsilon(CM)^4 - \gamma < C$. This condition holds if C is sufficiently large and ε is sufficiently small.

Remark 4.6. Both of these examples could easily be modified for higher dimensions by appropriate choice of subcritical power in place of the exponent 4 used here.

Remark 4.7. Other examples could easily be constructed. Here we have chosen to use the types of nonlinearities that are typically seen as model problems in the literature. The defining quality necessary to satisfy our conditions is a 'two-shelf' shape and superlinearity.

5. The quasi-monotone non-decreasing case: Hamiltonian systems

In this section, we consider the purely Hamiltonian case. Namely, we prove the following theorem.

Theorem 5.1. *Consider a system of the form*

$$\left.\begin{aligned} -\Delta u &= g_1(v) && \text{for all } x \in \Omega, \\ -\Delta v &= g_2(u) && \text{for all } x \in \Omega, \\ u, v &> 0 && \text{for all } x \in \Omega, \\ u = v &= 0 && \text{for all } x \in \partial\Omega, \end{aligned}\right\} \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$, $n \geqslant 3$, is convex with C^3 boundary. Here $g_1, g_2 : [0,\infty) \to \mathbb{R}$ are C^1 monotone non-decreasing functions satisfying the following conditions:

(A1) *there exist positive numbers η_1, η_2 such that*

$$\lim_{s\to\infty} \frac{g_1(s)}{s^p} = \eta_1 \quad \text{and} \quad \lim_{s\to\infty} \frac{g_2(s)}{s^q} = \eta_2,$$

where $p, q > 1$ and are subcritical in the sense that

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}, \quad n \geqslant 3;$$

(A2) *further assume that there exist $k > 0$ and $C > m_k/M$ such that*

(a) *$g_i(s) > -k$ for $0 \leqslant s \leqslant 1$,*

(b) *$g_i(s) > K$ for $1 \leqslant s \leqslant m_k$,*

(c) *$g_i(s) < C$ for $0 \leqslant s \leqslant CM$,*

where K and m_k are as defined in theorem 4.1. Then (5.1) has at least two solutions.

Remark 5.2. Observe that we do not assume (H1) here which states that the nonlinearities are subcritical. The condition (A1) represents subcritical behaviour in the Hamiltonian setting, and is used here to apply the result of Clement *et al*. [5].

Remark 5.3. Recall that we wish to apply theorem 2.8 to establish the existence of two solutions to (5.1). Clement *et al.* [5, theorem 2.1] proved that there exists an *a priori* bound for positive solutions to (5.1) when g_1 and g_2 are monotone non-decreasing and satisfy (A1). In fact, the result was proved in order to establish the existence of a positive solution to a certain Hamiltonian system using degree theory. Therefore, the conditions on the nonlinearities presented here can depend uniformly on a homotopy parameter t. To maintain the uniformity with the previous sections, we omit the explicit dependence on t in the statement of the theorem above; however, the proof here proceeds exactly as for theorem 4.1 because the *a priori* bound holds for the homotopy class of problems as required in definition 2.7.

Proof. As in § 4, $(\underline{u}, \underline{v}) := (\psi_k, \psi_k)$ is a strict lower solution and $(\bar{u}, \bar{v}) := (Cz, Cz)$ is a strict upper solution, and this pair is ordered. Hypothesis (A1) combined with the Hamiltonian structure and strong coupling ensures that every non-negative solution of (5.1) is *a priori* bounded [5, theorem 2.1]. This in turn implies that the condition (C2b) of theorem 2.8 is satisfied. Therefore, by theorem 2.8, there are two positive solutions to (5.1). □

Finally, we provide an example satisfying the hypotheses of theorem 5.1.

Example 5.4. Let

$$g_1(v) = \eta v^p + A v^\theta - \gamma,$$
$$g_2(u) = \eta u^q + A u^\theta - \gamma,$$

where η, γ and A are positive parameters and $0 < \theta < 1$, and $p, q > 1$ satisfy the subcriticality condition.

(A1) Obviously,

$$\lim_{v\to\infty} \frac{g_1(v)}{v^p} = \eta \quad \text{and} \quad \lim_{s\to\infty} \frac{g_2(u)}{u^q} = \eta$$

since $0 < \theta < 1$.

(A2) We will show below that the conditions (A2a)–(A2c) are satisfied for $g_1(v)$ (The same arguments work g_2.)

(a) Since $g_1(v) = \eta v^p + A v^\theta - \gamma$ is a non-decreasing function for $0 \leqslant v \leqslant 1$, to satisfy (A2a), it is enough to satisfy $g_1(0) = -\gamma > -k$. Thus, (A2a) is satisfied if $\gamma < k$.

(b) Since g_1 is a non-decreasing function for $1 \leqslant v \leqslant m_k$, to satisfy (A2b), it is enough to show $g_1(1) = \eta + A - \gamma > K$. This holds for g_1 if A is chosen large enough to satisfy $A > \gamma + K - \eta$.

(c) Similarly, to satisfy (A2c) it suffices to show $A(CM)^\theta + \eta(CM)^p - \gamma < C$. This condition holds true if η is sufficiently small and hence (A2c) holds.

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References

1 H. Amann. Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **18** (1976), 620–709.

2 A. Ambrosetti, H. Brézis and G. Cerami. Combined effects of concave and convex nonlinearities in some eliptic problem. *J. Funct. Analysis* **122** (1994), 519–543.

3 M. Chhetri and P. Girg. Existence and non-existence of positive solutions for a class of superlinear semi-positone systems. *Nonlin. Analysis* **71** (2009), 4984–4996.

4 M. Chhetri and S. Robinson. Existence and multiplicity of positive solutions for classes of singular elliptic PDEs. *J. Math. Analysis Applic.* **357** (2009), 176–182.

5 Ph. Clément, D. G. de Figueiredo and E. Mitidieri. Positive solutions of semilinear elliptic systems. *Commun. PDEs* **17** (1992), 923–940.

6 M. Crandall and P. Rabinowitz. Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems. *Arch. Ration. Mech. Analysis* **58** (1975), 207–218.

7 A. Castro and R. Shivaji. Non-negative solutions for a class of non-positone problems. *Proc. R. Soc. Edinb.* A **108** (1988), 291–302.

8 D. de Figueiredo. Semilinear elliptic systems. In *Nonlinear functional analysis and applications to differential equations*, pp. 122–152 (World Scientific, 1998).

9 D. G. de Figueiredo and P. Felmer. A Liouville-type theorem for elliptic systems. *Annali Scuola Norm. Sup. Pisa IV* **21** (1994), 387–397.

10 D. G. de Figueiredo and B. Sirakov. Liouville type theorems, monotonicity results and *a priori* bounds for positive solutions of elliptic systems. *Math. Annalen* **333** (2005), 231–260.

11 D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum. *A priori* estimates and existence of positive solutions of semilinear elliptic equations. *J. Math. Pures Appl.* **61** (1982), 41–63.

12 P. Drábek and S. Robinson. Multiple positive solutions for elliptic boundary value problems. *Rocky Mt. J. Math.* **36** (2006), 97–113.

13 B. Gidas and J. Spruck. *A priori* bounds for positive solutions of nonlinear elliptic equations. *Commun. PDEs* **6** (1981), 883–901.

14 B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Commun. Pure Appl. Math.* **34** (1981), 525–598.

15 D. Gilbarg and N. Trudinger. *Elliptic partial differential equations of second order*, 2nd edn (Springer, 1998).

16 Y. Guo and J. Liu. Liouville type theorems for positive solutions of elliptic systems in $\mathbb{R}^N$. *Commun. PDEs* **33** (2008), 263–284.

17 P. L. Lions. On the existence of positive solutions of semilinear elliptic equations. *SIAM Rev.* **24** (1982), 441–467.

18 C. V. Pao. *Nonlinear parabolic and elliptic equations* (New York: Plenum Press, 1992).

19 P. Quittner and Ph. Souplet. *A priori* estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces. *Arch. Ration. Mech. Analysis* **174** (2004), 49–81.

20 S. Robinson and M. Rudd. Multiplicity results for semi-positone problems on balls. *Dynam. Syst. Applic.* **15** (2006), 133–146.

21 R. Shivaji. A remark on the existence of three solutions via sub-super solutions. In *Nonlinear analysis and applications* (New York: Dekker, 1987).

22 Y. Sun, S. Wu and Y. Long. Combined effects of singular and superlinear nonlinearities in some singular boundary value problems. *J. Diff. Eqns* **176** (2001), 511–531.

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