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Abstract

In the 1970s, Dwork defined the logarithmic growth (log-growth for short) filtrations for p-adic differential equations $Dx = 0$ on the p-adic open unit disc $|t| < 1$, which measure the asymptotic behavior of solutions x as $|t| \to 1^-$. Then, Dwork calculated the log-growth filtration for p-adic Gaussian hypergeometric differential equation. In the late 2000s, Chiarellotto and Tsuzuki proposed a fundamental conjecture on the log-growth filtrations for (φ, ∇)-modules over $K[\![t]\!]_0$, which can be regarded as a generalization of Dwork's calculation. In this paper, we prove a generalization of the conjecture to (φ, ∇)-modules over the bounded Robba ring. As an application, we prove a generalization of Dwork's conjecture proposed by Chiarellotto and Tsuzuki on the specialization property for log-growth Newton polygons.

Contents

Introduction

In the introduction, let K be a complete discrete valuation field of mixed characteristic $(0, p)$ equipped with a valuation $|\cdot|$ such that $|p| = p^{-1}$, with integer ring $\mathcal{O}_K$ and residue field k. In this paper, we study the *logarithmic growth* (log-growth for short) filtrations for

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(φ, ∇)-modules over the bounded Robba ring $\mathcal{R}^{\mathrm{bd}}$. Our main result is a proof of a generalization of Chiarellotto–Tsuzuki conjecture (Theorem 0.1). Recall that (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$ naturally arise from Picard–Fuchs modules, in the sense of [Ked10, Definition 22.1.1], associated with nice families $f : X \to \mathbb{P}^1_K$ of algebraic varieties [Ked10, Theorem 22.2.1, Remark 22.2.2]. In the following, we recall Dwork's result of [Dwo82], which illustrates Chiarellotto–Tsuzuki conjecture in the case where f is given by Legendre family of elliptic curves.

Dwork's result

In this paragraph, we assume that k is algebraically closed of characteristic $p \neq 2$, and $\mathcal{O}_K$ is the ring of Witt vectors over k. Let $X \to \mathbb{P}^1_K$ be Legendre family of elliptic curves defined by $E_t : w^2 = z(z-1)(z-t)$. Then, the associated (first) Picard–Fuchs module corresponds to Gaussian hypergeometric differential operator $H := t(1-t)\partial^2 + (1-t)\partial - 1/4$ (see [Ked10, Examples 22.2.1, 22.2.3]). Let $a \in \mathcal{O}_K$ be a lift of $\bar{a} \in k\backslash\{0,1\}$. We put $S_a := \{x \in K[\![t-a]\!]; t(1-t)d^2x/d(t-a)^2 + (1-t)dx/d(t-a) - x/4 = 0\}$, which is regarded as the set of local solutions of $Hx = 0$ around $t = a$. We denote by $K\{t-a\}$ the subring of $K[\![t-a]\!]$ consisting of series converging on the open unit disc $|t-a| < 1$. Then, we have $\dim_K S_a = 2$, and $S_a \subset K\{t-a\}$. That is, the differential equation $Hx = 0$ is solvable on the open unit disc $|t-a| < 1$. However, one can prove that no nonzero $x \in S_a$ converges on the closed unit disc $|t-a| \leqslant 1$. To understand the asymptotic behavior of $x \in S_a$ as $|t-a| \to 1^-$, we introduce the *log-growth filtration* $\mathrm{Fil}_\bullet K\{t-a\}$ of $K\{t-a\}$. Let $\lambda \in \mathbb{R}_{\geqslant 0}$. A formal power series $x = \sum_{i\in\mathbb{N}} x_i(t-a)^i \in K[\![t-a]\!]$ with $x_i \in K$ has *log-growth* λ if there exists $C \in \mathbb{R}$ such that $|x_i| \leqslant Ci^\lambda$ for all $i \in \mathbb{N}$. Note that if x satisfies the condition, then $x \in K\{t-a\}$. We define $\mathrm{Fil}_\lambda K\{t-a\}$ as the K-subspace of $K[\![t-a]\!]$ consisting of series having log-growth λ. We put $\mathrm{Fil}_\lambda K\{t-a\} = 0$ for $\lambda \in \mathbb{R}_{<0}$. Then, we put $S_{a,\lambda} := S_a \cap \mathrm{Fil}_\lambda K\{t-a\}$ for $\lambda \in \mathbb{R}$. Dwork proves that the slope multiset of the filtration $S_{a,\bullet}$ is $\{0,1\}$ if $E_{\bar{a}}$ is ordinary, and $\{1/2, 1/2\}$ if $E_{\bar{a}}$ is supersingular (see [CT09, 7.4] for further explanation).

Chiarellotto–Tsuzuki conjecture

In [CT09], Chiarellotto and Tsuzuki formulated a conjecture called the *Chiarellotto–Tsuzuki conjecture* in this paper, which is regarded as a generalization of Dwork's result. Let us briefly recall the statement of the conjecture. Let q be a positive power of p, and φ_K an isometric ring endomorphism on K which lifts the q-power map on k. We define $K[\![t]\!]_0 := \mathcal{O}_K[\![t]\!] \otimes_{\mathcal{O}_K} K$, that is, the subring of $K\{t\}$ consisting of series $x = \sum_{i\in\mathbb{N}} x_i t^i$ with bounded coefficients, which is equipped with Gauss norm defined by $|x|_0 = \sup_{i\in\mathbb{N}} |x_i|$. We endow $K\{t\}$ with the K-linear derivation $\partial = d/dt : K\{t\} \to K\{t\}$ and a ring endomorphism φ of the form $\varphi(\sum_{i\in\mathbb{N}} x_i t^i) = \sum_{i\in\mathbb{N}} \varphi_K(x_i)(S')^i$ for some $S' \in K[\![t]\!]_0$ with $|S' - t^q|_0 < 1$. Then, both φ and ∂ restrict to $K[\![t]\!]_0$. Recall that for a ring R with a ring endomorphism φ, a *φ-module* over R is a finite free R-module M equipped with a φ-semilinear endomorphism φ_M such that the R-linear map $M \otimes_{R,\varphi} R \to M; m \otimes r \mapsto \varphi_M(m)r$ is an isomorphism. A *(φ, ∇)-module* over $K[\![t]\!]_0$ is a φ-module M over $K[\![t]\!]_0$ equipped with a K-linear differential operator ∂_M relative to ∂ satisfying the compatibility condition $\partial(\varphi(t)) \cdot \varphi_M \circ \partial_M = \partial_M \circ \varphi_M$. We define the sets of *analytic horizontal sections* and *analytic solutions* of M, respectively, by $V(M) := \ker(\partial_M \otimes \mathrm{id}_{K\{t\}} + \mathrm{id}_M \otimes \partial : M \otimes_{K[\![t]\!]_0} K\{t\} \to M \otimes_{K[\![t]\!]_0} K\{t\})$ and $\mathrm{Sol}(M) := \{f \in \mathrm{Hom}_{K[\![t]\!]_0}(M, K\{t\}); \partial \circ f = f \circ \partial_M\}$, which are equipped with a canonical perfect pairing $V(M) \otimes_K \mathrm{Sol}(M) \to K$. By Dwork's trick [Ked10, Corollary 17.2.2], $V(M)$ and $\mathrm{Sol}(M)$ are regarded as φ-modules over K of dimension equal to the rank of M, where $\varphi_{V(M)}$ and

$\varphi_{\mathrm{Sol}(M)}$ are defined as the restriction of $\varphi_M \otimes \varphi$ and the unique φ_K-semilinear endomorphism on $\mathrm{Sol}(M)$ satisfying $\varphi_{\mathrm{Sol}(M)}(f)(\varphi_M(m)) = \varphi(f(m))$ for $f \in \mathrm{Sol}(M), m \in M$, respectively. Consequently, $V(M)$ and $\mathrm{Sol}(M)$ are endowed with the *Frobenius slope filtrations* $S_\bullet(V(M))$ and $S_\bullet(\mathrm{Sol}(M))$, respectively (see Theorem 1.5(ii)). We also define the *growth filtrations* $\mathrm{Sol}_\bullet(M)$ and $V(M)^\bullet$ of $\mathrm{Sol}(M)$ and $V(M)$ respectively by $\mathrm{Sol}_\lambda(M) = \{f \in \mathrm{Sol}(M); f(M) \subset \mathrm{Fil}_\lambda K\{t\}\}$ and $V(M)^\lambda = \mathrm{Sol}_\lambda(M)^\perp$ for $\lambda \in \mathbb{R}$, where $\mathrm{Sol}_\lambda(M)^\perp$ denotes the orthogonal part of $\mathrm{Sol}_\lambda(M)$ with respect to the canonical pairing $V(M) \otimes_K \mathrm{Sol}(M) \to K$.

The Chiarellotto–Tsuzuki conjecture concerns a comparison between $V(M)^\bullet$ (respectively, $\mathrm{Sol}_\bullet(M)$) and $S_\bullet(V(M))$ (respectively, $S_\bullet(\mathrm{Sol}(M))$). To state the conjecture precisely, we briefly recall the notion of being *pure of bounded quotient* (PBQ), which is a condition on the *generic fiber* of M (see [CT11, 5.1] for details). Let $\mathcal{E}$ be the *Amice ring*, that is, the completion of the fraction field of $K[\![t]\!]_0$ for the norm $|\cdot|_0$, endowed with natural extensions of φ and ∂. We define $\mathcal{E}[\![X-t]\!]_0$ as before with $X-t$ a variable. We define the ring homomorphism $\tau : \mathcal{E} \to \mathcal{E}[\![X-t]\!]_0; f \mapsto \sum_{i\in\mathbb{N}} \partial^i(f)\cdot(X-t)^i/i!$. We endow $\mathcal{E}[\![X-t]\!]_0$ with the $\mathcal{E}$-linear derivation $\partial = d/d(X-t) : \mathcal{E}[\![X-t]\!]_0 \to \mathcal{E}[\![X-t]\!]_0$ and a ring endomorphism φ defined by $\varphi(\sum_{i\in\mathbb{N}} x_i(X-t)^i) = \sum_{i\in\mathbb{N}} \varphi(x_i)(\tau(\varphi(t)) - S')^i$, which commute with τ. Let M' be a (φ,∇)-module over $\mathcal{E}$. Let $\lambda_{\max}(M')$ denote the maximum Frobenius slope of M' (see Definition 1.3). Regarding $\tau^*(M') := M' \otimes_{\mathcal{E},\tau} \mathcal{E}[\![X-t]\!]_0$ as a (φ,∇)-module over $\mathcal{E}[\![X-t]\!]_0$ (see § 4.1), we define a φ-module $V(\tau^*(M'))$ over $\mathcal{E}$ equipped with a growth filtration $V(\tau^*(M'))^\bullet$ as before. For $\lambda \in \mathbb{R}$, by a theorem of Robba, we have a (unique) subobject $(M')^\lambda$ of M' in the category of (φ,∇)-modules over $\mathcal{E}$ such that there exists a canonical isomorphism $V(\tau^*(M'/(M')^\lambda)) \cong V(\tau^*(M'))^\lambda$, which forms a decreasing filtration of M' called the *log-growth filtration* of M' (see [CT09, 3.2 and 3.3]). We say that M' is PBQ if $M'/(M')^0$ is pure as a φ-module over $\mathcal{E}$, that is, the Frobenius Newton polygon of $M'/(M')^0$ is a straight line.

CHIARELLOTTO–TSUZUKI CONJECTURE [CT11, Conjecture 2.5]. Let M be a (φ,∇)-module over $K[\![t]\!]_0$. We regard $M_{\mathcal{E}} := M \otimes_{K[\![t]\!]_0} \mathcal{E}$ as a (φ,∇)-module over $\mathcal{E}$.

- (i) We have $V(M)^\lambda = \bigcup_{\mu>\lambda} V(M)^\mu$ and $\mathrm{Sol}_\lambda(M) = \bigcap_{\mu>\lambda} \mathrm{Sol}_\mu(M)$ for an arbitrary real number λ. Moreover, the slope multisets of $V(M)^\bullet$ and $\mathrm{Sol}_\bullet(M)$, which coincide by definition, are consisting of rational numbers.
- (ii) If $M_{\mathcal{E}}$ is PBQ, then

$$V(M)^\lambda = \bigcup_{\mu<\lambda_{\max}(M_{\mathcal{E}})-\lambda} S_\mu(V(M)),$$

or, equivalently,

$$\mathrm{Sol}_\lambda(M) = S_{\lambda-\lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M)),$$

for an arbitrary real number λ.

We quickly review known results on the Chiarellotto–Tsuzuki conjecture. If M is of rank 1, then the conjecture obviously holds as M is trivial as a ∇-module over $K[\![t]\!]_0$ (see [Ked10, Proposition 18.4.3]). If M arises from the Picard–Fuchs module associated with the Legendre family of elliptic curves, then Dwork's result implies part (ii) of the conjecture (see [CT09, 7.4] for details). Chiarellotto and Tsuzuki proved the conjecture in the rank 2 case [CT09, Theorem 7.1(1)]. The present author proved part (i) of the conjecture without assumptions [Ohk17, Theorem 3.7(i)].

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A generalization of Chiarellotto–Tsuzuki conjecture

In this paper, we prove Theorem 0.1, which is regarded as a generalization of the Chiarellotto–Tsuzuki conjecture to (φ, ∇)-modules over the bounded Robba ring $\mathcal{R}^{\mathrm{bd}}$. For $r \in \mathbb{R}_{\geqslant 0}$ and a formal Laurent series over K denoted by $x = \sum_{i\in\mathbb{Z}} x_i t^i$ with $x_i \in K$, we put $|x|_r = \sup_{i\in\mathbb{Z}} |x_i| e^{-ri} \in [0, \infty]$. We define the (respectively, *bounded*) *Robba ring* $\mathcal{R}$ (respectively, $\mathcal{R}^{\mathrm{bd}}$) as the ring consisting of series x such that there exists $r > 0$ such that $|x|_s < +\infty$ for all $s \in (0, r]$ (respectively, $s \in [0, r]$). We fix an element $X \in \mathcal{O}_K[\![t]\!]$ of the form $t^i v$ with $i \in \mathbb{N}_{\geqslant 1}$ and $v \in \mathcal{O}_K[\![t]\!]^{\times}$, and define $\mathcal{R}_{\log}$ as the polynomial ring $\mathcal{R}[\ell_X]$ with a variable ℓ_X. Then, the ring $\mathcal{R}^{\mathrm{bd}}$ is a field, and we have the following commutative diagram of rings, where the hooked arrows denote the inclusions.

$$\begin{array}{ccccccc}
\mathcal{E} & \hookleftarrow & K[\![t]\!]_0 & \hookrightarrow & K\{t\} & & \\
\| & & \downarrow & & \downarrow & & \\
\mathcal{E} & \hookleftarrow & \mathcal{R}^{\mathrm{bd}} & \hookrightarrow & \mathcal{R} & \hookrightarrow & \mathcal{R}_{\log}.
\end{array}$$

We endow $\mathcal{R}$ with the K-linear derivation $\partial = d/dt : \mathcal{R} \to \mathcal{R}$ and a ring endomorphism φ of the form $\varphi(\sum_{i\in\mathbb{N}} x_i t^i) = \sum_{i\in\mathbb{N}} \varphi_K(x_i) S^i$ for some $S \in \mathcal{R}^{\mathrm{bd}}$ with $|S - t^q|_0 < 1$. Then, φ and ∂ restrict to $\mathcal{R}^{\mathrm{bd}}$, and extend to $\mathcal{E}$, $\mathcal{R}_{\log}$ by setting $\varphi(\ell_X) = \log(\varphi(X)/X^q) + q\ell_X$ and $\partial(\ell_X) = \partial(X)/X$, respectively (see Definitions 2.1(ii), (iii), and 2.2). We define the *log-growth filtrations* of $\mathcal{R}$ and $\mathcal{R}_{\log}$ by $\mathrm{Fil}_\lambda \mathcal{R} = \{x \in \mathcal{R}; |x|_r = O(r^{-\lambda}) \text{ as } r \to +0\}$ if $\lambda \in \mathbb{R}_{\geqslant 0}$, $\mathrm{Fil}_\lambda \mathcal{R} = 0$ if $\lambda \in \mathbb{R}_{<0}$, and $\mathrm{Fil}_\lambda \mathcal{R}_{\log} := \bigoplus_{i\in\mathbb{N}} \mathrm{Fil}_{\lambda - i}\mathcal{R} \cdot \ell_X^i$ for $\lambda \in \mathbb{R}$. We can define the notion of (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$ as before. For a *literal* generalization of the Chiarellotto–Tsuzuki conjecture, we should study the whole category of (φ, ∇)-modules M over $\mathcal{R}^{\mathrm{bd}}$, where we encounter problems caused by the absence of a naïve analogue of Dwork's trick. To avoid complications, in this paper, we assume that $M_{\mathcal{R}} := M \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R}$ is unipotent as a ∇-module over $\mathcal{R}$. Thanks to the assumption, we can define $V(M), \mathrm{Sol}(M)$, etc., as before, where $K[\![t]\!]_0$ and $K\{t\}$ are replaced by $\mathcal{R}^{\mathrm{bd}}$ and $\mathcal{R}_{\log}$, respectively. The main result of this paper is as follows.

Theorem 0.1 (A generalization of the Chiarellotto–Tsuzuki conjecture). *Let M be a (φ, ∇)-module over $\mathcal{R}^{\mathrm{bd}}$ such that $M_{\mathcal{R}}$ is unipotent as a ∇-module over $\mathcal{R}$. We regard $M_{\mathcal{E}} := M \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{E}$ as a (φ, ∇)-module over $\mathcal{E}$.*

(i) *We have $V(M)^\lambda = \bigcup_{\mu<\lambda} V(M)^\mu$ and $\mathrm{Sol}_\lambda(M) = \bigcap_{\mu>\lambda} \mathrm{Sol}_\mu(M)$ for an arbitrary real number λ. Moreover, the slope multisets of $V(M)^\bullet$ and $\mathrm{Sol}_\bullet(M)$, which coincide by definition, are consisting of rational numbers.*

(ii) *If $M_{\mathcal{E}}$ is PBQ, then*

$$V(M)^\lambda = \bigcup_{\mu < \lambda_{\max}(M_{\mathcal{E}}) - \lambda} S_\mu(V(M)),$$

or, equivalently,

$$\mathrm{Sol}_\lambda(M) = S_{\lambda - \lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M))$$

for an arbitrary real number λ.

We prove part (ii) of Theorem 0.1 in § 8. We give two proofs of part (i) of Theorem 0.1; the first given in § 5 is done by a reduction to the case $X = t$, in which case the assertion is proved in [Ohk17, Theorem 4.19], and the second given in § 10 is simple and self-contained.

In the rest of this subsection, we put $S = S' \in K[\![t]\!]_0$ so that the inclusion $K[\![t]\!]_0 \to \mathcal{R}^{\mathrm{bd}}$ is φ-equivariant. Moreover, we admit part (ii) of Theorem 0.1 and the property of the log-growth filtrations of $\mathcal{R}$ and $\mathcal{R}_{\log}$ (see § 3.2). Under this setup, we will prove the following result.

THEOREM 0.2. *Part (ii) of the Chiarellotto–Tsuzuki conjecture holds.*

Note that even a log analogue of the Chiarellotto–Tsuzuki conjecture holds (Theorem 10.6).

LEMMA 0.3. *Let M be a (φ, ∇)-module over $K[\![t]\!]_0$. We put $M_{\mathcal{R}^{\mathrm{bd}}} := M \otimes_{K[\![t]\!]_0} \mathcal{R}^{\mathrm{bd}}$, which is regarded as a (φ, ∇)-module over $\mathcal{R}^{\mathrm{bd}}$. Then, $(M_{\mathcal{R}^{\mathrm{bd}}})_{\mathcal{R}}$ is unipotent as a ∇-module over $\mathcal{R}$. Moreover, there exist canonical isomorphisms of φ-modules over K*

$$\alpha : V(M) \to V(M_{\mathcal{R}^{\mathrm{bd}}}),$$
$$\beta : \mathrm{Sol}(M) \to \mathrm{Sol}(M_{\mathcal{R}^{\mathrm{bd}}}),$$

which are compatible with the canonical pairings. Furthermore, the maps α and β induce isomorphisms $V(M)^{\bullet} \to V(M_{\mathcal{R}^{\mathrm{bd}}})^{\bullet}$ and $\mathrm{Sol}_{\bullet}(M) \to \mathrm{Sol}_{\bullet}(M_{\mathcal{R}^{\mathrm{bd}}})$ of filtrations, respectively. In particular, the log-growth Newton polygon $\mathrm{NP}(M)$ of M (see [CT11, 2.3] or the next subsection) coincides with the log-growth Newton polygon $\mathrm{NP}(M_{\mathcal{R}^{\mathrm{bd}}})$ of $M_{\mathcal{R}^{\mathrm{bd}}}$ (Definition 10.1(i)).

Proof. As $M \otimes_{K[\![t]\!]_0} K\{t\}$ is trivial as a ∇-module over $K\{t\}$ by Dwork's trick [Ked10, Corollary 17.2.2], $(M_{\mathcal{R}^{\mathrm{bd}}})_{\mathcal{R}} \cong (M \otimes_{K[\![t]\!]_0} K\{t\}) \otimes_{K\{t\}} \mathcal{R}$ is unipotent (even trivial) as a ∇-module over $\mathcal{R}$. We define the maps

$$\alpha' : M \otimes_{K[\![t]\!]_0} K\{t\} \to (M \otimes_{K[\![t]\!]_0} \mathcal{R}^{\mathrm{bd}}) \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R}_{\log}; m \otimes r \mapsto m \otimes 1 \otimes r,$$
$$\beta' : \mathrm{Hom}_{K[\![t]\!]_0}(M, K\{t\}) \to \mathrm{Hom}_{\mathcal{R}^{\mathrm{bd}}}(M \otimes_{K[\![t]\!]_0} \mathcal{R}^{\mathrm{bd}}, \mathcal{R}_{\log}); f \mapsto (m \otimes r \mapsto f(m)r),$$

which are compatible with Frobenius actions and differential operators. Let $\alpha : V(M) \to V(M_{\mathcal{R}^{\mathrm{bd}}})$ and $\beta : \mathrm{Sol}(M) \to \mathrm{Sol}(M_{\mathcal{R}^{\mathrm{bd}}})$ be the morphisms of φ-modules over K induced by α' and β', respectively. As α' and β' are injective, the maps α and β are isomorphisms by comparing dimensions. By definition, α and β are compatible with the canonical pairings. To complete the proof, we have only to prove, by duality, that for $f \in \mathrm{Sol}(M)$, we have $f \in \mathrm{Sol}_{\lambda}(M)$ if and only if $\beta(f) \in \mathrm{Sol}_{\lambda}(M_{\mathcal{R}^{\mathrm{bd}}})$. If $f \in \mathrm{Sol}_{\lambda}(M)$, then $\beta(f) \in \mathrm{Sol}_{\lambda}(M_{\mathcal{R}^{\mathrm{bd}}})$ by

$$\beta(f)(M_{\mathcal{R}^{\mathrm{bd}}}) = f(M) \cdot \mathcal{R}^{\mathrm{bd}} \subset (\mathrm{Fil}_{\lambda} K\{t\}) \cdot \mathcal{R}^{\mathrm{bd}} \subset \mathrm{Fil}_{\lambda}\mathcal{R}_{\log} \cdot \mathrm{Fil}_0\mathcal{R}_{\log} \subset \mathrm{Fil}_{\lambda}\mathcal{R}_{\log},$$

where we use Corollary 3.7(i) and (iii). Conversely, if $\beta(f) \in \mathrm{Sol}_{\lambda}(M_{\mathcal{R}^{\mathrm{bd}}})$, then $f \in \mathrm{Sol}_{\lambda}(M)$ by

$$f(M) \subset K\{t\} \cap \beta(f)(M_{\mathcal{R}^{\mathrm{bd}}}) \subset K\{t\} \cap \mathrm{Fil}_{\lambda}\mathcal{R}_{\log} = K\{t\} \cap \mathrm{Fil}_{\lambda}\mathcal{R} = \mathrm{Fil}_{\lambda} K\{t\},$$

where we use Corollary 3.6. □

Proof of Theorem 0.2. Let M be a (φ, ∇)-module over $K[\![t]\!]_0$ such that $M_{\mathcal{E}}$ is PBQ. Then, we can apply part (ii) of Theorem 0.1 to $M_{\mathcal{R}^{\mathrm{bd}}}$ because $(M_{\mathcal{R}^{\mathrm{bd}}})_{\mathcal{R}}$ is unipotent as a ∇-module over $\mathcal{R}$ by Lemma 0.3, and $(M_{\mathcal{R}^{\mathrm{bd}}})_{\mathcal{E}} \cong M_{\mathcal{E}}$ is PBQ. Hence, we obtain the assertion by using Lemma 0.3. □

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Application to Dwork's conjecture

Recall that the Grothendieck–Katz specialization theorem for φ-modules M over $K[\![t]\!]_0$ asserts that the Frobenius Newton polygon of M/tM lies on or above the Frobenius Newton polygon of $M_{\mathcal{E}}$ with the same endpoints [Ked10, Theorem 15.3.2]. Chiarellotto and Tsuzuki formulated *Dwork's conjecture* as an analogue of the Grothendieck–Katz specialization theorem for (φ, ∇)-modules M over $K[\![t]\!]_0$ (see also Remark 10.5). Recall that a *slope* of $V(M)^{\bullet}$ is a real number λ such that $\bigcap_{\mu<\lambda} V(M)^{\mu} \neq \bigcup_{\mu>\lambda} V(M)^{\mu}$. Let $\{s_1(M) \leqslant \cdots \leqslant s_n(M)\}$ denote the multiset of slopes λ of $V(M)^{\bullet}$ with multiplicity $\dim_K \bigcap_{\mu<\lambda} V(M)^{\mu} - \dim_K \bigcup_{\mu>\lambda} V(M)^{\mu}$, where n denotes the rank of M. Then, we define the *log-growth Newton polygon* $\mathrm{NP}(M)$ of M as the boundary of the lower convex hull of the points $(0,0)$ and $(i, s_1(M) + \cdots + s_i(M))$ for $i \in \{1, \ldots, n\}$. Similarly, we define the Newton polygon $\mathrm{NP}(M)$ of a (φ, ∇)-module M over $\mathcal{E}$ (Definition 10.1(ii)).

Dwork's conjecture [CT11, Conjecture 2.7]. Let M be a (φ, ∇)-module over $K[\![t]\!]_0$. Then, $\mathrm{NP}(M)$ lies on or above $\mathrm{NP}(M_{\mathcal{E}})$ with the same endpoints.

Dwork's conjecture is previously known in the case where the rank is less than or equal to two [CT09, Corollary 7.3]. In § 10, we prove the following result.

Theorem 0.4. *(A generalization of Dwork's conjecture) Let M be a (φ, ∇)-module of dimension n over $\mathcal{R}^{\mathrm{bd}}$ such that $M_{\mathcal{R}}$ is unipotent as a ∇-module over $\mathcal{R}$. Then, the Newton polygon $\mathrm{NP}(M)$ of M (Definition 10.1(i)) lies on or above $\mathrm{NP}(M_{\mathcal{E}})$ with the same endpoints. In particular, we have $s_n(M) \leqslant s_n(M_{\mathcal{E}})$, where $s_n(M_{(\mathcal{E})})$ denotes the maximum slope of $\mathrm{NP}(M_{(\mathcal{E})})$.*

If we admit Theorem 0.4, then we have the following.

Theorem 0.5. *Dwork's conjecture holds.*

Proof. Let M be a (φ, ∇)-module over $K[\![t]\!]_0$. We can apply Theorem 0.4 to $M_{\mathcal{R}^{\mathrm{bd}}}$ by using Lemma 0.3 as in the proof of Theorem 0.2. Moreover, $\mathrm{NP}(M)$ (respectively, $\mathrm{NP}(M_{\mathcal{E}})$) coincides with $\mathrm{NP}(M_{\mathcal{R}^{\mathrm{bd}}})$ (respectively, $\mathrm{NP}((M_{\mathcal{R}^{\mathrm{bd}}})_{\mathcal{E}})$) by Lemma 0.3 (respectively, by definition), which implies the assertion. □

Structure of the paper

We prove part (ii) of Theorem 0.1 in § 8. If the reader admits the key ingredients, that is, Propositions 3.11 and 7.2, then the (short) proof can be easily understood. We prove Propositions 3.11 and 7.2 in §§ 3 and 7 respectively, after giving preparations in §§ 1, 2 and §§ 4–6, respectively. We prove part (i) of Theorem 0.1 and Theorem 0.4 in § 10 by using the PBQ filtrations of (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$, which are studied in § 9.

In § 6, we prove the invariance of the growth filtration under the base change of the coefficient field K, which can be skipped by assuming that k is algebraically closed. Parts of §§ 2 and 3 (specifically, §§ 2.2, 2.3, 3.1, and 3.3) are rather technical because the extended Robba ring is involved. Hence, the reader may postpone these parts by admitting results proved there.

Finally, we note that to make the paper concise, some results of an earlier version of this paper are not available in the current version, and will be included in a forthcoming paper(s).

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Notation and terminology

(1) In this paper, a ring R is a commutative ring with 1 unless otherwise mentioned. For a ring homomorphism $\psi : R \to S$, we define the base change functor $\psi^*(-)$ from the category of R-modules to the category of S-modules by $M \mapsto \psi^* M = M_S = M \otimes_R S$. When ψ is injective, we identify R with $\psi(R)$, and regard ψ as the inclusion.

(2) We recall some terminology of difference modules [Ked10, § 14]. A *difference ring* (respectively, *field*) is a ring (respectively, field) R equipped with a ring endomorphism ϕ. Let R be a difference ring. A *difference module* over R is an R-module M equipped with a ϕ-semilinear endomorphism ϕ_M. A *ϕ-module* over R is a difference module M over R such that M is finite free as an R-module, and the R-linear map $\phi^* M = M \otimes_{R,\phi} R \to M; m \otimes r \mapsto r\phi_M(m)$ is an isomorphism. The category of ϕ-modules over R is equipped with the following operations: the internal Hom $\mathrm{Hom}_R(-,-)$, the tensor product $(-) \otimes_R (-)$, the dual $(-)^\vee = \mathrm{Hom}_R(-, R)$, and the natural pairing $(-) \otimes_R (-)^\vee \to R$. For $c \in R^\times$, let $R(c) = Re_c$ be the ϕ-module over R defined by $\phi_{R(c)}(e_c) = ce_c$. We define the *twist* of a ϕ-module M over R by c to be $M(c) = M \otimes_R R(c)$. For $d \in \mathbb{N}_{>0}$, we define the *d-pushforward functor* $[d]_*(-)$ from the category of ϕ-modules over R to the category of ϕ^d-modules over R by $(M, \phi_M) \mapsto (M, \phi_M^d)$. For a morphism $\psi : R \to S$ of difference rings, we define a base change functor $\psi^*(-)$, as in part (1), from the category of ϕ-modules over R to the category of ϕ-modules over S. Then, the d-pushforward functor and the base change functor are compatible with internal Homs, tensor products, duals, and natural pairings in an obvious sense.

Let F be a difference field with respect to a ring endomorphism ϕ. We say that F is *inversive* if ϕ is bijective. We say that F is *weakly difference-closed* if any equation of the form $\phi(x) = cx$ with $c \in F^\times$ always has a solution $x \in F^\times$. We say that F is *strongly difference-closed* if F is weakly difference-closed and inversive.

(3) Let V be a finite-dimensional vector space over a field F with $n = \dim_F V$, equipped with a filtration of subspaces which is exhaustive and separated. We recall the definition of the Newton polygon associated with the filtration. For simplicity, we assume that our filtration is decreasing, which is denoted by $\{V^\lambda; \lambda \in \mathbb{R}\}$. We put $m(\lambda) = \dim_F(\bigcap_{\varepsilon>0} V^{\lambda-\varepsilon})/(\bigcup_{\varepsilon>0} V^{\lambda+\varepsilon}) = \lim_{\varepsilon\to 0^+} \dim_F V^{\lambda-\varepsilon} - \lim_{\varepsilon\to 0^+} \dim_F V^{\lambda+\varepsilon}$. Note that $m(\lambda) = 0$ for all but finitely many λ, and $\sum_{\lambda\in\mathbb{R}} m(\lambda) = n$. If $m(\lambda) \neq 0$, then we call λ a *slope* of $V^\bullet$. The *slope multiset* of $V^\bullet$ is the multiset consisting of slopes λ of $V^\bullet$ with multiplicity $m(\lambda)$, which is denoted by $\{s_1 \leqslant \cdots \leqslant s_n\}$. We define the *Newton polygon* $\mathrm{NP}(V^\bullet)$ of $V^\bullet$ by the boundary of the lower convex hull in the xy-plane of the set of points $(0, 0)$ and $(i, s_1 + \cdots + s_i)$ for $i \in \{1, \ldots, n\}$.

1. Basic results on ϕ-modules over a complete discrete valuation field

We recall the definition of the slope filtrations $S_\bullet(M)$ of ϕ-modules M over a complete discrete valuation field [CT09, CT11]: our slope filtration will be defined by renumbering a filtration of M as in [Ked10, Theorem 14.4.15] so that the indices of $S_\bullet(M)$ represent the *slopes* of M. Then, we give basic properties of slope filtrations by rephrasing results of [Ked10, § 14].

Definition 1.1. In this section, we fix a triple (F, ϕ, q), where F is a complete discrete valuation field of mixed characteristic $(0, p)$ together with a valuation $|\cdot|$, ϕ is an arbitrary isometric ring endomorphism on F, and q is a positive power of p. Let $\bar{\phi}$ denote the endomorphism on the residue field of F induced by ϕ. Unless otherwise mentioned, we regard F and its residue field as difference rings with respect to ϕ and $\bar{\phi}$, respectively. When $\bar{\phi}$ is the q-power map, we call ϕ

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a *q-power Frobenius lift*, and we put the adjective *Frobenius* everywhere (for example, a slope is called a Frobenius slope).

An *extension* of (F, ϕ, q) is a triple (E, ϕ, q) as previously equipped with an isometric ring homomorphism $F \to E$ which is ϕ-equivariant; in the case where $\phi : F \to F$ is a q-power Frobenius lift, we tacitly assume that $\phi : E \to E$ is also a q-power Frobenius lift.

In the rest of this section, let $c \in F^\times, d \in \mathbb{N}_{>0}$, $\lambda \in \mathbb{R}$, and let M, N be ϕ-modules over F. Moreover, let (E, ϕ, q) be an extension of (F, ϕ, q) .

Remark 1.2. There exists an extension E of F such that the value groups of E and F coincide, and the residue field of E is strongly difference-closed [Ked08, Proposition 3.2.4]. Moreover, if ϕ is a q-power Frobenius lift on F, then we may assume that the residue field of E is algebraically closed: for example, we consider the completion of a maximal unramified extension of the inductive limit $\varinjlim(F \xrightarrow{\phi} F \xrightarrow{\phi} \cdots)$ (see [Ked10, Hypothesis 14.4.1 and Proposition 14.3.4]).

Definition 1.3 [Ked10, Definitions 6.1.3, 14.4.6, and Remark 14.4.7]. We endow M with the supremum norm with respect to a basis of M (see [Ked10, Definition 1.3.2]). We define the *spectral radius* of ϕ_M as $\lim_{n\to\infty}(\sup_{v\in M, v\neq 0} |\phi_M^n(v)|/|v|)^{1/n}$, which is independent of the choice of the norm [Ked10, Theorem 1.3.6, Proposition 6.1.5]. Let $\{M_i\}$ be the Jordan–Hölder constituents of M. We call M *pure* if there exists $\lambda \in \mathbb{R}$ such that the ϕ-modules M_i have spectral radius $|q|^\lambda$; in this case, we call λ the *slope* of M. Note that $M = 0$ and any irreducible ϕ-module over F are pure by definition: for example, $F(c)$ is pure of slope $\log|c|/\log|q|$. We define the *slope multiset* of M as the multiset consisting of the slopes of the ϕ-modules M_i with multiplicity $\dim_F M_i$. We define $\lambda_{\max}(M)$ as the maximum element in the slope multiset of M if $M \neq 0$, and $\lambda_{\max}(M) = 0$ if $M = 0$. Unless otherwise mentioned, we calculate the slope multiset of the ϕ^d-module $[d]_*M$ over F (respectively, the ϕ-module $M \otimes_F E$ over E) with respect to the triple (F, ϕ^d, q^d) (respectively, (E, ϕ, q)).

Lemma 1.4.

(i) *If we replace (F, ϕ, q) in Definition 1.1 by (F, ϕ, q') with q' a power of p, then the slope multiset of M is multiplied by* $\log|q|/\log|q'|$.

(ii) *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of ϕ-modules over F, then the slope multiset of M is equal to the disjoint union of those of M' and M''. In particular, the slope multiset of $M \oplus N$ is equal to the disjoint union of those of M and N.*

(iii) *(Rationality) The slope multiset of M consists of rational numbers. In particular, $\lambda_{\max}(M)$ is a rational number.*

(iv) *(Base change) The slope multiset of $M \otimes_F E$ is equal to that of M.*

(v) *(d-pushforward) The slope multiset of $[d]_*M$ is equal to that of M.*

(vi) *(Tensor product) The slope multiset of $M \otimes_F N$ consists of $\lambda + \mu$ where λ, μ respectively run over the slope multisets of M, N with multiplicity.*

(vii) *(Dual) The slope multiset of $M^\vee$ is equal to (-1) times that of M. In particular, M is pure of slope λ if and only if $M^\vee$ is pure of slope $-\lambda$.*

Proof. Parts (i), (ii), and (v) hold by definition. By part (ii), to prove the rest of the assertion, we may reduce to the irreducible case. Then, parts (iii), (iv), (vi), and (vii) follow from [Ked10, Corollary 14.4.5, Lemma 14.4.3(c), Corollary 14.4.9, and Proposition 14.4.8] respectively. □

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Theorem 1.5.

(i) *[Ked10, Theorem 14.4.13] Assume that F is inversive. Then, there exists a unique internal direct sum decomposition $M = \bigoplus_{\lambda\in\mathbb{R}} M_\lambda$ of ϕ-modules, in which each nonzero M_λ is pure of slope λ. Moreover, the multiplicity of λ in the slope multiset of M is equal to $\dim_F M_\lambda$.*

(ii) *(Cf. [CT09, Definition 2.3].) There exists a unique increasing filtration $\{S_\lambda(M); \lambda \in \mathbb{R}\}$ of ϕ-submodules of M, called the slope filtration of M, satisfying the following:*
   (a) *the filtration $S_\bullet(M)$ is exhaustive and separated;*
   (b) *(right continuity) we have $S_\lambda(M) = \bigcap_{\mu>\lambda} S_\mu(M)$, i.e.$S_\lambda(M) = S_{\lambda+\varepsilon}(M)$ for all sufficiently small $\varepsilon \in \mathbb{R}_{>0}$;*
   (c) *if $S_\lambda(M)/\bigcup_{\mu<\lambda} S_\mu(M)$ is nonzero, then it is pure of slope λ.*

   Moreover, the multiset consisting of slopes of M less than or equal to (respectively, strictly greater than) λ coincides with the slope multiset of $S_\lambda(M)$ (respectively, $M/S_\lambda(M)$); the multiplicity of λ in the slope multiset of M is equal to $\dim_F(S_\lambda(M)/\bigcup_{\mu<\lambda} S_\mu(M))$.

(iii) *If F is inversive, then $S_\lambda(M) = \bigoplus_{\mu\leqslant\lambda} M_\mu$. In particular, $S_\lambda(M)/\bigcup_{\mu<\lambda} S_\mu(M) \cong M_\lambda$.*

(iv) *(Functoriality) Let $f : M \to N$ be a morphism of ϕ-modules over F. Then, we have $f(M_\lambda) \subset N_\lambda$ if F inversive, and $f(S_\lambda(M)) \subset S_\lambda(N)$.*

Lemma 1.6. *If the slope multisets of M and N are disjoint, then $\mathrm{Hom}(M, N) = 0$.*

Proof. By dévissage, we may assume that M and N are irreducible. By assumption, $M \ncong N$, hence, $\mathrm{Hom}(M, N) = 0$. □

Proof of Theorem 1.5. (ii) We prove the existence of $S_\bullet(M)$. By [Ked10, Theorem 14.4.15], there exists a unique filtration $0 = \mathcal{F}_0M \subset \mathcal{F}_1M \subset \cdots \subset \mathcal{F}_mM = M$ of ϕ-modules such that $\mathcal{F}_iM/\mathcal{F}_{i-1}M$ is pure of slope λ_i with $\lambda_1 < \cdots < \lambda_m$. We define $S_\lambda(M) = 0$ if $\lambda \in (-\infty, \lambda_1)$, $S_\lambda(M) = \mathcal{F}_iM$ if $\lambda \in [\lambda_i, \lambda_{i+1})$ for $i = 1, \ldots, m$, and $S_\lambda(M) = M$ if $\lambda \in [\lambda_m, +\infty)$. Then, $S_\bullet(M)$ satisfies the required conditions. Let $S'_\bullet(M)$ be another filtration satisfying conditions (a)–(c) as in part (ii). Then, the filtration $\mathcal{F}_\bullet(M)$ can be recovered by using $S'_\bullet(M)$ in an obvious way, which implies that $S'_\bullet(M) = S_\bullet(M)$.

(iii) As the filtration $\{\bigoplus_{\mu\leqslant\lambda} M_\mu; \lambda \in \mathbb{R}\}$ satisfies conditions (a)–(c) as in part (ii), the assertion follows from the uniqueness of $S_\bullet(M)$.

(iv) By Lemma 1.6, we have $\mathrm{Hom}(M_\lambda, N_\mu) = 0$ for $\mu \neq \lambda$ if F is inversive, and $\mathrm{Hom}(S_\lambda(M), N/S_\lambda(N)) = 0$, which implies the assertion. □

For example, the slope filtration of $M = F(c)$ is given by $S_\lambda(M) = M$ if $\lambda \geqslant \log|c|/\log|q|$, and $S_\lambda(M) = 0$ otherwise.

A *pairing* of M and N is a morphism $M \otimes_F N \to F$ of ϕ-modules over F, which bijectively corresponds to, an F-bilinear map $b : M \times N \to F$ such that $b(\phi_M(m), \phi_N(n)) = \phi(b(m, n))$ for $m \in M, n \in N$. Furthermore, the pairing is called *perfect* if b is nondegenerate. For example, the canonical map $M \otimes_F M^\vee \to F$ is a perfect pairing of M and $M^\vee$. We define the orthogonal part of a ϕ-submodule M_0 of M by $M_0^\perp := \{n \in N; b(m_0, n) = 0\ \forall m_0 \in M_0\}$, which is a ϕ-submodule of N. We similarly define the orthogonal part of a ϕ-submodule N_0 of N.

Lemma 1.7.

(I) *Assume that F is inversive.*
   (i) *(Base change) There exists a canonical isomorphism $M_\lambda \otimes_F E \cong (M \otimes_F E)_\lambda$.*

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(ii) *(d-pushforward) There exists a canonical isomorphism* $[d]_*(M_\lambda) \cong ([d]_* M)_\lambda$.

(iii) *(Strictness) If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of* ϕ*-modules over* F*, then there exists an induced exact sequence* $0 \to M'_\lambda \to M_\lambda \to M''_\lambda \to 0$.

(iv) *(Tensor product) There exists a canonical isomorphism* $(M \otimes_F N)_\lambda \cong \bigoplus_{\mu+\nu=\lambda}(M_\mu \otimes_F N_\nu)$.

(v) *(Twist) Put* $\mu := \log|c|/\log|q| \in \mathbb{Q}$*. Then, there exists a canonical isomorphism* $M(c)_\lambda \cong M_{\lambda-\mu}(c)$.

(vi) *(Duality) Let* $M \otimes_F N \to F$ *be a perfect pairing. Then, we have* $(N_{-\lambda})^\perp = \bigoplus_{\mu\neq\lambda} M_\mu$.

(II) (i) *(Base change) There exists a canonical isomorphism* $S_\lambda(M) \otimes_F E \cong S_\lambda(M \otimes_F E)$.

(ii) *(d-pushforward) There exists a canonical isomorphism* $[d]_*(S_\lambda(M)) \cong S_\lambda([d]_* M)$.

(iii) *(Strictness) If* $0 \to M' \to M \to M'' \to 0$ *is an exact sequence of* ϕ*-modules over* F*, then there exists an induced exact sequence* $0 \to S_\lambda(M') \to S_\lambda(M) \to S_\lambda(M'') \to 0$.

(iv) *(Tensor product) There exists a canonical isomorphism*

$$S_\lambda(M \otimes_F N) \cong \sum_{\mu+\nu=\lambda} S_\mu(M) \otimes_F S_\nu(N).$$

(v) *(Twist) Put* $\mu := \log|c|/\log|q| \in \mathbb{Q}$*. Then, there exists a canonical isomorphism* $S_\lambda(M(c)) \cong S_{\lambda-\mu}(M)(c)$.

(vi) *(Duality) Let* $M \otimes_F N \to F$ *be a perfect pairing. Then, we have* $S_{-\lambda}(N)^\perp = \bigcup_{\mu<\lambda} S_\mu(M)$.

Proof. (I) Parts (i), (ii), and (iv) follow from the uniqueness of the decomposition given in Theorem 1.5(i). Part (iii) follows from Theorem 1.5(iv). Part (v) is a special case of part (iv). We prove part (vi). By Lemma 1.6, the image of $M_\mu \otimes_F N_{-\lambda}$ under the given pairing vanishes if $\mu \neq \lambda$. Hence, $N_{-\lambda} \subset (\bigoplus_{\mu\neq\lambda} M_\mu)^\perp$. By Theorem 1.5(i), we have $(\bigoplus_{\mu\neq\lambda} M_\mu)^\perp \cong (M/(\bigoplus_{\mu\neq\lambda} M_\mu))^\vee \cong (M_\lambda)^\vee$. Hence, $(\bigoplus_{\mu\neq\lambda} M_\mu)^\perp$ is pure of slope $-\lambda$. Therefore, the image of $(\bigoplus_{\mu\neq\lambda} M_\mu)^\perp$ under the projection $N \to N/N_{-\lambda}$ vanishes by Lemma 1.6. Hence, $(\bigoplus_{\mu\neq\lambda} M_\mu)^\perp \subset N_{-\lambda}$, which implies the assertion.

(II) Part (i) follows from the uniqueness of the slope filtration. To prove the rest of the assertion, by performing a base change and using part (i), we may assume that F is inversive, in which case $S_\lambda(M), S_\lambda(N)$ etc., can be replaced by $\bigoplus_{\mu\leqslant\lambda} M_\lambda, \bigoplus_{\mu\leqslant\lambda} N_\lambda$, etc. Then, parts (ii)–(v) follow from parts (ii)–(v) of part (I), respectively. We prove part (vi). By part (I)(vi), we have $S_{-\lambda}(N)^\perp = (\bigoplus_{\nu\geqslant\lambda} N_{-\nu})^\perp = \bigcap_{\nu\geqslant\lambda}(N_{-\nu})^\perp = \bigcap_{\nu\geqslant\lambda}(\bigoplus_{\mu\neq\nu} M_\mu) = \bigoplus_{\mu<\lambda} M_\mu$. □

We gather miscellaneous results used later in this paper.

Definition 1.8 (Cf. [Ked05b, Definition 4.2.1]). A *d-eigenvector* of M is a nonzero element $v \in M$ such that $\phi_M^d(v) = cv$ for some $c \in F^\times$; we call the quotient $\log|c|/\log|q|^d$ the *slope* of v. Note that v is regarded as a 1-eigenvector of $[d]_* M$ whose slope is equal to that of v.

Lemma 1.9. *If* $v \in M$ *is a d-eigenvector of slope* λ*, then* $v \in S_\lambda(M)$ *and* $v \notin \bigcup_{\mu<\lambda} S_\mu(M)$*. Moreover, if* M *is pure, then the slope of* M *is equal to* λ.

Proof. By Lemma 1.7(II)(ii), we may replace M by $[d]_* M$. Thus, we may assume that $d = 1$. By Lemma 1.7(II)(iii), we may replace M by Fv. Thus, we may assume that $M = F(c)$ for some $c \in F^\times$ with $\lambda = \log|c|/\log|q|$, in which case the assertion is obvious. □

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Theorem 1.10 (Dieudonné–Manin classification theorem [Ked10, Theorem 14.6.3]). *Assume that the residue field of F is strongly difference closed. Then, there exists d such that M admits an F-basis $e_1, \dots, e_n$ consisting of d-eigenvectors, where $n = \dim_F M$; if M is pure of slope zero, i.e. étale, then we may assume that $\phi_M(e_i) = e_i$ for all i. Moreover, if μ_i denotes the slope of e_i, then the slope multiset of M coincides with the multiset $\{\mu_i; 1 \leqslant i \leqslant n\}$, and M_λ is spanned by $\{e_i; \mu_i = \lambda\}$.*

Definition 1.11 (Cf. [Liu13, Definition 1.6.6]). Assume that F is inversive. Let $M = \bigoplus_{\lambda \in \mathbb{R}} M_\lambda$ be the decomposition given in Theorem 1.5(i). We define the *reverse filtration* $S^\bullet(M)$ of M by $S^\lambda(M) = \bigoplus_{\mu \geqslant \lambda} M_\mu$ for $\lambda \in \mathbb{R}$.

Note that $S^\bullet(M)$ is a decreasing filtration of ϕ-submodules of M, which is exhaustive and separated. Moreover, the slope multiset of $S^\bullet(M)$ coincides with that of $M_\bullet$.

2. The Robba ring and the extended Robba ring

In this section, we recall basic facts on various analytic rings used in this paper, in particular, the extended Robba ring $\tilde{\mathcal{R}}$, which is introduced in [Ked08], and studied further in [Liu13]. We also define a log analogue $\tilde{\mathcal{R}}_{\log}$ of $\tilde{\mathcal{R}}$.

Notation 1. In the rest of the paper, let p be fixed a prime number. Let (K, φ_K, q) be a triple as in Definition 1.1. In §§ 2 and 3, as in [Ked08, Liu13], we do not assume that φ_K is a q-power Frobenius lift. Let $\mathcal{O}_K$, $\mathfrak{m}_K$, and k be the integer ring, maximal ideal, and residue field of K, respectively. Except in § 2, we normalize the valuation $|\cdot|$ of K by $|p| = p^{-1}$.

In this paper, when we denote rings associated to K, such as the Robba ring $\mathcal{R}_K$, we omit the subscript K if no confusion arises.

2.1 The Robba ring

Definition 2.1. (i) [Ked08, Definitions 1.1.1, 1.2.3] For $r > 0$, let $\mathcal{R}^r$ be the ring of formal Laurent series $\sum_{i \in \mathbb{Z}} a_i t^i$ with $a_i \in K$ converging on $e^{-r} \leqslant |t| < 1$. We define the *Robba ring* $\mathcal{R}$ (over K) as the union of the $\mathcal{R}^r$'s, which is a Bézout domain. Explicitly, we have

$$\mathcal{R}^r = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i; \lim_{i \to \pm\infty} |a_i| e^{-sn} = 0 \ (s \in (0, r]) \right\},$$

$$\mathcal{R} = \bigcup_{r \in (0, +\infty)} \mathcal{R}^r.$$

We equip $\mathcal{R}^r$ with the norm $|\cdot|_s$ for $s \in (0, r]$ defined by $|\sum_{i \in \mathbb{Z}} a_i t^i|_s = \sup_{i \in \mathbb{Z}} \{|a_i| e^{-si}\}$.

(ii) [Ked08, Definition 1.1.3] We define the *bounded* (respectively, *integral*) Robba ring $\mathcal{R}^{\mathrm{bd}}$ (respectively, $\mathcal{R}^{\mathrm{int}}$) as the subring of $\mathcal{R}$ consisting of series with bounded (respectively, integral) coefficients. We equip $\mathcal{R}^{\mathrm{bd}}$ with Gauss norm $|\cdot|_0$ defined by $|\sum_{i \in \mathbb{Z}} a_i t^i|_0 = \sup_{i \in \mathbb{Z}} |a_i|$. Then, the ring $\mathcal{R}^{\mathrm{bd}}$ is a henselian discrete valuation field of mixed characteristic $(0, p)$ with integer ring $\mathcal{R}^{\mathrm{int}}$ and residue field $k((t))$ (see [Ked08, Lemma 3.9]). We define the *Amice ring* $\mathcal{E}$ as the completion of $\mathcal{R}^{\mathrm{bd}}$ for Gauss norm. Explicitly, we have

$$\mathcal{R}^{\mathrm{int}} = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \in \mathcal{R}; a_i \in \mathcal{O}_K \right\},$$

$$\mathcal{R}^{\mathrm{bd}} = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i \in \mathcal{R}; \sup_{i\in\mathbb{Z}} |a_i| < \infty \right\} \cong \mathcal{R}^{\mathrm{int}} \otimes_{\mathcal{O}_K} K,$$

$$\mathcal{E} = \left\{ \sum_{i\in\mathbb{Z}} a_i t^i; \sup_{i\in\mathbb{Z}} |a_i| < \infty,\ \lim_{i\to-\infty} |a_i| = 0 \right\}.$$

(iii) Let $X = t^i v$ be an element of $\mathcal{O}_K[\![t]\!]$ with $i \in \mathbb{N}_{\geqslant 1}$ and $v \in \mathcal{O}_K[\![t]\!]^\times$ such that $\varphi_K(v|_{t=0})/(v|_{t=0})^q \in 1 + \mathfrak{m}_K$ (this always holds when φ_K is a q-power Frobenius lift). We define $\mathcal{R}_{\log}$ as the polynomial ring $\mathcal{R}[\ell_X]$ with a variable ℓ_X called the *branch of log* associated to X. We will endow $\mathcal{R}_{\log}$ with extra structures by regarding ℓ_X as $\log X$ (Definitions 2.2 and 4.2).

Definition 2.2 [Ked08, Definition 1.2.1]. A *relative* (q*-power*) *Frobenius lift* on $\mathcal{R}$ is a ring endomorphism $\varphi : \mathcal{R} \to \mathcal{R}$ of the form $\sum_{i\in\mathbb{Z}} a_i t^i \mapsto \sum_{i\in\mathbb{Z}} \varphi_K(a_i) S^i$ for some $S \in \mathcal{R}^{\mathrm{bd}}$ satisfying $|S - t^q|_0 < 1$. We show that φ naturally induces ring endomorphisms on $\mathcal{R}^{\mathrm{int}}, \mathcal{R}^{\mathrm{bd}}, \mathcal{R}_{\log}$, and $\mathcal{E}$, moreover, on $K[\![t]\!]_0, K\{t\}$ if $S \in K[\![t]\!]_0$, which we denote by φ for notational simplicity. Note that $\varphi(X)/X^q \in 1 + \mathfrak{m}_K\mathcal{R}^{\mathrm{int}}$ by $\varphi(t)/t^q, \varphi(v)/v^q \in 1 + \mathfrak{m}_K\mathcal{R}^{\mathrm{int}}$. We extend φ to $\mathcal{R}_{\log}$ by setting $\varphi(\ell_X) = \log(\varphi(X)/X^q) + q\ell_X$, where $\log : 1 + \mathfrak{m}_K\mathcal{R}^{\mathrm{int}} \to \mathcal{R}^{\mathrm{bd}}; f \mapsto \sum_{n=1}^{\infty}(-1)^{n-1}(f-1)^n/n$. Obviously, φ restricts to $\mathcal{R}^{\mathrm{int}}, \mathcal{R}^{\mathrm{bd}}$, and to $K[\![t]\!]_0, K\{t\}$ if $S \in K[\![t]\!]_0$. Note that $|\varphi(f)|_0 = |f|_0$ for $f \in \mathcal{R}^{\mathrm{bd}}$. We define $\varphi : \mathcal{E} \to \mathcal{E}$ as the completion of $\varphi : \mathcal{R}^{\mathrm{bd}} \to \mathcal{R}^{\mathrm{bd}}$ for Gauss norm.

Convention 1. In the rest of the paper, unless otherwise mentioned, we fix a relative Frobenius lift φ on $\mathcal{R}$, and a branch ℓ_X of log. When we speak of $K[\![t]\!]_0$ and $K\{t\}$, we tacitly assume that $\varphi(t) = S \in K[\![t]\!]_0$. Moreover, we apply results of § 1 to φ-modules over K, $\mathcal{E}$, and $\tilde{\mathcal{E}}$ (defined in § 2.2) with respect to the triples (K, φ_K, q), $(\mathcal{E}, \varphi, q)$, and $(\tilde{\mathcal{E}}, \varphi, q)$, respectively.

2.2 The extended Robba ring

Throughout this subsection, we make the following assumption.

Assumption 2.3. We assume that k is strongly difference-closed. As K is inversive [Liu13, Lemma 1.3.2], our assumption coincides with [Ked08, Hypothesis 2.1.1].

Definition 2.4 ([Ked08, Definition 2.2.4 and Remark 2.2.5], [Liu13, Definition 1.4.1]). For $r > 0$, let $\tilde{\mathcal{R}}^r$ be the abelian group of formal sums $\sum_{i\in\mathbb{Q}} a_i u^i$ with $a_i \in K$ satisfying the following conditions.

- (a) For each $c > 0$, the set of $i \in \mathbb{Q}$ such that $|a_i| \geqslant c$ is well-ordered.
- (b) We have $\lim_{i\to-\infty} |a_i| e^{-ri} = 0$.
- (c) We have $\sup_{i\in(-\infty,+\infty)} |a_i| e^{-ri} < \infty$.
- (d) For all $s > 0$, we have $\lim_{i\to+\infty} |a_i| e^{-si} = 0$.

The group $\tilde{\mathcal{R}}^r$ forms a ring with multiplication given by convolution. We call the union $\tilde{\mathcal{R}} = \bigcup_{r\in(0,+\infty)} \tilde{\mathcal{R}}^r$ the *extended Robba ring* over K, which is a Bézout domain. In the rest of this subsection, when we write $\sum_{i\in\mathbb{Q}} a_i u^i$, we tacitly assume that $a_i \in K$ for all $i \in \mathbb{Q}$. We equip $\tilde{\mathcal{R}}^r$

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with the norm $|\sum_{i\in\mathbb{Q}} a_i u^i|_r = \sup_{i\in\mathbb{Q}}\{|a_i|e^{-ri}\}$. We define $\tilde{\mathcal{R}}_{\log}$ as the polynomial ring $\tilde{\mathcal{R}}[\log u]$ over $\tilde{\mathcal{R}}$ with a variable $\log u$.

We define the *bounded* (respectively, *integral*) extended Robba ring $\tilde{\mathcal{R}}^{\mathrm{bd}}$ (respectively, $\tilde{\mathcal{R}}^{\mathrm{int}}$) as the subring of $\tilde{\mathcal{R}}$ consisting of series with bounded (respectively, integral) coefficients. We equip $\tilde{\mathcal{R}}^{\mathrm{bd}}$ with Gauss norm $|\cdot|_0$ defined by $|\sum_{i\in\mathbb{Q}} a_i u^i|_0 = \sup_{i\in\mathbb{Q}} |a_i|$. Then, the ring $\mathcal{R}^{\mathrm{bd}}$ is a henselian discrete valuation field with integer ring $\mathcal{R}^{\mathrm{int}}$ and residue field $k((u^{\mathbb{Q}}))$. Here, $k((u^{\mathbb{Q}}))$ denotes the field of Hahn series over k (see [Ked08, Definition 2.2.1, Notation 2.5.1]). We define $\tilde{\mathcal{E}}$ as the completion of $\tilde{\mathcal{R}}^{\mathrm{bd}}$ for Gauss norm, which coincides with the set of formal sums $\sum_{i\in\mathbb{Q}} a_i u^i$ satisfying condition (a) as previously, $\lim_{i\to-\infty} |a_i| = 0$, and $\sup_{i\in(-\infty,+\infty)} |a_i| < 0$ (see [Liu13, Definition 1.4.4 and Remark 1.4.5]).

We endow $\tilde{\mathcal{R}}$ with the ring automorphism φ defined by $\varphi(\sum_{i\in\mathbb{Q}} a_i u^i) = \sum_{i\in\mathbb{Q}} \varphi_K(a_i) u^{qi}$. Note that $\varphi(\tilde{\mathcal{R}}^{qr}) \subset \tilde{\mathcal{R}}^r$ and $|\varphi(f)|_r = |f|_{qr}$ for $f \in \tilde{\mathcal{R}}^{qr}$. As in Definition 2.2, φ induces ring endomorphisms on $\tilde{\mathcal{R}}^{\mathrm{int}}$, $\tilde{\mathcal{R}}^{\mathrm{bd}}$, $\tilde{\mathcal{E}}$, and $\tilde{\mathcal{R}}_{\log}$ by setting $\varphi(\log u) = q\log u$, which we denote by φ. Moreover, the field $k((u^{\mathbb{Q}}))$ is strongly difference-closed with respect to the ring endomorphism induced by $\varphi : \tilde{\mathcal{R}}^{\mathrm{int}} \to \tilde{\mathcal{R}}^{\mathrm{int}}$ (see [Ked08, Proposition 2.5.5]).

Lemma 2.5. *Let $f = \sum_{i\in\mathbb{Q}} a_i u^i \in \tilde{\mathcal{R}}^{r'}$ with $r' > 0$. Then, for any $r \in (0, r']$ and $i_0 \in \mathbb{Q}$, we have*

$$\sup_{i\in(-\infty,i_0]} |a_i|e^{-ri} \leqslant |f|_{r'} e^{(r'-r)i_0} (< \infty).$$

Proof. The assertion follows from

$$\sup_{i\in(-\infty,i_0]} |a_i|e^{-ri} = \sup_{i\in(-\infty,i_0]} |a_i|e^{-r'i+(r'-r)i} \leqslant \sup_{i\in(-\infty,i_0]} |a_i|e^{-r'i+(r'-r)i_0} \leqslant |f|_{r'} e^{(r'-r)i_0}. \qquad \square$$

Lemma 2.6.

(i) *For $0 < r \leqslant r'$, we have $\tilde{\mathcal{R}}^{r'} \subset \tilde{\mathcal{R}}^r$. Consequently, $\tilde{\mathcal{R}}^{r'}$ is endowed with the family of the norms $\{|\cdot|_r; r \in (0, r']\}$.*

(ii) *(Maximum modulus principle) Let $f \in \tilde{\mathcal{R}}^{r'}$. Then, for any closed interval $I = [r_1, r_2] \subset (0, r']$, we have*

$$\sup_{r\in I} |f|_r = \max\{|f|_{r_1}, |f|_{r_2}\},$$
$$\inf_{r\in I} |f|_r = \min\{|f|_{r_1}, |f|_{r_2}\}.$$

Proof. (i) Let $f = \sum_{i\in\mathbb{Q}} a_i u^i \in \tilde{\mathcal{R}}^{r'}$. It suffices to verify conditions (a)–(d) in Definition 2.4. We consider conditions (a)$'$–(d)$'$ as in Definition 2.4 with r replaced by r', which hold by $f \in \tilde{\mathcal{R}}^{r'}$. By definition, conditions (a) and (d) coincide with conditions (a)$'$ and (d)$'$ respectively, and condition (b) is weaker than condition (b)$'$. In particular, conditions (a), (b), and (d) hold. By condition (d)$'$, there exists $i_0 \in \mathbb{Q}$ such that $\sup_{i\in[i_0,+\infty)} |a_i|e^{-ri} < \infty$. As $\sup_{i\in(-\infty,i_0]} |a_i|e^{-ri} \leqslant |f|_{r'} e^{(r'-r)i_0} < \infty$ by Lemma 2.5, condition (c) holds.

(ii) We have only to prove $\min\{|f|_{r_1}, |f|_{r_2}\} \leqslant |f|_r \leqslant \max\{|f|_{r_1}, |f|_{r_2}\}$ for all $r \in I$. Then, we may assume that f is a monomial, in which case the assertion is obvious. $\square$

Lemma 2.7. *For $f \in \tilde{\mathcal{R}}^r$, we have $f \in \tilde{\mathcal{R}}^{\mathrm{bd}}$ if and only if $\sup_{s\in(0,r]} |f|_s < \infty$; in this case, $\lim_{s\to 0+} |f|_s$ exists and $\lim_{s\to 0+} |f|_s = |f|_0$.*

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Proof. Write $f=\sum_{i\in\mathbb{Q}}a_iu^i$, and put $|f|_0=\sup_{i\in\mathbb{Q}}|a_i|$ (it can be equal to ∞). Then, we have the equality in $\mathbb{R}\cup\{\infty\}$:

$$\sup_{s\in(0,r]}|f|_s=\max\{|f|_0,|f|_r\} \tag{2.7.1}$$

by

$$\sup_{s\in(0,r]}|f|_s=\sup_{s\in(0,r]}\sup_{i\in\mathbb{Q}}|a_i|e^{-si}=\sup_{i\in\mathbb{Q}}\sup_{s\in(0,r]}|a_i|e^{-si}=\sup_{i\in\mathbb{Q}}\max\{|a_i|,|a_i|e^{-ri}\}=\max\{|f|_0,|f|_r\}.$$

The first assertion is an immediate consequence of (2.7.1). We will prove the second assertion. Assume that $\sup_{s\in(0,r]}|f|_s<\infty$. Put $f_+=\sum_{i\geqslant 0}a_iu^i$, $f_-=\sum_{i<0}a_iu^i$. As $|f|_s=\max\{|f_+|_s,|f_-|_s\}$ for all $s\in[0,r]$, we may assume that $f=f_\pm$. In the case $f=f_+$, the function $s\mapsto|f|_s$ on $(0,r]$ is non-increasing and bounded above by $|f|_0$. Hence, $\lim_{s\to0+}|f|_s$ exists and $\lim_{s\to0+}|f|_s\leqslant|f|_0$. We also have $|f|_0\leqslant\lim_{s\to0+}|f|_s$ by (2.7.1), which implies the assertion. In the case $f=f_-$, the function $s\mapsto|f|_s$ on $(0,r]$ is non-decreasing and bounded below by $|f|_0$. Hence, $\lim_{s\to0+}|f|_s$ exists and $|f|_0\leqslant\lim_{s\to0+}|f|_s$. To complete the proof, it suffices to prove that $\lim_{s\to0+}|f|_s\leqslant|f|_0$. By condition (b) in Definition 2.4, there exists $i_0<0$ such that $\sup_{i\in(-\infty,i_0]}|a_i|e^{-ri}\leqslant|f|_0$. Then, for any $s\in(0,r]$, we have

$$\begin{aligned}|f|_s&=\max\left\{\sup_{i\in(-\infty,i_0)}|a_i|e^{-si},\sup_{i\in[i_0,0]}|a_i|e^{-si}\right\}\leqslant\max\left\{\sup_{i\in(-\infty,i_0)}|a_i|e^{-ri},\sup_{i\in[i_0,0]}|a_i|e^{-si_0}\right\}\\&\leqslant\max\{|f|_0,|f|_0e^{-si_0}\}.\end{aligned}$$

By taking the limit $s\to0+$, we obtain the assertion. □

Lemma 2.8. *For any $b\in\mathfrak{m}_K$, the map $\varphi-b:\tilde{\mathcal{R}}^{\mathrm{bd}}\to\tilde{\mathcal{R}}^{\mathrm{bd}}$ is bijective.*

Proof. To prove the injectivity, we suppose, by way of contradiction, that there exists $x\in\tilde{\mathcal{R}}^{\mathrm{bd}}\backslash\{0\}$ such that $\varphi(x)=bx$. Then, $|\varphi(x)|_0=|x|_0=|bx|_0<|x|_0$, which is a contradiction. To prove the surjectivity, we may replace $\varphi-b$ by $1-\varphi_K^{-1}(b)\varphi^{-1}(=\varphi^{-1}\circ(\varphi-b))$. We have only to prove that for a given $f=\sum_{i\in\mathbb{Q}}a_iu^i\in\tilde{\mathcal{R}}^{\mathrm{bd}}$, there exists $g\in\tilde{\mathcal{R}}^{\mathrm{bd}}$ such that $f=(1-\varphi_K^{-1}(b)\varphi^{-1})g$. For $i\in\mathbb{Q}$, we put $b_i:=\sum_{n\in\mathbb{N}}\varphi_K^{-1}(b)\cdot\dots\cdot\varphi_K^{-n}(b)\varphi_K^{-n}(a_{iq^n})\in K$. Since

$$|b_i|\leqslant\sup_{n\in\mathbb{N}}|b|^n|a_{iq^n}|\leqslant\sup_{n\in\mathbb{N}}|a_{iq^n}|\leqslant|f|_0, \tag{2.8.1}$$

the sequence $\{|b_i|\}_{i\in\mathbb{Q}}$ is bounded. Fix $r>0$ such that $f\in\tilde{\mathcal{R}}^r$. We define conditions (a)′–(d)′ as in Definition 2.4 with the a_i's replaced by the b_i's. We claim that conditions (a)′–(d)′ hold. If the claim holds, then $g:=\sum_{i\in\mathbb{Q}}b_iu^i$ belongs to $\tilde{\mathcal{R}}^{\mathrm{bd}}$, and $f=(1-\varphi_K^{-1}(b)\varphi^{-1})g$ by definition, which implies the assertion. Fix $c>0$. Put $I:=\{i\in\mathbb{Q};|a_i|\geqslant c\}$, which is well-ordered by $f\in\tilde{\mathcal{R}}^r$, and put $I':=\{i\in\mathbb{Q};|b_i|\geqslant c\}$. Choose $m\in\mathbb{N}$ sufficiently large such that $|b|^{m+1}\cdot|f|_0<c/2$. If $i\in I'$, then $\max\{|a_i|,|a_{iq}|,\dots,|a_{iq^m}|\}\geqslant c$, i.e. $i\in I\cup\dots\cup q^{-m}I$, by

$$c\leqslant|b_i|\leqslant\max\left\{\sup_{0\leqslant n\leqslant m}|b|^n|a_{iq^n}|,\sup_{m+1\leqslant n}|b|^n|a_{iq^n}|\right\}\leqslant\max\{|a_i|,|a_{iq}|,\dots,|a_{iq^m}|,|b|^{m+1}|f|_0\}.$$

where the second inequality follows from (2.8.1). Hence, $I'\subset I\cup\dots\cup q^{-m}I$, which implies that I' is well-ordered. Thus, condition (a)′ holds. For $i\leqslant0$, we have

$$|b_i|e^{-ri}\leqslant\sup_{n\in\mathbb{N}}|a_{iq^n}|e^{-ri}\leqslant\sup_{n\in\mathbb{N}}|a_{iq^n}|e^{-riq^n}\leqslant\sup_{j\in(-\infty,i]}|a_j|e^{-rj},$$

where the first inequality follows from (2.8.1). Hence, condition (b)$'$ holds by condition (b) in Definition 2.4. Moreover, by (2.8.1) and the displayed inequalities just above,

$$\sup_{i\in\mathbb{Q}} |b_i|e^{-ri} = \max\left\{\sup_{i\in(-\infty,0]} |b_i|e^{-ri}, \sup_{i\in[0,+\infty)} |b_i|e^{-ri}\right\}$$
$$\leqslant \max\left\{\sup_{j\in(-\infty,0]} |a_j|e^{-rj}, \sup_{i\in[0,+\infty)} |b_i|\right\} \leqslant \max\{|f|_r, |f|_0\} < \infty.$$

Hence, condition (c)$'$ holds. Finally, condition (d)$'$ holds by the boundedness of $\{|b_i|\}_{i\in\mathbb{Q}}$. □

Finally, we recall the reverse filtrations of φ-modules over $\tilde{\mathcal{R}}^{\mathrm{bd}}$ constructed in [Liu13].

DEFINITION 2.9 (Cf. Definition 1.8). Let $\tilde{M}$ be a φ-module over $\tilde{\mathcal{R}}^{\mathrm{bd}}$, and d a positive integer. A *d-eigenvector* of $\tilde{M}$ is a nonzero element $v \in \tilde{M}$ such that $\varphi_M^d(v) = cv$ for some $c \in (\tilde{\mathcal{R}}^{\mathrm{bd}})^\times$; we call the quotient $\log|c|_0/\log|q|^d$ the *slope* of v. We also define the notion of d-eigenvectors of $\tilde{\mathcal{R}}_{\log}$ with $\tilde{M}$ replaced by $\tilde{\mathcal{R}}_{\log}$.

PROPOSITION 2.10 [Liu13, Propositions 1.5.4, 1.6.9, and Theorem 1.5.8]. *Let $\tilde{M}$ be a φ-module over $\tilde{\mathcal{R}}^{\mathrm{bd}}$. Then, the reverse filtration of $\tilde{M}_{\tilde{\mathcal{E}}}$ (Definition 1.11) descends to a filtration $S^\bullet(\tilde{M})$ of $\tilde{M}$, which we call the reverse filtration of $\tilde{M}$. Moreover, there exists $d \in \mathbb{N}_{>0}$ such that each graded piece $S^\lambda(\tilde{M})/\bigcup_{\mu<\lambda} S^\mu(\tilde{M})$ admits a basis consisting of d-eigenvectors of slope λ. In particular, $S^{\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})}(\tilde{M})$ admits a basis consisting of d-eigenvectors of slope $\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})$.*

2.3 Embeddings to extended rings

PROPOSITION 2.11. *We choose an extension L of K such that the residue field of L is strongly difference-closed (Remark 1.2). We repeat the construction as in § 2.2 with K replaced by L. Then, there exist a φ-equivariant embedding $\tilde{\psi}_L : \mathcal{R} \hookrightarrow \tilde{\mathcal{R}}_L$ and $r_0 > 0$ such that for any $r \in (0, r_0)$, $\mathcal{R}^r$ maps to $\tilde{\mathcal{R}}_L^r$ preserving $|\cdot|_r$. Moreover, $\mathcal{R}^{\mathrm{int}}$ (respectively, $\mathcal{R}^{\mathrm{bd}}$) maps $\tilde{\mathcal{R}}_L^{\mathrm{int}}$ (respectively, $\tilde{\mathcal{R}}_L^{\mathrm{bd}}$) preserving $|\cdot|_0$.*

Proof. By embedding $\mathcal{R}$ into $\mathcal{R}_L$ via the inclusion, we may assume that $L = K$, in which case the assertion is proved in [Ked08, Proposition 2.2.6] and [Liu13, Remark 1.4.10]. □

LEMMA 2.12. *Let notation be as in Proposition 2.11.*

(i) *There exists a unique $c_0 \in \tilde{\mathcal{R}}_L^{\mathrm{bd}}$ such that $\varphi(c_0) = qc_0 + \tilde{\psi}_L(\log(\varphi(X)/X^q))$.*
(ii) *Let c_0 be as in part (i), and $c_1 \in (K^\times)^{\varphi_K=1}$ (for example, $c_1 = 1$). We extend $\tilde{\psi}_L : \mathcal{R} \hookrightarrow \tilde{\mathcal{R}}_L$ to $\mathcal{R}_{\log} \to \tilde{\mathcal{R}}_{L,\log}$ by setting $\tilde{\psi}_L(\ell_X) := c_0 + c_1 \log u$. Then, $\tilde{\psi}_L$ is injective and φ-equivariant.*

Proof. As $\tilde{\psi}_L(\log(\varphi(X)/X^q)) \in \tilde{\mathcal{R}}_L^{\mathrm{bd}}$ by Definitions 2.1(ii), 2.2, and Proposition 2.11, part (i) follows from Lemma 2.8 with $b = q$. Part (ii) follows from a direct computation. □

Notation 2. In the rest of § 2, let $L, \tilde{\psi}_L, r_0, c_0$, and c_1 be as previously. By abuse of notation, let $\tilde{\psi}_L$ denote the embedding $\mathcal{E} \hookrightarrow \tilde{\mathcal{E}}_L$ obtained by completing $\tilde{\psi}_L : \mathcal{R}^{\mathrm{bd}} \hookrightarrow \tilde{\mathcal{R}}_L^{\mathrm{bd}}$ for Gauss norms.

Lemma 2.13.

(i) *[[Ked08](#), Propositions 3.2.4 and 3.5.2] The multiplication map* $\tilde{\mathcal{R}}_L^{\mathrm{bd}} \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R} \to \tilde{\mathcal{R}}_L; x \otimes y \mapsto x\tilde{\psi}_L(y)$ *is injective and* φ*-equivariant.*

(ii) *The multiplication map* $\tilde{\mathcal{R}}_L^{\mathrm{bd}} \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R}_{\log} \to \tilde{\mathcal{R}}_{L,\log}; x \otimes y \mapsto x\tilde{\psi}_L(y)$ *is injective and* φ*-equivariant.*

(iii) *(Cf. [deJ98, Proposition 8.1]) The multiplication map* $\tilde{\mathcal{R}}_L^{\mathrm{bd}} \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{E} \to \tilde{\mathcal{E}}_L; x \otimes y \mapsto x\tilde{\psi}_L(y)$ *is injective and* φ*-equivariant.*

Proof. (ii) We have only to prove the injectivity. Assume that $z \in \tilde{\mathcal{R}}_L^{\mathrm{bd}} \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R}_{\log}$ belongs to the kernel of the multiplication map. Write $z = \sum_{i=0}^{n} z_i(1 \otimes \ell_X)^i$ with $z_i \in \tilde{\mathcal{R}}_L^{\mathrm{bd}} \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R}$. Then, z_n belongs to the kernel of the multiplication map in part (i) by assumption, hence, $z_n = 0$. By repeating this argument, we obtain $z = 0$.

(iii) We claim that there exists a $\mathbb{Q}_p$-linear map $f_\eta : \tilde{\mathcal{E}}_L \to \mathcal{E}$ such that:

(a)′ we have $f_\eta(\tilde{\psi}_L(y)w) = yf_\eta(w)$ for $y \in \mathcal{E}$, $w \in \tilde{\mathcal{E}}_L$.

Let $f : \tilde{\mathcal{R}}_L \to \mathcal{R}$ be a natural projection as in [Ked08, Definition 3.5.1]: recall that f restricts to $\tilde{\mathcal{R}}_L^r \to \mathcal{R}^r$ for any $r \in (0, r_0)$, and:

(a) we have $f(\tilde{\psi}_L(y)w) = yf(w)$ for $y \in \mathcal{R}^{\mathrm{bd}}$, $w \in \tilde{\mathcal{R}}_L$;

(b) we have $|w|_r = \sup_{\alpha \in [0,1), a \in L^\times} \{|a|^{-1}e^{-\alpha}|f(au^{-\alpha}w)|_r\}$ for $w \in \tilde{\mathcal{R}}_L^r$,

Fix $w \in \tilde{\mathcal{R}}_L^{\mathrm{bd}}$. We choose $r_1 \in (0, r_0)$ such that $w \in \tilde{\mathcal{R}}_L^{r_1}$. By part (b) with $\alpha = 0$ and $a = 1$ on the right-hand side, we obtain $|f(w)|_r \leqslant |w|_r$ for all $r \in (0, r_1]$. By Lemma 2.7, we have $f(w) \in \mathcal{R}^{\mathrm{bd}}$, and $|f(w)|_0 = \lim_{r \to 0+} |f(w)|_r \leqslant \lim_{r \to 0+} |w|_r = |w|_0$. Hence, f restricts to a $\mathbb{Q}_p$-linear map $\tilde{\mathcal{R}}_L^{\mathrm{bd}} \to \mathcal{R}^{\mathrm{bd}}$ bounded for Gauss norms. By taking the completion, we obtain a $\mathbb{Q}_p$-linear map $f_\eta : \tilde{\mathcal{E}}_L \to \mathcal{E}$, which satisfies part (a)′ by definition. Thus, we obtain the claim.

We suppose, by way of contradiction, that there exists a nonzero element z belonging to the kernel of the multiplication map. Write $z = \sum_{i=1}^{n} x_i \otimes y_i \in \tilde{\mathcal{R}}_L^{\mathrm{bd}} \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{E}$ with n minimal. Then, $y_1, \ldots, y_n$ are linearly independent over $\mathcal{R}^{\mathrm{bd}}$ by [Ked08, Corollary 3.4.3]. As $x_1 \neq 0$, there exist $\alpha \in (0, 1]$ and $a \in L^\times$ such that $f(au^{-\alpha}x_1) \neq 0$ by part (b). By applying f_η to $\sum_{i=1}^{n} au^{-\alpha}x_i \cdot \tilde{\psi}_L(y_i) = 0$ and then using part (a)′, we have $\sum_{i=1}^{n} f(au^{-\alpha}x_i) \cdot y_i = 0$, which is a nontrivial relation between the elements y_i over $\mathcal{R}^{\mathrm{bd}}$. This contradicts the choices of elements y_i. □

The following lemma will only be used in the proof of Proposition 9.6. Hence, the reader may skip it if they are interested only in the proof of part (ii) of Theorem 0.1.

Lemma 2.14 (Cf. [CT11, Lemma 5.7]). *Let* M *be a nonzero* φ*-module over* $\mathcal{R}^{\mathrm{bd}}$*, and* Q *a* φ*-module over* $\mathcal{E}$ *which is pure of slope* μ*. If there exists an* $\mathcal{R}^{\mathrm{bd}}$*-linear map* $f : M \to Q$ *which is injective and* φ*-equivariant, then* $\lambda_{\max}(M_{\mathcal{E}}) = \mu$*.*

Proof. Put $\tilde{M} := M \otimes_{\mathcal{R}^{\mathrm{bd}}} \tilde{\mathcal{R}}_L^{\mathrm{bd}}$ (respectively, $\tilde{Q} := Q \otimes_{\mathcal{E}} \tilde{\mathcal{E}}_L$), which is regarded as a φ-module over $\tilde{\mathcal{R}}_L^{\mathrm{bd}}$ (respectively, $\tilde{\mathcal{E}}_L$). We define the $\tilde{\mathcal{R}}^{\mathrm{bd}}$-linear map $\tilde{f} : \tilde{M} \to \tilde{Q}$ as the composition of the maps

$$M \otimes_{\mathcal{R}^{\mathrm{bd}}} \tilde{\mathcal{R}}_L^{\mathrm{bd}} \to Q \otimes_{\mathcal{E}} \mathcal{E} \otimes_{\mathcal{R}^{\mathrm{bd}}} \tilde{\mathcal{R}}_L^{\mathrm{bd}}; m \otimes x \mapsto f(m) \otimes 1 \otimes x,$$

$$Q \otimes_{\mathcal{E}} \mathcal{E} \otimes_{\mathcal{R}^{\mathrm{bd}}} \tilde{\mathcal{R}}_L^{\mathrm{bd}} \to Q \otimes_{\mathcal{E}} \tilde{\mathcal{E}}_L; w \otimes y \otimes x \mapsto w \otimes x\tilde{\psi}_L(y).$$

Then, $\tilde{f}$ is injective and φ-equivariant by assumption and Lemma 2.13(iii). By Proposition 2.10, there exists a d-eigenvector $\tilde{v}\in\tilde{M}$ of slope $\lambda_{\max}(M_{\mathcal{E}})(=\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}}))$ for some $d>0$. Hence, $\tilde{f}(\tilde{v})\in\tilde{Q}$ is also a d-eigenvector of slope $\lambda_{\max}(M_{\mathcal{E}})$ (in the sense of Definition 1.8). Therefore, the assertion follows from Lemma 1.9. □

3. Logarithmic growth filtration

In § 3.1 (respectively, § 3.2), we define the log-growth filtrations of the rings $\mathcal{R},\mathcal{R}_{\log}$ (respectively, $\tilde{\mathcal{R}}$, $\tilde{\mathcal{R}}_{\log}$). In § 3.3, we study, in the viewpoint of log-growth, the images of certain homomorphisms of the forms $M\to\mathcal{R}_{\log}$ and $\tilde{M}\to\tilde{\mathcal{R}}_{\log}$, where M and $\tilde{M}$ denote φ-modules over $\mathcal{R}^{\mathrm{bd}}$ and $\tilde{\mathcal{R}}^{\mathrm{bd}}$ respectively.

3.1 Logarithmic growth filtrations of $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}_{\log}$

In this subsection, we keep Assumption 2.3.

DEFINITION 3.1. Let $f\in\tilde{\mathcal{R}}$ and $\lambda\in\mathbb{R}_{\geqslant 0}$. We choose $r'\in\mathbb{R}_{>0}$ such that $f\in\tilde{\mathcal{R}}^{r'}$. We say that f has *log-growth* λ if there exists a constant C such that

$$r^{\lambda}|f|_r\leqslant C \text{ for all } r\in(0,r'].$$

The definition does not depend on the choice of r' by Lemma 2.6(ii). We define the λth *log-growth filtration* $\mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}$ as the K-subspace of $\tilde{\mathcal{R}}$ consisting of series having log-growth λ. For $\lambda\in\mathbb{R}_{<0}$, we set $\mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}=0$. Note that $\mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}$ for $\lambda>0$ is not closed under multiplication, hence, not a subring of $\tilde{\mathcal{R}}$.

We define the *log-growth filtration* of $\mathcal{R}_{\log}$ by

$$\mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log}=\bigoplus_{n=0}^{\infty}\mathrm{Fil}_{\lambda-n}\tilde{\mathcal{R}}\cdot(\log u)^n\subset\tilde{\mathcal{R}}_{\log}\quad\text{for } \lambda\in\mathbb{R}.$$

Note that we have $\mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log}=0$ for $\lambda<0$ by definition. We say that $f\in\tilde{\mathcal{R}}_{\log}$ has *log-growth* λ if $f\in\mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log}$. Furthermore, if $f\notin\mathrm{Fil}_{\mu}\tilde{\mathcal{R}}_{\log}$ for all $\mu<\lambda$, then we say that f has *exact* log-growth λ.

This definition may be rephrased in terms of the behavior of the norms of the coefficients of $f\in\tilde{\mathcal{R}}$.

LEMMA 3.2. *Let $f=\sum_{i\in\mathbb{Q}}a_iu^i\in\tilde{\mathcal{R}}$ with $a_i\in K$, and $\lambda\in\mathbb{R}_{\geqslant 0}$. Then, the following are equivalent.*

(i) *We have $f\in\mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}$.*
(ii) *We have $|a_i|=O(i^{\lambda})$ as $i\to+\infty$.*

Proof. In the following, fix $r'>0$ such that $f\in\tilde{\mathcal{R}}^{r'}$. Assume that condition (i) holds, i.e. there exists $C\in\mathbb{R}$ such that $r^{\lambda}|f|_r\leqslant C$ for $r\in(0,r']$. For arbitrary $i\in\mathbb{Q}$ and $r\in(0,r']$, we have

$$|a_i|=r^{\lambda}|a_i|e^{-ri}e^{ri}/r^{\lambda}\leqslant r^{\lambda}|f|_re^{ri}/r^{\lambda}\leqslant Ce^{ri}/r^{\lambda}. \tag{3.2.1}$$

Hence, $|a_i|\leqslant Ce\cdot i^{\lambda}$ for all $i\geqslant 1/r'$ by (3.2.1) with $r=1/i$. Thus, condition (ii) holds.

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Assume that condition (ii) holds, i.e. there exist $C \in \mathbb{R}$ and $i_0 > 0$ such that $|a_i| \leqslant Ci^{\lambda}$ for all $i \in [i_0, +\infty)$. If we put $g_1 = \sum_{i\in(-\infty,i_0]} a_i u^i$ and $g_2 = \sum_{i\in[i_0,+\infty)} a_i u^i$, then $g_1, g_2 \in \tilde{\mathcal{R}}^{r'}$ and $|f|_r = \max\{|g_1|_r, |g_2|_r\}$ for all $r \in (0, r']$. Hence, to verify condition (i), we have only to prove $g_1, g_2 \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}$. Since $|g_1|_r \leqslant |g_1|_{r'} e^{(r'-r)i_0} \leqslant |g_1|_{r'} e^{r' i_0}$ for $r \in (0, r']$ by Lemma 2.5, we have $g_1 \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}$. For $r \in (0, r']$, we have

$$r^{\lambda}|g_2|_r = r^{\lambda} \sup_{i\in[i_0,+\infty)} |a_i|e^{-ri} \leqslant C \sup_{i\in[i_0,+\infty)} (ri)^{\lambda} e^{-ri} \leqslant C \sup_{x\in(0,+\infty} x^{\lambda} e^{-x} < \infty,$$

which implies $g_2 \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}$. □

Lemma 3.3 (Cf. [Ohk17, Lemma 4.7]). *Let $R \in \{\tilde{\mathcal{R}}, \tilde{\mathcal{R}}_{\log}\}$.*

(i) *The filtration $\mathrm{Fil}_{\bullet}R$ is an increasing filtration of K-subspaces of R satisfying*

$$\mathrm{Fil}_{\lambda}R \cdot \mathrm{Fil}_{\mu}R \subset \mathrm{Fil}_{\lambda+\mu}R \text{ for } \lambda, \mu \in \mathbb{R}.$$

(ii) *We have $\varphi(\mathrm{Fil}_{\lambda}R) \subset \mathrm{Fil}_{\lambda}R$ for $\lambda \in \mathbb{R}$.*
(iii) *We have $\mathrm{Fil}_0 R = \tilde{\mathcal{R}}^{\mathrm{bd}}$.*

Proof. By definition, we may reduce to the case $R = \tilde{\mathcal{R}}$. Then, part (i) follows from the multiplicativity of $|\cdot|_r$. To prove part (ii), we may assume $\lambda \geqslant 0$. Let $f \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}$. We choose $r' > 0$ such that $f \in \tilde{\mathcal{R}}^{qr'}$. Then, $r^{\lambda}|\varphi(f)|_r = q^{-\lambda}(qr)^{\lambda}|f|_{qr}$ for $r \in (0, r']$, which implies $\varphi(f) \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}$. In part (iii), we have $\tilde{\mathcal{R}}^{\mathrm{bd}} \subset \mathrm{Fil}_0\tilde{\mathcal{R}}$ by Lemma 3.2, and the converse follows from Lemma 2.7. □

3.2 Logarithmic growth filtrations of $\mathcal{R}$ and $\mathcal{R}_{\log}$

Definition 3.4. Let $\star \in \{\,,\log\}$. We define the *log-growth filtration* $\mathrm{Fil}_{\bullet}\mathcal{R}_{\star}$ of $\mathcal{R}_{\star}$ by repeating the construction as in Definition 3.1 after replacing $\tilde{\mathcal{R}}$ and $\log u$ by $\mathcal{R}$ and ℓ_X, respectively. Note that $\mathrm{Fil}_{\bullet}\mathcal{R}_{\star}$ is independent of the choice of φ by definition.

By the following lemma, the embedding $\tilde{\psi}_L : \mathcal{R}_{\star} \to \tilde{\mathcal{R}}_{L,\star}$ for $\star \in \{\,,\log\}$ as in § 2.3 respects the log-growth filtration. Hence, some properties of $\mathrm{Fil}_{\bullet}\tilde{\mathcal{R}}_{L,\star}$ will automatically transmit to $\mathrm{Fil}_{\bullet}\mathcal{R}_{\star}$.

Lemma 3.5. *Let notation be as in § 2.3. Let $\star \in \{\,,\log\}$, let $f \in \mathcal{R}_{\star}$, and $\lambda \in \mathbb{R}$. Then, $f \in \mathrm{Fil}_{\lambda}\mathcal{R}_{\star}$ if and only if $\tilde{\psi}_L(f) \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{L,\star}$.*

Proof. In the case $\star =$ (*empty*), the assertion immediately follows from the fact that $\tilde{\psi}_L : \mathcal{R}^r \to \tilde{\mathcal{R}}^r_L$ preserves $|\cdot|_r$ (Proposition 2.11). In the case $\star = \log$, we write $f = \sum_{i=0}^n f_i \ell_X^i$ with $f_i \in \mathcal{R}$, and prove by induction on n. In the base case $n = 0$, the assertion follows from the previous result. In the induction step, we first assume $f \in \mathrm{Fil}_{\lambda}\mathcal{R}_{\log}$. Then, for all i, we have $f_i \in \mathrm{Fil}_{\lambda-i}\mathcal{R}$ by assumption, which implies $\tilde{\psi}_L(f_i) \in \mathrm{Fil}_{\lambda-i}\tilde{\mathcal{R}}_L$ by the induction hypothesis. As $\tilde{\psi}_L(\ell_X) \in \mathrm{Fil}_1\tilde{\mathcal{R}}_{L,\log}$, we have $\tilde{\psi}_L(f) = \sum_{i=0}^n \tilde{\psi}_L(f_i)\tilde{\psi}_L(\ell_X)^i \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{L,\log}$ by Lemma 3.3(i). Conversely, we assume $\tilde{\psi}_L(f) \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{L,\log}$. As $\tilde{\psi}_L(f) = c_1^n\tilde{\psi}_L(f_n)(\log u)^n + \cdots \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{L,\log}$ with $c_1 \in K^{\times}$ as in Lemma 2.12(ii), we have $c_1^n\tilde{\psi}_L(f_n) \in \mathrm{Fil}_{\lambda-n}\tilde{\mathcal{R}}_L$ by definition. Hence, $\tilde{\psi}_L(f_n) \in \mathrm{Fil}_{\lambda-n}\tilde{\mathcal{R}}_L$, which implies $f_n \in \mathrm{Fil}_{\lambda-n}\mathcal{R}$ by the previous result. By applying the induction hypothesis to $\tilde{\psi}_L(f - f_n\ell_X^n) = \tilde{\psi}_L(f) - \tilde{\psi}_L(f_n)\tilde{\psi}_L(\ell_X)^n \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log}$, we have $f - f_n\ell_X^n \in \mathrm{Fil}_{\lambda}\mathcal{R}_{\log}$, which implies $f = f - f_n\ell_X^n + f_n\ell_X^n \in \mathrm{Fil}_{\lambda}\mathcal{R}_{\log}$. □

Corollary 3.6. *Let $f = \sum_{i\in\mathbb{Z}} a_i t^i \in \mathcal{R}$ with $a_i \in K$, and $\lambda \in \mathbb{R}_{\geqslant 0}$. Then, $f \in \mathrm{Fil}_{\lambda}\mathcal{R}$ if and only if $|a_i| = O(i^{\lambda})$ as $i \to +\infty$.*

Proof. As $\mathrm{Fil}_\bullet\mathcal{R}$ is independent of the choice of φ, we may assume that $\varphi(t) = S = t^q$, in which case $\tilde{\psi}_L : \mathcal{R} \to \tilde{\mathcal{R}}_L$ is regarded as the inclusion. Then, the assertion follows from Lemmas 3.2 and 3.5. An alternative proof is given by simply repeating the argument as in the proof of Lemma 3.2. □

COROLLARY 3.7. *Let $R \in \{\mathcal{R}, \mathcal{R}_{\log}\}$.*

(i) *The filtration $\mathrm{Fil}_\bullet R$ is an increasing filtration of K-subspaces of R satisfying*

$$\mathrm{Fil}_\lambda R \cdot \mathrm{Fil}_\mu R \subset \mathrm{Fil}_{\lambda+\mu} R \quad \text{for } \lambda, \mu \in \mathbb{R}.$$

(ii) *We have $\varphi(\mathrm{Fil}_\lambda R) \subset \mathrm{Fil}_\lambda R$ for $\lambda \in \mathbb{R}$.*

(iii) *We have $\mathrm{Fil}_0 R = \mathcal{R}^{\mathrm{bd}}$.*

Proof. The assertion is reduced to Lemma 3.3 by Lemma 3.5. □

3.3 Technical results

In this subsection, we keep Assumption 2.3.

LEMMA 3.8 (Cf. [CT11, Lemma 4.8]). *Assume that $g, h \in \tilde{\mathcal{R}}_{\log}$ satisfy a relation of the form*

$$(\varphi^d - c)g = h \quad \text{with } c \in (\tilde{\mathcal{R}}^{\mathrm{bd}})^\times \text{ and } d \in \mathbb{N}_{\geqslant 1}.$$

If we have $h \in \mathrm{Fil}_\lambda \tilde{\mathcal{R}}_{\log}$ for some $\lambda \geqslant \log|c|_0/\log|q|^d$, then $g \in \mathrm{Fil}_\lambda \tilde{\mathcal{R}}_{\log}$.

Proof. After replacing φ by φ^d, we may assume $d = 1$. We reduce to the case where $g, h \in \tilde{\mathcal{R}}$ and $c \in K^\times$. Write $g = \sum_{i=0}^n g_i(\log u)^i$, $h = \sum_{i=0}^n h_i(\log u)^i$ with $g_i, h_i \in \tilde{\mathcal{R}}$. Then, $(\varphi - c/q^i)g_i = h_i/q^i$, and $h_i \in \mathrm{Fil}_{\lambda-i}\tilde{\mathcal{R}}$. Moreover, $g \in \mathrm{Fil}_\lambda \tilde{\mathcal{R}}_{\log}$ if and only if $g_i \in \mathrm{Fil}_{\lambda-i}\tilde{\mathcal{R}}$ for all i. Hence, after replacing (g, h, c, λ) by $(g_i, h_i/q^i, c/q^i, \lambda - i)$, we may assume $g, h \in \tilde{\mathcal{R}}$. We can choose $c' \in K^\times$ such that $|c'| = |c|_0$, since K is discretely valued. As $\tilde{\mathcal{E}}(c'/c)$ is pure of slope 0 as a φ-module over $\tilde{\mathcal{E}}$, there exists $b \in \tilde{\mathcal{E}}^\times$ such that $b = (c'/c)\varphi(b)$ by Dieudonné–Manin classification theorem 1.10 (recall that the residue field $k((t^{\mathbb{Q}}))$ of $\tilde{\mathcal{E}}$ is strongly difference-closed). We have $b \in (\tilde{\mathcal{R}}^{\mathrm{bd}})^\times$ by [Ked08, Proposition 2.5.8], and $(\varphi - c')(g/b) = c'h/cb$. Hence, by Lemma 3.3(i) and (iii), we have $g, h \in \mathrm{Fil}_\lambda \tilde{\mathcal{R}}_{\log}$, respectively, if and only if $g/b, c'h/cb \in \mathrm{Fil}_\lambda \tilde{\mathcal{R}}_{\log}$, respectively. Therefore, after replacing (g, h, c) by $(g/b, c'h/cb, c')$, we may assume $c \in K^\times$ as desired.

We choose $r_0 \in (0, +\infty)$ such that $g, h \in \tilde{\mathcal{R}}^{r_0}$. Then,

$$\sup_{r \in [r_0/q, r_0]} r^\lambda |g|_r \leqslant \sup_{r \in [r_0/q, r_0]} r^\lambda \cdot \sup_{r \in [r_0/q, r_0]} |g|_r < \infty$$

by Lemma 2.6(ii). Hence, we can choose $C \in \mathbb{R}$ sufficiently large such that

$$|g|_r \leqslant C/|c| \cdot r^{-\lambda} \ \forall r \in [r_0/q, r_0],$$
$$|h|_r \leqslant C \cdot r^{-\lambda} \ \forall r \in (0, r_0].$$

Then, it suffices to prove

$$|g|_r \leqslant C/|c| \cdot r^{-\lambda} \ \forall r \in [r_0/q^{n+1}, r_0/q^n]$$

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by induction on $n \in \mathbb{N}$. In the base case $n = 0$, we have nothing to prove. In the induction step, the assertion follows from

$$|cg|_r = |\varphi(g) - h|_r \leqslant \max\{|\varphi(g)|_r, |h|_r\} = \max\{|g|_{qr}, |h|_r\} \leqslant \max\{C/|c| \cdot (qr)^{-\lambda}, Cr^{-\lambda}\} = Cr^{-\lambda}.$$

□

Lemma 3.9 (Cf. [Ohk17, Theorem 6.1]). *If $g \in \tilde{\mathcal{R}}_{\log}$ is a d-eigenvector of slope λ (Definition 2.9), then we have $\lambda \geqslant 0$, and g has exact log-growth λ.*

Proof. By definition, there exists $c \in (\tilde{\mathcal{R}}^{\mathrm{bd}})^{\times}$ such that $(\varphi^d - c)g = 0$ and $\lambda = \log|c|_0/\log|q|^d$. By Lemma 3.8, we have $g \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log}$. Hence, $\lambda \geqslant 0$ by $g \neq 0$. To prove the second assertion, we have only to prove that g does not have log-growth μ for any $\mu < \lambda$. As in the proof of Lemma 3.8, we may assume $d = 1$, $g \in \tilde{\mathcal{R}}$, and $c \in K^{\times}$. We choose $r_0 > 0$ such that $g \in \tilde{\mathcal{R}}^{r_0}$. As we have

$$\inf_{r \in [r_0/q, r_0]} r^{\lambda}|g|_r \geqslant \inf_{r \in [r_0/q, r_0]} r^{\lambda} \cdot \inf_{r \in [r_0/q, r_0]} |g|_r > 0$$

by Lemma 2.6(ii), we can choose $C' > 0$ sufficiently small such that $|g|_r \geqslant C'/|c| \cdot r^{-\lambda}$ for all $r \in [r_0/q, r_0]$. Then, it suffices to prove

$$|g|_r \geqslant C'/|c| \cdot r^{-\lambda} \ \forall r \in [r_0/q^{n+1}, r_0/q^n]$$

by induction on $n \in \mathbb{N}$. In the base case $n = 0$, we have nothing to prove. In the induction step, the assertion follows from $|cg|_r = |\varphi(g)|_r = |g|_{qr} \geqslant C'/|c| \cdot (qr)^{-\lambda} = C'r^{-\lambda}$. □

Proposition 3.10. *Let $\tilde{M}$ be a nonzero φ-module over $\tilde{\mathcal{R}}^{\mathrm{bd}}$. Put $\tilde{M}_{\tilde{\mathcal{E}}} := \tilde{M} \otimes_{\tilde{\mathcal{R}}^{\mathrm{bd}}} \tilde{\mathcal{E}}$, which is regarded as a φ-module over $\tilde{\mathcal{E}}$. Let $\tilde{f} : \tilde{M} \to \tilde{\mathcal{R}}_{\log}$ be an $\tilde{\mathcal{R}}^{\mathrm{bd}}$-linear map such that $\varphi \circ \tilde{f} = c\tilde{f} \circ \varphi_{\tilde{M}}$ for some $c \in (\tilde{\mathcal{R}}^{\mathrm{bd}})^{\times}$. Put $\mu := \log|c|_0/\log|q|$.*

(i) *We have $\tilde{f}(\tilde{M}) \subset \mathrm{Fil}_{\mu + \lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})}\tilde{\mathcal{R}}_{\log}$.*

(ii) *Assume that $\tilde{f}$ is injective. If $\tilde{f}(\tilde{M}) \subset \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log}$, then $\mu + \lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}}) \leqslant \lambda$.*

Proof. We reduce to the case $c = 1$, where $\tilde{f}$ is φ-equivariant, and $\mu = 0$. Put $\tilde{N} = \tilde{M}(c)$, which is a φ-module over $\tilde{\mathcal{R}}^{\mathrm{bd}}$ such that $\lambda_{\max}(\tilde{N}_{\tilde{\mathcal{E}}}) = \mu + \lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})$. We define the $\tilde{\mathcal{R}}^{\mathrm{bd}}$-linear map $\tilde{f}' : \tilde{N} \to \mathcal{R}_{\log}; m \otimes e_c \mapsto \tilde{f}(m)$. It is straightforward to see that $\varphi \circ \tilde{f}' = \tilde{f}' \circ \varphi_{\tilde{N}}$ and $\tilde{f}'(\tilde{N}) = \tilde{f}(\tilde{M})$. Moreover, $\tilde{f}'$ is injective if and only if so is $\tilde{f}$. Hence, after replacing $\tilde{f}$ by $\tilde{f}'$, we may assume $c = 1$ as desired.

(i) Let $F_0 = 0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_m = \tilde{M}$ denote the reverse filtration of $\tilde{M}$ (Proposition 2.10). We have only to prove $\tilde{f}(F_i) \subset \mathrm{Fil}_{\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})}\tilde{\mathcal{R}}_{\log}$ by induction on $i \in \{0, \dots, m\}$. In the base case $i = 0$, the assertion is trivial. In the induction step, for some d, there exists a set of generator of F_{i+1}/F_i consisting of d-eigenvectors of slopes less than or equal to $\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})$. We choose a lift S of such a set to F_{i+1}. For $v \in S$, we have $v' := (\varphi_{\tilde{M}}^d - c')v \in F_i$ for some $c' \in (\tilde{\mathcal{R}}^{\mathrm{bd}})^{\times}$ (depending on v), and $\log|c'|_0/\log|q|^d \leqslant \lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})$. Then, $(\varphi^d - c')(\tilde{f}(v)) = \tilde{f}((\varphi_{\tilde{M}}^d - c')v) = \tilde{f}(v') \in \tilde{f}(F_i) \subset \mathrm{Fil}_{\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})}\tilde{\mathcal{R}}_{\log}$ by the induction hypothesis. Hence, by applying Lemma 3.8 to $(g, h) = (\tilde{f}(v), \tilde{f}(v'))$, we have $\tilde{f}(v) \in \mathrm{Fil}_{\lambda_{\max}(M_{\tilde{\mathcal{E}}})}\tilde{\mathcal{R}}_{\log}$. We complete the proof by $\tilde{f}(F_{i+1}) = \tilde{f}(F_i + \tilde{\mathcal{R}}^{\mathrm{bd}} \cdot S) = \tilde{f}(F_i) + \tilde{\mathcal{R}}^{\mathrm{bd}} \cdot \tilde{f}(S) \subset \mathrm{Fil}_{\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})}\tilde{\mathcal{R}}_{\log}$.

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(ii) By Proposition 2.10, there exists a d-eigenvector $v \in \tilde{M}$ of slope $\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})$. As $\tilde{f}$ is φ-equivariant and injective, $\tilde{f}(v) \in \tilde{\mathcal{R}}_{\log}$ is also a d-eigenvector of slope $\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})$. Hence, $\tilde{f}(v)$ has exact log-growth $\lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})$ by Lemma 3.9. As $\tilde{f}(v) \in \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log}$, we have $\lambda \geqslant \lambda_{\max}(\tilde{M}_{\tilde{\mathcal{E}}})$. □

PROPOSITION 3.11. *Let M be a nonzero φ-module over $\mathcal{R}^{\mathrm{bd}}$. Let $f : M \to \mathcal{R}_{\log}$ be an $\mathcal{R}^{\mathrm{bd}}$-linear map such that $\varphi \circ f = cf \circ \varphi_M$ for some $c \in (\mathcal{R}^{\mathrm{bd}})^{\times}$. Put $\mu := \log |c|_0 / \log |q|$.*

(i) *We have $f(M) \subset \mathrm{Fil}_{\mu+\lambda_{\max}(M_{\mathcal{E}})}\mathcal{R}_{\log}$.*
(ii) *Assume that f is injective. If $f(M) \subset \mathrm{Fil}_{\lambda}\mathcal{R}_{\log}$, then $\mu + \lambda_{\max}(M_{\mathcal{E}}) \leqslant \lambda$.*

Proof. (i) Let $\tilde{\psi}_K : \mathcal{R}_{\log} \to \tilde{\mathcal{R}}_{\log}$ be an embedding as in § 2.3 with $L = K$. Put $\tilde{M} = M \otimes_{\mathcal{R}^{\mathrm{bd}}} \tilde{\mathcal{R}}^{\mathrm{bd}}$, which is regarded as a φ-module over $\tilde{\mathcal{R}}^{\mathrm{bd}}$. Let $\tilde{f} : \tilde{M} \to \tilde{\mathcal{R}}_{\log}$ be the composition

$$M \otimes_{\mathcal{R}^{\mathrm{bd}}} \tilde{\mathcal{R}}^{\mathrm{bd}} \xrightarrow{f \otimes \mathrm{id}_{\tilde{\mathcal{R}}^{\mathrm{bd}}}} \mathcal{R}_{\log} \otimes_{\mathcal{R}^{\mathrm{bd}}} \tilde{\mathcal{R}}^{\mathrm{bd}} \hookrightarrow \tilde{\mathcal{R}}_{\log},$$

where the second morphism is a multiplication map as in Lemma 2.13(ii). Then, $\varphi^d \circ \tilde{f} = c\tilde{f} \circ \varphi^d_{\tilde{M}}$ by assumption. Hence, $\tilde{f}(\tilde{M}) \subset \mathrm{Fil}_{\mu+\lambda_{\max}(M_{\mathcal{E}})}\tilde{\mathcal{R}}_{\log}$ by Proposition 3.10(i). As $\tilde{\psi}_K(f(M)) = \tilde{f}(M \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R}^{\mathrm{bd}}) \subset \tilde{f}(\tilde{M})$ by the definition of $\tilde{f}$, we have $f(M) \subset \mathrm{Fil}_{\mu+\lambda_{\max}(M_{\mathcal{E}})}\mathcal{R}_{\log}$ by Lemma 3.5.

(ii) Let notation be as previously. As f is injective, so is $\tilde{f}$ by definition. Moreover, we have $\tilde{f}(\tilde{M}) \subset \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log}$ by

$$\tilde{f}(\tilde{M}) = \tilde{\psi}_K(f(M)) \cdot \tilde{\mathcal{R}}^{\mathrm{bd}} \subset \tilde{\psi}_K(\mathrm{Fil}_{\lambda}\mathcal{R}_{\log}) \cdot \tilde{\mathcal{R}}^{\mathrm{bd}} \subset \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log} \cdot \mathrm{Fil}_0\tilde{\mathcal{R}}_{\log} \subset \mathrm{Fil}_{\lambda}\tilde{\mathcal{R}}_{\log},$$

where we use Lemmas 3.3(i), 3.3(iii), and 3.5. Hence, we obtain the assertion by Proposition 3.10(ii). □

4. Preliminaries on (φ, ∇)-modules over the Robba ring

We give the formalism of ∇-modules and (ϕ, ∇)-modules over certain rings in § 4.1, and recall basic results on unipotent ∇-modules over $\mathcal{R}$ in § 4.2.

4.1 Definition of (φ, ∇)-modules over the Robba ring

DEFINITION 4.1. Let R be a ring, and $A \subset R$ a subring. Let $\partial : R \to R$ be an A-linear derivation, and x an element of R such that $\partial(x) = 1$.

(i) We define a ∇-*module* over R (relative to A) as a finite free R-module M endowed with an A-linear differential operator ∂_M relative to ∂, that is, $\partial_M : M \to M$ is an A-linear map satisfying $\partial_M(rm) = \partial(r)m + r\partial_M(m)$ for $r \in R$ and $m \in M$. We may regard a ∇-module over R as a differential module over R in the sense of [Ked10, Definition 5.1.2]. We define basic operations in the category of ∇-modules over R, such as the internal Hom $\mathrm{Hom}_R(-,-)$, the tensor product $- \otimes_R -$, and the dual $(-)^{\vee}$ as in [Ked10, 5.3]. We also define the natural pairing $M \otimes_R M^{\vee} \to R; v \otimes f \mapsto f(v)$. We put $M^{\nabla} := \ker \partial_M$, which is regarded as an A-submodule of M.

(ii) Let ϕ be a ring endomorphism on R such that $\partial(\phi(x))\phi \circ \partial = \partial \circ \phi$, and $\phi(A) \subset A$. We define a (ϕ, ∇)-*module* over R (relative to A) as a ϕ-module M over R endowed with an A-linear differential operator ∂_M relative to ∂ satisfying the compatibility condition $\partial(\phi(x)) \cdot \phi_M \circ \partial_M = \partial_M \circ \phi_M$. We define the forgetful functor from the category of (ϕ, ∇)-modules over R to the category of ϕ-modules (respectively, ∇-modules) over R by $(M, \phi_M, \partial_M) \mapsto (M, \phi_M)$ (respectively,

$(M, \phi_M, \partial_M) \mapsto (M, \partial_M)$). In the category of (ϕ, ∇)-modules over R, we can naturally define the internal Hom, the tensor product, the dual, and the natural pairing in such a way that these operations are preserved under the above forgetful functors. Let M be a (ϕ, ∇)-module over R. We define the *d-pushforward* $[d]_*M$ of M to be (ϕ^d, ∇)-module over R given by $(M, \phi_M^d, \partial_M)$. For $c \in A^\times$, let $R(c) = Re_c$ be the (ϕ, ∇)-module over R defined by $\phi_{R(c)}(e_c) = ce_c$ and $\partial_{R(c)}(e_c) = 0$, and $M(c) := M \otimes_R R(c)$. As in part (i), we put $M^\nabla := \ker \partial_M$, which is regarded as a difference module over $A = (A, \phi|_A)$.

Let $(R, A, \partial, \phi, x)$ be a 5-tuple as above. We give a few remarks on base changes of (ϕ, ∇)-modules over R, which also applies to ∇-modules over R by regarding the category of ∇-modules over R as a full subcategory of (id_R, ∇)-modules over R via the correspondence $(M, \partial_M) \mapsto (M, \mathrm{id}_M, \partial_M)$. Let $(R', A', \partial', \phi', x')$ be another 5-tuple as in Definition 4.1, and $\psi : R \to R'$ a ring homomorphism such that $\psi \circ \phi = \phi' \circ \psi$, $\partial' \circ \psi = \psi \circ \partial$, $\psi(x) \in x' + A'$, and $\psi(A) \subset A'$. We define the *base change functor* $\psi^*(-)$ from the category of (ϕ, ∇)-modules over R to the category of (ϕ, ∇)-modules over R' by the correspondence $(M, \phi_M, \partial_M) \mapsto (\psi^* M, \phi_M \otimes \phi', \partial_M \otimes \mathrm{id}_{R'} + \mathrm{id}_M \otimes \partial')$. Note that the base change functor is compatible with compositions. Precisely speaking, let $(R'', A'', \partial'', \phi'', x'')$ be a 5-tuple with a ring homomorphism $\psi' : R' \to R''$ as previously. Then, we have a natural isomorphism $(\psi' \circ \psi)^* \cong (\psi')^* \circ \psi^*$.

Definition 4.2. Let R denote any one of $K[\![t]\!]_0, K\{t\}, \mathcal{R}^{\mathrm{bd}}, \mathcal{R}, \mathcal{R}_{\log}$, and $\mathcal{E}$. Let $\partial := d/dt : R \to R$ be the natural derivation for $R \neq \mathcal{R}_{\log}$. We extend $\partial : R \to R$ to $\mathcal{R}_{\log} \to \mathcal{R}_{\log}$ by setting $\partial(\ell_X) := \partial(X)/X = i/t + \partial(v)/v$ with notation as in Definition 2.1(iii). We can see that $\partial(\varphi(t))\varphi \circ \partial = \partial \circ \varphi$. We define the category of (φ, ∇)-modules over R by applying Definition 4.1(ii) to the 5-tuple $(R, K, \partial, \varphi, t)$. Any inclusion between the rings $K[\![t]\!]_0, K\{t\}, \mathcal{R}^{\mathrm{bd}}, \mathcal{R}, \mathcal{R}_{\log}$, and $\mathcal{E}$ (see the commutative diagram in the Appendix) induces a base change functor between the corresponding categories of (φ, ∇)-modules.

We endow the ring $\mathcal{E}[\![X - t]\!]_0$ with φ and ∂ as in the introduction. Then, we can define the notion of (φ, ∇)-modules over $\mathcal{E}[\![X - t]\!]_0$ as previously. Moreover, the ring homomorphism $\tau : \mathcal{E} \to \mathcal{E}[\![X - t]\!]_0$ as in the introduction induces the base change functor from the category of (φ, ∇)-modules over $\mathcal{E}$ to the category of (φ, ∇)-modules over $\mathcal{E}[\![X - t]\!]_0$.

4.2 Unipotent ∇-modules over the Robba ring

We start with proving that in the category of differential rings, the isomorphism class of the differential ring $(\mathcal{R}_{\log}, \partial)$ is independent of the choice of ℓ_X. Let $\ell_{X'}$ be another branch of log associated to $X' = t^j v'$ with $v' \in \mathcal{O}_K[\![t]\!]^\times$ and $j \in \mathbb{N}_{\geqslant 1}$. Put $c'_1 = i/j$, and $g := \partial(X)/X - c'_1\partial(X')/X' = \partial(v)/v - c'_1\partial(v')/v' \in K[\![t]\!]_0$. Then, there exists $c'_0 \in K\{t\}$ (unique up to modulo K) such that $\partial(c'_0) = g$ by considering an antidifferential of g. Note that we have $c'_0 \in K\{t\} \cap \mathrm{Fil}_1\mathcal{R}$ by $g \in K[\![t]\!]_0$ and Corollary 3.6(or by using [CT09, Proposition 1.2(5)]). We define the $\mathcal{R}$-algebra homomorphism $T_{X,X'} : \mathcal{R}[\ell_X] \to \mathcal{R}[\ell_{X'}]; \ell_X \mapsto c'_0 + c'_1\ell_{X'}$.

Lemma 4.3. *Let notation be as previously.*

(i) *We have the following commutative diagram.*

$$\begin{array}{ccc} \mathcal{R}[\ell_X] & \xrightarrow{\partial} & \mathcal{R}[\ell_X] \\ \downarrow{\scriptstyle T_{X,X'}} & & \downarrow{\scriptstyle T_{X,X'}} \\ \mathcal{R}[\ell_{X'}] & \xrightarrow{\partial} & \mathcal{R}[\ell_{X'}] \end{array}$$

(ii) *The map $T_{X,X'}$ is bijective.*
(iii) *The map $T_{X,X'}$ induces a bijection* $\mathrm{Fil}_\lambda\mathcal{R}[\ell_X] \to \mathrm{Fil}_\lambda\mathcal{R}[\ell_{X'}]$ *for* $\lambda \in \mathbb{R}$.

Proof. Part (i) is verified directly. We define the $\mathcal{R}$-algebra homomorphism $T_{X',X} : \mathcal{R}[\ell_{X'}] \to \mathcal{R}[\ell_X]; \ell_{X'} \mapsto -c_0'/c_1' + 1/c_1' \cdot \ell_X$. Then, $T_{X',X}$ gives an inverse of $T_{X,X'}$, which implies part (ii). Since $T_{X,X'}(\ell_X) \in \mathrm{Fil}_1\mathcal{R}[\ell_{X'}]$, we have $T_{X,X'}(\mathrm{Fil}_\lambda\mathcal{R}[\ell_X]) \subset \mathrm{Fil}_\lambda\mathcal{R}[\ell_{X'}]$ for $\lambda \in \mathbb{R}$. Similarly, we have $T_{X',X}(\mathrm{Fil}_\lambda\mathcal{R}[\ell_{X'}]) \subset \mathrm{Fil}_\lambda\mathcal{R}[\ell_X]$ for $\lambda \in \mathbb{R}$, which implies part (iii). ☐

Lemma 4.4. *The K-linear derivation $\partial : \mathcal{R}_{\log} \to \mathcal{R}_{\log}$ is surjective.*

Proof. By Lemma 4.3(i) and (ii), we may assume $X = t$, in which case the assertion is trivial. ☐

Definition 4.5. Let M be a ∇-module over $\mathcal{R}$. We put $\mathbf{V}(M) := (M \otimes_{\mathcal{R}} \mathcal{R}_{\log})^\nabla$, which is regarded as a vector space over K. We define the *monodromy operator* N_X as the K-linear endomorphism on $\mathbf{V}(M)$ induced by $\mathrm{id}_M \otimes d/d\ell_X$, where $d/d\ell_X : \mathcal{R}_{\log} \to \mathcal{R}_{\log}$ is the unique (locally nilpotent) $\mathcal{R}$-linear derivation sending ℓ_X to 1.

Let $\mathcal{C}$ denote the category of ∇-modules over $\mathcal{R}$. Recall that $\mathcal{C}$ is an abelian tensor category whose objects admit composition series [Cre98, 6.2]. Moreover, the correspondence $M \mapsto \mathbf{V}(M)$ gives a left-exact covariant functor from $\mathcal{C}$ to the category of vector spaces over K. Recall that $M \in \mathcal{C}$ is *trivial* (respectively, *unipotent*) if M is isomorphic to a finite direct sum (respectively, a successive extension) of copies of $\mathcal{R}$, where $\mathcal{R}$ is equipped with the trivial differential operator ∂ (see [Ked05a, Definition 4.27]). Then, the unipotent objects form an abelian full subcategory $\mathcal{C}^{\mathrm{unip}}$ of $\mathcal{C}$, which is closed under formation of direct sums, tensor products, duals, subobjects, quotients, and extensions [Ked04, Definition 4.27].

Lemma 4.6. *Let M be a ∇-module over $\mathcal{R}$.*

(i) *We have $M^\nabla = 0$ if and only if $\mathbf{V}(M) = 0$.*
(ii) *If M is irreducible in $\mathcal{C}$, then we have either $\mathbf{V}(M) = 0$ or $M \cong \mathcal{R}$.*
(iii) *The canonical maps*

$$M^\nabla \otimes_K \mathcal{R} \to M, \quad \mathbf{V}(M) \otimes_K \mathcal{R}_{\log} \to M \otimes_{\mathcal{R}} \mathcal{R}_{\log}$$

are injective. In particular, $\dim_K \mathbf{V}(M) \leqslant \mathrm{rank}_{\mathcal{R}} M$, *and N_X is nilpotent.*

Proof. Part (i) follows from $\mathbf{V}(M)^{N_X=0} = M^\nabla$ and the local nilpotency of N_X. Part (ii) can be proved by using part (i) and a canonical isomorphism $M^\nabla \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{R}, M)$ (see [Ked10, Deifnition 5.3.2]). To prove part (iii), by dévissage, we may assume that M is irreducible in $\mathcal{C}$, in which case the assertion immediately follows from parts (i) and (ii). ☐

Corollary 4.7. *The restriction of the functor $\mathbf{V}(-)$ to $\mathcal{C}^{\mathrm{unip}}$ is faithfully exact, compatible with tensor products and direct sums, and preserves the rank. Moreover, if $M \in \mathcal{C}^{\mathrm{unip}}$, then there exists a functorial isomorphism $\mathbf{V}(M) \otimes_K \mathcal{R}_{\log} \cong M \otimes_{\mathcal{R}} \mathcal{R}_{\log}$ of ∇-modules over $\mathcal{R}_{\log}$.*

Proof. Note that if $M \in \mathcal{C}^{\mathrm{unip}}$, then the differential operator $\partial_{M\otimes_{\mathcal{R}}\mathcal{R}_{\log}} : M \otimes_{\mathcal{R}} \mathcal{R}_{\log} \to M \otimes_{\mathcal{R}} \mathcal{R}_{\log}$ is surjective, which is easily seen by reducing to Lemma 4.4. This fact implies the faithful exactness of the restriction of $\mathbf{V}(-)$ to $\mathcal{C}^{\mathrm{unip}}$. Hence, by dévissage, we can easily see that $\mathbf{V}(M) \otimes_K \mathcal{R}_{\log} \cong M \otimes_{\mathcal{R}} \mathcal{R}_{\log}$ for $M \in \mathcal{C}^{\mathrm{unip}}$, which implies the rest of the assertion. ☐

Lemma 4.8. *Let M be a ∇-module over $\mathcal{R}$.*

(i) *There exists a maximum unipotent subobject $U(M)$ of M. Moreover, we have a canonical isomorphism $\mathbf{V}(U(M)) \cong \mathbf{V}(M)$, and $\mathbf{V}(M/U(M)) = 0$.*
(ii) *Let $n := \mathrm{rank}_{\mathcal{R}} M$. Then, M is unipotent (respectively, trivial) if and only if $\dim_K \mathbf{V}(M) = n$ (respectively, $\dim_K M^{\nabla} = n$).*

Proof. (i) Let N is any maximal unipotent subobject of M. If N' is a unipotent subobject of M, then $N + N'$ is also unipotent, hence, $N + N' = N$, i.e. $N' \subset N$. Hence, N is a maximum unipotent subobject of M. For the rest of the assertion, it suffices to prove that $\mathbf{V}(M/U(M)) = 0$. The inverse image U' of $U(M/U(M))$ under the projection $M \to M/U(M)$ is a unipotent subobject of M. Hence, $U' = U(M)$, that is, $U(M/U(M)) = 0$. Therefore, any morphism $\mathcal{R} \to M/U(M)$ of ∇-modules over $\mathcal{R}$ has image 0, i.e.$(M/U(M))^{\nabla} \cong \mathrm{Hom}_{\mathcal{C}}(\mathcal{R}, M/U(M)) = 0$, which implies the assertion by Lemma 4.6(ii).

(ii) As $\dim_K \mathbf{V}(M) = \mathrm{rank}_{\mathcal{R}} U(M)$ by part (i) and Corollary 4.7, M is unipotent if and only if $\dim_K \mathbf{V}(M) = n$. If M is trivial, then we obviously have $\dim_K M^{\nabla} = n$. Conversely, if $\dim_K M^{\nabla} = n$, then M is trivial since the injection $M^{\nabla} \otimes_K \mathcal{R} \hookrightarrow M$ in Lemma 4.6(iii) is an isomorphism by comparing ranks. □

In the rest of this subsection, we study the behavior of ∇-modules over $\mathcal{R}$ under base changes of the coefficient field K.

Notation 3. Let $(K', \varphi_{K'}, q)$ be an extension of (K, φ_K, q) in the sense of Definition 1.1. We put $\mathcal{R}_{K',\log} = \mathcal{R}_{K'}[\ell_X]$, where we regard X as an element of $\mathcal{O}_{K'}[\![t]\!]$. By abuse of notation, we denote the inclusions $\mathcal{R} \to \mathcal{R}_{K'}$ and $\mathcal{R}_{\log} \to \mathcal{R}_{K',\log}$ by ψ.

Lemma 4.9. *Let M be a ∇-module over $\mathcal{R}$. If $M^{\nabla} = 0$, then $(\psi^* M)^{\nabla} = 0$.*

Proof. Assume that $(\psi^* M) \neq 0$. We prove that $M^{\nabla} \neq 0$. We extend $\mathrm{id}_K : K \to K$ to a bounded K-linear map $\chi : K' \to K$ by the p-adic Hahn–Banach theorem [Rob00, 4.7]. We define the $\mathcal{R}$-linear map $\chi' : \mathcal{R}_{K'} \to \mathcal{R}; \sum_{i\in\mathbb{Z}} x_i t^i \mapsto \sum_{i\in\mathbb{Z}} \chi(x_i) t^i$. Then, the map $M \otimes_{\mathcal{R}} \mathcal{R}_{K'} \to M; m \otimes r' \mapsto \chi'(r')m$ commutes with differential operators. Thus, we obtain a map $\chi'' : (\psi^* M)^{\nabla} \to M^{\nabla}$. We have only to prove $\chi'' \neq 0$. Let x be a nonzero element of $(\psi^* M)^{\nabla}$, and $e_1, \ldots, e_n$ a basis of M. We write $x = e_1 \otimes r_1 + \cdots + e_n \otimes r_n$ with $r_j = \sum_{i\in\mathbb{Z}} r_{ij} t^i \in \mathcal{R}_{K'}$. By $x \neq 0$, we have $r_{ij} \in (K')^{\times}$ for some i, j. Then, $\chi''(x/r_{ij}) = \chi'(r_1/r_{ij})e_1 + \cdots + \chi'(r_j/r_{ij})e_j + \cdots + \chi'(r_n/r_{ij})e_n$ is nonzero because the coefficient of t^i in $\chi'(r_j/r_{ij})$ is equal to 1. □

Proposition 4.10. *Let M be a ∇-module over $\mathcal{R}$. Then, the canonical maps*

$$M^{\nabla} \otimes_K K' \to (\psi^* M)^{\nabla}, \quad \mathbf{V}(M) \otimes_K K' \to \mathbf{V}(\psi^* M)$$

are isomorphisms. In particular, M is unipotent (respectively, trivial) if and only if $\psi^* M$ is unipotent (respectively, trivial).

Proof. Since $M^{\nabla} \cong \mathbf{V}(M)^{N_X=0}$ and $(\psi^* M)^{\nabla} \cong \mathbf{V}(\psi^* M)^{N_X=0}$, we have only to prove that the second map is an isomorphism. We have $\mathbf{V}(M/U(M)) = 0$ by Lemma 4.8(i), hence, $\mathbf{V}(\psi^* M/\psi^*(U(M))) = 0$ by Lemma 4.9. Therefore, we have a canonical isomorphism $\mathbf{V}(\psi^*(U(M))) \cong \mathbf{V}(\psi^* M)$. Hence, after replacing M by $U(M)$, we may assume that $M \in \mathcal{C}^{\mathrm{unip}}$. Then, by dévissage, we may reduce to the case $M = \mathcal{R}$, in which case the assertion is trivial. □

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5. Logarithmic growth filtrations for (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$

Assumption 5.1. In the rest of the paper, we assume that φ_K is a q-power Frobenius lift on K. We also fix an extension L of K such that the residue field of L is algebraically closed (Remark 1.2). Note that the residue field of L is strongly difference-closed [Ked10, Proposition 14.3.4], i.e. satisfies Assumption 2.3.

Convention 2. In the rest of this paper, unless otherwise is mentioned, a subquotient of a (φ, ∇)-module M over $\mathcal{R}^{\mathrm{bd}}$ (respectively, $\mathcal{E}$) means a subquotient of M in the category of (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$ (respectively, $\mathcal{E}$).

In the following, we study (φ, ∇)-modules M over $\mathcal{R}^{\mathrm{bd}}$ such that $M_{\mathcal{R}}$ is unipotent as a ∇-module over $\mathcal{R}$. Note that these M form an abelian full subcategory of the category of (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$, which is closed under formation of direct sums, extensions, internal Homs, duals, and tensor products. We define $V(M)$ and $\mathrm{Sol}(M)$, which are φ-modules over K called the sets of *analytic horizontal sections* and *analytic solutions*, respectively, equipped with a perfect pairing $V(M) \otimes_K \mathrm{Sol}(M) \to K$ called the *canonical pairing*. We put

$$V(M) := (M \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R}_{\log})^{\nabla},$$
$$\mathrm{Sol}(M) := \{f \in \mathrm{Hom}_{\mathcal{R}^{\mathrm{bd}}}(M, \mathcal{R}_{\log}); \partial \circ f = f \circ \partial_M\}.$$

Then, both $V(M)$ and $\mathrm{Sol}(M)$ are regarded as φ-modules over K, where $\varphi_{\mathrm{Sol}(M)}$ is defined as the unique φ_K-semilinear map satisfying $\varphi_{\mathrm{Sol}(M)}(f)(\varphi_M(m)) = \varphi(f(m))$ for $f \in \mathrm{Sol}(M)$ and $m \in M$. We define the canonical pairing as that induced by the K-bilinear map

$$V(M) \times \mathrm{Sol}(M) \to \mathcal{R}_{\log}^{\nabla} \cong K; (m, f) \mapsto f'(m),$$

where f' denotes the linear extension of f to $M \otimes_{\mathcal{R}^{\mathrm{bd}}} \mathcal{R}_{\log} \to \mathcal{R}_{\log}$. It is straightforward to check that the canonical pairing is perfect, and $V(-)$ (respectively, $\mathrm{Sol}(-)$) forms a covariant (respectively, contravariant) functor, which is faithfully exact, compatible with direct sums and tensor products, and preserves the rank. In the following, unless otherwise mentioned, when S is a subset of $\mathrm{Sol}(M)$, let $S^{\perp}$ denote the orthogonal part of S with respect to the above bilinear map $V(M) \times \mathrm{Sol}(M) \to K$.

We define the *Frobenius slope filtration* $S_{\bullet}(V(M))$ and $S_{\bullet}(\mathrm{Sol}(M))$ of $V(M)$ and $\mathrm{Sol}(M)$, respectively, as in § 1. For $\lambda \in \mathbb{R}$, we have the natural duality

$$\bigcup_{\lambda < \mu} S_{-\mu}(V(M)) = S_{\lambda}(\mathrm{Sol}(M))^{\perp}.$$

We define the *growth filtrations* $\mathrm{Sol}_{\bullet}(M)$ and $V(M)^{\bullet}$ of $\mathrm{Sol}(M)$ and $V(M)$, respectively, by $\mathrm{Sol}_{\lambda}(M) := \{f \in \mathrm{Sol}(M); f(M) \subset \mathrm{Fil}_{\lambda}\mathcal{R}_{\log}\}$ and $V(M)^{\lambda} = \mathrm{Sol}_{\lambda}(M)^{\perp}$ for $\lambda \in \mathbb{R}$. As in the literature, we may call $V(M)^{\bullet}$ the *special log-growth filtration* of M. Then, $V(M)^{\bullet}$ (respectively, $\mathrm{Sol}_{\bullet}(M)$) is a decreasing (respectively, increasing) filtration of φ-submodules of $V(M)$ (respectively, $\mathrm{Sol}(M)$) by Corollary 3.7(ii), and $V(-)/V(-)^{\lambda}$ (respectively, $\mathrm{Sol}_{\lambda}(-)$) is regarded as a right-exact quotient functor (respectively, left-exact subfunctor) of $V(-)$ (respectively, $\mathrm{Sol}(-)$), which commutes with direct sums and d-pushforwards.

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Remark 5.2 (Independence of the choice of the branch of log). As a consequence of Lemma 4.3, the isomorphism class of $V(M)^\bullet$ (respectively, $\mathrm{Sol}_\bullet(M)$) in the category of vector spaces over K with decreasing (respectively, increasing) filtrations is independent of the choice of the branch ℓ_X of log. Note that the growth filtrations for M in the case $X = t$ are already studied in [Ohk17], where $\mathcal{R}_{\log}, V(M)^{(\bullet)}$, and $\mathrm{Sol}_{(\bullet)}(M)$ are denoted by $\Gamma_{\log,\mathrm{an},\mathrm{con}}, \mathcal{V}(M)^{(\bullet)}$, and $\mathfrak{Sol}_{(\bullet)}(M)$, respectively.

First proof of part (i) of Theorem 0.1. By Remark 5.2, we may assume that $X = t$, in which case the assertion is nothing but [Ohk17, Theorem 4.19]. □

We have the following relations between growth filtrations and Frobenius slope filtrations, which are analogues of [CT11, Theorem 2.3 (2)] for (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$.

Proposition 5.3 (Cf. [Ohk17, Proposition 4.1(i)]). *Let M be a (φ, ∇)-module over $\mathcal{R}^{\mathrm{bd}}$ such that $M_{\mathcal{R}}$ is unipotent as a ∇-module over $\mathcal{R}$. Then, we have*

$$V(M)^\lambda \subset \bigcup_{\mu < \lambda_{\max}(M_{\mathcal{E}}) - \lambda} S_\mu(V(M)),$$

or, equivalently,

$$\mathrm{Sol}_\lambda(M) \supset S_{\lambda - \lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M))$$

for an arbitrary real number λ.

See § 8 for the proof.

Remark 5.4. (i) The first displayed relation may be written as $V(M)^\lambda \subset S_{\lambda - \lambda_{\max}(M_{\mathcal{E}})}(V(M^\vee))^\perp$ à la [CT11, Theorem 2.3(2)] by Lemma 1.7(II)(vi), where the orthogonal part is taken with respect to the natural K-bilinear map $V(M) \times V(M^\vee) \to K$.

(ii) In terms of Newton polygons, Proposition 5.3 says that the Newton polygon of $V(M)^\bullet$ lies on or below that of $S_{\lambda_{\max}(M_{\mathcal{E}}) - \bullet}(V(M))$ (Proposition 10.3(i)).

Corollary 5.5. *Let notation be as in Proposition 5.3. Then, the filtrations $V(M)^\bullet$ and $\mathrm{Sol}_\bullet(M)$ are exhaustive and separated.*

Proof. By duality, we have only to prove the exhaustivity of $\mathrm{Sol}_\bullet(M)$. Let μ denote the maximum Frobenius slope of $\mathrm{Sol}(M)$. Then, we have $\mathrm{Sol}_{\mu + \lambda_{\max}(M_{\mathcal{E}})}(M) = \mathrm{Sol}(M)$ by the second displayed relation in Proposition 5.3 with $\lambda = \mu + \lambda_{\max}(M_{\mathcal{E}})$. □

Finally, we briefly recall some results on the log-growth filtrations of (φ, ∇)-modules over $\mathcal{E}$. In the rest of this section, let M be a (φ, ∇)-module over $\mathcal{E}$. The log-growth filtration $M^\bullet$ (see the introduction) is a decreasing filtration of subobjects of M indexed by $\mathbb{R}$, which is exhaustive and separated. Moreover, the correspondence $M \mapsto M^\lambda$ forms a covariant endofunctor on the category of (φ, ∇)-modules over $\mathcal{E}$ (see [CT09, Proposition 3.6]). Furthermore, $M \neq 0$ implies $M^\lambda \neq M$ for $\lambda \geqslant 0$ (see [CT09, Theorem 3.2(4)]). Recall that we say that M is PBQ if M/M^0 is pure as a φ-module over $\mathcal{E}$ (see [CT11, Definition 5.1(1)]). As a consequence of functoriality, if M is PBQ, then so is any quotient of M (see [CT11, Proposition 5.3]). We define the *Frobenius slope filtration* $S_\bullet(M)$ of M as in § 1. Then, we have analogues of Proposition 5.3 and Theorem 0.1.

Theorem 5.6 [CT11, Theorems 2.3(1) and 7.1].

(i) *We have* $M^\lambda \subset \bigcup_{\mu<\lambda_{\max}(M)-\lambda} S_\mu(M)$ *for an arbitrary real number* λ *with equality when* M *is PBQ.*

(ii) *We have* $M^\lambda = \bigcup_{\mu>\lambda} M^\mu$ *for an arbitrary real number* λ. *Moreover, the slope multiset of* $M^\bullet$ *consists of rational numbers.*

6. Base change of the coefficient field

In this section, let notation be as in Notation 3. By abuse of notation, we denote the inclusions $\mathcal{R}^{\mathrm{bd}} \to \mathcal{R}^{\mathrm{bd}}_{K'}$ and $\mathcal{E} \to \mathcal{E}_{K'}$ by ψ. We prove that the base change functor $\psi^*(-)$ respects the growth filtration (Lemma 6.2).

Lemma 6.1 (Cf. [CT11, Proposition 2.1]). *Let* $\star \in \{\,,\log\}$.

(i) *The canonical ring homomorphism* $\mathcal{R}_\star \otimes_K K' \to \mathcal{R}_{K',\star}$ *is injective. Moreover,* $(\mathrm{Fil}_\lambda \mathcal{R}_\star) \otimes_K K'$ *maps* $\mathrm{Fil}_\lambda \mathcal{R}_{K',\star}$.

(ii) *Let* $c_1, \dots, c_n \in K'$ *be* K*-linearly independent, and* $r_1, \dots, r_n \in \mathcal{R}_\star$. *If* $\sum_{j=1}^n c_j r_j \in \mathrm{Fil}_\lambda \mathcal{R}_{K',\star}$, *then we have* $r_j \in \mathrm{Fil}_\lambda \mathcal{R}_\star$ *for all* j.

Proof. We may easily reduce to the case $\star =$ (*empty*).

(i) The second assertion follows from Corollary 3.7. We prove the first. Assume that x belongs to the kernel of the map $\mathcal{R} \otimes_K K' \to \mathcal{R}_{K'}$. Write $x = \sum_{j=1}^n r_j \otimes c_j \in \mathcal{R} \otimes_K K'$ with n minimal. Then, the c_j's are K-linearly independent. Write $r_j = \sum_{i\in\mathbb{Z}} r_{ij} t^i \in \mathcal{R}_K$ with $r_{ij} \in K$. By $0 = \sum_{j=1}^n c_j r_j = \sum_{i\in\mathbb{Z}} (\sum_{j=1}^n c_j r_{ij}) t^i$, we have $\sum_{j=1}^n c_j r_{ij} = 0$, which implies $r_{ij} = 0$ for all i, j, i.e. $x = 0$.

(ii) Write $r_j \in \mathcal{R}$ as above. Then, $\sup_{i\geqslant 1} i^{-\lambda} |\sum_{j=1}^n c_j r_{ij}| < \infty$ by assumption and Corollary 3.6. By [Ked10, Theorem 1.3.6], there exists a constant C such that

$$\sup\{|\alpha_1|, \dots, |\alpha_n|\} \leqslant C \left| \sum_{j=1}^n c_j \alpha_j \right| \forall \alpha_1, \dots, \alpha_n \in K.$$

Then, $r_1, \dots, r_n \in \mathrm{Fil}_\lambda \mathcal{R}$ by

$$\sup_{j=1,\dots,n} \sup_{i\geqslant 1} i^{-\lambda} |r_{ij}| = \sup_{i\geqslant 1} \sup_{j=1,\dots,n} i^{-\lambda} |r_{ij}| \leqslant \sup_{i\geqslant 1} i^{-\lambda} C \left| \sum_{j=1}^n c_j r_{ij} \right| < \infty. \qquad \square$$

Lemma 6.2 (Cf. [CT09, Proposition 1.10]).

(i) *Let* M *be a* (φ, ∇)*-module over* $\mathcal{R}^{\mathrm{bd}}$ *such that* $M_{\mathcal{R}}$ *is unipotent as a* ∇*-module over* $\mathcal{R}$. *Then,* $(\psi^* M)_{\mathcal{R}_{K'}}$ *is unipotent as a* ∇*-module over* $\mathcal{R}_{K'}$, *and there exist canonical isomorphisms of* φ*-modules over* K'

$$V(M) \otimes_K K' \to V(\psi^* M),$$
$$\mathrm{Sol}(M) \otimes_K K' \to \mathrm{Sol}(\psi^* M),$$

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which are compatible with the canonical pairings. Moreover, these isomorphisms respectively induce isomorphisms

$$V(M)^{\bullet}\otimes_K K'\cong V(\psi^*M)^{\bullet},$$
$$\mathrm{Sol}_{\bullet}(M)\otimes_K K'\cong \mathrm{Sol}_{\bullet}(\psi^*M).$$

(ii) *(See [CT09, Proposition 1.10] and [CT11, Proposition 2.1].) Let M be a (φ,∇)-module over $K[\![t]\!]_0$. Then, we have analogous isomorphisms as in part (i).*

(iii) *Let M be a (φ,∇)-module over $\mathcal{E}$. Then, there exists a canonical isomorphism*

$$\psi^*(M^{\bullet})\cong(\psi^*M)^{\bullet}.$$

Proof. (i) The first assertion follows from Proposition 4.10. To prove the second assertion, by duality, we have only to prove that $\mathrm{Sol}_{\lambda}(M)\otimes_K K'\cong\mathrm{Sol}_{\lambda}(\psi^*M)$ for $\lambda\in\mathbb{R}$. For $f\in\mathrm{Sol}(M)$, let $\beta(f)\in\mathrm{Sol}(\psi^*M)$ be the image of $f\otimes 1$ under the map $\mathrm{Sol}(M)\otimes_K K'\to\mathrm{Sol}(\psi^*M)$. We claim that $f\in\mathrm{Sol}_{\lambda}(M)$ if and only if $\beta(f)\in\mathrm{Sol}_{\lambda}(\psi^*M)$. By definition, $\beta(f)$ coincides with the composition

$$M\otimes_{\mathcal{R}^{\mathrm{bd}}}\mathcal{R}^{\mathrm{bd}}_{K'}\xrightarrow{f\otimes\mathrm{id}_{\mathcal{R}^{\mathrm{bd}}_{K'}}}\mathcal{R}_{\log}\otimes_{\mathcal{R}^{\mathrm{bd}}}\mathcal{R}^{\mathrm{bd}}_{K'}\longrightarrow\mathcal{R}_{K',\log}$$

where the second map is the multiplication map. Hence, $\beta(f)(\psi^*M)=f(M)\cdot\mathcal{R}^{\mathrm{bd}}_{K'}=f(M)\cdot\mathrm{Fil}_0\mathcal{R}_{K',\log}$ by Corollary 3.7(iii). Therefore, we obtain the claim by using Corollary 3.7(i) and the fact $\mathcal{R}_{\log}\cap\mathrm{Fil}_{\lambda}\mathcal{R}_{K',\log}=\mathrm{Fil}_{\lambda}\mathcal{R}_{\log}$.

We fix a basis $\{c_j\}_{j\in J}$ of K' as a vector space over K. Let $F\in\mathrm{Sol}(\psi^*M)$. We can uniquely write $F=\sum_{j\in J}c_j\beta(f_j)$ with $f_j\in\mathrm{Sol}(M)$ such that $f_j=0$ for all but finitely many j. By Lemma 6.1(ii), $F\in\mathrm{Sol}_{\lambda}(\psi^*M)$ if and only if $\beta(f_j)\in\mathrm{Sol}_{\lambda}(\psi^*M)$ for all $j\in J$. Moreover, by the claim, the latter condition is equivalent to $f_j\in\mathrm{Sol}_{\lambda}(M)$ for all $j\in J$, which implies the assertion.

(ii) We may reduce to part (i) by Lemma 0.3.

(iii) Let notation be as in the introduction. We naturally extend $\tau:\mathcal{E}\to\mathcal{E}[\![X-t]\!]_0$ (respectively, $\psi:\mathcal{E}\to\mathcal{E}_{K'}$) to $\mathcal{E}_{K'}\to\mathcal{E}_{K'}[\![X-t]\!]_0$ (respectively, $\mathcal{E}[\![X-t]\!]_0\to\mathcal{E}_{K'}[\![X-t]\!]_0; X-t\mapsto X-t$), which is denoted by τ (respectively, ψ) for simplicity. As $\psi\circ\tau=\tau\circ\psi$, we have a natural isomorphism $\psi^*\circ\tau^*\cong\tau^*\circ\psi^*$. Let $\lambda\in\mathbb{R}$. By the definition of the log-growth filtration, $(\psi^*M)^{\lambda}$ is characterized as a unique subobject U of ψ^*M such that there exists a canonical isomorphism $V(\tau^*U)\cong V(\tau^*\psi^*M)^{\lambda}$. Hence, we have only to prove $V(\tau^*\psi^*(M^{\lambda}))\cong V(\tau^*\psi^*M)^{\lambda}$. We have $V(\tau^*\psi^*(M^{\lambda}))\cong V(\psi^*\tau^*(M^{\lambda}))$ and $V(\tau^*\psi^*M)^{\lambda}\cong V(\psi^*\tau^*M)^{\lambda}$. By applying part (ii) to $\psi:\mathcal{E}[\![X-t]\!]_0\to\mathcal{E}_{K'}[\![X-t]\!]_0$, we have $V(\psi^*\tau^*(M^{\lambda}))\cong V(\tau^*(M^{\lambda}))\otimes_{\mathcal{E}}\mathcal{E}_{K'}\cong V(\tau^*M)^{\lambda}\otimes_{\mathcal{E}}\mathcal{E}_{K'}\cong V(\psi^*\tau^*M)^{\lambda}$, which implies the assertion. □

Corollary 6.3. *Let M be a (φ,∇)-module over $\mathcal{E}$. Then, M is PBQ if and only if ψ^*M is PBQ.*

Proof. As we have a canonical isomorphism $\psi^*M/(\psi^*M)^0\cong\psi^*(M/M^0)$ by Lemma 6.2(iii), M/M^0 is pure as a φ-module over $\mathcal{E}$ if and only if $\psi^*M/(\psi^*M)^0$ is pure as a φ-module over $\mathcal{E}_{K'}$. □

7. Slope criterion

LEMMA 7.1 (Cf. [Ohk17, Lemma 7.3]). *Let M be a (φ, ∇)-module over $\mathcal{E}$. Then, $\lambda_{\max}(M/M^0) = \lambda_{\max}(M)$.*

Proof. It suffices to prove $\lambda_{\max}(M/M^0) \geqslant \lambda_{\max}(M)$. As we have a canonical surjection $M/M^0 \to M/\bigcup_{\mu<\lambda_{\max}(M)} S_\mu(M)$ by Theorem 5.6(i), we have $\lambda_{\max}(M/M^0) \geqslant \lambda_{\max}(M/\bigcup_{\mu<\lambda_{\max}(M)} S_\mu(M)) = \lambda_{\max}(M)$. □

PROPOSITION 7.2 (Slope criterion). *For a (φ, ∇)-module M over $\mathcal{E}$, we consider the following conditions.*

(i) *The (φ, ∇)-module M over $\mathcal{E}$ is PBQ.*
(ii) *For any nonzero quotient Q of M, we have $\lambda_{\max}(Q) = \lambda_{\max}(M)$.*

Then, condition (i) implies condition (ii). Moreover, if k is perfect, then condition (ii) implies condition (i).

Proof. Let (i)′ and (ii)′ denote conditions (i) and (ii) for M/M^0 respectively, that is:

(i)′ the (φ, ∇)-module M/M^0 over $\mathcal{E}$ is PBQ;
(ii)′ for any nonzero quotient Q of M/M^0, we have $\lambda_{\max}(Q) = \lambda_{\max}(M/M^0)$.

We claim that conditions (i) and (i)′ (respectively, (ii) and (ii)′) are equivalent. Note that $(M/M^0)^0 = 0$ because $\tau^*(M/M^0)$ is trivial as a ∇-module over $\mathcal{E}[\![X-t]\!]_0$. Hence, conditions (i) and (i)′ are equivalent. By Lemma 7.1, condition (ii) implies condition (ii)′. Conversely, assume that condition (ii)′ holds. Let Q be a nonzero quotient of M. We have $\lambda_{\max}(Q) = \lambda_{\max}(Q/Q^0), \lambda_{\max}(M) = \lambda_{\max}(M/M^0)$ by Lemma 7.1, and $\lambda_{\max}(Q/Q^0) = \lambda_{\max}(M/M^0)$ by condition (ii)′, where Q/Q^0 is regarded as a nonzero quotient of M/M^0. Hence, condition (ii) holds. Thus, we obtain the claim.

To prove the assertion, we may replace M by M/M^0 in virtue of the claim. Thus, we may assume that $M^0 = 0$. Then, condition (i) is equivalent to saying that M is pure as a φ-module over $\mathcal{E}$. Hence, condition (i) implies condition (ii). We assume that k is perfect and condition (ii) holds. Let μ_1 be the least Frobenius slope of M. Then, $S_{\mu_1}(M)$ is regarded as a quotient of M by Theorem 7.3. By applying condition (ii) to $Q = S_{\mu_1}(M)$, we obtain $\mu_1 = \lambda_{\max}(M)$, which implies that $M = S_{\lambda_{\max}(M)}(M) = S_{\mu_1}(M)$ is pure (of slope μ_1) as a φ-module over $\mathcal{E}$. □

THEOREM 7.3 (Splitting theorem [CT11, Theorem 4.1]). *Let M be a (φ, ∇)-module over $\mathcal{E}$. Assume that k is perfect. Then, M is bounded in the sense of [CT11, 2.2], that is, $M^0 = 0$ if and only if the Frobenius slope filtration of M splits, that is, there exists an isomorphism of (φ, ∇)-modules over $\mathcal{E}$*

$$M \cong \bigoplus_{i=1,\dots,m} S_{\mu_i}(M) \Big/ \bigcup_{\mu<\mu_i} S_\mu(M),$$

where $\mu_1 < \cdots < \mu_m$ denote the Frobenius slopes of M without multiplicity. In particular, $S_{\mu_1}(M)$ is regarded a quotient of M.

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8. Proofs of Proposition 5.3 and part (ii) of Theorem 0.1

Proof of Proposition 5.3. We may assume that $M \neq 0$. By duality, we have only to prove that $\mathrm{Sol}_{\lambda-\lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M)) \subset \mathrm{Sol}(M)$ for $\lambda \in \mathbb{R}$. By Lemma 6.2(i), we may reduce to the case $K = L$. Thus, we may assume that k is algebraically closed. As the functors $S_{\bullet}(-)$ and $\mathrm{Sol}_{\bullet}(-)$ commute with d-pushforwards, after replacing M by $[d]_*M$, we may assume that $S_{\lambda-\lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M))$ admits a K-basis B consisting of Frobenius 1-eigenvectors f of slopes μ_f. Let $f \in B$. Then, we have $\varphi_{\mathrm{Sol}(M)}(f) = c_f f$ with $c_f \in K^{\times}$, i.e. $\varphi \circ f = c_f f \circ \varphi_M$, and $\mu_f = \log|c_f|/\log|q| \leqslant \lambda - \lambda_{\max}(M_{\mathcal{E}})$. By applying Proposition 3.11(i) to f, we have $f \in \mathrm{Sol}_{\mu_f+\lambda_{\max}(M_{\mathcal{E}})}(M) \subset \mathrm{Sol}_{\lambda}(M)$. Hence, $S_{\lambda-\lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M)) = \sum_{f\in B} Kf \subset \mathrm{Sol}_{\lambda}(M)$. □

Proof of part (ii) of Theorem 0.1. We may assume that $M \neq 0$. By duality and Proposition 5.3, we have only to prove that $\mathrm{Sol}(M) \subset \mathrm{Sol}_{\lambda-\lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M))$ for $\lambda \in \mathbb{R}$. By Corollary 6.3 and a reduction argument as previously, we may assume that k is algebraically closed, and $\mathrm{Sol}_{\lambda}(M)$ admits a K-basis B consisting of Frobenius 1-eigenvectors f of slopes μ_f. Let $f \in B$, and $c_f \in K^{\times}$ be as previously. Let N be the kernel of the map $f : M \to \mathcal{R}_{\log}$. Then, N is regarded as a subobject of M because N is stable under φ_M and ∂_M. Moreover, we have $N \neq M$ by $f \neq 0$. Hence, we may regard $Q := M/N$ as a nonzero quotient of M. By Slope criterion 7.2, $\lambda_{\max}(Q_{\mathcal{E}}) = \lambda_{\max}(M_{\mathcal{E}})$. Let $f' \in \mathrm{Sol}(Q)$ be the solution induced by f. By definition, f' is injective, $\varphi \circ f' = c_f f' \circ \varphi_{\mathrm{Sol}(Q)}$, and $f'(Q) = f(M) \subset \mathrm{Fil}_{\lambda}\mathcal{R}_{\log}$. By applying Proposition 3.11(ii) to f', we obtain $\mu_f \leqslant \lambda - \lambda_{\max}(Q_{\mathcal{E}}) = \lambda - \lambda_{\max}(M_{\mathcal{E}})$. Hence, we have $f \in S_{\mu_f}(\mathrm{Sol}(M)) \subset S_{\lambda-\lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M))$. Therefore, $\mathrm{Sol}_{\lambda}(M) = \sum_{f\in B} Kf \subset S_{\lambda-\lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M))$. □

9. The PBQ filtration

The PBQ filtration of an arbitrary (φ, ∇)-module over $K[\![t]\!]_0$ is constructed in [CT11]. We generalize the construction to an arbitrary (φ, ∇)-module over $\mathcal{R}^{\mathrm{bd}}$.

Assumption 9.1. In this section, as in [CT11, 5.2 and 5.3] we assume that k is perfect.

Definition 9.2. Let M be a (φ, ∇)-module over $\mathcal{E}$, and $N \subset M$ a subobject. Then, $N^0 \subset M^0$ by the functoriality of the log-growth filtration, and $M^0 \subset \bigcup_{\mu<\lambda_{\max}(M)} S_{\mu}(M)$ by Theorem 5.6(i). Thus, we obtain the canonical morphism

$$\sigma_{N,M} : N/N^0 \to M \Big/ \bigcup_{\mu<\lambda_{\max}(M)} S_{\mu}(M).$$

Proposition 9.3 (Cf. [CT11, Proposition 5.4]). *Let M be a (φ, ∇)-module over $\mathcal{E}$. Then, there exists a unique subobject N of M such that $\sigma_{N,M}$ is an isomorphism. Moreover, N is PBQ with $\lambda_{\max}(N) = \lambda_{\max}(M)$, and $N \neq 0$ if $M \neq 0$. We call N the maximally PBQ submodule of M.*

Proof. We may assume that $M \neq 0$. The first assertion is proved in the reference. As $\sigma_{N,M}$ is an isomorphism, N/N^0 is pure of slope $\lambda_{\max}(M)$. Hence, N is PBQ, and $\lambda_{\max}(N) = \lambda_{\max}(N/N^0) = \lambda_{\max}(M)$ by Lemma 7.1. □

Definition 9.4 [CT11, Corollary 5.5]. Let M be a (φ, ∇)-module over $\mathcal{E}$. We define $P_i(M)$ for $i \in \mathbb{N}$ as follows. We put $P_0(M) = 0$, $P_1(M) = N$ with notation as in Proposition 9.3.

For $i \geqslant 2$, we inductively define $P_i(M)$ as the inverse image of $P_1(M/P_{i-1}(M))$ under the canonical projection $M \to M/P_{i-1}(M)$. By definition, we have a canonical isomorphism $P_i(M)/P_{i-1}(M) \cong P_1(M/P_{i-1}(M))$. Thus, $P_\bullet(M)$ forms an increasing filtration of subobjects of M, which is called the *PBQ filtration* of M.

LEMMA 9.5. *Let M be a (φ, ∇)-module over $\mathcal{E}$.*

(i) *The (φ, ∇)-module $P_1(M)$ over $\mathcal{E}$ is PBQ with $\lambda_{\max}(P_1(M)) = \lambda_{\max}(M)$, and $P_1(M) \neq 0$ if $M \neq 0$.*
(ii) *We have $P_1(M) = M$ if and only if M is PBQ. In particular, if M is not PBQ, then $P_1(M) \neq M$.*
(iii) *The filtration $P_\bullet(M)$ is separated and exhaustive, and each graded piece $P_{i+1}(M)/P_i(M)$ is PBQ.*
(iv) *If M is not PBQ, then $\lambda_{\max}(M/P_1(M)) < \lambda_{\max}(M)$.*

Proof. Part (i) is a part of Proposition 9.3. We prove part (ii). If $P_1(M) = M$, then M is PBQ by part (i). Conversely, if M is PBQ, then $\sigma_{M,M}$ is an isomorphism by using Theorem 5.6(i) with $\lambda = 0$, which implies that $P_1(M) = M$ by uniqueness. In part (iii), the separatedness is trivial, and the rest of the assertion follows from $P_i(M)/P_{i-1}(M) \cong P_1(M/P_{i-1}(M))$ and part (i). We prove part (iv). As $M = P_1(M) + \bigcup_{\mu<\lambda_{\max}(M)} S_\mu(M)$ by the surjectivity of $\sigma_{P_1(M),M}$, we have $M/P_1(M) = \bigcup_{\mu<\lambda_{\max}(M)} S_\mu(M/P_1(M))$ by Lemma 1.7(II)(iii), which implies the assertion. □

PROPOSITION 9.6. *Let M be a (φ, ∇)-module over $\mathcal{R}^{\mathrm{bd}}$. Then, $P_\bullet(M_\mathcal{E})$ descends to a filtration $P_\bullet(M)$ of subobjects of M. We call $P_1(M)$ (respectively, $P_\bullet(M)$) the maximally PBQ submodule (respectively, the PBQ filtration) of M.*

Proof. By the definition of $P_\bullet(M_\mathcal{E})$, we have only to prove that $P_1(M_\mathcal{E})$ descends to $\mathcal{R}^{\mathrm{bd}}$. We may assume that $M \neq 0$. We proceed by induction on $n = \dim_{\mathcal{R}^{\mathrm{bd}}} M$. In the case where $M_\mathcal{E}$ is PBQ (including the base case $n = 1$), we have $P_1(M) = M$ by $P_1(M_\mathcal{E}) = M_\mathcal{E}$ (Lemma 9.5(ii)). In the induction step, we may assume that $M_\mathcal{E}$ is not PBQ. It suffices to construct a subobject N of M satisfying the following conditions: (a) $N \neq M$; (b) $\lambda_{\max}(N_\mathcal{E}) = \lambda_{\max}(M_\mathcal{E})$; (c) the canonical morphism $i : N_\mathcal{E}/\bigcup_{\mu<\lambda_{\max}(M_\mathcal{E})} S_\mu(N_\mathcal{E}) \to M_\mathcal{E}/\bigcup_{\mu<\lambda_{\max}(M_\mathcal{E})} S_\mu(M_\mathcal{E})$ is an isomorphism. Indeed, if this is the case, then $P_1(N_\mathcal{E})$ descends to $P_1(N)$ by condition (a) and the induction hypothesis. As $\sigma_{P_1(N_\mathcal{E}),M_\mathcal{E}} = i \circ \sigma_{P_1(N_\mathcal{E}),N_\mathcal{E}}$ by condition (b), the morphism $\sigma_{P_1(N_\mathcal{E}),M_\mathcal{E}}$ is an isomorphism by condition (c). Hence, $P_1(M_\mathcal{E}) = P_1(N_\mathcal{E})$ by uniqueness. Thus, $P_1(M_\mathcal{E})$ descends to $P_1(N)$. We construct N, and then verify conditions (a)–(c). Let μ denote the minimum Frobenius slope of $M_\mathcal{E}/M^0_\mathcal{E}$. Then, $\mu < \lambda_{\max}(M_\mathcal{E})$ by assumption, and $Q := S_\mu(M_\mathcal{E}/M^0_\mathcal{E})$ is regarded as a quotient of $M_\mathcal{E}/M^0_\mathcal{E}$ by Theorem 7.3. Let $f : M \to Q$ be the composition of the canonical map $M \to M_\mathcal{E}$ followed by the projection $M_\mathcal{E} \to Q$. We have $f \neq 0$ because $f(M)$ generates $Q(\neq 0)$ as a vector space over $\mathcal{E}$. Let N denote the kernel of f. Then, N is a nonzero subobject of M. By applying Lemma 2.14 to the map $M/N \hookrightarrow Q$ induced by f, we have $\lambda_{\max}((M/N)_\mathcal{E}) = \lambda_{\max}(Q) = \mu < \lambda_{\max}(M_\mathcal{E})$. Hence, $\lambda_{\max}(N_\mathcal{E}) = \lambda_{\max}(M_\mathcal{E})$ and $(M/N)_\mathcal{E} = \bigcup_{\mu<\lambda_{\max}(M_\mathcal{E})} S_\mu((M/N)_\mathcal{E})$. Therefore, the morphism i is an isomorphism by Lemma 1.7(II)(iii). □

We record the following consequence though it is not used in the paper.

COROLLARY 9.7. *If M is an irreducible object in the category of (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$, then $M_\mathcal{E}$ is PBQ.*

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Proof. If $M_{\mathcal{E}}$ is not PBQ, then $0 \subsetneq P_1(M_{\mathcal{E}}) \subsetneq M_{\mathcal{E}}$ by Lemma 9.5(i) and (ii), which implies that $0 \subsetneq P_1(M) \subsetneq M$. In particular, M is reducible. $\square$

Lemma 9.8. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of (φ, ∇)-modules over $\mathcal{R}^{\mathrm{bd}}$ such that $M_{\mathcal{R}}$ is unipotent as a ∇-module over $\mathcal{R}$. Let $\lambda \in \mathbb{R}$.*

(i) *Assume that $V(M')^{\lambda} = \bigcup_{\mu < \lambda_{\max}(M_{\mathcal{E}}) - \lambda} S_{\mu}(V(M'))$, or, equivalently, $\mathrm{Sol}_{\lambda}(M') = S_{\lambda - \lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M'))$. Then, there exist canonical exact sequences*

$$0 \to V(M')^{\lambda} \to V(M)^{\lambda} \to V(M'')^{\lambda} \to 0,$$
$$0 \to \mathrm{Sol}_{\lambda}(M'') \to \mathrm{Sol}_{\lambda}(M) \to \mathrm{Sol}_{\lambda}(M') \to 0.$$

(ii) *There exist canonical exact sequences*

$$0 \to V(P_1(M))^{\lambda} \to V(M)^{\lambda} \to V(M/P_1(M))^{\lambda} \to 0,$$
$$0 \to \mathrm{Sol}_{\lambda}(M/P_1(M)) \to \mathrm{Sol}_{\lambda}(M) \to \mathrm{Sol}_{\lambda}(P_1(M)) \to 0.$$

In particular, the slope multisets of $V(M)^{\bullet}$ and $\mathrm{Sol}_{\bullet}(M)$, which coincide by definition, are equal to the disjoint union of those of $V(P_1(M))^{\bullet}$ and $V(M/P_1(M))^{\bullet}$.

Proof. (i) We proceed as in the proof of [CT11, Proposition 2.6(1)]. By duality and the left exactness of $\mathrm{Sol}_{\lambda}(-)$, we have only to prove the surjectivity of $\mathrm{Sol}_{\lambda}(M) \to \mathrm{Sol}_{\lambda}(M')$. By assumption and Proposition 5.3, we obtain the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Sol}_{\lambda}(M) & \longrightarrow & \mathrm{Sol}_{\lambda}(M') \\ \big\downarrow{\scriptstyle \text{inc.}} & & \big\| \\ S_{\lambda - \lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M)) & \longrightarrow & S_{\lambda - \lambda_{\max}(M_{\mathcal{E}})}(\mathrm{Sol}(M')) \end{array}$$

Then, the bottom horizontal arrow is surjective by Lemma 1.7(II)(iii), which implies the assertion.

(ii) As $P_1(M)_{\mathcal{E}}$ is PBQ with $\lambda_{\max}(P_1(M)_{\mathcal{E}}) = \lambda_{\max}(M_{\mathcal{E}})$ (Lemma 9.5(i)), we have $V(P_1(M)) = \bigcup_{\mu < \lambda_{\max}(M_{\mathcal{E}}) - \lambda} S_{\mu}(V(P_1(M)))$ by Theorem 0.1(ii). Hence, we obtain the assertion by applying part (i) to $M' = P_1(M)$. $\square$

Lemma 9.9.

(i) *[CT11, Proposition 2.6 (1)] Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of (φ, ∇)-modules over $\mathcal{E}$, and $\lambda \in \mathbb{R}$. Assume that $(M')^{\lambda} = \bigcup_{\mu < \lambda_{\max}(M) - \lambda} S_{\mu}(M')$. Then, there exists a canonical exact sequence $0 \to (M')^{\lambda} \to M^{\lambda} \to (M'')^{\lambda} \to 0$.*

(ii) *Let M be a (φ, ∇)-module over $\mathcal{E}$, and $\lambda \in \mathbb{R}$. Then, there exists a canonical exact sequence $0 \to P_1(M)^{\lambda} \to M^{\lambda} \to (M/P_1(M))^{\lambda} \to 0$. In particular, the slope multiset of $M^{\bullet}$ is equal to the disjoint union of those of $P_1(M)^{\bullet}$ and $(M/P_1(M))^{\bullet}$.*

Proof. We can prove part (ii) as in the proof of Lemma 9.8(ii) with Theorem 0.1(ii) replaced by Theorem 5.6(i). $\square$

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10. Proof of Theorem 0.4

We start with a simple alternative proof of part (i) of Theorem 0.1 by using the PBQ filtration.

Second proof of part (i) of Theorem 0.1. We may assume that $M \neq 0$. By duality, we have only to prove that the filtration $\mathrm{Sol}_\bullet(M)$ is right continuous, and has rational slopes. As in the proof of Proposition 5.3, we may assume that k is algebraically closed. We prove by induction on $n = \dim_{\mathcal{R}^{\mathrm{bd}}} M$. In the case where M is PBQ (including the base case $n = 1$), we have $\mathrm{Sol}_\lambda(M) = S_{\lambda-\lambda_{\max}(M_\mathcal{E})}(\mathrm{Sol}(M))$ by Theorem 0.1(ii). Hence, the assertion follows from Lemma 1.4(iii) and Theorem 1.5(ii)(b). In the induction step, we may assume that $M_\mathcal{E}$ is not PBQ. Then, we have $0 \subsetneq P_1(M) \subsetneq M$ by Lemma 9.5(i) and (ii). Hence, by the second exact sequence in Lemma 9.8(ii), the assertion follows from the induction hypothesis for $P_1(M)$ and $M/P_1(M)$. $\square$

DEFINITION 10.1. (i) Let M be a (φ, ∇)-module over $\mathcal{R}^{\mathrm{bd}}$ of dimension n such that $M_\mathcal{R}$ is unipotent as a ∇-module over $\mathcal{R}$. We define the *log-growth Newton polygon* $\mathrm{NP}(M)$ of M as the Newton polygon of $V(M)^\bullet$. Let $\{s_1(M) \leqslant \cdots \leqslant s_n(M)\}$ denote the slope multiset of $V(M)^\bullet$. Put $h_i(M) := s_1(M) + \cdots + s_i(M)$ for $i \in \{1, \ldots, n\}$, and $h_0(M) := 0$.

(ii) Let M be a (φ, ∇)-module over $\mathcal{E}$ of dimension n. We define the *log-growth Newton polygon* $\mathrm{NP}(M)$ of M as the Newton polygon of $M^\bullet$. We also define $s_i(M)$ for $i \in \{1, \ldots, n\}$, and $h_i(M)$ for $i \in \{0, \ldots, n\}$ as previously.

Notation 4. For M a (φ, ∇)-module over $\mathcal{R}$, let $S_\bullet(M)$ be the slope filtration of M in the sense of [Tsu98, Definition 5.1.1], whose existence is proved in [Ked04]. Note that Tsuzuki's definition of the slopes [Tsu98, Definition 3.1.5(1)] is compatible with that given in Definition 1.3.

LEMMA 10.2. *Let M be a (φ, ∇)-module over $\mathcal{R}$ which is unipotent as a ∇-module over $\mathcal{R}$. Let notation be as in § 4.2. We regard $\mathbf{V}(M)$ as a φ-module over K as is the case of $V(M)$ (see § 5). Then, $\dim_K S_\lambda(\mathbf{V}(M)) = \mathrm{rank}_\mathcal{R} S_\lambda(M)$ for $\lambda \in \mathbb{R}$.*

Proof. By dévissage, we may assume that M is irreducible in the category of (φ, ∇)-modules over $\mathcal{R}$. The difference module M^∇ over K is a φ-module over K since M^∇ is isomorphic to $\mathbf{V}(M)^{N_X=0}$, which is a φ-submodule of $\mathbf{V}(M)$ by $N_X \circ \varphi_{\mathbf{V}(M)} = q\varphi_{\mathbf{V}(M)} \circ N_X$. Hence, the canonical injection $i : M^\nabla \otimes_K \mathcal{R} \to M$ as in Lemma 4.6(ii) is an isomorphism of (φ, ∇)-modules over $\mathcal{R}$ by the irreducibility of M. Hence, i induces isomorphisms $M^\nabla \cong \mathbf{V}(M^\nabla \otimes_K \mathcal{R}) \cong \mathbf{V}(M)$ of φ-modules over K, and an isomorphism $S_\lambda(M^\nabla) \otimes_K \mathcal{R} \cong S_\lambda(M)$ of $\mathcal{R}$-modules by the uniqueness of the slope filtration of M. Hence, $\dim_K S_\lambda(\mathbf{V}(M)) = \dim_K S_\lambda(M^\nabla) = \mathrm{rank}_\mathcal{R} S_\lambda(M)$. $\square$

PROPOSITION 10.3. (i) *Let M be a (φ, ∇)-module over $\mathcal{R}^{\mathrm{bd}}$ of dimension n such that $M_\mathcal{R}$ is unipotent as a ∇-module over $\mathcal{R}$. Let $\{s'_1(M) \leqslant \cdots \leqslant s'_n(M)\}$ denote the slope multiset of $S_\bullet(M_\mathcal{R})$, which coincides with that of $S_\bullet(V(M))$ by Lemma 10.2. Then, for $i \in \{1, \ldots, n\}$, we have $s_i(M) \leqslant \lambda_{\max}(M_\mathcal{E}) - s'_{n-i+1}(M)$ with equality when $M_\mathcal{E}$ is PBQ. In particular, $\mathrm{NP}(M)$ lies on or below the Newton polygon of $S_{\lambda_{\max}(M_\mathcal{E})-\bullet}(M_\mathcal{R})$, and the two polygons coincide if $M_\mathcal{E}$ is PBQ.*

(ii) *Let M be a (φ, ∇)-module over $\mathcal{E}$ of dimension n. Let $\{s'_1(M) \leqslant \cdots \leqslant s'_n(M)\}$ denote the slope multiset of $S_\bullet(M)$. Then, for $i \in \{1, \ldots, n\}$, we have $s_i(M) \leqslant \lambda_{\max}(M) - s'_{n-i+1}(M)$ with equality when M is PBQ. In particular, $\mathrm{NP}(M)$ lies on or below the Newton polygon of $S_{\lambda_{\max}(M)-\bullet}(M)$, and the two polygons coincide if M is PBQ.*

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Proof. (i) We remark that as a consequence of the right continuity of $V(M)^\bullet$ (respectively, $S_\bullet(V(M))$) proved in Theorem 0.1(i), we have $s_i(M) \leqslant \lambda$ (respectively, $s'_i(M) \leqslant \lambda$) if and only if $n - \dim_K V(M)^\lambda \geqslant i$ (respectively, $\dim_K S_\lambda(V(M)) \geqslant i$); moreover, we have $s_i(M) \geqslant \lambda$ (respectively, $s'_i(M) \geqslant \lambda$) if and only if for any $\mu \in (-\infty, \lambda)$, we have $i > n - \dim_K V(M)^\mu$ (respectively, $i > \dim_K S_\mu(V(M))$).

We have

$$\dim_K V(M)^{\lambda_{\max}(M_{\mathcal{E}}) - s'_{n-i+1}(M)} \leqslant \dim_K \left(\bigcup_{\mu < s'_{n-i+1}(M)} S_\mu(V(M)) \right) < n - i + 1,$$

where the first (respectively, second) inequality follows from Proposition 5.3 (respectively, the above remark). Hence, $s_i(M) \leqslant \lambda_{\max}(M_{\mathcal{E}}) - s'_{n-i+1}(M)$ by the previous remark. Assume that $M_{\mathcal{E}}$ is PBQ. Then, for any $\mu \in (-\infty, \lambda_{\max}(M_{\mathcal{E}}) - s'_{n-i+1}(M))$, we have

$$\dim_K V(M)^\mu = \dim_K \left(\bigcup_{\eta < \lambda_{\max}(M_{\mathcal{E}}) - \mu} S_\eta(V(M)) \right) \geqslant \dim_K S_{s'_{n-i+1}(M)}(V(M)) = n - i + 1,$$

where the first (respectively, second) equality follows from Theorem 0.1(ii) (respectively, the previous remark). Hence, $s_i(M) \geqslant \lambda_{\max}(M_{\mathcal{E}}) - s'_{n-i+1}(M)$ by the previous remark, which implies the assertion.

(ii) We can prove as previously with Proposition 5.3 and Theorem 0.1(ii) replaced by Theorem 5.6. $\square$

Remark 10.4 (Non-coincidence of the right endpoints). In part (i) (respectively, part (ii)) of Proposition 10.3, the right endpoints of the two polygons may not coincide if $M_{\mathcal{E}}$ (respectively, M) is not PBQ. For example, let M be the (φ, ∇)-module over $\mathcal{R}^{\mathrm{bd}}$ defined by $\mathcal{R}^{\mathrm{bd}} \oplus \mathcal{R}^{\mathrm{bd}}(q)$. As M is trivial as a ∇-module over $\mathcal{R}^{\mathrm{bd}}$, we have $s_1(M_{(\mathcal{E})}) = s_2(M_{(\mathcal{E})}) = 0$. Moreover, we obviously have $s'_1(M_{(\mathcal{E})}) = 0, s'_2(M_{(\mathcal{E})}) = 1$, and $\lambda_{\max}(M_{\mathcal{E}}) = 1$. Hence, $\mathrm{NP}(M_{(\mathcal{E})})$ has right endpoint $(2, 0)$, and the Newton polygons of $S_{\lambda_{\max}(M_{\mathcal{E}}) - \bullet}(M_{\mathcal{R}})$ and $S_{\lambda_{\max}(M_{\mathcal{E}}) - \bullet}(M_{\mathcal{E}})$ have right endpoints $(2, 1)$. Note that $M_{\mathcal{E}}$ is not PBQ by $M^0_{\mathcal{E}} = 0$, and the PBQ filtration of $M_{(\mathcal{E})}$ is given by $P_1(M) = \mathcal{R}^{\mathrm{bd}}(q), P_2(M) = M$ (respectively, $P_1(M_{\mathcal{E}}) = \mathcal{E}(q), P_2(M_{\mathcal{E}}) = M_{\mathcal{E}}$).

Proof of Theorem 0.4. As in the proof of Proposition 5.3, we may assume that k is algebraically closed. We have only to prove that $h_i(M) \geqslant h_i(M_{\mathcal{E}})$ for all i with equality when $i = n$. We prove by induction on n. In the case where $M_{\mathcal{E}}$ is PBQ (including the base case $n = 1$), the assertion follows from Proposition 10.3 and the specialization theorem for Frobenius Newton polygons [Ked10, Theorem 16.4.6], that is, $s'_{n-i+1}(M) + \cdots + s'_n(M) \leqslant s'_{n-i+1}(M_{\mathcal{E}}) + \cdots + s'_n(M_{\mathcal{E}})$ with equality when $i = n$. In the induction step, we may assume that $M_{\mathcal{E}}$ is not PBQ. Then, $0 \subsetneq P_1(M) \subsetneq M$ by Lemma 9.5(i) and (ii). As $h_i(M)$ is equal to the minimum of the sums of i elements in the slope multiset of $V(M)^\bullet$, we have $h_i(M) = \min\{h_j(M/P_1(M)) + h_l(P_1(M)); j, l \in \mathbb{N}, j + l = i\}$ by Lemma 9.8(ii). Similarly, we have $h_i(M_{\mathcal{E}}) = \min\{h_j((M/P_1(M))_{\mathcal{E}}) + h_l(P_1(M)_{\mathcal{E}}); j, l \in \mathbb{N}, j + l = i\}$ by Lemma 9.9(ii). Hence, the assertion follows from the induction hypothesis for $M/P_1(M)$ and $P_1(M)$. $\square$

Note that the inequality $s_n(M) \leqslant s_n(M_{\mathcal{E}})$ in Theorem 0.4 may be regarded as a partial generalization of Christol's transfer theorem [CT09, Proposition 4.3], which asserts an analogous

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inequality for a ∇-module over $K[\![t]\!]_0$ such that $M \otimes_{K[\![t]\!]_0} K\{t\}$ is trivial as a ∇-module over $K\{t\}$.

Remark 10.5 (André's theorem, cf. [CT11, Remark 2.8]). In [Dwo73, pp. 45–46], Dwork gives a primitive version of Proposition 10.3 as a conjectural statement for p-adic differential equations. Then, as an analogue of Grothendieck–Katz theorem in the sense of the introduction, Dwork states the following.

CONJECTURE [Dwo73, Conjecture 2]. The [log-growth] Newton polygon (at b) rises under specialization.

André proved this conjecture. Precisely speaking, let M be a ∇-module of rank n over $K[\![t]\!]_0$ such that $M \otimes_{K[\![t]\!]_0} K\{t\}$ is trivial as a ∇-module over $K\{t\}$. We can define $\mathrm{NP}(M_{(\mathcal{E})})$ and $h_n(M_{(\mathcal{E})})$ as in Definition 10.1. Let $NP_{\log,\bar{0}}(M)$ (respectively, $NP_{\log,\bar{t}}(M_{\bar{t}})$) be the translation of $\mathrm{NP}(M)$ (respectively, $\mathrm{NP}(M_{\mathcal{E}})$) by $-h_n(M)$ (respectively, $-h_n(M_{\mathcal{E}})$) along the y-axis so that the right endpoint coincides with $(n, 0)$, which is called the *special* (respectively, *generic) log-growth Newton polygon* of M in [And08, 3.4]. Then, André proves that $NP_{\log,\bar{0}}(M)$ lies on or above $NP_{\log,\bar{t}}(M_{\bar{t}})$ (see [And08, Theorem 4.1.1]); we consequently have $h_n(M) \leqslant h_n(M_{\mathcal{E}})$. Dwork's conjecture in the introduction is an analogue of the above conjecture for (φ, ∇)-modules M over $K[\![t]\!]_0$, which is equivalent to saying, under André's notation, that $NP_{\log,\bar{0}}(M)$ lies on or above $NP_{\log,\bar{t}}(M_{\bar{t}})$ with the same left endpoints, i.e. $h_n(M) = h_n(M_{\mathcal{E}})$; note that the equality $h_n(M) = h_n(M_{\mathcal{E}})$ does not always hold for ∇-modules M over $K[\![t]\!]_0$ such that $M \otimes_{K[\![t]\!]_0} K\{t\}$ are trivial as ∇-modules over $K\{t\}$ (see [Ohk15]). To the best of the author's knowledge, at this point, there is no alternative proof of Dwork's conjecture or $h_n(M) = h_n(M_{\mathcal{E}})$ exploiting Andre's theorem in an essential way.

Finally, we give applications to log-(φ, ∇)-modules over $K[\![t]\!]_0$. We put $K\{t\}_{\log} := K\{t\}[\ell_X]$, which is regarded as a $K\{t\}$-subalgebra of $\mathcal{R}_{\log}$. Let R denote either $K[\![t]\!]_0$ or $K\{t\}_{\log}$. As in Convention 2, we assume that φ on $\mathcal{R}$ satisfies $\varphi(t) = S \in K[\![t]\!]_0$. Then, R is stable under both φ and $\partial_{\log} := t\partial = td/dt$ by $\partial_{\log}(\ell_X) = \partial_{\log}(X)/X = i + \partial_{\log}(v)/v \in K[\![t]\!]_0$, where we write $X = t^i v$ as in Definition 2.1(iii). Moreover, we have $\partial_{\log}(\varphi(t))/\varphi(t) \cdot \varphi \circ \partial_{\log} = \partial_{\log} \circ \varphi$. We define a *log-$\nabla$-module* over R as a finite free R-module equipped with a differential operator relative to $\partial_{\log}$. We define a *log-(φ, ∇)-module* over R as a φ-module M over R equipped with a differential operator $\partial_{\log,M}$ relative to $\partial_{\log}$ satisfying the compatibility condition $\partial_{\log}(\varphi(t))/\varphi(t) \cdot \varphi_M \circ \partial_{\log,M} = \partial_{\log,M} \circ \varphi_M$ (cf. Definition 4.1(ii)). We put $R' = \mathcal{R}^{\mathrm{bd}}$ if $R = K[\![t]\!]_0$, and $R' = \mathcal{R}_{\log}$ if $R = K\{t\}_{\log}$. Then, $M' := M \otimes_R R'$ is regarded as a (φ, ∇)-module over R' by setting $\varphi_{M'} = \varphi_M \otimes \varphi$ and $\partial_{M'} = \partial_{\log,M} \otimes t \cdot \mathrm{id}_{R'} + \mathrm{id}_M \otimes \partial$. Moreover, we have an obvious base change functor from the category of log-(φ, ∇)-modules over $K[\![t]\!]_0$ to the category of log-(φ, ∇)-modules over $K\{t\}_{\log}$ as in § 4.1. Thus, we obtain the following diagram of functors, which is commutative up to a natural isomorphism.

$$\begin{array}{ccc}
\{\text{log-}(\varphi, \nabla)\text{-modules over } K[\![t]\!]_0\} & \longrightarrow & \{(\varphi, \nabla)\text{-modules over } \mathcal{R}^{\mathrm{bd}}\} \\
\downarrow & & \downarrow \\
\{\text{log-}(\varphi, \nabla)\text{-modules over } K\{t\}_{\log}\} & \longrightarrow & \{(\varphi, \nabla)\text{-modules over } \mathcal{R}_{\log}\}
\end{array}$$

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We claim that if M is a log-(φ, ∇)-module over $K[\![t]\!]_0$, then $M \otimes_{K[\![t]\!]_0} K\{t\}_{\log}$ is trivial as a log-∇-module over $K\{t\}_{\log}$. As in Remark 5.2, we may assume that $X = t$, in which case the claim follows from a nilpotent analogue of Dwork's trick [Ked10, Corollary 17.2.4]: note that a log-(φ, ∇)-module over $K[\![t]\!]_0$ is regarded as a finite differential module over the differential ring $(K[\![t]\!]_0, td/dt)$ with a Frobenius structure in the sense of [Ked10, Remark 17.1.2].

We define the *log-growth filtration* of $K\{t\}_{\log}$ by $\mathrm{Fil}_\lambda K\{t\}_{\log} := K\{t\}_{\log} \cap \mathrm{Fil}_\lambda \mathcal{R}_{\log}$ for $\lambda \in \mathbb{R}$ (cf.[Ohk17, Definition 4.11]). By using the claim, we can define $V(M), \mathrm{Sol}(M)$, etc., for log-(φ, ∇)-modules M over $K[\![t]\!]_0$ as in § 5. Moreover, an analogue of Lemma 0.3 holds: note that the unipotence of $(M_{\mathcal{R}^{\mathrm{bd}}})_{\mathcal{R}}$ follows from the above claim and Lemma 4.8(ii). Hence, we can prove the following result as in the proof of Theorem 0.2 given in the introduction.

Theorem 10.6. *Analogues of Proposition 5.3, Chiarellotto–Tsuzuki conjecture, and Dwork's conjecture hold for log-(φ, ∇)-modules M over $K[\![t]\!]_0$, where we define $M_{\mathcal{E}} := (M \otimes_{K[\![t]\!]_0} \mathcal{R}^{\mathrm{bd}})_{\mathcal{E}}$.*

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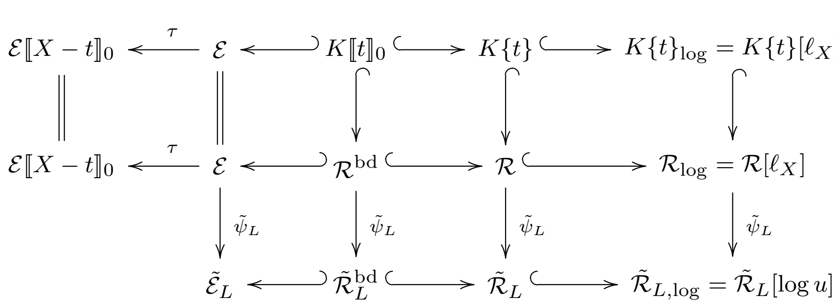
Appendix

The following is a list of notation used in the paper.

§ 1 (F, ϕ, q), $\lambda_{\max}(-)$, $S_\bullet(-)$, $S^\bullet(-)$
§ 2 (K, φ_K, q), $\mathcal{O}_K$, $\mathfrak{m}_K$, k
§ 2.1 $\mathcal{R}^r$, $\mathcal{R}$, $|\cdot|_r : \mathcal{R}^r \to \mathbb{R}_{\geqslant 0}$, $\mathcal{R}^{\mathrm{bd}}$, $\mathcal{R}^{\mathrm{int}}$, $|\cdot|_0 : \mathcal{R}^{\mathrm{bd}} \to \mathbb{R}_{\geqslant 0}$, $\mathcal{E}$, X, ℓ_X, $\mathcal{R}_{\log}$, $\varphi : \mathcal{R} \to \mathcal{R}$, S
§ 2.2 $\tilde{\mathcal{R}}^r$, $\tilde{\mathcal{R}}$, $|\cdot|_r : \tilde{\mathcal{R}}^r \to \mathbb{R}_{\geqslant 0}$, $\tilde{\mathcal{R}}^{\mathrm{bd}}$, $\tilde{\mathcal{R}}^{\mathrm{int}}$, $|\cdot|_0 : \tilde{\mathcal{R}}^{\mathrm{bd}} \to \mathbb{R}_{\geqslant 0}$, $k((u^{\mathbb{Q}}))$, $\tilde{\mathcal{E}}$, $\tilde{\mathcal{R}}_{\log}$, $\varphi : \tilde{\mathcal{R}} \to \tilde{\mathcal{R}}$
§ 2.3 L, $\tilde{\psi}_L$, r_0, c_0, c_1
§ 3.1 $\mathrm{Fil}_\bullet \tilde{\mathcal{R}}$, $\mathrm{Fil}_\bullet \tilde{\mathcal{R}}_{\log}$
§ 3.2 $\mathrm{Fil}_\bullet \mathcal{R}$, $\mathrm{Fil}_\bullet \mathcal{R}_{\log}$
§ 4.1 ∂, ∂_M, ϕ_M, $(-)^\nabla$
§ 4.2 $T_{X,X'}$, $\mathbf{V}(-)$, N_X, K', ψ, $U(-)$
§ 5 $V(-)$, $\mathrm{Sol}(-)$, $V(-)^\bullet$, $\mathrm{Sol}_\bullet(-)$, $M^\bullet$
§ 9 $\sigma_{N,M}$, $P_\bullet(-)$
§ 10 $\mathrm{NP}(-)$, $s_\bullet(-)$, $h_\bullet(-)$, $s'_\bullet(-)$, $K\{t\}_{\log}$, $\partial_{\log}$, $\partial_{\log,M}$, $\mathrm{Fil}_\bullet K\{t\}_{\log}$

The following is a commutative diagram of rings, where the hooked arrows denote the inclusions, and all morphisms are injective.



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