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Embedding theorems for LTL and its variants†

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In this paper, we prove some embedding theorems for LTL (linear-time temporal logic) and its variants: *viz.* some generalisations, extensions and fragments of LTL. Using these embedding theorems, we give uniform proofs of the completeness, cut-elimination and/or decidability theorems for LTL and its variants. The proposed embedding theorems clarify the relationships between some LTL-variations (for example, LTL, a dynamic topological logic, a fixpoint logic, a spatial logic, Prior's logic, Davies' logic and an NP-complete LTL) and some traditional logics (for example, classical logic, intuitionistic logic and infinitary logic).

1. Introduction

1.1. *Proposed embedding theorems*

In this paper, we prove some theorems for embedding *linear-time temporal logic* (LTL) and its variants into *infinitary logic* (IL), classical logic and intuitionistic logic. We will then use these embedding theorems to give uniform proofs of the completeness, cut-elimination and/or decidability theorems for some LTL-variations. The LTL-variations studied in this paper include:

- a *generalised first-order LTL* – see Section 4;
- two infinitary extensions that subsume some *dynamic topological logics* – see Section 5;
- a *3-dimensional spatial logic* – see Section 6.1;
- some next-time only fragments, which include *Prior's logic* and *Davies' logic* – see Section 6.2;
- an NP-complete fragment with a bounded time domain – see Section 6.3.

In the following explanation of the embedding theorems, we will just focus on two theorems for syntactically and semantically embedding LTL into IL (see Sections 2 and 3).

LTL has the temporal operators X (next), G (globally) and F (eventually) and is considered to be one of the most useful temporal logics in Computer Science (Clarke *et al.* 1999; Emerson 1990; Holzmann 2006; Kröger 1977; Pnueli 1977), while IL has the connectives of infinitary conjunction $\bigwedge$ and infinitary disjunction $\bigvee$, and has been

† Sections 2, 4, 5, 6.1 and 6.3 in the current paper are based on some refinements of technical parts of the conference presentations Kamide (2009; 2010b; 2010d; 2010e; 2011).

studied by many logicians (Feferman 1968; Lorenzen 1951; Novikov 1961; Takeuti 1985). However, the research fields concerned with LTL and IL have evolved independently, so the relationship between them is still to be discovered. Hence, one of the aims of the current paper is to clarify the relationship between them by giving two theorems for syntactically and semantically embedding LTL into IL. These embedding theorems show that G and F in LTL can be represented by $\bigwedge$ and $\bigvee$, respectively, in IL. The syntactical embedding theorem is based on Gentzen-type sequent calculi, and the semantical embedding theorem is based on Kripke semantics. We go on to prove the cut-elimination theorem for Kawai's sequent calculus LT_ω (Kawai 1987) for LTL using the syntactical embedding theorem, and the completeness theorem for LT_ω using the syntactical and semantical embedding theorems together.

The syntactical embedding theorem says that 'a sequent $\Rightarrow \alpha$ is provable in LT_ω if and only if the sequent $\Rightarrow f(\alpha)$ is provable in a sequent calculus LK_ω for IL, where f is a certain mapping'. The essential idea of f is to represent the following informal interpretations:

$$\begin{aligned} f(p) &= p \\ f(\mathrm{X}^i p) &= p_i \\ f(\mathrm{X}^i \mathrm{G}\alpha) &= \bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\} \\ f(\mathrm{X}^i \mathrm{F}\alpha) &= \bigvee\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\} \end{aligned}$$

for any propositional variable p of LTL.

Although the syntactical embedding theorem gives a proof-theoretical interpretation of the connection between LTL and IL, we cannot get a direct semantic interpretation of the same connection. On the other hand, LTL is usually defined semantically. Indeed, the *model checking* methods using LTL are based on a purely semantic expression. Moreover, most LTL users are unfamiliar with Gentzen-type proof theory. Thus, a semantic interpretation of the connection between LTL and IL is required. In order to obtain such an interpretation, we will present a semantical version of the syntactical embedding theorem, which we call the semantical embedding theorem.

We will discuss the syntactical and semantical embedding theorems of LTL into IL in Section 2. We begin by introducing LT_ω and LK_ω, and then prove the syntactical embedding theorem of LT_ω into LK_ω. The cut-elimination theorem for LT_ω is then derived from this syntactical embedding theorem. We then introduce the semantics of LTL and IL, and prove the semantical embedding theorem of LTL into IL. The completeness theorem with respect to the semantics for LTL is proved for LT_ω using both the syntactical and semantical embedding theorems. We conclude Section 2 by presenting an indexed formulation $2S_\omega$ of LTL, which is Baratella and Masini's 2-sequent calculus for LTL, and show that LT_ω and $2S_\omega$ are equivalent.

In Section 3, we present the syntactical and semantical embedding theorems of first-order LTL into first-order IL using a similar approach to Section 2.

1.2. *Generalised first-order LTL*

We will introduce a Gentzen-type sequent calculus GLT_ω for a generalised first-order LTL (GLTL) in Section 4, and we will prove cut-elimination and completeness theorems for GLT_ω using some embedding theorems. GLT_ω has modal operators

$$\heartsuit_i \ (i \in \{1, 2, \ldots, n\})$$
$$\heartsuit_{\mathrm{G}} \ \text{(generalised G)}$$
$$\heartsuit_{\mathrm{F}} \ \text{(generalised F)},$$

and it includes (first-order) LT_ω as a special case.

In the following, we will explain that the proposed new modal operators $\heartsuit_i$, $\heartsuit_{\mathrm{G}}$ and $\heartsuit_{\mathrm{F}}$ can be regarded as generalisations of X, G and F. These operators are intended to characterise the axiom scheme

$$\heartsuit_{\mathrm{G}}\alpha \leftrightarrow \bigwedge\{\iota\alpha \mid \iota \in K^*\}$$

where K^* is the set of all words of finite length of the alphabet

$$K := \{\heartsuit_i \mid i \in \{1, 2, \ldots, n\}\}.$$

We now suppose that for any formula α, we have f_α is a mapping on the set of formulas such that

$$f_\alpha(x) := \bigwedge\{\heartsuit_i(x \wedge \alpha) \mid i \in \omega\}.$$

$\heartsuit_{\mathrm{G}}\alpha$ then becomes a fixpoint of f_α. This axiom scheme just corresponds to the so-called iterative interpretation of common knowledge. On the other hand, if we take

$$K := \{\heartsuit_1\},$$

we can understand $\heartsuit_1$ and $\heartsuit_{\mathrm{G}}$, respectively, as the temporal operators X and G in LTL. The corresponding axiom scheme for the singleton case represents the LTL-axiom scheme

$$\mathrm{G}\alpha \leftrightarrow \bigwedge\{\mathrm{X}^i\alpha \mid i \in \omega\}.$$

So the operator $\heartsuit_{\mathrm{G}}$ can be regarded as a natural generalisation of G. Similarly, $\heartsuit_{\mathrm{F}}$ can be regarded as a generalisation of F.

1.3. *Infinitary extensions of LTL*

In Section 5, we will introduce two Gentzen-type sequent calcului L_ω and L_ω^-, which are infinitary extensions of LTL:

— L_ω includes a variant of dynamic topological logics;
— L_ω^-, which is a subsystem of L_ω, is an integration of both LT_ω and LK_ω.

We will then prove cut-elimination theorems for L_ω and L_ω^- and a completeness theorem for L_ω^-. There is no completeness theorem for L_ω since a subsystem $\mathrm{S4}_\omega$ of L_ω is known to be Kripke-incomplete. $\mathrm{S4}_\omega$ is an extension of both IL and the normal modal logic S4, and has been used as a base logic for formalising game theory (Kaneko and Nagashima 1997).

We will also introduce the X-only fragment L^x_ω of L^-_ω, and prove an equivalence between L^x_ω and L^-_ω based on some appropriate interpretations of G and F.

We will now briefly review some previous work on dynamic topological logic (DTL). DTL is a combination of S4 and temporal logic and has been studied recently by several researchers (Artemov *et al.* 1997; Kremer and Mints 2005; Konev *et al.* 2006; Mints 2006)†. DTL provides a context for studying the confluence of the topological semantics for S4, topological dynamics and temporal logic (Kremer and Mints 2005). Two bimodal (next-interior) fragments of DTL, which are called S4F (functional) and S4C (continuous), were first introduced in Artemov *et al.* (1997). Some Gentzen-type cut-free sequent calculi and Hilbert-type axiom schemes were introduced in Artemov *et al.* (1997) for S4F and S4C, and the complete topological and Kripke-type semantics were also obtained for S4F and S4C. An alternative formulation of cut-free sequent calculus for S4C was presented in Mints (2006). Trimodal DTLs were formalised semantically in Kremer and Mints (2005) by combining the S4 modal operator ♡ (interior) and the temporal operators X and G. Although sequent calculi for S4F and S4C have been studied, sequent calculi for trimodal DTLs have not been. The reason may be that Konev *et al.* (2006) showed that the trimodal DTL of homeomorphism is not recursively axiomatisable, and that this logic requires the infinitary axiom scheme

$$G\alpha \leftrightarrow \alpha \wedge X\alpha \wedge XX\alpha \wedge XXX\alpha \wedge \cdots \infty,$$

which is also an axiom scheme of L_ω and L^-_ω.

1.4. *Spatial extensions of LTL*

In Section 6.1, we will introduce a spatial logic as a Gentzen-type sequent calculus. This logic is called 3-dimensional spatial logic (3SL) and it can be used to give an appropriate representation of the 3-Cartesian product ω^3 of the set ω of natural numbers. We will prove completeness and cut-elimination theorems for 3SL using some embedding theorems.

We will now give some motivation for studying 3SL. A central issue for spatial reasoning in practice is to formalise reasoning about 3-dimensional space. It is known that time can be appropriately modelled by ω, and, similarly, in a very simple case, space can be naturally modelled by the 3-Cartesian product ω^3 of ω. Although temporal logics based on ω have been studied successfully, spatial logics based on ω^3 have not yet been studied in detail. The aim of introducing 3SL is thus to obtain a sound, complete and cut-free sequent calculus for reasoning about ω^3.

The proposed semantics for 3SL has 3-dimensional satisfaction relations

$$\models_{i_x;i_y;i_z}$$

where i_x, i_y and i_z represent points on the x-axis, y-axis and z-axis, respectively. Intuitively,

$$\models_{i_x;i_y;i_z} \alpha$$

† See also `http://individual.utoronto.ca/philipkremer/DynamicTopologicalLogic.html`.

means 'α is true at the point $(i_x, i_y, i_z) \in \omega^3$'. The spatial operators P_x (position in x-axis), P_y (position in y-axis), P_z (position in z-axis), A (anywhere) and A^- (converse anywhere) are then defined semantically by:

(1) $\models_{i_x;i_y;i_z} P_x\alpha$ if and only if $\models_{i_x+1;i_y;i_z} \alpha$;

(2) $\models_{i_x;i_y;i_z} P_y\alpha$ if and only if $\models_{i_x;i_y+1;i_z} \alpha$;

(3) $\models_{i_x;i_y;i_z} P_z\alpha$ if and only if $\models_{i_x;i_y;i_z+1} \alpha$;

(4) $\models_{i_x;i_y;i_z} A\alpha$ if and only if $\models_{j_x;j_y;j_z} \alpha$ for any $j_x, j_y, j_z \in \omega$ with

$$j_x \geqslant i_x$$
$$j_y \geqslant i_y$$
$$j_z \geqslant i_z;$$

(5) $\models_{i_x;i_y;i_z} A^-\alpha$ if and only if $\models_{j_x;j_y;j_z} \alpha$ for any $j_x, j_y, j_z \in \omega$ with

$$0 \leqslant j_x \leqslant i_x$$
$$0 \leqslant j_y \leqslant i_y$$
$$0 \leqslant j_z \leqslant i_z.$$

1.5. *Next-time only fragments of LTL*

In Section 6.2, we present a Gentzen-type sequent calculus SDL for an extension of Davies' logic (Davies 1996) from the point of view of embedding theorems. SDL is regarded as an intuitionistic (or constructive) version of a Gentzen-type sequent calculus for Prior's tomorrow tense logic (Prior 1957), which is just the next-time only part of LTL. We obtain the cut-elimination and Kripke-completeness theorems for SDL using two theorems for embedding SDL into intuitionistic logic.

We will now briefly review some previous work on Davies' logic. Davies (1996) introduced a constructive temporal logic, which is defined as a judgement system based on the connectives X and $\rightarrow$. He pointed out that this logic can be related to a type system for *binding-time analysis*. Kojima and Igarashi (2008) studied logical aspects of Davies' logic by adding the connectives $\wedge$ and $\vee$, and obtained a Kripke-complete natural deduction system, NJ$^\circ$, and a cut-free sequent calculus, LJ$^\circ$, as proof systems for such an extended logic. Kamide (2010c) introduced an alternative to Kojima and Igarashi's logic, and obtained a cut-free and Kripke-complete sequent calculus, SDL, and a strongly normalisable and confluent natural deduction system, NDL, as proof systems. Kamide's logic is almost the same as Kojima and Igarashi's logic, the difference being that Kamide's logic has the axiom scheme

$$X(\alpha \vee \beta) \rightarrow X\alpha \vee X\beta,$$

while Kojima and Igarashi's logic has no such axiom scheme. Kamide (2010c) gave a direct proof of the Kripke-completeness theorem for SDL without using embedding theorems.

1.6. *NP-complete fragments of LTL*

In Section 6.3, we obtain an NP-complete fragment of LTL, called *bounded linear-time temporal logic* (BLTL), by restricting the time domain of temporal operators. We show that BLTL is NP-complete using a theorem for semantically embedding BLTL into classical logic. This semantical embedding theorem can also be justified through the *bounded model checking* technique (Biere *et al.* 2003), which uses a propositional satisfiability checking method.

The satisfiability problems for LTL fragments are known to be an important issue for constructing efficiently executable temporal logics. The satisfiability problem for LTL is PSPACE-complete (Sistla and Clarke 1985), and finding NP-complete fragments of LTL has been the subject of considerable study (Demri and Schnoebelen 2002; Etessami *et al.* 1997; Muscholl and Walukiewicz 2005; Walukiewicz 1998). We try to construct an alternative to such an NP-complete fragment by restricting the time domain of temporal operators. Although the standard LTL temporal operators have an infinite (unbounded) time domain, that is, the set ω of natural numbers, the bounded operators we consider have a *bounded time domain*, which is restricted by a fixed positive integer l, that is, the set

$$\omega_l := \{x \in \omega \mid x \leqslant l\}.$$

Despite this restriction, these bounded operators can derive almost all of the typical LTL axioms, including the temporal induction axiom.

We will now give a brief review of some related work on LTL fragments. Sistla and Clarke (1985) showed that the LTL fragment endowed with the standard operators X, G and F are PSPACE-complete and that satisfiability for the fragments endowed with either X or (F and G) is NP-complete. As mentioned above, some NP-complete fragments of LTL have been well studied. In particular, Demri and Schnoebelen (2002) proposed some restrictions on the nesting of operators and on the number of propositions. Also, Muscholl and Walukiewicz (2005) suggested restricting X to operators X_a ($a \in \Sigma$) that enforce the current letter to be a, where the formula $X\alpha$ is expressed in the form

$$\bigvee_{a\in\Sigma} X_a\alpha$$

where Σ is the alphabet. They then proved that the satisfiability problem for the LTL fragment with X_a ($a \in \Sigma$), F and G is NP-complete.

2. LTL

2.1. *Syntactical embedding*

The *formulas of propositional linear-time temporal logic* (LTL) are constructed from countably many propositional variables, $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation), G (globally), F (eventually) and X (next). We use:

— lower-case letters $p, q, \ldots$ to denote propositional variables;

— Greek lower-case letters $\alpha, \beta, \ldots$ to denote formulas;

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— Greek capital letters $\Gamma, \Delta, \ldots$ to denote finite (possibly empty) sets of formulas;
— for any $\sharp \in \{G, F, X\}$, the expression $\sharp\Gamma$ to denote the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$;
— the symbol $\equiv$ to denote the equality of sets of symbols;
— the symbol ω to represent the set of natural numbers;
— lower-case letters i, j and k to denote any natural numbers.

An expression $X^i\alpha$ for any $i \in \omega$ is defined inductively by

$$X^0\alpha \equiv \alpha$$
$$X^{n+1}\alpha \equiv X^n X\alpha.$$

An expression of the form $\Gamma \Rightarrow \Delta$ is called a *sequent*, and we write $L \vdash S$ to denote the fact that a sequent S is provable in a sequent calculus L. An inference rule R is said to be *admissible* in a sequent calculus L if for any instance

$$\frac{S_1 \quad \cdots \quad S_n}{S}$$

of R, if $L \vdash S_i$ for all i, then $L \vdash S$.

We will now give a brief presentation of Kawai's LT_ω for LTL.

Definition 2.1 (LT_ω). The initial sequents of LT_ω are of the form

$$X^i p \Rightarrow X^i p$$

for any propositional variable p.

The structural rules of LT_ω are of the form

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \text{ (we-right).}$$

The logical inference rules of LT_ω are of the form

$$\frac{\Gamma \Rightarrow \Sigma, X^i\alpha \quad X^i\beta, \Delta \Rightarrow \Pi}{X^i(\alpha \to \beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi} \text{ (}{\to}\text{left)} \qquad \frac{X^i\alpha, \Gamma \Rightarrow \Delta, X^i\beta}{\Gamma \Rightarrow \Delta, X^i(\alpha \to \beta)} \text{ (}{\to}\text{right)}$$

$$\frac{X^i\alpha, \Gamma \Rightarrow \Delta}{X^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left1)} \qquad \frac{X^i\beta, \Gamma \Rightarrow \Delta}{X^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left2)}$$

$$\frac{\Gamma \Rightarrow \Delta, X^i\alpha \quad \Gamma \Rightarrow \Delta, X^i\beta}{\Gamma \Rightarrow \Delta, X^i(\alpha \wedge \beta)} \text{ (}\wedge\text{right)} \qquad \frac{X^i\alpha, \Gamma \Rightarrow \Delta \quad X^i\beta, \Gamma \Rightarrow \Delta}{X^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta} \text{ (}\vee\text{left)}$$

$$\frac{\Gamma \Rightarrow \Delta, X^i\alpha}{\Gamma \Rightarrow \Delta, X^i(\alpha \vee \beta)} \text{ (}\vee\text{right1)} \qquad \frac{\Gamma \Rightarrow \Delta, X^i\beta}{\Gamma \Rightarrow \Delta, X^i(\alpha \vee \beta)} \text{ (}\vee\text{right2)}$$

$$\frac{\Gamma \Rightarrow \Delta, X^i\alpha}{X^i\neg\alpha, \Gamma \Rightarrow \Delta} \text{ (}\neg\text{left)} \qquad \frac{X^i\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, X^i\neg\alpha} \text{ (}\neg\text{right)}$$

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$$\frac{X^{i+k}\alpha, \Gamma \Rightarrow \Delta}{X^{i}G\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, X^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, X^{i}G\alpha}\ (\text{Gright})$$

$$\frac{\{\ X^{i+j}\alpha, \Gamma \Rightarrow \Delta\ \}_{j\in\omega}}{X^{i}F\alpha, \Gamma \Rightarrow \Delta}\ (\text{Fleft}) \qquad \frac{\Gamma \Rightarrow \Delta, X^{i+k}\alpha}{\Gamma \Rightarrow \Delta, X^{i}F\alpha}\ (\text{Fright}).$$

Note that (Gright) and (Fleft) have an infinite number of premises. The fact that sequents of the form $X^i\alpha \Rightarrow X^i\alpha$ for any formula α are provable in LT_ω can be proved by induction on the complexity of α.

Kawai (1987) proved the cut-elimination theorem for LT_ω; we will give an alternative embedding-based proof of this theorem.

The following propositions give some examples of admissible rules and provable sequents. We will write the expression $\alpha \Leftrightarrow \beta$ to mean we have both the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$.

Proposition 2.2. The following rule is admissible in cut-free LT_ω:

$$\frac{\Gamma \Rightarrow \Delta}{X\Gamma \Rightarrow X\Delta}\ (\text{Xregu}).$$

Proof. We use induction on the proof P of $\Gamma \Rightarrow \Delta$ in cut-free LT_ω. We distinguish the cases according to the last inference of P. We will only show the case for (→left) as an example.

— Case (→left):
The last inference of P is of the form

$$\frac{\Pi \Rightarrow X^i\alpha \quad X^i\beta, \Sigma \Rightarrow \Delta}{X^i(\alpha\to\beta), \Pi, \Sigma \Rightarrow \Delta}\ (\to\text{left}).$$

By the induction hypothesis, we get

$$\frac{\begin{matrix}\vdots & & \vdots\\ X\Pi \Rightarrow XX^i\alpha & \quad & XX^i\beta, X\Sigma \Rightarrow X\Delta\end{matrix}}{XX^i(\alpha\to\beta), X\Pi, X\Sigma \Rightarrow X\Delta}\ (\to\text{left}).$$

□

Proposition 2.3. For any formulas α and β and any $i \in \omega$, the following sequents are provable in cut-free LT_ω:

(1) $X^i(\alpha \circ \beta) \Leftrightarrow X^i\alpha \circ X^i\beta$ where $\circ \in \{\to, \wedge, \vee\}$,
(2) $X^i(\neg\alpha) \Leftrightarrow \neg(X^i\alpha)$,
(3) $G\alpha \Rightarrow X\alpha$,
(4) $G\alpha \Rightarrow XG\alpha$,
(5) $G\alpha \Rightarrow GG\alpha$,
(6) $\alpha, G(\alpha\to X\alpha) \Rightarrow G\alpha$ (temporal induction).

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Proof. We will only show (6) as an example. So we have

$$\dfrac{\begin{array}{c}\vdots\\ \{\ \alpha, \mathrm{G}(\alpha{\rightarrow}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^k\alpha\ \}_{k\in\omega}\end{array}}{\alpha, \mathrm{G}(\alpha{\rightarrow}\mathrm{X}\alpha) \Rightarrow \mathrm{G}\alpha}\ (\mathrm{Gright})$$

where

$$\mathrm{LT}_\omega \vdash \alpha, \mathrm{G}(\alpha{\rightarrow}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^k\alpha,$$

for any $k \in \omega$, can be shown by mathematical induction on k. The base step is obvious, and the induction step can be shown by

$$\dfrac{\dfrac{\begin{array}{c}\vdots \text{ induction hypothesis}\\ \alpha, \mathrm{G}(\alpha{\rightarrow}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^k\alpha\end{array} \qquad \mathrm{X}^{k+1}\alpha \Rightarrow \mathrm{X}^{k+1}\alpha}{\alpha, \mathrm{G}(\alpha{\rightarrow}\mathrm{X}\alpha), \mathrm{X}^k(\alpha{\rightarrow}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^{k+1}\alpha}\ ({\rightarrow}\mathrm{left})}{\alpha, \mathrm{G}(\alpha{\rightarrow}\mathrm{X}\alpha), \mathrm{G}(\alpha{\rightarrow}\mathrm{X}\alpha) \Rightarrow \mathrm{X}^{k+1}\alpha}\ (\mathrm{Gleft})$$

□

The formulas of *propositional infinitary logic (IL)* are constructed from countably many propositional variables, $\rightarrow$, $\neg$, $\bigwedge$ (infinitary conjunction) and $\bigvee$ (infinitary disjunction). If Φ is a countable non-empty set of formulas of IL, then $\bigwedge\Phi$ and $\bigvee\Phi$ are also formulas of IL. Note that $\bigwedge\{\alpha\}$ and $\bigvee\{\alpha\}$ are equivalent to α, and that the standard binary connectives $\wedge$ (conjunction) and $\vee$ (disjunction) are regarded as special cases of $\bigwedge$ and $\bigvee$, which are assumed here to be a countable infinitary conjunction and a countable infinitary disjunction, respectively.

We will now define the sequent calculus LK_ω for IL.

Definition 2.4 (LK$_\omega$). The initial sequents of LK_ω are of the form

$$p \Rightarrow p$$

for any propositional variable p.

The structural rules for LK_ω are (cut), (we-left) and (we-right) as in Definition 2.1.

The logical inference rules of LK_ω are of the form

$$\dfrac{\Gamma \Rightarrow \Sigma, \alpha \quad \beta, \Delta \Rightarrow \Pi}{\alpha{\rightarrow}\beta, \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ ({\rightarrow}\mathrm{left}^0) \qquad \dfrac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha{\rightarrow}\beta}\ ({\rightarrow}\mathrm{right}^0)$$

$$\dfrac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\mathrm{left}^0) \qquad \dfrac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}\ (\neg\mathrm{right}^0)$$

$$\dfrac{\alpha, \Gamma \Rightarrow \Delta\ (\alpha \in \Theta)}{\bigwedge\Theta, \Gamma \Rightarrow \Delta}\ (\textstyle\bigwedge\mathrm{left}) \qquad \dfrac{\{\ \Gamma \Rightarrow \Delta, \alpha\ \}_{\alpha\in\Theta}}{\Gamma \Rightarrow \Delta, \bigwedge\Theta}\ (\textstyle\bigwedge\mathrm{right})$$

$$\dfrac{\{\ \alpha, \Gamma \Rightarrow \Delta\ \}_{\alpha\in\Theta}}{\bigvee\Theta, \Gamma \Rightarrow \Delta}\ (\textstyle\bigvee\mathrm{left}) \qquad \dfrac{\Gamma \Rightarrow \Delta, \alpha\ (\alpha \in \Theta)}{\Gamma \Rightarrow \Delta, \bigvee\Theta}\ (\textstyle\bigvee\mathrm{right})$$

where Θ denotes a non-empty countable set of formulas.

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The superscript '0' in some of the rule names in LK_ω means that these rules are the special cases of the corresponding LT_ω rules, that is, the case where i in X^i is 0.

It is well known that LK_ω enjoys cut-elimination – see, for example, Feferman (1968), Tanaka (1999), Tanaka (2001) and Takeuti (1985).

Definition 2.5. We fix a countable non-empty set Φ of propositional variables, and define the sets

$$\Phi_i := \{p_i \mid p \in \Phi\} \quad (i \in \omega)$$

of propositional variables where $p_0 := p$, that is, $\Phi_0 = \Phi$. The language $\mathcal{L}_{\mathrm{LTL}}$ of LTL is defined using Φ, $\rightarrow, \wedge, \vee, \neg$, X, G and F. The language $\mathcal{L}_{\mathrm{IL}}$ of IL is defined using $\bigcup_{i\in\omega} \Phi_i$, $\rightarrow, \neg$, $\bigwedge$ and $\bigvee$. For convenience, the binary versions of $\bigwedge$ and $\bigvee$ are also denoted by $\wedge$ and $\vee$, respectively, and these binary symbols are included in the definition of $\mathcal{L}_{\mathrm{IL}}$.

A mapping f from $\mathcal{L}_{\mathrm{LTL}}$ to $\mathcal{L}_{\mathrm{IL}}$ is defined as follows:

$$
\begin{aligned}
f(\mathrm{X}^i p) &:= p_i \in \Phi_i \quad (i \in \omega) && \text{for any } p \in \Phi \text{ (in particular, } f(p) := p \in \Phi) \\
f(\mathrm{X}^i(\alpha \circ \beta)) &:= f(\mathrm{X}^i\alpha) \circ f(\mathrm{X}^i\beta) && \text{where } \circ \in \{\rightarrow, \wedge, \vee\} \\
f(\mathrm{X}^i\neg\alpha) &:= \neg f(\mathrm{X}^i\alpha) \\
f(\mathrm{X}^i\mathrm{G}\alpha) &:= \bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\} && (\dagger) \\
f(\mathrm{X}^i\mathrm{F}\alpha) &:= \bigvee\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}. && (\ddagger)
\end{aligned}
$$

An expression $f(\Gamma)$ denotes the result of replacing every occurrence of a formula α in Γ by an occurrence of $f(\alpha)$.

In Definition 2.5, conditions (†) and (‡) correspond to the axiom schemes

$$\mathrm{G}\alpha \leftrightarrow \bigwedge_{i\in\omega} \mathrm{X}^i\alpha$$

$$\mathrm{F}\alpha \leftrightarrow \bigvee_{i\in\omega} \mathrm{X}^i\alpha,$$

respectively, which mean 'G and F in LTL can be represented by $\bigwedge$ and $\bigvee$, respectively, in IL'.

Theorem 2.6 (syntactical embedding). Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{LTL}}$, and f be the mapping defined in Definition 2.5. Then:

(1) If

$$\mathrm{LT}_\omega \vdash \Gamma \Rightarrow \Delta,$$

then

$$\mathrm{LK}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta).$$

(2) If

$$\mathrm{LK}_\omega - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta),$$

then

$$\mathrm{LT}_\omega - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta.$$

Proof.

(1) We use induction on the proof P of $\Gamma \Rightarrow \Delta$ in LT_ω, distinguishing the cases according to the last inference of P. We will just show some example cases:

— Case $\left(\mathrm{X}^i p \Rightarrow \mathrm{X}^i p\right)$:
In this case, we obtain

$$\mathrm{LK}_\omega \vdash f(\mathrm{X}^i p) \Rightarrow f(\mathrm{X}^i p),$$

that is,

$$\mathrm{LK}_\omega \vdash p_i \Rightarrow p_i \quad (p_i \in \Phi_i),$$

by the definition of f.

— Case ($\rightarrow$left):
So the last inference of P has the form

$$\frac{\Gamma \Rightarrow \Sigma, \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Delta \Rightarrow \Pi}{\mathrm{X}^i(\alpha{\rightarrow}\beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ (\rightarrow\text{left}).$$

By the induction hypothesis, we have

$$\mathrm{LK}_\omega \vdash f(\Gamma) \Rightarrow f(\Sigma), f(\mathrm{X}^i\alpha)$$

and

$$\mathrm{LK}_\omega \vdash f(\mathrm{X}^i\beta), f(\Delta) \Rightarrow f(\Pi),$$

so we get

$$\frac{\begin{array}{c}\vdots \\ f(\Gamma) \Rightarrow f(\Sigma), f(\mathrm{X}^i\alpha)\end{array} \quad \begin{array}{c}\vdots \\ f(\mathrm{X}^i\beta), f(\Delta) \Rightarrow f(\Pi)\end{array}}{f(\mathrm{X}^i\alpha){\rightarrow}f(\mathrm{X}^i\beta), f(\Gamma), f(\Delta) \Rightarrow f(\Sigma), f(\Pi)}\ (\rightarrow\text{left}^0)$$

where $f(\mathrm{X}^i\alpha){\rightarrow}f(\mathrm{X}^i\beta)$ coincides with $f(\mathrm{X}^i(\alpha{\rightarrow}\beta))$ by the definition of f.

— Case (Gleft):
So the last inference of P has the form

$$\frac{\mathrm{X}^{i+k}\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i\mathrm{G}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Gleft}).$$

By the induction hypothesis, we have

$$\mathrm{LK}_\omega \vdash f(\mathrm{X}^{i+k}\alpha), f(\Gamma) \Rightarrow f(\Delta),$$

and hence obtain

$$\frac{\begin{array}{c}\vdots \\ f(\mathrm{X}^{i+k}\alpha), f(\Gamma) \Rightarrow f(\Delta)\end{array} \qquad (f(\mathrm{X}^{i+k}\alpha) \in \{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\})}{\bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}, f(\Gamma) \Rightarrow f(\Delta)}\ (\textstyle\bigwedge\text{left})$$

where

$$\bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}$$

coincides with $f(\mathrm{X}^i\mathrm{G}\alpha)$ by the definition of f.

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— Case (Gright):
So the last inference of P has the form

$$\frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^{i}\mathrm{G}\alpha}\ \text{(Gright)}.$$

By the induction hypothesis, we have

$$LK_{\omega} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^{i+j}\alpha)$$

for all $j \in \omega$. Let Φ be

$$\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}.$$

So we have

$$\frac{\begin{array}{c}\vdots\\ \{\ f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^{i+j}\alpha)\ \}_{f(\mathrm{X}^{i+j}\alpha)\in\Phi}\end{array}}{f(\Gamma) \Rightarrow f(\Delta), \bigwedge\Phi}\ (\bigwedge \text{right})$$

where $\bigwedge\Phi$ coincides with $f(\mathrm{X}^{i}\mathrm{G}\alpha)$ by the definition of f.

(2) We use induction on the proof Q of $f(\Gamma) \Rightarrow f(\Delta)$ in LK_{ω}, distinguishing the cases according to the last inference of Q. We will just show one case as an example:

— Case ($\bigwedge$right):
Let Φ be

$$\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}.$$

The last inference of Q has the form

$$\frac{\{\ f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^{i+j}\alpha)\ \}_{f(\mathrm{X}^{i+j}\alpha)\in\Phi}}{f(\Gamma) \Rightarrow f(\Delta), \bigwedge\Phi}\ (\bigwedge \text{right})$$

where $\bigwedge\Phi$ coincides with $f(\mathrm{X}^{i}\mathrm{G}\alpha)$ by the definition of f. By the induction hypothesis, we have

$$\mathrm{LT}_{\omega} \vdash \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha$$

for all $j \in \omega$, so

$$\frac{\begin{array}{c}\vdots\\ \{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}\end{array}}{\Gamma \Rightarrow \Delta, \mathrm{X}^{i}\mathrm{G}\alpha}\ \text{(Gright)}.$$

This completes the proof. □

Note that we cannot give a direct proof of the converse of Theorem 2.6 (1) in a similar way. In order to prove the converse, we have to consider induction on the proofs Q of $f(\Gamma) \Rightarrow f(\Delta)$ in LK_{ω}. Hence, we must consider the case where the last inference of Q has the form

$$\frac{f(\Gamma_1) \Rightarrow f(\Delta_1), \beta \quad \beta, f(\Gamma_2) \Rightarrow f(\Delta_2)}{f(\Gamma_1), f(\Gamma_2) \Rightarrow f(\Delta_1), f(\Delta_2)}\ \text{(cut)}$$

where β is unknown if it can be expressed as $\beta = f(\gamma)$ for a formula γ, and the condition 'β can be expressed as $\beta = f(\gamma)$' is required to apply the induction†.

Theorem 2.7 (cut-elimination). The rule (cut) is admissible in cut-free LT_ω.

Proof. Suppose

$$\mathrm{LT}_\omega \vdash \Gamma \Rightarrow \Delta.$$

Then we have

$$\mathrm{LK}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta)$$

by Theorem 2.6 (1), and hence

$$\mathrm{LK}_\omega - (\mathrm{cut}) \ \vdash f(\Gamma) \Rightarrow f(\Delta)$$

by the cut-elimination theorem for LK_ω. By Theorem 2.6 (2), we then obtain

$$\mathrm{LT}_\omega - (\mathrm{cut}) \ \vdash \Gamma \Rightarrow \Delta,$$

which completes the proof. □

Note that because of the cut-elimination theorem for LK_ω, we can strengthen the statements of Theorem 2.6 by replacing 'if ... then' with 'if and only if', so we have

$$\mathrm{LT}_\omega \vdash \Gamma \Rightarrow \Delta \quad \text{if and only if} \quad \mathrm{LK}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta).$$

This fact will be used to prove the completeness theorem for LT_ω.

2.2. *Semantical embedding*

The symbol $\geqslant$ or $\leqslant$ is used to represent a linear order on ω. In the following, we will define LTL semantically as a satisfaction relation.

Let Γ be a set

$$\{\alpha_1, \ldots, \alpha_m\} \qquad (m \geqslant 0)$$

of formulas. Then:

— Γ^* means $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geqslant 1$, and $\neg(p \rightarrow p)$, where p is a fixed propositional variable, otherwise.
— Γ_* means $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geqslant 1$, and $p \rightarrow p$, where p is a fixed propositional variable, otherwise.

Definition 2.8 (LTL). Let S be a non-empty set of states. A structure (σ, I) is a *model* if:

(1) σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S;
(2) I is a mapping from the set Φ of propositional variables to the power set of S.

In this definition, σ is called a *computation*, and I is called an *interpretation*.

A satisfaction relation $(\sigma, I, i) \models_{LTL} \alpha$ for any formula α, where (σ, I) is a model, and i ($\in \omega$) represents some position within σ, is defined inductively by:

† The proof of the syntactical embedding theorem (of LT_ω into LK_ω) given in Kamide (2009) has such an error, which is corrected here.

(1) $(\sigma, I, i) \models_{LTL} p$ if and only if $s_i \in I(p)$ for any $p \in \Phi$.
(2) $(\sigma, I, i) \models_{LTL} \alpha \wedge \beta$ if and only if $(\sigma, I, i) \models_{LTL} \alpha$ and $(\sigma, I, i) \models_{LTL} \beta$.
(3) $(\sigma, I, i) \models_{LTL} \alpha \vee \beta$ if and only if $(\sigma, I, i) \models_{LTL} \alpha$ or $(\sigma, I, i) \models_{LTL} \beta$.
(4) $(\sigma, I, i) \models_{LTL} \alpha \rightarrow \beta$ if and only if not-$[(\sigma, I, i) \models_{LTL} \alpha]$ or $(\sigma, I, i) \models_{LTL} \beta$.
(5) $(\sigma, I, i) \models_{LTL} \neg\alpha$ if and only if not-$[(\sigma, I, i) \models_{LTL} \alpha]$.
(6) $(\sigma, I, i) \models_{LTL} \mathrm{X}\alpha$ if and only if $(\sigma, I, i+1) \models_{LTL} \alpha$.
(7) $(\sigma, I, i) \models_{LTL} \mathrm{G}\alpha$ if and only if $(\sigma, I, j) \models_{LTL} \alpha$ for any $j \geqslant i$.
(8) $(\sigma, I, i) \models_{LTL} \mathrm{F}\alpha$ if and only if $(\sigma, I, j) \models_{LTL} \alpha$ for some $j \geqslant i$.

A formula α is said to be *LTL-valid* if

$$(\sigma, I, 0) \models_{LTL} \alpha$$

for any model (σ, I). A sequent $\Gamma \Rightarrow \Delta$ is said to be LTL-valid if the formula $\Gamma_* \rightarrow \Delta^*$ is LTL-valid.

The following definition gives a semantics for IL.

Definition 2.9 (IL). Let Θ be a countable (non-empty) set of formulas. V is a mapping from the set Φ of propositional variables to the set $\{t, f\}$ of truth values. V is called a *valuation.* A valuation V is extended to a mapping from the set of formulas to $\{t, f\}$ by:
(1) $V(\alpha \rightarrow \beta) = t$ if and only if $V(\alpha) = f$ or $V(\beta) = t$.
(2) $V(\neg\alpha) = t$ if and only if $V(\alpha) = f$.
(3) $V(\bigwedge \Theta) = t$ if and only if $V(\alpha) = t$ for all $\alpha \in \Theta$.
(4) $V(\bigvee \Theta) = t$ if and only if $V(\alpha) = t$ for some $\alpha \in \Theta$.

In order to make a comparison between LTL and IL, a satisfaction relation $V \models_{IL} \alpha$ for any formula α is inductively defined by
(1) $V \models_{IL} p$ if and only if $V(p) = t$ for any $p \in \Phi$.
(2) $V \models_{IL} \alpha \rightarrow \beta$ if and only if not-$[V \models_{IL} \alpha]$ or $V \models_{IL} \beta$.
(3) $V \models_{IL} \neg\alpha$ if and only if not-$[V \models_{IL} \alpha]$.
(4) $V \models_{IL} \bigwedge \Theta$ if and only if $V \models_{IL} \alpha$ for any $\alpha \in \Theta$.
(5) $V \models_{IL} \bigvee \Theta$ if and only if $V \models_{IL} \alpha$ for some $\alpha \in \Theta$.

A formula α is said to be *IL-valid* if $V \models_{IL} \alpha$ (or equivalently $V(\alpha) = t$) for any valuation V. A sequent $\Gamma \Rightarrow \Delta$ is said to be IL-valid if the formula $\Gamma_* \rightarrow \Delta^*$ is IL-valid.

It is well known that the completeness theorem with respect to the semantics of IL is true for LK_ω (see, for example, Feferman (1968), Tanaka (1999), Tanaka (2001) and Takeuti (1985)).

We will need the following lemma for our proof of the semantical embedding theorem.

Lemma 2.10. Let Φ, Φ_i ($i \in \omega$) and f be the same as those in Definition 2.5. We suppose:
— V is a valuation from $\bigcup_{i \in \omega} \Phi_i$ to $\{t, f\}$.
— S is a non-empty set of states.
— (σ, I) is a model such that σ is a computation $s_0, s_1, s_2, \ldots$ ($s_i \in S, i \in \omega$), and I is an interpretation from Φ to the power set of S satisfying

$$\forall i \in \omega, \forall p \in \Phi \; [s_i \in I(p) \quad \text{if and only if} \quad V(p_i) = t].$$

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Then, for any formula α in $\mathcal{L}_{\mathrm{LTL}}$,

$$(\sigma, I, i) \models_{LTL} \alpha \quad \text{if and only if} \quad V \models_{IL} f(\mathrm{X}^i \alpha).$$

Proof. We use induction on α, and to simplify the notation, we will omit the V when we write $V \models_{IL}$ here.

— Base case $\alpha \equiv p \in \Phi$:
We have

$$\begin{aligned}(\sigma, I, i) \models_{LTL} p \quad &\text{iff} \quad s_i \in I(p)\\ &\text{iff} \quad V(p_i) = t\\ &\text{iff} \quad \models_{IL} p_i\\ &\text{iff} \quad \models_{IL} f(\mathrm{X}^i p).\end{aligned}$$

— Induction step:
We will only show some cases as examples.

– Case $\alpha \equiv \beta \wedge \gamma$:

$$\begin{aligned}(\sigma, I, i) \models_{LTL} \beta \wedge \gamma \quad &\text{iff} \quad (\sigma, I, i) \models_{LTL} \beta \quad \text{and} \quad (\sigma, I, i) \models_{LTL} \gamma\\ &\text{iff} \quad \models_{IL} f(\mathrm{X}^i \beta) \quad \text{and} \quad \models_{IL} f(\mathrm{X}^i \gamma)\\ &\qquad\qquad \text{(by the induction hypothesis)}\\ &\text{iff} \quad \models_{IL} f(\mathrm{X}^i \beta) \wedge f(\mathrm{X}^i \gamma)\\ &\text{iff} \quad \models_{IL} f(\mathrm{X}^i(\beta \wedge \gamma)) \qquad \text{(by the definition of } f\text{)}\end{aligned}$$

– Case $\alpha \equiv \neg\beta$:

$$\begin{aligned}(\sigma, I, i) \models_{LTL} \neg\beta \quad &\text{iff} \quad \text{not-}[(\sigma, I, i) \models_{LTL} \beta]\\ &\text{iff} \quad \text{not-}[\models_{IL} f(\mathrm{X}^i \beta)] \qquad \text{(by the induction hypothesis)}\\ &\text{iff} \quad \models_{IL} \neg f(\mathrm{X}^i \beta)\\ &\text{iff} \quad \models_{IL} f(\mathrm{X}^i \neg\beta) \qquad \text{(by the definition of } f\text{)}\end{aligned}$$

– Case $\alpha \equiv \mathrm{X}\beta$:

$$\begin{aligned}(\sigma, I, i) \models_{LTL} \mathrm{X}\beta \quad &\text{iff} \quad (\sigma, I, i+1) \models_{LTL} \beta\\ &\text{iff} \quad \models_{IL} f(\mathrm{X}^{i+1} \beta) \qquad \text{(by the induction hypothesis)}\\ &\text{iff} \quad \models_{IL} f(\mathrm{X}^i(\mathrm{X}\beta)) \qquad \text{(by the definition of } f\text{)}\end{aligned}$$

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– Case $\alpha \equiv G\beta$:

$$\begin{aligned}(\sigma, I, i) \models_{LTL} G\beta \quad & \text{iff} \quad \forall j \geqslant i\ [(\sigma, I, j) \models_{LTL} \beta] \\ & \text{iff} \quad \forall j \geqslant i\ [\models_{IL} f(X^j\beta)] \qquad \text{(by the induction hypothesis)} \\ & \text{iff} \quad \forall k \in \omega\ [\models_{IL} f(X^{i+k}\beta)] \\ & \text{iff} \quad \models_{IL} \gamma \quad \text{for all} \quad \gamma \in \{f(X^{i+k}\beta) \mid k \in \omega\} \\ & \text{iff} \quad \models_{IL} \bigwedge\{f(X^{i+k}\beta) \mid k \in \omega\} \\ & \text{iff} \quad \models_{IL} f(X^i G\beta). \qquad \text{(by the definition of } f)\end{aligned}$$

□

We then obtain the following theorem as a special case.

Theorem 2.11 (semantical embedding). Let f be the mapping defined in Definition 2.5. Then, for any formula α in $\mathcal{L}_{\mathrm{LTL}}$,

$$\alpha \text{ is LTL-valid} \quad \text{if and only if} \quad f(\alpha) \text{ is IL-valid.}$$

Theorem 2.12 (completeness). For any sequent S,

$$\mathrm{LT}_\omega \vdash S \quad \text{if and only if} \quad S \text{ is LTL-valid.}$$

Proof. Let S be $\Gamma \Rightarrow \Delta$ and α be $\Gamma_* {\rightarrow} \Delta^*$. Then:

$$\begin{aligned}\mathrm{LT}_\omega \vdash S \quad & \text{iff} \quad \mathrm{LT}_\omega \vdash \Rightarrow \alpha \\ & \text{iff} \quad \mathrm{LK}_\omega \vdash \Rightarrow f(\alpha) \\ & \qquad \text{(by Theorem 2.6 and the cut-elimination theorem for } \mathrm{LK}_\omega) \\ & \text{iff} \quad f(\alpha) \text{ is IL-valid} \qquad \text{(by the completeness theorem for } \mathrm{LK}_\omega) \\ & \text{iff} \quad \alpha \text{ is LTL-valid} \qquad \text{(by Theorem 2.11)} \\ & \text{iff} \quad S \text{ is LTL-valid.}\end{aligned}$$

□

2.3. *Indexed formulation*

Baratella and Masini's 2-sequent calculi $2S_\omega$ and $2SP_\omega$ for the propositional and first-order predicate LTLs were introduced in Baratella and Masini (2004), where they also proved the cut-elimination and completeness theorems for these calculi, presenting an analogy between LTL and Peano arithmetic with the ω-rule. Kamide (2006b) showed an equivalence between Kawai's LT_ω and Baratella and Masini's $2S_\omega$. We will now give an alternative proof of the cut-elimination theorems for LT_ω and $2S_\omega$ using this equivalence.

The language of $2S_\omega$ and the notation used are almost the same as those of LT_ω.

Definition 2.13. An expression α^i (α is a formula and $i \in \omega$) is called an indexed formula. An expression $\Gamma \Rightarrow^2 \Delta$ where Γ and Δ are finite (possibly empty) sets of indexed formulas is called a 2-sequent.

Definition 2.14 ($2S_\omega$). The initial sequents of $2S_\omega$ have the form

$$\alpha^i \Rightarrow^2 \alpha^i.$$

The structural rules of $2S_\omega$ have the form

$$\frac{\Gamma \Rightarrow^2 \Delta, \alpha^i \quad \alpha^i, \Sigma \Rightarrow^2 \Pi}{\Gamma, \Sigma \Rightarrow^2 \Delta, \Pi}\ (\text{cut2})$$

$$\frac{\Gamma \Rightarrow^2 \Delta}{\alpha^i, \Gamma \Rightarrow^2 \Delta}\ (\text{we-left2}) \qquad \frac{\Gamma \Rightarrow^2 \Delta}{\Gamma \Rightarrow^2 \Delta, \alpha^i}\ (\text{we-right2}).$$

The logical inference rules of $2S_\omega$ have the form

$$\frac{\Gamma \Rightarrow^2 \Sigma, \alpha^i \quad \beta^i, \Delta \Rightarrow^2 \Pi}{(\alpha{\rightarrow}\beta)^i, \Gamma, \Delta \Rightarrow^2 \Sigma, \Pi}\ ({\rightarrow}\text{left2}) \qquad \frac{\alpha^i, \Gamma \Rightarrow^2 \Delta, \beta^i}{\Gamma \Rightarrow^2 \Delta, (\alpha{\rightarrow}\beta)^i}\ ({\rightarrow}\text{right2})$$

$$\frac{\alpha^i, \Gamma \Rightarrow^2 \Delta}{(\alpha \wedge \beta)^i, \Gamma \Rightarrow^2 \Delta}\ (\wedge\text{left12}) \qquad \frac{\beta^i, \Gamma \Rightarrow^2 \Delta}{(\alpha \wedge \beta)^i, \Gamma \Rightarrow^2 \Delta}\ (\wedge\text{left22})$$

$$\frac{\Gamma \Rightarrow^2 \Delta, \alpha^i \quad \Gamma \Rightarrow^2 \Delta, \beta^i}{\Gamma \Rightarrow^2 \Delta, (\alpha \wedge \beta)^i}\ (\wedge\text{right2}) \qquad \frac{\alpha^i, \Gamma \Rightarrow^2 \Delta \quad \beta^i, \Gamma \Rightarrow^2 \Delta}{(\alpha \vee \beta)^i, \Gamma \Rightarrow^2 \Delta}\ (\vee\text{left2})$$

$$\frac{\Gamma \Rightarrow^2 \Delta, \alpha^i}{\Gamma \Rightarrow^2 \Delta, (\alpha \vee \beta)^i}\ (\vee\text{right12}) \qquad \frac{\Gamma \Rightarrow^2 \Delta, \beta^i}{\Gamma \Rightarrow^2 \Delta, (\alpha \vee \beta)^i}\ (\vee\text{right22})$$

$$\frac{\Gamma \Rightarrow^2 \Delta, \alpha^i}{(\neg\alpha)^i, \Gamma \Rightarrow^2 \Delta}\ (\neg\text{left2}) \qquad \frac{\alpha^i, \Gamma \Rightarrow^2 \Delta}{\Gamma \Rightarrow^2 \Delta, (\neg\alpha)^i}\ (\neg\text{right2})$$

$$\frac{\alpha^{i+1}, \Gamma \Rightarrow^2 \Delta}{(\mathrm{X}\alpha)^i, \Gamma \Rightarrow^2 \Delta}\ (\text{Xleft}) \qquad \frac{\Gamma \Rightarrow^2 \Delta, \alpha^{i+1}}{\Gamma \Rightarrow^2 \Delta, (\mathrm{X}\alpha)^i}\ (\text{Xright})$$

$$\frac{\alpha^{i+k}, \Gamma \Rightarrow^2 \Delta}{(\mathrm{G}\alpha)^i, \Gamma \Rightarrow^2 \Delta}\ (\text{Gleft2}) \qquad \frac{\{\ \Gamma \Rightarrow^2 \Delta, \alpha^{i+j}\ \}_{j\in\omega}}{\Gamma \Rightarrow^2 \Delta, (\mathrm{G}\alpha)^i}\ (\text{Gright2})$$

$$\frac{\{\ \alpha^{i+j}, \Gamma \Rightarrow^2 \Delta\ \}_{j\in\omega}}{(\mathrm{F}\alpha)^i, \Gamma \Rightarrow^2 \Delta}\ (\text{Fleft2}) \qquad \frac{\Gamma \Rightarrow^2 \Delta, \alpha^{i+k}}{\Gamma \Rightarrow^2 \Delta, (\mathrm{F}\alpha)^i}\ (\text{Fright2}).$$

Definition 2.15. Let $\mathcal{L}_1$ be the set of formulas of LT_ω and $\mathcal{L}_2$ be the set of indexed formulas of $2S_\omega$. Then:

(1) A mapping f from $\mathcal{L}_1$ to $\mathcal{L}_2$ is defined by

$$f(\mathrm{X}^i\alpha) := \alpha^i$$

for any formula α.

(2) A mapping g from $\mathcal{L}_2$ to $\mathcal{L}_1$ is defined by

$$g(\alpha^i) := X^i\alpha$$

for any formula α.

Note that

$$fg(\alpha^i) = \alpha^i$$
$$gf(X^i\alpha) = X^i\alpha$$

hold for any formula α.

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Theorem 2.16 (equivalence). We have:

(1) For any 2-sequent $\Gamma \Rightarrow^2 \Delta$, if

$$2S_\omega \vdash \Gamma \Rightarrow^2 \Delta,$$

then

$$LT_\omega \vdash g(\Gamma) \Rightarrow g(\Delta).$$

(2) For any sequent $\Gamma \Rightarrow \Delta$, if

$$LT_\omega\text{-(cut)} \vdash \Gamma \Rightarrow \Delta,$$

then

$$2S_\omega\text{(-cut2)} \vdash f(\Gamma) \Rightarrow^2 f(\Delta).$$

Proof. We will only show (1) as an example. We use induction on a cut-free proof P of $\Gamma \Rightarrow^2 \Delta$ in $2S_\omega$, and will only show one case as an example:

— Case (Xleft):
Hence, the last inference of P is of the form

$$\frac{\alpha^{i+1}, \Sigma \Rightarrow^2 \Pi}{(X\alpha)^i, \Sigma \Rightarrow^2 \Pi}\ \text{(Xleft)}.$$

By the induction hypothesis, we obtain

$$LT_\omega \vdash g(\alpha^{i+1}), g(\Sigma) \Rightarrow g(\Pi),$$

and thus

$$LT_\omega \vdash g((X\alpha)^i), g(\Sigma) \Rightarrow g(\Pi)$$

because

$$\begin{aligned} g(\alpha^{i+1}) &= X^{i+1}\alpha \\ &= X^i(X\alpha) \\ &= g((X\alpha)^i). \end{aligned}$$

□

Using the cut-elimination theorem for LT_ω and Theorem 2.16, we can now give an alternative proof of the following theorem (Baratella and Masini 2004).

Theorem 2.17 (cut-elimination). The rule (cut2) is admissible in cut-free $2S_\omega$.

Proof. Suppose

$$2S_\omega \vdash \Gamma \Rightarrow^2 \Delta$$

for an arbitrary 2-sequent $\Gamma \Rightarrow^2 \Delta$. Then we have

$$LT_\omega \vdash g(\Gamma) \Rightarrow g(\Delta)$$

by Theorem 2.16 (1). By the cut-elimination theorem for LT_ω, we obtain

$$LT_\omega\text{(-cut)} \vdash g(\Gamma) \Rightarrow g(\Delta),$$

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and thus

$$2S_\omega(-cut2) \vdash fg(\Gamma) \Rightarrow^2 fg(\Delta)$$

by Theorem 2.16 (2). Hence, we have

$$2S_\omega(-cut2) \vdash \Gamma \Rightarrow^2 \Delta$$

as required. □

Using Theorem 2.17 and an appropriate modification of Theorem 2.16, we can now get an alternative proof of the cut-elimination theorem for LT_ω.

The equivalence and cut-elimination results for $2S_\omega$ and LT_ω given above can be extended naturally to the first-order versions $2PS_\omega$ and FLT_ω, so we will omit them from the next section.

Baratella and Masini (2004) presented some extended results for some mathematical theories (that is, a set of extra-logical axioms) over $2S_\omega$ and $2PS_\omega$. We can also obtain some similar results over LT_ω and FLT_ω by using and extending the equivalence between LT_ω and S_ω and the equivalence between FLT_ω and $2PS_\omega$, respectively.

3. First-order LTL

3.1. *Syntactical embedding*

We use the following list of symbols for the language $\mathcal{L}$ of the underlying logic:

— $a_0, a_1, \ldots$ for free variables;
— $x_0, x_1, \ldots$ for bound variables;
— $f_0, f_1, \ldots$ for functions;
— $p_0, p_1, \ldots$ for predicates;
— $\rightarrow$, $\neg$, $\wedge$, $\vee$, $\forall$ (any), $\exists$ (exists), G (globally), F (eventually) and X (next) for the logical connectives.

The numbers of free and bound variables are assumed to be countable, as are the numbers of functions and predicates. We also assume that there is at least one predicate. A 0-ary function is an individual constant, and a 0-ary predicate is a propositional variable. We use lower case letters $p, q, \ldots$ to denote atomic formulas. We will continue to use a similar notation to that used in the previous section for the current section.

A sequent calculus FLT_ω for first-order LTL (called FLTL) is given by the following definition.

Definition 3.1. The initial sequents of FLT_ω are of the form

$$X^i p \Rightarrow X^i p$$

for any atomic formula p.

The structural rules of FLT_ω are (cut), (we-left) and (we-right) as in Definition 2.1.

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The logical inference rules of FLT_ω are obtained from those of LT_ω by adding logical inference rules of the form

$$\frac{\mathrm{X}^i\alpha(t), \Gamma \Rightarrow \Delta}{\mathrm{X}^i\forall x\alpha(x), \Gamma \Rightarrow \Delta}\ (\forall\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha(a)}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\forall x\alpha(x)}\ (\forall\text{right})$$

$$\frac{\mathrm{X}^i\alpha(a), \Gamma \Rightarrow \Delta}{\mathrm{X}^i\exists x\alpha(x), \Gamma \Rightarrow \Delta}\ (\exists\text{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha(t)}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\exists x\alpha(x)}\ (\exists\text{right})$$

where a is a free variable that must not occur in the lower sequents of ($\forall$right) and ($\exists$left), and t is an arbitrary term.

The fact that sequents of the form $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ for any formula α are provable in cut-free FLT_ω can be proved by induction on the complexity of α.

A language of first-order IL (called FIL) is obtained from $\mathcal{L}$ by removing $\{\wedge, \vee, \mathrm{G}, \mathrm{F}, \mathrm{X}\}$ and adding $\bigwedge$ and $\bigvee$.

Definition 3.2. We assume that the notion of a *term* is defined in the usual way. Let F_0 be the set of all formulas generated from the set of atomic formulas by the standard finitely inductive definition with respect to $\{\rightarrow, \neg, \forall, \exists\}$. We now suppose that F_t is already defined with respect to $t = 0, 1, 2, \ldots$. A non-empty countable subset Θ_t of F_t is said to be an *allowable set* if it contains a finite number of free variables. The expressions $\bigwedge\Theta$ and $\bigvee\Theta$ for an allowable set Θ are considered below. We define F_{t+1} from

$$F_t \cup \left\{\bigwedge\Theta, \bigvee\Theta \mid \Theta \text{ is an allowable set in } F_t\right\}$$

by the standard finitely inductive definition with respect to $\{\rightarrow, \neg, \forall, \exists\}$. We define F_ω, which is called the set of formulas, by $\bigcup_{t<\omega} F_t$, and an expression in F_ω is called a *formula*.

A sequent calculus FLK_ω for FIL is given by the following definition.

Definition 3.3. The initial sequents of FLK_ω are of the form

$$p \Rightarrow p$$

for any atomic formula p.

The structural rules of FLK_ω are (cut), (we-left) and (we-right) as in Definition 2.1.

The logical inference rules of FLK_ω are obtained from those of LK_ω by adding logical inference rules of the form

$$\frac{\alpha(t), \Gamma \Rightarrow \Delta}{\forall x\alpha(x), \Gamma \Rightarrow \Delta}\ (\forall\text{left}^0) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha(a)}{\Gamma \Rightarrow \Delta, \forall x\alpha(x)}\ (\forall\text{right}^0)$$

$$\frac{\alpha(a), \Gamma \Rightarrow \Delta}{\exists x\alpha(x), \Gamma \Rightarrow \Delta}\ (\exists\text{left}^0) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha(t)}{\Gamma \Rightarrow \Delta, \exists x\alpha(x)}\ (\exists\text{right}^0)$$

where a is a free variable that must not occur in the lower sequents of ($\forall$right^0) and ($\exists$left^0), and t is an arbitrary term, and

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad (\alpha \in \Theta)}{\bigwedge \Theta, \Gamma \Rightarrow \Delta} \ (\bigwedge \text{left}^f) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \alpha\ \}_{\alpha \in \Theta}}{\Gamma \Rightarrow \Delta, \bigwedge \Theta} \ (\bigwedge \text{right}^f)$$

$$\frac{\{\ \alpha, \Gamma \Rightarrow \Delta\ \}_{\alpha \in \Theta}}{\bigvee \Theta, \Gamma \Rightarrow \Delta} \ (\bigvee \text{left}^f) \qquad \frac{\Gamma \Rightarrow \Delta, \alpha \quad (\alpha \in \Theta)}{\Gamma \Rightarrow \Delta, \bigvee \Theta} \ (\bigvee \text{right}^f)$$

where Θ is an allowable set.

The superscript '0' in some of the rule names in FLK_ω means that these rules are the special cases of the corresponding rules of FLT_ω, that is, the case where i is 0. The superscript 'f' in some of the rule names in FLK_ω distinguishes them from the propositional case, that is, Θ in the rules is an allowable set. The sequents of the form $\alpha \Rightarrow \alpha$ for any formula α are provable in cut-free FLK_ω. It is well known that FLK_ω enjoys cut-elimination.

Definition 3.4. We fix a countable non-empty set Φ of atomic formulas, and define the sets

$$\Phi_i := \{p_i \mid p \in \Phi\} \quad (i \in \omega)$$

of atomic formulas where $p_0 = p$ (that is, $\Phi_0 = \Phi$). The language $\mathcal{L}_{\text{FLTL}}$ (or the set of formulas) of FLTL is defined using $\Phi, \rightarrow, \neg, \wedge, \vee, \forall, \exists$, X, G and F. The language $\mathcal{L}_{\text{FIL}}$ of FIL is defined using $\bigcup_{i \in \omega} \Phi_i, \rightarrow, \neg, \bigwedge, \bigvee, \forall$ and $\exists$ in a similar way to Definition 3.2. For convenience, the binary versions of $\bigwedge$ and $\bigvee$ are also denoted by $\wedge$ and $\vee$, respectively, and these binary symbols are assumed to be included in $\mathcal{L}_{\text{FIL}}$.

A mapping f from $\mathcal{L}_{\text{FLTL}}$ to $\mathcal{L}_{\text{FIL}}$ is obtained using the same conditions in Definition 2.5 with the addition of the following conditions:

$$f\left(\text{X}^i Qx\alpha(x)\right) := Qxf\left(\text{X}^i\alpha(x)\right)$$

where $Q \in \{\forall, \exists\}$.

Theorem 3.5 (syntactical embedding). Let Γ and Δ be sets of formulas in $\mathcal{L}_{\text{FLTL}}$, and f be the mapping defined in Definition 3.4.

(1) If

$$\text{FLT}_\omega \vdash \Gamma \Rightarrow \Delta,$$

then

$$\text{FLK}_\omega \vdash f(\Gamma) \Rightarrow f(\Delta).$$

(2) If

$$\text{FLK}_\omega - (\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta),$$

then

$$\text{FLT}_\omega - (\text{cut}) \vdash \Gamma \Rightarrow \Delta.$$

Proof. The proof is similar to the proof of Theorem 2.6. □

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Theorem 3.6 (cut-elimination). The rule (cut) is admissible in cut-free FLT_ω.

Proof. The statement follows from Theorem 3.5 and the cut-elimination theorem for FLK_ω. □

Note that we can strengthen the statements of Theorem 3.5 by replacing 'if … then' by 'if and only if'. This fact will be used to prove the completeness theorem for FLT_ω.

3.2. *Semantical embedding*

We write $\alpha[y/x]$ to denote the formula obtained from a formula α by replacing all free occurrences of an individual variable x in α by an arbitrary individual variable y, but avoiding any clash of variable names. Let Γ be a set $\{\alpha_1, \ldots, \alpha_m\}$ $(m \geqslant 0)$ of formulas. Then:

— Γ^* means $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geqslant 1$, and $\neg(p \rightarrow p)$ where p is a fixed atomic formula otherwise.
— Γ_* means $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geqslant 1$, and $p \rightarrow p$ where p is a fixed atomic formula otherwise.

For simplicity, we adopt a first-order language $\mathcal{L}_{\mathrm{FLTL}}$ without individual constants and function symbols for FLTL.

Definition 3.7. A structure

$$A := \langle U, \{I^i\}_{i \in \omega} \rangle$$

is said to be an *FLTL-model* if the following conditions hold:

(1) U is a non-empty set;
(2) I^i $(i \in \omega)$ are mappings such that

$$p^{I^i} \subseteq U^n$$

(that is, p^{I^i} are n-ary relations on U) for each n-ary predicate symbol p.

We write $\underset{\sim}{u}$ for the name of $u \in U$, and write $\mathcal{L}_{\mathrm{FLTL}}[A]$ to denote the language obtained from $\mathcal{L}_{\mathrm{FLTL}}$ by adding the names of all the elements of U. A formula α is said to be a *closed formula* if α has no free individual variables. A formula of the form $\forall x_1 \cdots \forall x_m \alpha$ is said to be the *universal closure* of α if the free variables of α are $x_1, \ldots, x_m$. We write $cl(\alpha)$ for the universal closure of α.

Definition 3.8. Let

$$A := \langle U, \{I^i\}_{i \in \omega} \rangle$$

be an FLTL-model. The satisfaction relations

$$A \models_i \alpha \quad (i \in \omega)$$

for any closed formula α of $\mathcal{L}_{\mathrm{FLTL}}[A]$ are defined inductively by:

(1) $A \models_i p(\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n)$ if and only if $(u_1, \ldots, u_n) \in p^{I^i}$ for each n-ary atomic formula $p(\underset{\sim}{u}_1, \ldots, \underset{\sim}{u}_n)$;
(2) $A \models_i \alpha \wedge \beta$ if and only if $A \models_i \alpha$ and $A \models_i \beta$;
(3) $A \models_i \alpha \vee \beta$ if and only if $A \models_i \alpha$ or $A \models_i \beta$;
(4) $A \models_i \alpha \rightarrow \beta$ if and only if not-($A \models_i \alpha$) or $A \models_i \beta$;

(5) $A \models_i \neg\alpha$ if and only if not-($A \models_i \alpha$);
(6) $A \models_i \forall x\alpha$ if and only if $A \models_i \alpha[\underline{u}/x]$ for all $u \in U$;
(7) $A \models_i \exists x\alpha$ if and only if $A \models_i \alpha[\underline{u}/x]$ for some $u \in U$;
(8) $A \models_i \mathrm{X}\alpha$ if and only if $A \models_{i+1} \alpha$;
(9) $A \models_i \mathrm{G}\alpha$ if and only if $A \models_j \alpha$ for any $j \geqslant i$;
(10) $A \models_i \mathrm{F}\alpha$ if and only if $A \models_j \alpha$ for some $j \geqslant i$.

The satisfaction relations

$$A \models_i \alpha \qquad (i \in \omega)$$

for any formula α of $\mathcal{L}_{\mathrm{FLTL}}$ are defined by

$$A \models_i \alpha \quad \text{if and only if} \quad A \models_i cl(\alpha).$$

A formula α of $\mathcal{L}_{\mathrm{FLTL}}$ is said to be *FLTL-valid* if $A \models_0 \alpha$ holds for each model A. A sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\mathrm{FLTL}}$ is said to be FLTL-valid if the formula $\Gamma_* \rightarrow \Delta^*$ is FLTL-valid.

In the following, we adopt a first-order language $\mathcal{L}_{\mathrm{FIL}}$ without individual constants and function symbols for FIL. We also assume that $\mathcal{L}_{\mathrm{FIL}}$ has uncountably many individual variables. This assumption is known to be necessary to get a completeness theorem for FIL, and is used to rename bound variables in the completeness proof.

Definition 3.9. A structure $B := \langle U, I \rangle$ is said to be an *FIL-model* if the following conditions hold:

(1) U is a non-empty set;
(2) I is a mapping such that $p^I \subseteq U^n$ (that is, p^I is a n-ary relation on U) for each n-ary predicate symbol p.

We write $\mathcal{L}_{\mathrm{FIL}}[B]$ to denote the language obtained from $\mathcal{L}_{\mathrm{FIL}}$ by adding the names of all the elements of U.

Definition 3.10. Let Θ be an allowable set and $B := \langle U, I \rangle$ be an FIL-model. The satisfaction relation $B \models \alpha$ for any closed formula α of $\mathcal{L}_{\mathrm{FIL}}[B]$ is defined inductively by

(1) $B \models p(\underline{u}_1, \ldots, \underline{u}_n)$ if and only if

$$(u_1, \ldots, u_n) \in p^I$$

for each n-ary atomic formula $p(\underline{u}_1, \ldots, \underline{u}_n)$,

(2) $B \models \bigwedge \Theta$ if and only if $B \models \alpha$ for all $\alpha \in \Theta$;
(3) $B \models \bigvee \Theta$ if and only if $B \models \alpha$ for some $\alpha \in \Theta$;
(4) $B \models \alpha \rightarrow \beta$ if and only if not-($B \models \alpha$) or $B \models \beta$;
(5) $B \models \neg\alpha$ if and only if not-($B \models \alpha$);
(6) $B \models \forall x\alpha$ if and only if $B \models \alpha[\underline{u}/x]$ for all $u \in U$;
(7) $B \models \exists x\alpha$ if and only if $B \models \alpha[\underline{u}/x]$ for some $u \in U$.

The satisfaction relation $B \models \alpha$ for any formula α of $\mathcal{L}_{\mathrm{FIL}}$ is defined by

$$B \models \alpha \quad \text{if and only if} \quad B \models cl(\alpha).$$

A formula α of $\mathcal{L}_{\mathrm{FIL}}$ is said to be *FIL-valid* if $B \models \alpha$ holds for each FIL-model B. A sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\mathrm{FIL}}$ is said to be FIL-valid if the formula $\Gamma_* \rightarrow \Delta^*$ is FIL-valid.

It is well known that the completeness theorem with respect to the FIL-model holds for FLK_ω.

In the following, we will use the same languages $\mathcal{L}_{\mathrm{FLTL}}$ and $\mathcal{L}_{\mathrm{FIL}}$ as in the previous discussion (that is, they have no individual constants or function symbols), and, for compatibility between $\mathcal{L}_{\mathrm{FIL}}$ and $\mathcal{L}_{\mathrm{FLTL}}$, we will also assume that $\mathcal{L}_{\mathrm{FLTL}}$ has uncountably many individual variables. In order to apply the mapping f in Definition 3.4, we assume the languages $\mathcal{L}_{\mathrm{FLTL}}$ and $\mathcal{L}_{\mathrm{FIL}}$ based on Φ and $\bigcup_{i\in\omega}\Phi_i$, respectively.

Lemma 3.11. Let f be the mapping defined in Definition 3.4. For any FLTL-model

$$A = \langle U, \{I^i\}_{i\in\omega}\rangle,$$

we can construct an FIL-model

$$B = \langle U, I\rangle$$

such that for any formula α in $\mathcal{L}_{\mathrm{FLTL}}$,

$$A \models_i \alpha \quad \text{if and only if} \quad B \models f(\mathrm{X}^i\alpha).$$

Proof. Let Φ be a set of atomic formulas and Φ_i be the set $\{p_i \mid p \in \Phi\}$ of atomic formulas with $p_0 := p$. Let U be given (that is, U is common in A and B). We now suppose that A is an FLTL-model

$$\langle U, \{I^i\}_{i\in\omega}\rangle$$

such that I^i $(i \in \omega)$ are mappings satisfying

$$p^{I^i} \subseteq U^n$$

for all $p \in \Phi$. Let B be an FIL-model $\langle U, I\rangle$ such that I is a mapping satisfying

$$p^I \subseteq U^n$$

for all

$$p \in \bigcup_{i\in\omega}\Phi_i$$

and that

$$(x_1, x_2, \ldots, x_n) \in p^{I^i}$$

if and only if

$$(x_1, x_2, \ldots, x_n) \in p_i^I.$$

The claim then follows by induction on the complexity of α. □

Lemma 3.12. Let f be the mapping defined in Definition 3.4. For any FIL-model

$$B = \langle U, I\rangle,$$

we can construct an FLTL-model

$$A = \langle U, \{I^i\}_{i\in\omega}\rangle$$

such that for any formula α in $\mathcal{L}_{\mathrm{FLTL}}$,

$$B \models f(\mathrm{X}^i\alpha) \quad \text{if and only if} \quad A \models_i \alpha.$$

Proof. The proof is similar to the proof of Lemma 3.11. □

Theorem 3.13 (semantical embedding). Let f be the mapping defined in Definition 3.4. For any formula α in $\mathcal{L}_{\mathrm{FLTL}}$, α is FLTL-valid if and only if $f(\alpha)$ is FIL-valid.

Proof. The statement follows from Lemmas 3.11 and 3.12. □

Theorem 3.14 (completeness). For any sequent S,

$$\mathrm{FLT}_\omega \vdash S$$

if and only if S is FLTL-valid.

Proof. Let S be $\Gamma \Rightarrow \Delta$ and α be $\Gamma_* \rightarrow \Delta^*$. Then,

$$\begin{array}{lll} \mathrm{FLT}_\omega \vdash S & \text{iff} & \mathrm{FLT}_\omega \vdash \Rightarrow \alpha \\ & \text{iff} & \mathrm{FLK}_\omega \vdash \Rightarrow f(\alpha) \\ & & \quad \text{(by Theorem 3.5 and the cut-elimination theorem for } \mathrm{FLK}_\omega) \\ & \text{iff} & f(\alpha) \text{ is FIL-valid} \qquad \text{(by the completeness theorem for } \mathrm{FLK}_\omega) \\ & \text{iff} & \alpha \text{ is FLTL-valid.} \qquad \text{(by Theorem 3.13)} \end{array}$$

□

4. Generalised first-order LTL

4.1. *Syntactical embedding*

Let n be a fixed positive integer. We use the symbol N to represent the set $\{1, 2, \ldots, n\}$ of indexes of modal operators. We use the following list of symbols for the language $\mathcal{L}$ of the underlying logic:

— $a_0, a_1, \ldots$ for free variables;
— $x_0, x_1, \ldots$ for bound variables;
— $f_0, f_1, \ldots$ for functions;
— $p_0, p_1, \ldots$ for predicates;
— $\rightarrow$, $\neg$, $\wedge$, $\vee$, $\forall$ (any), $\exists$ (exists) for the logical connectives;
— $\heartsuit_i$ ($i \in N$), $\heartsuit_{\mathrm{G}}$ (generalised G) and $\heartsuit_{\mathrm{F}}$ (generalised F) for the modal operators.

We assume the numbers of free and bound variables are countable, and that the numbers of functions and predicates are also countable. We also assume that there is at least one predicate. A 0-ary function is an individual constant, and a 0-ary predicate is a propositional variable. We use lower case letters $p, q, \ldots$ to denote atomic formulas. We use $\heartsuit\Gamma$ where

$$\heartsuit \in \{\heartsuit_i \mid i \in N\} \cup \{\heartsuit_{\mathrm{G}}, \heartsuit_{\mathrm{F}}\}$$

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to denote the set

$$\{\heartsuit\gamma \mid \gamma \in \Gamma\}.$$

We use the symbol K to represent the set

$$\{\heartsuit_i \mid i \in N\},$$

and the symbol K^* to represent the set of all words of finite length of the alphabet K. Note that K^* includes $\emptyset$, so $\{\imath\alpha \mid \imath \in K^*\}$ includes α. We use the lower-case Greek letters $\imath$ and κ to denote any members of K^*.

A sequent calculus GLT_ω for a generalised first-order LTL (called GLTL) is given by the following definition.

Definition 4.1. The initial sequents of GLT_ω are of the form

$$\imath p \Rightarrow \imath p$$

for any atomic formula p.

The structural rules of GLT_ω are (cut), (we-left) and (we-right) as in Definition 2.1.

The logical inference rules of GLT_ω are of the form

$$\frac{\Gamma \Rightarrow \Sigma, \imath\alpha \quad \imath\beta, \Delta \Rightarrow \Pi}{\imath(\alpha{\rightarrow}\beta), \Gamma, \Delta \Rightarrow \Sigma, \Pi}\ ({\rightarrow}\mathrm{left}^g) \qquad \frac{\imath\alpha, \Gamma \Rightarrow \Delta, \imath\beta}{\Gamma \Rightarrow \Delta, \imath(\alpha{\rightarrow}\beta)}\ ({\rightarrow}\mathrm{right}^g)$$

$$\frac{\imath\alpha, \Gamma \Rightarrow \Delta}{\imath(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\mathrm{left1}^g) \qquad \frac{\imath\beta, \Gamma \Rightarrow \Delta}{\imath(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\mathrm{left2}^g)$$

$$\frac{\Gamma \Rightarrow \Delta, \imath\alpha \quad \Gamma \Rightarrow \Delta, \imath\beta}{\Gamma \Rightarrow \Delta, \imath(\alpha \wedge \beta)}\ (\wedge\mathrm{right}^g) \qquad \frac{\imath\alpha, \Gamma \Rightarrow \Delta \quad \imath\beta, \Gamma \Rightarrow \Delta}{\imath(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\mathrm{left}^g)$$

$$\frac{\Gamma \Rightarrow \Delta, \imath\alpha}{\Gamma \Rightarrow \Delta, \imath(\alpha \vee \beta)}\ (\vee\mathrm{right1}^g) \qquad \frac{\Gamma \Rightarrow \Delta, \imath\beta}{\Gamma \Rightarrow \Delta, \imath(\alpha \vee \beta)}\ (\vee\mathrm{right2}^g)$$

$$\frac{\Gamma \Rightarrow \Delta, \imath\alpha}{\imath\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\mathrm{left}^g) \qquad \frac{\imath\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \imath\neg\alpha}\ (\neg\mathrm{right}^g)$$

$$\frac{\imath\alpha(t), \Gamma \Rightarrow \Delta}{\imath\forall x\alpha(x), \Gamma \Rightarrow \Delta}\ (\forall\mathrm{left}^g) \qquad \frac{\Gamma \Rightarrow \Delta, \imath\alpha(a)}{\Gamma \Rightarrow \Delta, \imath\forall x\alpha(x)}\ (\forall\mathrm{right}^g)$$

$$\frac{\imath\alpha(a), \Gamma \Rightarrow \Delta}{\imath\exists x\alpha(x), \Gamma \Rightarrow \Delta}\ (\exists\mathrm{left}^g) \qquad \frac{\Gamma \Rightarrow \Delta, \imath\alpha(t)}{\Gamma \Rightarrow \Delta, \imath\exists x\alpha(x)}\ (\exists\mathrm{right}^g)$$

where a is a free variable that must not occur in the lower sequents of $(\forall\mathrm{right}^g)$ and $(\exists\mathrm{left}^g)$, and t is an arbitrary term, and

$$\frac{\imath\kappa\alpha, \Gamma \Rightarrow \Delta}{\imath\heartsuit_{\mathrm{G}}\alpha, \Gamma \Rightarrow \Delta}\ (\heartsuit_{\mathrm{G}}\mathrm{left}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \imath\kappa\alpha\ \}_{\kappa \in K^*}}{\Gamma \Rightarrow \Delta, \imath\heartsuit_{\mathrm{G}}\alpha}\ (\heartsuit_{\mathrm{G}}\mathrm{right})$$

$$\frac{\{\ \imath\kappa\alpha, \Gamma \Rightarrow \Delta\ \}_{\kappa \in K^*}}{\imath\heartsuit_{\mathrm{F}}\alpha, \Gamma \Rightarrow \Delta}\ (\heartsuit_{\mathrm{F}}\mathrm{left}) \qquad \frac{\Gamma \Rightarrow \Delta, \imath\kappa\alpha}{\Gamma \Rightarrow \Delta, \imath\heartsuit_{\mathrm{F}}\alpha}\ (\heartsuit_{\mathrm{F}}\mathrm{right}).$$

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The fact that sequents of the form $\iota\alpha \Rightarrow \iota\alpha$ for any formula α are provable in cut-free GLT_ω can be proved by induction on the complexity of α. Note that GLT_ω includes FLT_ω as a special case.

The inference rules ($\heartsuit_{\mathrm{G}}$left) and ($\heartsuit_{\mathrm{G}}$right) are intended to imply the axiom scheme

$$\heartsuit_{\mathrm{G}}\alpha \leftrightarrow \bigwedge\{\iota\alpha \mid \iota \in K^*\}.$$

We now suppose that for any formula α, we have f_α is a mapping on the set of formulas such that

$$f_\alpha(x) := \bigwedge\{\heartsuit_i(x \wedge \alpha) \mid i \in \omega\}.$$

$\heartsuit_{\mathrm{G}}\alpha$ then becomes a fixpoint of f_α. The axiom scheme presented above just corresponds to the so-called iterative interpretation of common knowledge. On the other hand, if we take $K := \{\heartsuit_1\}$, we can understand $\heartsuit_1$ and $\heartsuit_{\mathrm{G}}$, respectively, as the temporal operators X and G in LTL. The corresponding axiom scheme for the singleton case represents the LTL-axiom scheme

$$\mathrm{G}\alpha \leftrightarrow \bigwedge\{\mathrm{X}^i\alpha \mid i \in \omega\}.$$

The operator $\heartsuit_{\mathrm{G}}$ can thus be regarded as a natural generalisation of G. Similarly, $\heartsuit_{\mathrm{F}}$ can be regarded as a generalisation of F.

The language of FIL and the sequent calculus FLK_ω follow from Definitions 3.2 and 3.3.

Definition 4.2. We fix a countable non-empty set Φ of atomic formulas, and define the sets

$$\Phi_\kappa := \{p_\kappa \mid p \in \Phi\} \qquad (\kappa \in K^*)$$

of atomic formulas where $p_\varnothing = p$ (that is, $\Phi_\varnothing := \Phi$). The language $\mathcal{L}_{\mathrm{GLTL}}$ (or the set of formulas) of GLTL is defined using Φ, $\rightarrow, \neg, \wedge, \vee, \forall, \exists$, $\heartsuit_i$ ($i \in N$), $\heartsuit_{\mathrm{G}}$ and $\heartsuit_{\mathrm{F}}$. The language $\mathcal{L}_{\mathrm{FIL}}$ of FIL is defined using $\bigcup_{\kappa\in K^*}\Phi_\kappa$, $\rightarrow, \neg$, $\bigwedge, \bigvee, \forall$ and $\exists$ in a similar way to Definition 3.2. For convenience, the binary versions of $\bigwedge$ and $\bigvee$ are also denoted by $\wedge$ and $\vee$, respectively, and these binary symbols are assumed to be included in $\mathcal{L}_{\mathrm{FIL}}$.

A mapping f from $\mathcal{L}_{\mathrm{GLTL}}$ to $\mathcal{L}_{\mathrm{FIL}}$ is defined as follows:

$$\begin{aligned}
f(\iota p) &:= p_\iota \in \Phi_\iota \quad (\iota \in K^*) && \text{for any } p \in \Phi \text{ (in particular, } f(p) := p \in \Phi_\varnothing) \\
f(\iota(\alpha \circ \beta)) &:= f(\iota\alpha) \circ f(\iota\beta) && \text{where } \circ \in \{\rightarrow, \wedge, \vee\} \\
f(\iota\neg\alpha) &:= \neg f(\iota\alpha) && \\
f(\iota Qx\alpha(x)) &:= Qxf(\iota\alpha(x)) && \text{where } Q \in \{\forall, \exists\} \\
f(\iota\heartsuit_{\mathrm{G}}\alpha) &:= \bigwedge\{f(\iota\kappa\alpha) \mid \kappa \in K^*\} && \\
f(\iota\heartsuit_{\mathrm{F}}\alpha) &:= \bigvee\{f(\iota\kappa\alpha) \mid \kappa \in K^*\}. &&
\end{aligned}$$

Theorem 4.3 (syntactical embedding). Let Γ and Δ be sets of formulas in $\mathcal{L}_{\mathrm{GLTL}}$, and f be the mapping defined in Definition 4.2. Then:

(1) If

$$\mathrm{GLT}_\omega \vdash \Gamma \Rightarrow \Delta,$$

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then

$$\mathrm{FLK}_{\omega} \vdash f(\Gamma) \Rightarrow f(\Delta).$$

(2) If

$$\mathrm{FLK}_{\omega} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta),$$

then

$$\mathrm{GLT}_{\omega} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta.$$

Proof. We can show part (1) by induction on the proof P of $\Gamma \Rightarrow \Delta$ in GLT_{ω}. We can show part (2) by induction on the proof Q of $f(\Gamma) \Rightarrow f(\Delta)$ in $\mathrm{FLK}_{\omega} - (\mathrm{cut})$.

We will just show the following case for (1) as an example.

— The last inference of P is of the form

$$\frac{\{\ \Gamma \Rightarrow \Delta, \iota\kappa\alpha\ \}_{\kappa \in K^*}}{\Gamma \Rightarrow \Delta, \iota\heartsuit_{\mathrm{G}}\alpha}\ (\heartsuit_{\mathrm{G}}\mathrm{right}).$$

By the induction hypothesis, we have

$$\mathrm{FLK}_{\omega} \vdash f(\Gamma) \Rightarrow f(\Delta), f(\iota\kappa\alpha)$$

for all $\kappa \in K^*$. Let Φ be $\{f(\iota\kappa\alpha) \mid \kappa \in K^*\}$. We then get the required result

$$\frac{\begin{array}{c}\vdots\\ \{\ f(\Gamma) \Rightarrow f(\Delta), f(\iota\kappa\alpha)\ \}_{f(\iota\kappa\alpha) \in \Phi}\end{array}}{f(\Gamma) \Rightarrow f(\Delta), \bigwedge \Phi}\ (\bigwedge \mathrm{right}^f)$$

where $\bigwedge \Phi$ coincides with $f(\iota\heartsuit_{\mathrm{G}}\alpha)$ by the definition of f. □

Theorem 4.4 (cut-elimination). The rule (cut) is admissible in cut-free GLT_{ω}.

Proof. The statement follows from Theorem 4.3 and the cut-elimination theorem for FLK_{ω}. □

Note that we can strengthen the statements of Theorem 4.3 by replacing 'if … then' by 'if and only if'. This fact will be used to prove the completeness theorem for GLT_{ω}.

4.2. *Semantical embedding*

In this section we will use similar notation to that used in the previous section, such as $\alpha[y/x]$. For simplicity, we adopt a first-order language $\mathcal{L}_{\mathrm{GLTL}}$ for GLTL without individual constants and function symbols. We write $\hat{\imath}$ for $i_1 i_2 \cdots i_k$ if

$$\iota = \heartsuit_{i_1}\heartsuit_{i_2}\cdots\heartsuit_{i_k}$$

and $\emptyset$ if $\iota = \emptyset$.

Definition 4.5. A structure

$$A := \left\langle U, \{I^{\hat{\imath}}\}_{\iota \in K^*} \right\rangle$$

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is said to be a *GLTL-model* if the following conditions hold:

(1) U is a non-empty set;

(2) $I^{\hat{\imath}}$ $(\imath \in K^*)$ are mappings such that

$$p^{I^{\hat{\imath}}} \subseteq U^n$$

(that is, $p^{I^{\hat{\imath}}}$ are n-ary relations on U) for each n-ary predicate symbol p.

We write $\underline{u}$ for the name of $u \in U$, and write $\mathcal{L}_{\mathrm{GLTL}}[A]$ to denote the language obtained from $\mathcal{L}_{\mathrm{GLTL}}$ by adding the names of all the elements of U. A formula α is said to be a *closed formula* if α has no free individual variable. A formula of the form $\forall x_1 \cdots \forall x_m \alpha$ is said to be the *universal closure* of α if the free variables of α are $x_1, \ldots, x_m$. We write $cl(\alpha)$ for the universal closure of α.

Definition 4.6. Let

$$A := \langle U, \{I^{\hat{\imath}}\}_{\imath \in K^*} \rangle$$

be a GLTL-model. The satisfaction relations

$$A \models_{\hat{\imath}} \alpha \qquad (\imath \in K^*)$$

for any closed formula α of $\mathcal{L}_{\mathrm{GLTL}}[A]$ are defined inductively by:

(1) $A \models_{\hat{\imath}} p(\underline{u}_1, \ldots, \underline{u}_n)$ if and only if $(u_1, \ldots, u_n) \in p^{I^{\hat{\imath}}}$ for each n-ary atomic formula $p(\underline{u}_1, \ldots, \underline{u}_n)$,

(2) $A \models_{\hat{\imath}} \alpha \wedge \beta$ if and only if $A \models_{\hat{\imath}} \alpha$ and $A \models_{\hat{\imath}} \beta$;

(3) $A \models_{\hat{\imath}} \alpha \vee \beta$ if and only if $A \models_{\hat{\imath}} \alpha$ or $A \models_{\hat{\imath}} \beta$;

(4) $A \models_{\hat{\imath}} \alpha \rightarrow \beta$ if and only if not-$(A \models_{\hat{\imath}} \alpha)$ or $A \models_{\hat{\imath}} \beta$;

(5) $A \models_{\hat{\imath}} \neg\alpha$ if and only if not-$(A \models_{\hat{\imath}} \alpha)$;

(6) $A \models_{\hat{\imath}} \forall x \alpha$ if and only if $A \models_{\hat{\imath}} \alpha[\underline{u}/x]$ for all $u \in U$;

(7) $A \models_{\hat{\imath}} \exists x \alpha$ if and only if $A \models_{\hat{\imath}} \alpha[\underline{u}/x]$ for some $u \in U$;

(8) for any $k \in N$, $A \models_{\hat{\imath}} \heartsuit_k \alpha$ if and only if $A \models_{\widehat{\imath k}} \alpha$;

(9) $A \models_{\hat{\imath}} \heartsuit_{\mathrm{G}} \alpha$ if and only if $A \models_{\widehat{\imath\kappa}} \alpha$ for all $\kappa \in K^*$;

(10) $A \models_{\hat{\imath}} \heartsuit_{\mathrm{F}} \alpha$ if and only if $A \models_{\widehat{\imath\kappa}} \alpha$ for some $\kappa \in K^*$.

The satisfaction relations

$$A \models_{\hat{\imath}} \alpha \qquad (\imath \in K^*)$$

for any formula α of $\mathcal{L}_{\mathrm{GLTL}}$ are defined by

$$A \models_{\hat{\imath}} \alpha \quad \text{if and only if} \quad A \models_{\hat{\imath}} cl(\alpha).$$

A formula α of $\mathcal{L}_{\mathrm{GLTL}}$ is called *GLTL-valid* if $A \models_{\varnothing} \alpha$ holds for each model A. A sequent $\Gamma \Rightarrow \Delta$ of $\mathcal{L}_{\mathrm{GLTL}}$ is called GLTL-valid if so is the formula $\Gamma_* \rightarrow \Delta^*$.

Note that

$$A \models_{\hat{\imath}} \kappa\alpha \quad \text{if and only if} \quad A \models_{\widehat{\imath\kappa}} \alpha$$

holds for any satisfaction relation $\models_{\hat{\imath}}$, any formula α and any $\kappa \in K^*$.

In the following, we use the same languages $\mathcal{L}_{\mathrm{GLTL}}$ and $\mathcal{L}_{\mathrm{FIL}}$ as in the previous discussion (that is, they have no individual constants or function symbols), and also assume for compatibility between $\mathcal{L}_{\mathrm{FIL}}$ and $\mathcal{L}_{\mathrm{GLTL}}$ that $\mathcal{L}_{\mathrm{GLTL}}$ has uncountably many individual variables. In order to apply the mapping f in Definition 4.2, we assume the languages $\mathcal{L}_{\mathrm{GLTL}}$ and $\mathcal{L}_{\mathrm{FIL}}$ based on Φ and $\bigcup_{\kappa\in K^*}\Phi_\kappa$, respectively.

Lemma 4.7. Let f be the mapping defined in Definition 4.2. For any GLTL-model

$$A = \langle U, \{I^{\hat{\imath}}\}_{\imath\in K^*}\rangle,$$

we can construct an FIL-model $B = \langle U, I\rangle$ such that for any formula α in $\mathcal{L}_{\mathrm{GLTL}}$,

$$A \models_{\hat{\imath}} \alpha \quad \text{if and only if} \quad B \models f(\imath\alpha).$$

Proof. Let Φ be a set of atomic formulas and Φ_κ be the set $\{p_\kappa \mid p \in \Phi\}$ of atomic formulas with $p_\varnothing := p$. Let U be given (that is, U is common in A and B). Suppose that A is a GLTL-model

$$\langle U, \{I^{\hat{\imath}}\}_{\imath\in K^*}\rangle$$

such that $I^{\hat{\imath}}$ ($\imath \in K^*$) are mappings satisfying

$$p^{I^{\hat{\imath}}} \subseteq U^n$$

for all $p \in \Phi$. Let B be an FIL-model $\langle U, I\rangle$ such that I is a mapping satisfying

$$p^I \subseteq U^n$$

for all

$$p \in \bigcup_{\kappa\in K^*}\Phi_\kappa$$

and that

$$(x_1, x_2, \ldots, x_n) \in p^{I^{\hat{\imath}}}$$

if and only if

$$(x_1, x_2, \ldots, x_n) \in p_\imath^I.$$

The proof then follows by induction on the complexity of α:

— Base step:

- Case ($\alpha \equiv p(\underline{x}_1, \ldots, \underline{x}_n) \in \Phi$):

$$\begin{aligned} A \models_{\hat{\imath}} p(\underline{x}_1, \ldots, \underline{x}_n) &\text{ iff } (x_1, x_2, \ldots, x_n) \in p^{I^{\hat{\imath}}} \\ &\text{ iff } (x_1, x_2, \ldots, x_n) \in p_\imath^I \\ &\text{ iff } B \models p_\imath \\ &\text{ iff } B \models f(\imath p(\underline{x}_1, \ldots, \underline{x}_n)). \qquad \text{(by the definition of } f\text{)} \end{aligned}$$

— Induction step:
We will only show the following cases as examples:

- Case ($\alpha \equiv \heartsuit_i\beta$):

$$\begin{aligned} A \models_{\hat{\imath}} \heartsuit_i\beta &\text{ iff } A \models_{\hat{\imath}i} \beta \\ &\text{ iff } B \models f(\imath\heartsuit_i\beta). \qquad \text{(by the induction hypothesis)} \end{aligned}$$

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– Case ($\alpha \equiv \heartsuit_G\beta$):

$$
\begin{aligned}
A \models_{\hat{\imath}} \heartsuit_G\beta \text{ iff } & A \models_{\hat{\imath}\hat{\kappa}} \beta \quad \text{for all} \quad \kappa \in K^* \\
\text{iff } & B \models f(\iota\kappa\beta) \quad \text{for all} \quad \kappa \in K^* \qquad \text{(by the induction hypothesis)} \\
\text{iff } & B \models \bigwedge\{f(\iota\kappa\beta) \mid \kappa \in K^*\} \\
\text{iff } & B \models f(\iota\heartsuit_G\beta). \qquad \text{(by the definition of } f\text{)}
\end{aligned}
$$

□

Lemma 4.8. Let f be the mapping defined in Definition 4.2. For any FIL-model

$$B = \langle U, I \rangle,$$

we can construct a GLTL-model

$$A = \langle U, \{I^{\hat{\imath}}\}_{\iota \in K^*} \rangle$$

such that for any formula α in $\mathcal{L}_{\mathrm{GLTL}}$,

$$B \models f(\iota\alpha) \quad \text{if and only if} \quad A \models_{\hat{\imath}} \alpha.$$

Proof. The proof is similar to the proof of Lemma 4.7. □

Theorem 4.9 (semantical embedding). Let f be the mapping defined in Definition 4.2. For any formula α in $\mathcal{L}_{\mathrm{GLTL}}$,

$$\alpha \text{ is GLTL-valid} \quad \text{if and only if} \quad f(\alpha) \text{ is FIL-valid.}$$

Proof. The statement follows from Lemmas 4.7 and 4.8. □

Theorem 4.10 (completeness). For any sequent S,

$$\mathrm{GLT}_\omega \vdash S \quad \text{if and only if} \quad S \text{ is GLTL-valid.}$$

Proof. The statement follows from Theorems 4.3 and 4.9 and the completeness theorem for FLK_ω. □

5. Infinitary extensions of LTL

We adopt the following list of symbols for the language of the underlying logic: countably many propositional variables $p_0, p_1, \ldots$, $\rightarrow$, $\neg$, $\bigwedge$, $\bigvee$, X, G, F and $\heartsuit$ (interior).

Definition 5.1 (L_ω and L_ω^-). The initial sequents of L_ω are of the form

$$\mathrm{X}^i p \Rightarrow \mathrm{X}^i p$$

for any propositional variable p.

The structural rules of L_ω are (cut), (we-left) and (we-right) as in Definition 2.1.

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The logical inference rules of L_ω are (→left), (→right), (¬left), (¬right), (Gleft), (Gright), (Fleft), (Fright) as in Definition 2.1 and the inference rules are of the form

$$\frac{X^i\alpha, \Gamma \Rightarrow \Delta \quad (\alpha \in \Theta)}{X^i(\bigwedge \Theta), \Gamma \Rightarrow \Delta}\ (\bigwedge \text{left}^l) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, X^i\alpha\ \}_{\alpha \in \Theta}}{\Gamma \Rightarrow \Delta, X^i(\bigwedge \Theta)}\ (\bigwedge \text{right}^l)$$

$$\frac{\{\ X^i\alpha, \Gamma \Rightarrow \Delta\ \}_{\alpha \in \Theta}}{X^i(\bigvee \Theta), \Gamma \Rightarrow \Delta}\ (\bigvee \text{left}^l) \qquad \frac{\Gamma \Rightarrow \Delta, X^i\alpha \quad (\alpha \in \Theta)}{\Gamma \Rightarrow \Delta, X^i(\bigvee \Theta)}\ (\bigvee \text{right}^l)$$

where Θ denotes a non-empty countable set of formulas, and

$$\frac{X^i\alpha, \Gamma \Rightarrow \Delta}{X^i\heartsuit\alpha, \Gamma \Rightarrow \Delta}\ (\heartsuit\text{left}) \qquad \frac{X^i\heartsuit\Gamma \Rightarrow X^k\alpha}{X^i\heartsuit\Gamma \Rightarrow X^k\heartsuit\alpha}\ (\heartsuit\text{right}).$$

where L_ω^- is obtained from L_ω by deleting {(♡left), (♡right)}.

Definition 5.2 ($S4_\omega$). A sequent calculus $S4_\omega$ for an infinitary version of the modal logic S4 can be obtained from L_ω by deleting (Gleft), (Gright), (Fleft), (Fright) and replacing i and k by 0 (that is, we delete every occurrence of X). The modified inference obtained rules for $S4_\omega$ by replacing i and k by 0 are denoted by a '0' superscript.

It is well known that $S4_\omega$ enjoys the cut-elimination property – see, for example, Kaneko and Nagashima (1997).

It can be shown by induction on the complexity of α that sequents of the form $X^i\alpha \Rightarrow X^i\alpha$ for any formula α are provable in cut-free L_ω and cut-free L_ω^-.

Proposition 5.3. Let L be L_ω or L_ω^-. The rule (Xregu) is admissible in cut-free L.

Proposition 5.4. For any formulas α and β, any non-empty countable set Θ of formulas and any $i \in \omega$, the following sequents are provable in cut-free L_ω and cut-free L_ω^-:

(1) $X^i(\alpha \to \beta) \Leftrightarrow X^i\alpha \to X^i\beta$,
(2) $X^i\neg\alpha \Leftrightarrow \neg X^i\alpha$,
(3) $X^i(\sharp\Theta) \Leftrightarrow \sharp(X^i\Theta)$ where $\sharp \in \{\bigwedge, \bigvee\}$,
(4) $G\alpha \Leftrightarrow \bigwedge\{X^i\alpha \mid i \in \omega\}$,
(5) $F\alpha \Leftrightarrow \bigvee\{X^i\alpha \mid i \in \omega\}$.

And for any formula α and any $i \in \omega$, the following sequents are provable in cut-free L_ω:

(6) $X^i\heartsuit\alpha \Leftrightarrow \heartsuit X^i\alpha$.

Proof. We will just show (4) and (6) as examples.

(4) (⇒)

$$\dfrac{\dfrac{\{\ X^i\alpha \Rightarrow X^i\alpha\ \}_{X^i\alpha \in \{X^i\alpha \mid i \in \omega\}}}{\{\ G\alpha \Rightarrow X^i\alpha\ \}_{X^i\alpha \in \{X^i\alpha \mid i \in \omega\}}}\ (\text{Gleft})}{G\alpha \Rightarrow \bigwedge\{X^i\alpha \mid i \in \omega\}}\ (\bigwedge \text{right}^l)$$

($\Leftarrow$)

$$\dfrac{\dfrac{\{\ \mathrm{X}^j\alpha \Rightarrow \mathrm{X}^j\alpha\ \}_{j\in\omega}}{\{\ \bigwedge\{\mathrm{X}^i\alpha \mid i\in\omega\} \Rightarrow \mathrm{X}^j\alpha\ \}_{j\in\omega}}\ (\bigwedge \text{left}^l)}{\bigwedge\{\mathrm{X}^i\alpha \mid i\in\omega\} \Rightarrow \mathrm{G}\alpha}\ (\text{Gright}).$$

(6) ($\Rightarrow$)

$$\dfrac{\dfrac{\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha}{\mathrm{X}^i\heartsuit\alpha \Rightarrow \mathrm{X}^i\alpha}\ (\heartsuit\text{left})}{\mathrm{X}^i\heartsuit\alpha \Rightarrow \heartsuit\mathrm{X}^i\alpha}\ (\heartsuit\text{right})$$

($\Leftarrow$)

$$\dfrac{\dfrac{\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha}{\heartsuit\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha}\ (\heartsuit\text{left})}{\heartsuit\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\heartsuit\alpha}\ (\heartsuit\text{right}).$$

□

Remarks 5.5 (on Proposition 5.4).

(i) The sequents listed in (1), (2), (3) and (6) correspond to the characteristic axioms for some next-interior fragments of DTL. In fact, a Hilbert-style axiomatisation of S4C can be obtained from that of S4 by adding the axiom schemes

(a) $\mathrm{X}(\alpha \circ \beta) \leftrightarrow \mathrm{X}\alpha \circ \mathrm{X}\beta$ where $\circ \in \{\rightarrow, \wedge, \vee\}$,

(b) $\mathrm{X}\neg\alpha \leftrightarrow \neg\mathrm{X}\alpha$,

(c) $\mathrm{X}\heartsuit\alpha \rightarrow \heartsuit\mathrm{X}\alpha$,

and the inference rule

$$\frac{\alpha}{\mathrm{X}\alpha}\ .$$

(ii) In particular, the sequents of the forms $\mathrm{X}\heartsuit\alpha \Rightarrow \heartsuit\mathrm{X}\alpha$ and $\heartsuit\mathrm{X}\alpha \Rightarrow \mathrm{X}\heartsuit\alpha$ listed in (6) correspond, respectively, to the *continuous axiom*, which characterises the continuity property of the function f on the topological space X of the underlying dynamic topological system (X, f), and the *homeomorphism axiom*, which characterises the open mapping property of f in (X, f). If a function is a continuous open bijection, the function is called a *homeomorphism*.

(iii) In order to prove the sequents listed in (6), we need the fact that the parameters i and k in ($\heartsuit$right) and ($\heartsuit$left) can be different from each other.

(iv) The sequents listed in (4) and (5) correspond to the characteristic axioms for a full DTL with a homeomorphism f on a topological space X. Intuitively, (4) and (5) are interpreted (Konev *et al.* 2006), for a given subset V of X, by

$$\mathrm{G}V := \bigcap\{f^{-i}(V) \mid i \in \omega\}$$

and

$$\mathrm{F}V := \bigcup\{f^{-i}(V) \mid i \in \omega\},$$

respectively, where f^{-i} means the i-times iteration of the inverse mapping of f.

Remarks 5.6 (comparison with other sequent systems).

(i) A sequent calculus $S4F_G$ (Artemov *et al.* 1997) for S4F can be obtained from a standard sequent system for S4 by adding (Xregu). The rules (→right), (→left) and (♡left) have been shown to be admissible in cut-free $S4F_G$ (Artemov *et al.* 1997).

(ii) A sequent calculus $S4C_G$ (Artemov *et al.* 1997) for S4C can be obtained from $S4F_G$ by adding a rule of the form

$$\frac{\heartsuit X \heartsuit \alpha, \Gamma \Rightarrow \Delta}{X \heartsuit \alpha, \Gamma \Rightarrow \Delta} .$$

(iii) Mints' sequent calculus for S4C (Mints 2006) is similar to $S4C_G$, and uses a rule of the form

$$\frac{\mathbf{B} \Rightarrow \alpha}{\mathbf{B} \Rightarrow \heartsuit \alpha}$$

where **B** is a set of formulas of the form $X^i \heartsuit \alpha$. Note that this rule does not allow us to derive the sequent $\heartsuit X \alpha \Rightarrow X \heartsuit \alpha$ of homeomorphisms. In order to derive such a sequent, we require the rule (♡right) proposed in this paper.

(iv) A sequent calculus for a bimodal version of DTL with a homeomorphism, called S4H on Kremer's *Dynamic Topological Logic* web page†, can then be regarded as the $\{\to, \wedge, \vee, \heartsuit, X\}$-fragment of L_ω.

Definition 5.7. We fix a countable non-empty set Φ of propositional variables, and define the sets

$$\Phi_i := \{p_i \mid p \in \Phi\} \qquad (1 \leqslant i \in \omega)$$
$$\Phi_0 := \Phi$$

of propositional variables where $p_0 = p$. The language $\mathcal{L}_{L_\omega}$ of L_ω is defined using Φ, $\to, \neg, \bigwedge, \bigvee, \heartsuit$, X, G and F. The language $\mathcal{L}_{S4_\omega}$ of $S4_\omega$ is defined using $\bigcup_{i\in\omega} \Phi_i$, $\to, \neg, \bigwedge, \bigvee$ and ♡.

A mapping f from $\mathcal{L}_{L_\omega}$ to $\mathcal{L}_{S4_\omega}$ is defined as follows:

$$f(X^i p) := p_i \in \Phi_i \qquad (i \in \omega) \qquad \text{for any } p \in \Phi \text{ (in particular, } f(p) := p \in \Phi)$$
$$f(X^i(\alpha \to \beta)) := f(X^i\alpha) \to f(X^i\beta)$$
$$f(X^i \neg\alpha) := \neg f(X^i\alpha)$$
$$f(X^i(\sharp\Theta)) := \sharp f(X^i\Theta) \qquad \text{where } \Theta \text{ is a non-empty countable set of formulas, and } \sharp \in \left\{\bigwedge, \bigvee\right\}$$
$$f(X^i \heartsuit \alpha) := \heartsuit f(X^i\alpha) \qquad (\dagger)$$
$$f(X^i G\alpha) := \bigwedge\{f(X^{i+j}\alpha) \mid j \in \omega\}$$
$$f(X^i F\alpha) := \bigvee\{f(X^{i+j}\alpha) \mid j \in \omega\}.$$

We also define the languages $\mathcal{L}_{L^-_\omega}$ (for L^-_ω) and $\mathcal{L}_{LK_\omega}$ (for LK_ω) as the ♡-less sublanguages of $\mathcal{L}_{L_\omega}$ and $\mathcal{L}_{S4_\omega}$, respectively. A mapping f from $\mathcal{L}_{L^-_\omega}$ to $\mathcal{L}_{LK_\omega}$ is then

† See http://individual.utoronto.ca/philipkremer/DynamicTopologicalLogic.html.

obtained from the above mapping by deleting the condition (†). We will also use the same name f for this mapping.

Theorem 5.8 (syntactical embedding). Let Γ and Δ be sets of formulas in $\mathcal{L}_{L_\omega}$, and f be the mapping defined in Definition 5.7.

(1) If

$$L_\omega \vdash \Gamma \Rightarrow \Delta,$$

then

$$S4_\omega \vdash f(\Gamma) \Rightarrow f(\Delta).$$

(2) If

$$S4_\omega - (\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta),$$

then

$$L_\omega - (\text{cut}) \vdash \Gamma \Rightarrow \Delta.$$

Let Γ and Δ be sets of formulas in $\mathcal{L}_{L_\omega^-}$, and f be the mapping defined secondly in Definition 5.7.

(1) If $L_\omega^- \vdash \Gamma \Rightarrow \Delta$, then $LK_\omega \vdash f(\Gamma) \Rightarrow f(\Delta)$.
(2) If $LK_\omega - (\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$, then $L_\omega^- - (\text{cut}) \vdash \Gamma \Rightarrow \Delta$.

Proof. We will only give the proof for the L_ω case as an example.

(1) We use induction on the proof P of $\Gamma \Rightarrow \Delta$ in L_ω. We distinguish the cases according to the last inference of P, and will only show the following case as an example:

— Case ($\heartsuit$right):
So the final inference of P is of the form

$$\frac{X^i \heartsuit \Gamma \Rightarrow X^k \alpha}{X^i \heartsuit \Gamma \Rightarrow X^k \heartsuit \alpha}\ (\heartsuit\text{right}).$$

By the induction hypothesis, we have

$$_\omega \vdash f(X^i \heartsuit \Gamma) \Rightarrow f(X^k \alpha),$$

that is,

$$_\omega \vdash \heartsuit f(X^i \Gamma) \Rightarrow f(X^k \alpha).$$

So we obtain

$$\frac{\begin{array}{c}\vdots\\ \heartsuit f(X^i \Gamma) \Rightarrow f(X^k \alpha)\end{array}}{\heartsuit f(X^i \Gamma) \Rightarrow \heartsuit f(X^k \alpha)}\ (\heartsuit\text{left}^0)$$

where

$$\heartsuit f(X^i \Gamma) \Rightarrow \heartsuit f(X^k \alpha)$$

coincides with

$$f(X^i \heartsuit \Gamma) \Rightarrow f(X^k \heartsuit \alpha)$$

by the definition of f.

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(2) We use induction on the proof Q of $f(\Gamma) \Rightarrow f(\Delta)$ in $\mathrm{S4}_\omega$ − (cut). We distinguish the cases according to the last inference of Q and will only show the following case as an example:

— Case ($\bigwedge$right^0):
We consider two subcases:

(a) The last inference of Q is of the form

$$\frac{\{\ f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^i\alpha)\ \}_{f(\mathrm{X}^i\alpha)\in f(\mathrm{X}^i\Theta)}}{f(\Gamma) \Rightarrow f(\Delta), \bigwedge f(\mathrm{X}^i\Theta)}\ (\bigwedge \mathrm{right}^0)$$

where $\bigwedge f(\mathrm{X}^i\Theta)$ coincides with $f(\mathrm{X}^i(\bigwedge\Theta))$ by the definition of f. By the induction hypothesis, we have

$$\mathrm{L}_\omega \vdash \Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha$$

for all $\mathrm{X}^i\alpha \in \mathrm{X}^i\Theta$, that is, for all $\alpha \in \Theta$. We then obtain

$$\frac{\begin{matrix}\vdots\\ \{\ \Gamma \Rightarrow \Delta, \mathrm{X}^i\alpha\ \}_{\alpha\in\Theta}\end{matrix}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i(\bigwedge\Theta)}\ (\bigwedge \mathrm{right}^l).$$

(b) The last inference of Q is of the form

$$\frac{\{\ f(\Gamma) \Rightarrow f(\Delta), f(\mathrm{X}^{i+j}\alpha)\ \}_{f(\mathrm{X}^{i+j}\alpha)\in\{f(\mathrm{X}^{i+j}\alpha)\ \mid\ j\in\omega\}}}{f(\Gamma) \Rightarrow f(\Delta), \bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}}\ (\bigwedge \mathrm{right}^0)$$

where

$$\bigwedge\{f(\mathrm{X}^{i+j}\alpha) \mid j \in \omega\}$$

coincides with $f(\mathrm{X}^i\mathrm{G}\alpha)$ by the definition of f. By the induction hypothesis, we have

$$\mathrm{L}_\omega \vdash \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha$$

for all

$$\mathrm{X}^{i+j}\alpha \in \{\mathrm{X}^{i+j}\alpha \mid j \in \omega\},$$

that is, for all $j \in \omega$. So we obtain

$$\frac{\begin{matrix}\vdots\\ \{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}\end{matrix}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\mathrm{Gright}).$$

□

Theorem 5.9 (cut-elimination). Let L be L_ω or L_ω^-. The rule (cut) is admissible in cut-free L.

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Proof. We will only show the case for L_ω as an example. Suppose

$$L_\omega \vdash \Gamma \Rightarrow \Delta.$$

Then, we have

$$S4_\omega \vdash f(\Gamma) \Rightarrow f(\Delta)$$

by Theorem 5.8 (1), and hence

$$S4_\omega - (\text{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta)$$

by the cut-elimination theorem for $S4_\omega$. By Theorem 5.8 (2), we then obtain

$$L_\omega - (\text{cut}) \vdash \Gamma \Rightarrow \Delta,$$

to complete the proof. □

We now define a semantics for L_ω^-.

Definition 5.10. Let Θ be a non-empty countable set of formulas. Timed valuations I^i ($i \in \omega$) are mappings from the set of all propositional variables to the set $\{t, f\}$ of truth values. Then, timed satisfaction relations $\models_i \alpha$ ($i \in \omega$) for any formula α are defined inductively by

(1) $\models_i p$ if and only if $I^i(p) = t$ for any propositional variable p.
(2) $\models_i \bigwedge \Theta$ if and only if $\models_i \alpha$ for any $\alpha \in \Theta$.
(3) $\models_i \bigvee \Theta$ if and only if $\models_i \alpha$ for some $\alpha \in \Theta$.
(4) $\models_i \alpha \rightarrow \beta$ if and only if not-($\models_i \alpha$) or $\models_i \beta$.
(5) $\models_i \neg\alpha$ if and only if not-($\models_i \alpha$).
(6) $\models_i X\alpha$ if and only if $\models_{i+1} \alpha$.
(7) $\models_i G\alpha$ if and only if $\models_j \alpha$ for any $j \geqslant i$.
(8) $\models_i F\alpha$ if and only if $\models_j \alpha$ for some $j \geqslant i$.

A formula α is said to be *L_ω^--valid* if $\models_0 \alpha$ holds for any timed satisfaction relations

$$\models^i \quad (i \in \omega).$$

A sequent $\Gamma \Rightarrow \Delta$ is said to be L_ω^--valid if the formula $\Gamma_* \rightarrow \Delta^*$ is L_ω^--valid.

In the following, we use Definition 6.7 as a semantics for LK_ω.

In order to apply the embedding function f in Definition 5.7, we assume the languages based on $\mathcal{L}_{L_\omega^-}$ and $\mathcal{L}_{LK_\omega}$ by constructing Φ and $\bigcup_{i\in\omega} \Phi_i$, respectively.

Lemma 5.11. Let f be the mapping defined in Definition 5.7. For any timed satisfaction relation $\models_i$ ($i \in \omega$), we can construct a satisfaction relation $\models$ such that for any formula α in $\mathcal{L}_{L_\omega^-}$, we have

$$\models_i \alpha \quad \text{if and only if} \quad \models f(X^i\alpha).$$

Proof. The proof is similar to the proof of Lemma 4.7. □

Lemma 5.12. Let f be the mapping defined in Definition 5.7. For any satisfaction relation $\models$ and any $i \in \omega$, we can construct a timed satisfaction relation $\models_i$ such that for any formula α in $\mathcal{L}_{\mathrm{L}_\omega^-}$,

$$\models f(\mathrm{X}^i\alpha) \quad \text{if and only if} \quad \models_i \alpha.$$

Proof. The proof is similar to the proof of Lemma 5.11. □

Theorem 5.13 (semantical embedding). Let f be the mapping defined in Definition 5.7. For any formula α in $\mathcal{L}_{\mathrm{L}_\omega^-}$,

$$\alpha \text{ is } \mathrm{L}_\omega^-\text{-valid} \quad \text{if and only if} \quad f(\alpha) \text{ is } \mathrm{LK}_\omega\text{-valid}.$$

Proof. The statement follows from Lemmas 5.11 and 5.12, where we take 0 for i. □

Theorem 5.14 (completeness). For any sequent S,

$$\mathrm{L}_\omega^- \vdash S \quad \text{if and only if} \quad S \text{ is } \mathrm{L}_\omega^-\text{-valid}.$$

Proof. The statement follows from Theorems 5.8 and 5.13. □

We will now show that the $\{\mathrm{G},\mathrm{F}\}$-free fragment (that is, X-only fragment) of L_ω^- is equivalent to L_ω^- under some appropriate interpretations of G and F.

Definition 5.15 (L_ω^x). A system L_ω^x is defined as the $\{\mathrm{G},\mathrm{F}\}$-free fragment of L_ω^-.

Theorem 5.16 (equivalence). Let G and F in L_ω^- be interpreted in L_ω^x by

$$\mathrm{G}\alpha := \bigwedge\{\mathrm{X}^i\alpha \mid i \in \omega\}$$
$$\mathrm{F}\alpha := \bigvee\{\mathrm{X}^i\alpha \mid i \in \omega\}.$$

L_ω^- and L_ω^x are theorem equivalent under this interpretation.

Proof. It is sufficient to show that the rules (Gleft), (Gright), (Fleft) and (Fright) in L_ω^- are derivable in L_ω^x under the interpretations of G and F.

We will only show the case of

$$\frac{\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}}{\Gamma \Rightarrow \Delta, \mathrm{X}^i\mathrm{G}\alpha}\ (\text{Gright}).$$

as an example. Let Θ be $\{\mathrm{X}^j\alpha \mid j \in \omega\}$. Then the set

$$\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^{i+j}\alpha\ \}_{j\in\omega}$$

means the set

$$\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta\ \}_{\beta\in\Theta}.$$

We assume that the sequents in

$$\{\ \Gamma \Rightarrow \Delta, \mathrm{X}^i\beta\ \}_{\beta\in\Theta}$$

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are provable in L^x_ω. We then have a proof in L^x_ω with

$$\dfrac{\begin{array}{c}\vdots\\ \{\ \Gamma \Rightarrow \Delta, X^i\beta\ \}_{\beta\in\Theta}\end{array}}{\Gamma \Rightarrow \Delta, X^i(\bigwedge \Theta)}\ (\bigwedge \text{right}^l)$$

where the sequent

$$\Gamma \Rightarrow \Delta, X^i(\bigwedge \Theta)$$

means the required sequent

$$\Gamma \Rightarrow \Delta, X^i G\alpha$$

by the interpretation of G. □

6. Other LTL-variations

6.1. *Spatial extensions of LTL*

We adopt the following for the language of the underlying logic:

— countably many propositional variables;
— $\to$, $\neg$, $\wedge$, $\vee$;
— P_x (position in x-axis);
— P_y (position in y-axis);
— P_z (position in z-axis);
— A (anywhere);
— S (somewhere);
— A^- (converse anywhere);
— S^- (converse somewhere).

An expression $\circ\Gamma$ where

$$\circ \in \{P_x, P_y, P_z, A, S, A^-, S^-\}$$

is used to denote the set

$$\{\circ\gamma \mid \gamma \in \Gamma\}.$$

We use the symbol P to represent

$$\{P_x, P_y, P_z\},$$

and the symbol P^* to represent the set of all words of finite length of the alphabet P. Note that P^* includes $\emptyset$. We use the Greek lower-case letter ι to denote any member of P^*.

ι' ($\in P^*$) is called a *permutation* of ι ($\in P^*$) if ι' is obtained from ι by a permutation of the symbols in ι. For example, $\iota' \equiv P_x P_y P_x$ is a permutation of $\iota \equiv P_x P_x P_y$, but $\iota'' \equiv P_x P_y P_y$ is not a permutation of ι. Note that ι is itself a permutation of ι. We sometimes use lower-case letters $i, j, k, i_x, i_y, i_z, \ldots$ to denote any natural numbers. An expression $P^i_m\alpha$ with $m \in \{x, y, z\}$ for any $i \in \omega$ is defined inductively by

$$P^0_m\alpha \equiv \alpha$$
$$P^{i+1}_m\alpha \equiv P_m P^i_m\alpha.$$

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Definition 6.1 (3SL). The initial sequents of 3SL are of the form

$$\iota p \Rightarrow \iota p$$

for any propositional variable p.

The structural inference rules of 3SL are the same as (cut), (we-left) and (we-right) in Definition 2.1.

The logical inference rules of 3SL are of the form for any $m, n \in \{x, y, z\}$,

$$\frac{\Gamma \Rightarrow \Delta, \iota\alpha \quad \iota\beta, \Sigma \Rightarrow \Pi}{\iota(\alpha{\rightarrow}\beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ ({\rightarrow}\text{left}^3) \qquad \frac{\iota\alpha, \Gamma \Rightarrow \Delta, \iota\beta}{\Gamma \Rightarrow \Delta, \iota(\alpha{\rightarrow}\beta)}\ ({\rightarrow}\text{right}^3)$$

$$\frac{\iota\alpha, \iota\beta, \Gamma \Rightarrow \Delta}{\iota(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\wedge\text{left}^3) \qquad \frac{\Gamma \Rightarrow \Delta, \iota\alpha \quad \Gamma \Rightarrow \Delta, \iota\beta}{\Gamma \Rightarrow \Delta, \iota(\alpha \wedge \beta)}\ (\wedge\text{right}^3)$$

$$\frac{\iota\alpha, \Gamma \Rightarrow \Delta \quad \iota\beta, \Gamma \Rightarrow \Delta}{\iota(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\vee\text{left}^3) \qquad \frac{\Gamma \Rightarrow \Delta, \iota\alpha, \iota\beta}{\Gamma \Rightarrow \Delta, \iota(\alpha \vee \beta)}\ (\vee\text{right}^3)$$

$$\frac{\Gamma \Rightarrow \Delta, \iota\alpha}{\iota\neg\alpha, \Gamma \Rightarrow \Delta}\ (\neg\text{left}^3) \qquad \frac{\iota\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \iota\neg\alpha}\ (\neg\text{right}^3)$$

$$\frac{\iota \mathrm{P}_m\mathrm{P}_n\alpha, \Gamma \Rightarrow \Delta}{\iota \mathrm{P}_n\mathrm{P}_m\alpha, \Gamma \Rightarrow \Delta}\ (\text{Pleft}) \qquad \frac{\Gamma \Rightarrow \Delta, \iota \mathrm{P}_m\mathrm{P}_n\alpha}{\Gamma \Rightarrow \Delta, \iota \mathrm{P}_n\mathrm{P}_m\alpha}\ (\text{Pright})$$

$$\frac{\iota \mathrm{P}_x^{k_x}\mathrm{P}_y^{k_y}\mathrm{P}_z^{k_z}\alpha, \Gamma \Rightarrow \Delta}{\iota \mathrm{A}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Aleft}) \qquad \frac{\{\ \Gamma \Rightarrow \Delta, \iota \mathrm{P}_x^{j_x}\mathrm{P}_y^{j_y}\mathrm{P}_z^{j_z}\alpha \mid j_x, j_y, j_z \in \omega\ \}}{\Gamma \Rightarrow \Delta, \iota \mathrm{A}\alpha}\ (\text{Aright})$$

$$\frac{\{\ \iota \mathrm{P}_x^{j_x}\mathrm{P}_y^{j_y}\mathrm{P}_z^{j_z}\alpha, \Gamma \Rightarrow \Delta \mid j_x, j_y, j_z \in \omega\ \}}{\iota \mathrm{S}\alpha, \Gamma \Rightarrow \Delta}\ (\text{Sleft})$$

$$\frac{\Gamma \Rightarrow \Delta, \iota \mathrm{P}_x^{k_x}\mathrm{P}_y^{k_y}\mathrm{P}_z^{k_z}\alpha}{\Gamma \Rightarrow \Delta, \iota \mathrm{S}\alpha}\ (\text{Sright}) \qquad \frac{\iota \mathrm{P}_x^{j_x}\mathrm{P}_y^{j_y}\mathrm{P}_z^{j_z}\alpha, \Gamma \Rightarrow \Delta}{\iota \mathrm{P}_x^{i_x}\mathrm{P}_y^{i_y}\mathrm{P}_z^{i_z}\mathrm{A}^-\alpha, \Gamma \Rightarrow \Delta}\ (\mathrm{A}^-\text{left})$$

with the conditions

$$0 \leqslant j_x \leqslant i_x$$
$$0 \leqslant j_y \leqslant i_y$$
$$0 \leqslant j_z \leqslant i_z,$$

and

$$\frac{\{\ \Gamma \Rightarrow \Delta, \iota \mathrm{P}_x^{j_x}\mathrm{P}_y^{j_y}\mathrm{P}_z^{j_z}\alpha \mid 0 \leqslant j_x \leqslant i_x, 0 \leqslant j_y \leqslant i_y, 0 \leqslant j_z \leqslant i_z\ \}}{\Gamma \Rightarrow \Delta, \iota \mathrm{P}_x^{i_x}\mathrm{P}_y^{i_y}\mathrm{P}_z^{i_z}\mathrm{A}^-\alpha}\ (\mathrm{A}^-\text{right})$$

$$\frac{\{\ \iota \mathrm{P}_x^{j_x}\mathrm{P}_y^{j_y}\mathrm{P}_z^{j_z}\alpha, \Gamma \Rightarrow \Delta \mid 0 \leqslant j_x \leqslant i_x, 0 \leqslant j_y \leqslant i_y, 0 \leqslant j_z \leqslant i_z\ \}}{\iota \mathrm{P}_x^{i_x}\mathrm{P}_y^{i_y}\mathrm{P}_z^{i_z}\mathrm{S}^-\alpha, \Gamma \Rightarrow \Delta}\ (\mathrm{S}^-\text{left})$$

$$\frac{\Gamma \Rightarrow \Delta, \iota \mathrm{P}_x^{j_x}\mathrm{P}_y^{j_y}\mathrm{P}_z^{j_z}\alpha}{\Gamma \Rightarrow \Delta, \iota \mathrm{P}_x^{i_x}\mathrm{P}_y^{i_y}\mathrm{P}_z^{i_z}\mathrm{S}^-\alpha}\ (\mathrm{S}^-\text{right})$$

with the conditions

$$0 \leqslant j_x \leqslant i_x$$
$$0 \leqslant j_y \leqslant i_y$$
$$0 \leqslant j_z \leqslant i_z.$$

Note that (Aright) and (Sleft) have infinite premises.

Proposition 6.2. The rules of the form

$$\frac{\Gamma \Rightarrow \Delta}{\mathrm{P}_m\Gamma \Rightarrow \mathrm{P}_m\Delta}\ (\mathrm{P}_m\mathrm{regu})$$

for any $m \in \{x, y, z\}$ are admissible in cut-free 3SL.

We can prove by induction on the complexity of α that sequents of the form $\iota\alpha \Rightarrow \iota\alpha$ for any formula α are provable in cut-free 3SL – we have to use Proposition 6.2 in the proof of this fact.

Definition 6.3. We fix a countable non-empty set Φ of propositional variables, and define the sets

$$\Phi_\iota := \{p_\iota \mid p \in \Phi\} \qquad (\iota \in P^*)$$

of propositional variables where $p_\varnothing = p$ (that is, $\Phi_\varnothing := \Phi$). The language $\mathcal{L}_{3\mathrm{SL}}$ (or the set of formulas) of 3SL is defined using Φ, $\rightarrow, \neg, \wedge, \vee$, P_x, P_y, P_z, A, S, A^- and S^-. The language $\mathcal{L}_{\mathrm{IL}}$ of LK_ω is defined using $\bigcup_{\iota \in P^*} \Phi_\iota$, $\rightarrow, \neg$, $\bigwedge$ and $\bigvee$. For convenience, the binary versions of $\bigwedge$ and $\bigvee$ are also denoted by $\wedge$ and $\vee$, respectively, and these binary symbols are assumed to be included in $\mathcal{L}_{\mathrm{IL}}$. For any permutations ι_1 and ι_2 of ι ($\in P^*$) and any $p \in \Phi$, we assume $p_{\iota_1} = p_{\iota_2}$, that is, $\Phi_{\iota_1} = \Phi_{\iota_2}$.

A mapping f from $\mathcal{L}_{3\mathrm{SL}}$ to $\mathcal{L}_{\mathrm{IL}}$ is defined as follows.

$$f(\iota p) := p_\iota \in \Phi_\iota \quad (\iota \in P^*) \qquad \text{for any } p \in \Phi$$
$$\text{(in particular, } f(p) := p \in \Phi_\varnothing)$$
$$f(\iota(\alpha \circ \beta)) := f(\iota\alpha) \circ f(\iota\beta) \qquad \text{where } \circ \in \{\rightarrow, \wedge, \vee\}$$
$$f(\iota\neg\alpha) := \neg f(\iota\alpha)$$
$$f(\iota \mathrm{P}_l \mathrm{P}_m \alpha) := f(\iota \mathrm{P}_m \mathrm{P}_l \alpha) \qquad \text{for any } l, m \in \{x, y, z\}$$
$$f(\iota \mathrm{A}\alpha) := \bigwedge\{f(\iota \mathrm{P}_x^{j_x} \mathrm{P}_y^{j_y} \mathrm{P}_z^{j_z} \alpha) \mid j_x, j_y, j_z \in \omega\}$$
$$f(\iota \mathrm{S}\alpha) := \bigvee\{f(\iota \mathrm{P}_x^{j_x} \mathrm{P}_y^{j_y} \mathrm{P}_z^{j_z} \alpha) \mid j_x, j_y, j_z \in \omega\}$$
$$f(\iota \mathrm{P}_x^{i_x} \mathrm{P}_y^{i_y} \mathrm{P}_z^{i_z} \mathrm{A}^-\alpha) := \bigwedge\{f(\iota \mathrm{P}_x^{j_x} \mathrm{P}_y^{j_y} \mathrm{P}_z^{j_z} \alpha) \mid 0 \leqslant j_x \leqslant i_x, 0 \leqslant j_y \leqslant i_y, 0 \leqslant j_z \leqslant i_z\}$$
$$\text{for any } i_x, i_y, i_z \in \omega$$
$$f(\iota \mathrm{P}_x^{i_x} \mathrm{P}_y^{i_y} \mathrm{P}_z^{i_z} \mathrm{S}^-\alpha) := \bigvee\{f(\iota \mathrm{P}_x^{j_x} \mathrm{P}_y^{j_y} \mathrm{P}_z^{j_z} \alpha) \mid 0 \leqslant j_x \leqslant i_x, 0 \leqslant j_y \leqslant i_y, 0 \leqslant j_z \leqslant i_z\}$$
$$\text{for any } i_x, i_y, i_z \in \omega.$$

Theorem 6.4 (syntactical embedding). Let Γ and Δ be sets of formulas in $\mathcal{L}_{3SL}$, and f be the mapping defined in Definition 6.3. Then:

(1) If
$$3SL \vdash \Gamma \Rightarrow \Delta,$$
then
$$LK_{\omega} \vdash f(\Gamma) \Rightarrow f(\Delta).$$

(2) If
$$LK_{\omega} - (cut) \vdash f(\Gamma) \Rightarrow f(\Delta),$$
then
$$3SL - (cut) \vdash \Gamma \Rightarrow \Delta.$$

Theorem 6.5 (cut-elimination). The rule (cut) is admissible in cut-free 3SL.

The following definition gives a semantics for 3SL.

Definition 6.6. Space-indexed valuations $I^{i_x;i_y;i_z}$ $(i_x, i_y, i_z \in \omega)$ are mappings from the set of all propositional variables to the set $\{t, f\}$ of truth values. Then, space-indexed satisfaction relations $\models_{i_x;i_y;i_z} \alpha$ $(i_x, i_y, i_z \in \omega)$ for any formula α are defined inductively by:

(1) $\models_{i_x;i_y;i_z} p$ if and only if $I^{i_x;i_y;i_z}(p) = t$ for any propositional variable p.
(2) $\models_{i_x;i_y;i_z} \alpha \wedge \beta$ if and only if $\models_{i_x;i_y;i_z} \alpha$ and $\models_{i_x;i_y;i_z} \beta$.
(3) $\models_{i_x;i_y;i_z} \alpha \vee \beta$ if and only if $\models_{i_x;i_y;i_z} \alpha$ or $\models_{i_x;i_y;i_z} \beta$.
(4) $\models_{i_x;i_y;i_z} \alpha{\rightarrow}\beta$ if and only if not-($\models_{i_x;i_y;i_z} \alpha$) or $\models_{i_x;i_y;i_z} \beta$.
(5) $\models_{i_x;i_y;i_z} \neg\alpha$ if and only if not-($\models_{i_x;i_y;i_z} \alpha$).
(6) $\models_{i_x;i_y;i_z} P_x\alpha$ if and only if $\models_{i_x+1;i_y;i_z} \alpha$.
(7) $\models_{i_x;i_y;i_z} P_y\alpha$ if and only if $\models_{i_x;i_y+1;i_z} \alpha$.
(8) $\models_{i_x;i_y;i_z} P_z\alpha$ if and only if $\models_{i_x;i_y;i_z+1} \alpha$.
(9) $\models_{i_x;i_y;i_z} A\alpha$ if and only if $\models_{i_x+j_x;i_y+j_y;i_z+j_z} \alpha$ for any $j_x, j_y, j_z \in \omega$.
(10) $\models_{i_x;i_y;i_z} S\alpha$ if and only if $\models_{i_x+j_x;i_y+j_y;i_z+j_z} \alpha$ for some $j_x, j_y, j_z \in \omega$.
(11) $\models_{i_x;i_y;i_z} A^{-}\alpha$ if and only if $\models_{j_x;j_y;j_z} \alpha$ for any $j_x, j_y, j_z \in \omega$ with
$$0 \leqslant j_x \leqslant i_x$$
$$0 \leqslant j_y \leqslant i_y$$
$$0 \leqslant j_z \leqslant i_z.$$
(12) $\models_{i_x;i_y;i_z} S^{-}\alpha$ if and only if $\models_{j_x;j_y;j_z} \alpha$ for some $j_x, j_y, j_z \in \omega$ with
$$0 \leqslant j_x \leqslant i_x$$
$$0 \leqslant j_y \leqslant i_y$$
$$0 \leqslant j_z \leqslant i_z.$$

A formula α is said to be *3SL-valid* if $\models_{0;0;0} \alpha$ holds for any space-indexed satisfaction relations $\models_{i_x;i_y;i_z}$ $(i_x, i_y, i_z \in \omega)$. A sequent $\Gamma \Rightarrow \Delta$ is said to be 3SL-valid if the formula $\Gamma_{*}{\rightarrow}\Delta^{*}$ is 3SL-valid.

To ensure compatibility of proofs, we will now redefine a semantics for LK_ω. This semantics is essentially the same as the semantics of IL defined in Definition 2.9.

Definition 6.7. Let Θ be a countable non-empty set of formulas. A valuation I is a mapping from the set of all propositional variables to the set $\{t, f\}$ of truth values. A satisfaction relation $\models \alpha$ for any formula α is defined inductively by:

(1) $\models p$ if and only if $I(p) = t$ for any propositional variable p.
(2) $\models \bigwedge \Theta$ if and only if $\models \alpha$ for any $\alpha \in \Theta$.
(3) $\models \bigvee \Theta$ if and only if $\models \alpha$ for some $\alpha \in \Theta$.
(4) $\models \alpha \rightarrow \beta$ if and only if not-($\models \alpha$) or $\models \beta$.
(5) $\models \neg\alpha$ if and only if not-($\models \alpha$).

A formula α is said to be *LK_ω-valid* if $\models \alpha$ holds for any satisfaction relation $\models$. A sequent $\Gamma \Rightarrow \Delta$ is said to be LK_ω-valid if the formula $\Gamma_* \rightarrow \Delta^*$ is LK_ω-valid.

Lemma 6.8. Let f be the mapping defined in Definition 6.3. For any space-indexed satisfaction relation $\models_{i_x;i_y;i_z}$ $(i_x, i_y, i_z \in \omega)$, we can construct a satisfaction relation $\models$ such that for any formula α in $\mathcal{L}_{3SL}$,

$$\models_{i_x;i_y;i_z} \alpha \quad \text{if and only if} \quad \models f(P_x^{i_x} P_y^{i_y} P_z^{i_z} \alpha).$$

Lemma 6.9. Let f be the mapping defined in Definition 6.3. For any satisfaction relation $\models$, we can construct a space-indexed satisfaction relation $\models_{i_x;i_y;i_z}$ such that for any formula α in $\mathcal{L}_{3SL}$,

$$\models f(P_x^{i_x} P_y^{i_y} P_z^{i_z} \alpha) \quad \text{if and only if} \quad \models_{i_x;i_y;i_z} \alpha.$$

Theorem 6.10 (semantical embedding). Let f be the mapping defined in Definition 6.3. For any formula α in $\mathcal{L}_{3SL}$,

$$\alpha \text{ is 3SL-valid} \quad \text{if and only if} \quad f(\alpha) \text{ is } LK_\omega\text{-valid}.$$

Proof. The statement follows from Lemmas 6.8 and 6.9, where we take 0 for i_x, i_y and i_z. □

Theorem 6.11 (completeness). For any sequent S,

$$3SL \vdash S \quad \text{if and only if} \quad S \text{ is 3SL-valid}.$$

Proof. The statement follows from Theorems 6.4, 6.5 and 6.10. □

6.2. *Next-time only fragments of LTL*

The *formulas* of SDL are constructed from countably many propositional variables, $\bot$ (falsity constant), $\rightarrow$, $\wedge$, $\vee$ and X. In Kamide (2010c), the constant $\bot$ was used to help give a direct proof (that is, without using embedding) of the Kripke-completeness theorem for SDL. We use Greek capital letters $\Gamma, \Delta, \ldots$ to represent finite (possibly empty) *sequences* of formulas, and an expression $X\Gamma$ to denote the sequence $\langle X\gamma \mid \gamma \in \Gamma \rangle$. We use the symbol

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$\equiv$ to denote the equality of sequences of symbols and the symbol N to represent the set of natural numbers. An expression of the form $\Gamma \Rightarrow \Delta$ where Δ is empty or a single formula is called an *intuitionistic sequent* (or just *sequent* for short).

We will now define SDL, where without risk of confusion, we will use the same rule names as we used for LT_ω.

Definition 6.12 (SDL)**.** In the following definition, Δ represents the empty sequence or a single formula.

The initial sequents of SDL are of the form

$$\mathrm{X}^i p \Rightarrow \mathrm{X}^i p$$
$$\mathrm{X}^i \bot \Rightarrow$$

for any propositional variable p.

The structural rules of SDL are of the form

$$\frac{\Gamma \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \Delta}{\Gamma, \Sigma \Rightarrow \Delta} \text{ (cut)}$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (we-left)} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \text{ (we-right)}$$

$$\frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \text{ (co)} \qquad \frac{\Gamma, \alpha, \beta, \Sigma \Rightarrow \Delta}{\Gamma, \beta, \alpha, \Sigma \Rightarrow \Delta} \text{ (ex).}$$

The logical inference rules of SDL are of the form

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \mathrm{X}^i\beta, \Sigma \Rightarrow \Delta}{\mathrm{X}^i(\alpha{\to}\beta), \Gamma, \Sigma \Rightarrow \Delta} \text{ (}{\to}\text{left)} \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha{\to}\beta)} \text{ (}{\to}\text{right)}$$

$$\frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left1)} \qquad \frac{\mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \wedge \beta), \Gamma \Rightarrow \Delta} \text{ (}\wedge\text{left2)}$$

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha \quad \Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \wedge \beta)} \text{ (}\wedge\text{right)} \qquad \frac{\mathrm{X}^i\alpha, \Gamma \Rightarrow \Delta \quad \mathrm{X}^i\beta, \Gamma \Rightarrow \Delta}{\mathrm{X}^i(\alpha \vee \beta), \Gamma \Rightarrow \Delta} \text{ (}\vee\text{left)}$$

$$\frac{\Gamma \Rightarrow \mathrm{X}^i\alpha}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \vee \beta)} \text{ (}\vee\text{right1)} \qquad \frac{\Gamma \Rightarrow \mathrm{X}^i\beta}{\Gamma \Rightarrow \mathrm{X}^i(\alpha \vee \beta)} \text{ (}\vee\text{right2).}$$

Sequents of the form $\mathrm{X}^i\alpha \Rightarrow \mathrm{X}^i\alpha$ for any formula α are provable in cut-free SDL.

Proposition 6.13. Let Δ be the empty sequence or a single formula. The following rule is admissible in cut-free SDL:

$$\frac{\Gamma \Rightarrow \Delta}{\mathrm{X}\Gamma \Rightarrow \mathrm{X}\Delta} \text{ (Xregu).}$$

Definition 6.14 (LJ)**.** A sequent calculus LJ for propositional intuitionistic logic can be obtained from SDL by replacing X^i with X^0.

Definition 6.15. We fix a countable non-empty set Φ of propositional variables and define the sets

$$\Phi_i := \{p_i \mid p \in \Phi\} \quad (i \in \omega)$$

of propositional variables where $p_0 := p$, that is, $\Phi_0 = \Phi$. The language $\mathcal{L}_{\mathrm{SDL}}$ of SDL is defined using Φ, $\bot, \to, \wedge, \vee$ and X. The language $\mathcal{L}_{\mathrm{LJ}}$ of LJ is defined using $\bigcup_{i\in\omega} \Phi_i$, $\bot, \to, \wedge$ and $\vee$.

A mapping f from $\mathcal{L}_{\mathrm{SDL}}$ to $\mathcal{L}_{\mathrm{LJ}}$ is defined by

$$\begin{aligned} f(\mathrm{X}^i\bot) &:= \bot \\ f(\mathrm{X}^i p) &:= p_i \in \Phi_i && \text{for any } p \in \Phi \text{ (in particular, } f(p) := p \in \Phi) \\ f(\mathrm{X}^i(\alpha \circ \beta)) &:= f(\mathrm{X}^i\alpha) \circ f(\mathrm{X}^i\beta) && \text{where } \circ \in \{\to, \wedge, \vee\}. \end{aligned}$$

Theorem 6.16 (syntactical embedding). Let Γ be a sequence of formulas in $\mathcal{L}_{\mathrm{SDL}}$, Δ be the empty sequence or a formula in $\mathcal{L}_{\mathrm{SDL}}$, and f be the mapping defined in Definition 6.15. Then:

(1) If

$$\mathrm{SDL} \vdash \Gamma \Rightarrow \Delta,$$

then

$$\mathrm{LJ} \vdash f(\Gamma) \Rightarrow f(\Delta).$$

(2) If

$$\mathrm{LJ} - (\mathrm{cut}) \vdash f(\Gamma) \Rightarrow f(\Delta),$$

then

$$\mathrm{SDL} - (\mathrm{cut}) \vdash \Gamma \Rightarrow \Delta.$$

Theorem 6.17 (cut-elimination). The rule (cut) is admissible in cut-free SDL.

Theorem 6.18 (decidability). SDL is decidable.

Proof. By Theorem 6.16 and the cut-elimination theorem for LJ, we have

$$\mathrm{SDL} \vdash \Gamma \Rightarrow \Delta$$

if and only if

$$\mathrm{LJ} \vdash f(\Gamma) \Rightarrow f(\Delta).$$

Hence, provability in SDL can be reduced to provability in LJ, and since LJ is decidable, SDL is also decidable. □

We will now introduce a Kripke semantics for SDL.

Definition 6.19. A *timed Kripke frame* is a structure $\langle M, N, R\rangle$ satisfying the following conditions:

(1) M is a non-empty set.
(2) N is the set of natural numbers.
(3) R is a reflexive and transitive binary relation on M.

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The set M can be understood as a set of information states, and the set N as a set of time points.

Definition 6.20. A *timed valuation* $\models$ on a Kripke frame $\langle M, N, R\rangle$ is a mapping from the set Ψ of all propositional variables to the power set $2^{M\times N}$ of the direct product $M \times N$ such that for any $p \in \Psi$, any $i \in N$, and any $x, y \in M$, if $(x, i) \in \models(p)$ and xRy, then $(y, i) \in \models(p)$. We will write $(x, i) \models p$ for $(x, i) \in \models(p)$. Each timed valuation $\models$ is extended to a mapping from the set Φ of all formulas to $2^{M\times N}$ by the following clauses:

(1) $(x, i) \models \bot$ does not hold.
(2) $(x, i) \models \alpha \to \beta$ if and only if $\forall y \in M$ [xRy and $(y, i) \models \alpha$ imply $(y, i) \models \beta$].
(3) $(x, i) \models \alpha \wedge \beta$ if and only if $(x, i) \models \alpha$ and $(x, i) \models \beta$.
(4) $(x, i) \models \alpha \vee \beta$ if and only if $(x, i) \models \alpha$ or $(x, i) \models \beta$.
(5) $(x, i) \models \mathrm{X}\alpha$ if and only if $(x, i+1) \models \alpha$.

The statement $(x, i) \models \alpha$ can be read as 'α is true at the information state x and the time i'.

Proposition 6.21. Let $\models$ be a timed valuation on a timed Kripke frame $\langle M, N, R\rangle$. For any formula α, any $i \in N$ and any $x, y \in M$, if $(x, i) \models \alpha$ and xRy, then $(y, i) \models \alpha$.

An expression $\Gamma^{\wedge}$ means $\gamma_1 \wedge \gamma_2 \wedge \cdots \wedge \gamma_n$ if $\Gamma \equiv \langle \gamma_1, \gamma_2, \ldots, \gamma_n\rangle$ $(0 \leqslant n)$. Let Δ be the empty sequence or a sequence consisting of a single formula. An expression Δ^* means α or $\bot$ if $\Delta \equiv \langle\alpha\rangle$ or $\emptyset$, respectively. An expression $(\Gamma \Rightarrow \Delta)^*$ means $\Gamma^{\wedge} \to \Delta^*$ if Γ is not empty, and means Δ^* if it is.

Definition 6.22. A *timed Kripke model* is a structure $\langle M, N, R, \models\rangle$ such that:

(1) $\langle M, N, R\rangle$ is a timed Kripke frame;
(2) $\models$ is a timed valuation on $\langle M, N, R\rangle$.

A formula α is *true* in a timed Kripke model $\langle M, N, R, \models\rangle$ if

$$(x, 0) \models \alpha$$

for any $x \in M$, and it is *SDL-valid* in a timed Kripke frame $\langle M, N, R\rangle$ if it is true for any timed valuation $\models$ on the timed Kripke frame. A sequent $\Gamma \Rightarrow \Delta$ is true in a timed Kripke model $\langle M, N, R, \models\rangle$ if the formula $(\Gamma \Rightarrow \Delta)^*$ is true in the timed Kripke model, and it is SDL-valid in a timed Kripke frame $\langle M, N, R\rangle$ if it is true for any timed valuation $\models$ on the timed Kripke frame.

The (non-timed) Kripke semantics for LJ is defined as usual, that is, it is obtained from that of SDL by deleting N. The notion of being *LJ-valid* for formulas and sequents are also defined as usual.

We have the following theorems.

Theorem 6.23 (semantical embedding). Let f be the mapping defined in Definition 6.15. For any formula α in $\mathcal{L}_{\mathrm{SDL}}$,

$$\alpha \text{ is SDL-valid} \quad \text{if and only if} \quad f(\alpha) \text{ is LJ-valid}.$$

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Theorem 6.24 (completeness). For any sequent S,

$$\text{SDL} \vdash S \quad \text{if and only if} \quad S \text{ is SDL-valid.}$$

6.3. *NP-complete fragments of LTL*

In this section we introduce BLTL. The *formulas* of BLTL are constructed from countably many propositional variables, $\rightarrow$, $\wedge$, $\vee$, $\neg$, X, G and F where X, G and F are bounded versions of the standard operators of LTL (and we will use the same symbols). If l is a fixed positive integer, we use the symbol ω_l to denote the set $\{i \in \omega \mid i \leqslant l\}$. In the following discussion, l is fixed as a certain positive integer.

Definition 6.25 (BLTL). Let S be a non-empty set of states. A structure $M := (\sigma, I)$ is a *model* if:

(1) σ is an infinite sequence $s_0, s_1, s_2, \ldots$ of states in S;
(2) I is a mapping from the set Φ of propositional variables to the power set of S.

A satisfaction relation $(M, i) \models \alpha$ for any formula α, where M is a model (σ, I) and i $(\in \omega)$ represents some position within σ, is defined inductively by

(1) $(M, i) \models p$ if and only if $s_i \in I(p)$, for any $p \in \Phi$.
(2) $(M, i) \models \alpha \wedge \beta$ if and only if $(M, i) \models \alpha$ and $(M, i) \models \beta$.
(3) $(M, i) \models \alpha \vee \beta$ if and only if $(M, i) \models \alpha$ or $(M, i) \models \beta$.
(4) $(M, i) \models \alpha \rightarrow \beta$ if and only if $(M, i) \models \alpha$ implies $(M, i) \models \beta$.
(5) $(M, i) \models \neg\alpha$ if and only if not-$[(M, i) \models \alpha]$.
(6) $(M, i) \models \mathrm{X}\alpha$ if and only if $(M, i+1) \models \alpha$ for any $i \leqslant l-1$.
(7) $(M, i) \models \mathrm{X}\alpha$ if and only if $(M, l) \models \alpha$ for any $i \geqslant l$.
(8) $(M, i) \models \mathrm{G}\alpha$ if and only if $\forall j \geqslant i$ with $j \in \omega_l$ we have $(M, j) \models \alpha$.
(9) $(M, i) \models \mathrm{F}\alpha$ if and only if $\exists j \geqslant i$ with $j \in \omega_l$ such that $(M, j) \models \alpha]$.
(10) $(M, l+m) \models \alpha$ if and only if $(M, l) \models \alpha$ for any $m \in \omega$.

A formula α is said to be *BLTL-valid* (*BLTL-satisfiable*) if $(M, 0) \models \alpha$ for any (some) model $M := (\sigma, I)$.

In the following, an expression $\alpha \leftrightarrow \beta$ means $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$, and expressions $\bigwedge C$ and $\bigvee C$ are used to represent the finite conjunction and disjunction, respectively, of the formulas in C.

Proposition 6.26. For any formulas α and β, the following formulas are BLTL-valid:

(1) $\mathrm{X}(\alpha \circ \beta) \leftrightarrow \mathrm{X}\alpha \circ \mathrm{X}\beta$ where $\circ \in \{\rightarrow, \wedge, \vee\}$.
(2) $\mathrm{X}(\neg\alpha) \leftrightarrow \neg(\mathrm{X}\alpha)$.
(3) $\mathrm{G}\alpha \rightarrow \alpha$.
(4) $\mathrm{G}\alpha \rightarrow \mathrm{X}\alpha$.
(5) $\mathrm{G}\alpha \rightarrow \mathrm{XG}\alpha$.
(6) $\mathrm{G}\alpha \rightarrow \mathrm{GG}\alpha$.
(7) $\alpha \wedge \mathrm{G}(\alpha \rightarrow \mathrm{X}\alpha) \rightarrow \mathrm{G}\alpha$ (temporal induction).
(8) for any $m \in \omega$, $\mathrm{X}^{l+m}\alpha \leftrightarrow \mathrm{X}^{l}\alpha$.

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(9) $G\alpha \leftrightarrow \bigwedge\{X^i\alpha \mid i \in \omega_l\}$.
(10) $F\alpha \leftrightarrow \bigvee\{X^i\alpha \mid i \in \omega_l\}$.

Note that (8), (9) and (10) in Proposition 6.26 can be regarded as characteristic axioms for the time bound l. Note also that (9) and (10) become the axioms of LTL if ω_l is replaced by ω. Thus, BLTL can be used quite naturally as a bounded time formalism.

The formulas of propositional classical logic (CL) are constructed from countably many propositional variables, $\rightarrow$, $\neg$, $\bigwedge$ (finite conjunction) and $\bigvee$ (finite disjunction).

Definition 6.27 (CL). Let Θ be a finite (non-empty) set of formulas. V is a mapping from the set Φ of propositional variables to the set $\{t, f\}$ of truth values. V is called a *valuation*. A satisfaction relation $V \models \alpha$ for any formula α is defined inductively by:

(1) $V \models p$ if and only if $V(p) = t$ for any $p \in \Phi$.
(2) $V \models \neg\alpha$ if and only if not-($V \models \alpha$).
(3) $V \models \alpha\rightarrow\beta$ if and only if $V \models \alpha$ implies $V \models \beta$.
(4) $V \models \bigwedge\Theta$ if and only if $V \models \alpha$ for any $\alpha \in \Theta$.
(5) $V \models \bigvee\Theta$ if and only if $V \models \alpha$ for some $\alpha \in \Theta$.

A formula α is said to be *CL-valid* (*CL-satisfiable*) if $V \models \alpha$ for any (some) valuation V.

Definition 6.28. We fix a countable non-empty set Φ of propositional variables and define the sets

$$\Phi_i := \{p_i \mid p \in \Phi\} \qquad (1 \leqslant i \in \omega)$$
$$\Phi_0 := \Phi$$

of propositional variables where $p_0 = p$. The language $\mathcal{L}_{\mathrm{BLTL}}$ of BLTL is defined using Φ, $\rightarrow, \wedge, \vee, \neg$, X, G and F. The language $\mathcal{L}_{\mathrm{CL}}$ of CL is defined using

$$\bigcup_{i\in\omega}\Phi_i,\ \rightarrow,\ \neg,\ \bigwedge\ \text{and}\ \bigvee.$$

For convenience, the binary versions of $\bigwedge$ and $\bigvee$ are also denoted by $\wedge$ and $\vee$, respectively, and these binary symbols are included in the definition of $\mathcal{L}_{\mathrm{CL}}$.

A mapping f from $\mathcal{L}^b$ to $\mathcal{L}$ is defined by

$$\begin{aligned}
f(X^i p) &:= p_i \in \Phi_i && \text{for any } p \in \Phi, \text{ (in particular, } f(p) := p \in \Phi)\\
f(X^i(\alpha \circ \beta)) &:= f(X^i\alpha) \circ f(X^i\beta) && \text{where } \circ \in \{\rightarrow, \wedge, \vee\}\\
f(X^i\neg\alpha) &:= \neg f(X^i\alpha) &&\\
f(X^m X\alpha) &:= f(X^l\alpha) && \text{for any } m \geqslant l\\
f(X^i G\alpha) &:= \bigwedge\{f(X^{i+j}\alpha) \mid j \in \omega_l\} &&\\
f(X^i F\alpha) &:= \bigvee\{f(X^{i+j}\alpha) \mid j \in \omega_l\}. &&
\end{aligned}$$

Lemma 6.29. Let f be the mapping defined in Definition 6.28, and S be a non-empty set of states. For any model $M := (\sigma, I)$ of BLTL, any satisfaction relation $\models$ on M and any state s_i in σ, we can construct a valuation V of CL and a satisfaction relation $\models$ of CL

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such that for any formula α in $\mathcal{L}_{\text{BLTL}}$,

$$(M, i) \models \alpha \quad \text{if and only if} \quad V \models f(\mathrm{X}^i\alpha).$$

Lemma 6.30. Let f be the mapping defined in Definition 6.28, and S be a non-empty set of states. For any valuation V of CL and any satisfaction relation $\models$ of CL, we can construct a model $M := (\sigma, I)$ of BLTL and satisfaction relation $\models$ on M such that for any formula α in $\mathcal{L}_{\text{BLTL}}$,

$$V \models f(\mathrm{X}^i\alpha) \quad \text{if and only if} \quad (M, i) \models \alpha.$$

Theorem 6.31 (semantical embedding). Let f be the mapping defined in Definition 6.28. For any formula α,

$$\alpha \text{ is BLTL-valid (BLTL-satisfiable)} \quad \text{if and only if} \quad f(\alpha) \text{ is CL-valid (CL-satisfiable).}$$

Proof. The statement follows from Lemmas 6.29 and 6.30. □

Theorem 6.32 (NP-completeness). The validity and satisfiability problems of BLTL are Co-NP-complete and NP-complete, respectively.

Proof. The validity and satisfiability problems of CL are known to be Co-NP-complete and NP-complete, respectively. By the decidability of CL, it is possible to decide for each α if $f(\alpha)$ is valid (satisfiable) in BLTL. Hence, by Theorem 6.31, the validity and satisfiability problems of BLTL are decidable. Since f is a polynomial-time reduction, the validity and satisfiability problems of BLTL are also Co-NP-complete and NP-complete, respectively. □

7. Concluding remarks

In this paper, we have shown syntactical and/or semantical embedding theorems for:

— LTL;
— GLTL (a fixpoint generalisation of LTL);
— L_ω (an extension of dynamic topological logic);
— L_ω^- (a combination of LTL and infinitary logic);
— 3SL (a spatial extension of LTL);
— SDL (an extension of Davies' logic); and
— BLTL (an NP-complete fragment of LTL).

Using these embedding theorems, the completeness, cut-elimination and/or decidability theorems for these LTL-variations were obtained uniformly. The proposed embedding theorems clarified the relationships between the LTL-variations and traditional logics such as infinitary logic and classical logic.

We will now briefly review some recent closely related work. Kamide and Wansing (2010) introduced a *paraconsistent constructive bounded LTL*, and proved some syntactical embedding and related theorems for this logic. Kamide (2010a) proposed a logic called *linear-time computation tree logic* (LCTL), which is a combination of a

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restricted version of LTL and *computation tree logic*, and showed the decidability of LCTL through the semantical embedding theorem for LCTL. Kamide (2010f) and Kaneiwa and Kamide (2010) introduced a logic called *sequence-indexed LTL* (SLTL), and obtained some theorems using the syntactical and semantical embedding theorems for SLTL.

As future work, it may be interesting to apply the embedding theorem approach to *temporal substructural logics* since proving the completeness, cut-elimination and/or decidability theorems for some substructural logics is known (Kamide 2006a) to be rather difficult. We also believe that the proposed embedding theorems are useful for proving and disproving some important logical properties such as Craig interpolation, but studying this issue remains for future work. We have seen that LT_ω has some inference rules that have an infinite number of premises. This infinite formulation just corresponds to LK_ω for infinitary logic. On the other hand, LTL is known to be finitely axiomatisable. Thus, we would like to obtain a theorem for embedding a finitely Gentzen-type formulation (sequent-style or natural deduction-style) of LTL into a standard logic. However, we have not obtained such a result yet.

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