COMPLEMENTED BANACH ALGEBRAS

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1. Introduction. Let A be a complex Banach algebra and L_r (L_l) be the lattice of all closed right (left) ideals in A. Following Tomiuk **(5)**, we say that A is a *right complemented algebra* if there exists a mapping $I \rightarrow I^p$ of L_r into L_r such that if $I \in L_r$, then $I \cap I^p = (0)$, $(I^p)^p = I$, $I \oplus I^p = A$ and if $I_1, I_2 \in L_r$ with $I_1 \subseteq I_2$, then $I_2{}^p \subseteq I_1{}^p$.

If in a Banach algebra A every proper closed right ideal has a non-zero left annihilator, then A is called a *left annihilator algebra.* If, in addition, the corresponding statement holds for every proper closed left ideal and $r(A) = (0) = l(A)$, A is called an *annihilator algebra* **(1)**.

A Banach algebra A is called a $B^{\#}$-algebra if, for each $a \in A$, there exists $a^{\#} \neq 0$ such that

$$||a^{\#}||\,||a|| = \lim_{n\to\infty} ||(a^{\#}a)^n||^{1/n};$$

and finally, the norm $||\cdot||$ in A is said to be minimal if, given any other norm $|\cdot|$ in A satisfying $|a| \leqslant ||a||$ for every $a \in A$, we have $|\cdot| = ||\cdot||$ **(2)**.

The following structure theorem has been proved by Tomiuk **(5**, Theorem 10**)**.

THEOREM. *If a simple annihilator right complemented algebra A has the minimal norm property or is a $B^{\#}$-algebra, then A is bicontinuously isomorphic to the algebra of all completely continuous operators on a Hilbert space.*

The purpose of the present note is to prove Tomiuk's result without assuming that A is an annihilator algebra. Our proof depends essentially on the fact that if e is a primitive idempotent in A, then the minimal norm property already guarantees that the set Φ of all continuous linear functionals in $(Ae)^*$ corresponding to the elements of eA **(**as described in **5**, p. 656**)** is in fact the whole of $(Ae)^*$.

2. THEOREM. *If a simple right complemented algebra A has the minimal norm property, then A is bicontinuously isomorphic to the algebra of all completely continuous operators on a Hilbert space.*

Proof. A semi-simple, right complemented algebra contains minimal ideals **(5**, Theorem 1**)**. Let $I = Ae$ be a minimal left ideal of A, e a primitive idempotent. We represent A as an algebra $\mathfrak{A}$ of operators on Ae, defining, for each

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$a \in A$, an operator $\bar{a} \in \mathfrak{A}$ by $\bar{a} : x \to ax$, $x \in Ae$. The correspondence $a \to \bar{a}$ is obviously an isomorphism and if we take

$$||\bar{a}|| = \sup_{||x|| \leqslant 1} ||ax||, \qquad x \in Ae,$$

as a new norm in A, the correspondence is, by the minimal norm property, an isometry.

Let Φ denote the subspace of $(Ae)^*$ corresponding to the elements of eA (**5**, p. 656). Then Φ is a closed subspace of $(Ae)^*$. In fact, for $a \in eA$, $ax = \phi_a(x)e$, $x \in Ae$. Then using the isometry established above, we have

$$\begin{aligned} ||a|| = ||\bar{a}|| &= \sup_{||x|| \leqslant 1} ||ax||, \qquad a \in eA, \\ &= \sup_{||x|| \leqslant 1} ||\phi_a(x)e|| \\ &= \sup_{||x|| \leqslant 1} |\phi_a(x)|\, ||e|| \\ &= ||\phi_a|| \cdot ||e||, \end{aligned}$$

from which it follows that Φ is homeomorphic with eA and, therefore, closed. By (**5**, Lemma 8), Φ is dense in $(Ae)^*$, and so $\Phi = (Ae)^*$.

From this it follows that $\mathfrak{A}$ contains all the operators of finite rank in Ae and the proof is completed as in (**5**, Theorems 7 and 10).

COROLLARY 1. *A simple, right complemented algebra with the minimal norm property is a dual algebra.*

This follows from a result due to Kaplansky (**3**, Cor. to Theorem 8.4).

COROLLARY 2. *A simple, right complemented, $B^{\#}$-algebra is bicontinuously isomorphic to the algebra of all completely continuous operators on a Hilbert space.*

Proof. Any semi-simple, right complemented algebra has a dense socle (**5**, Lemma 5), and a $B^{\#}$-algebra with a dense socle has the minimal norm property (**4**, Lemma 3.2). The result now follows from the theorem.

REFERENCES

1. F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc. (3), *4* (1954), 154–167.

2. F. F. Bonsall, *A minimal property of the norm in some Banach algebras*, J. London Math. Soc., *29* (1954), 157–163.

3. I. Kaplansky, *Normed algebras*, Duke Math. J., *16* (1949), 399–418.

4. A. Olubummo, *Weakly compact $B^{\#}$-algebras*, To appear in Proc. Amer. Math. Soc.

5. B. J. Tomiuk, *Structure theory of complemented Banach algebras*, Can. J. Math., *14* (1962), 651–659.

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