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Topological entropy of semi-dispersing billiards

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Abstract. In this paper we continue to explore the applications of the connections between singular Riemannian geometry and billiard systems that were first used in [**6**] to prove estimates on the number of collisions in non-degenerate semi-dispersing billiards.

In this paper we show that the topological entropy of a compact non-degenerate semi-dispersing billiard on any manifold of non-positive sectional curvature is finite. Also, we prove exponential estimates on the number of periodic points (for the first return map to the boundary of a simple-connected billiard table) and the number of periodic trajectories (for the billiard flow). In §5 we prove some estimates for the topological entropy of Lorentz gas.

1. *Summary of results*

The results of this paper rely on the connection between the singular Riemannian geometry and semi-dispersing billiard systems. Namely, for every billiard trajectory one can construct a singular Riemannian space such that the trajectory corresponds to a geodesic in this space. Moreover, the Alexandrov curvature of this space is not bigger than the curvature of the original billiard manifold. Thus, if we start with a billiard on a manifold of non-positive curvature, then the corresponding space also has non-positive curvature. The proof of the finiteness of entropy for semi-dispersing billiards is based on a singular analog of the following well known for regular manifolds of non-positive curvature statement: if two geodesics in a simply connected manifold of non-positive curvature have 'close' end points, then they are 'close' to each other everywhere, and not only on the manifold but also in its tangent bundle.

It is interesting to notice that our proof fails immediately if there are any regions of positive curvature inside the billiard. Moreover, we strongly suspect that if we allow even

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arbitrarily small portions of positive curvature, then it is possible to construct examples of semi-dispersing billiards with infinite topological entropy.

Let us proceed with the rigorous formulations of our results.

Let M be an arbitrary Riemannian manifold of non-positive bounded sectional curvature without boundary. Consider a collection of n geodesically convex compact subsets (*walls*) $B_i \subset M$, $i = 1, \ldots, n$, of M, such that their boundaries are C^1 submanifolds of codimension one. Let $B = M \setminus (\bigcup_{i=1}^n \mathrm{Int}(B_i))$, where $\mathrm{Int}(B_i)$ denotes the interior of the set B_i. The set $B \subset M$ will be called a billiard table. A semi-dispersing billiard flow $\{T^t\}_{t=-\infty}^{\infty}$ acts on a certain subset $\widetilde{TB}$ of full Liouville measure of the unit tangent bundle to B. To be more precise, $\widetilde{TB}$ consists of such points $(x, v) \in TM$, $x \in B$, $v \in T_x M$, that for every $x \in \partial B$, vector v is directed 'strictly inside of B', and the orbit of (x, v) is defined for all $t \in (-\infty, \infty)$ (see, for example, [**5**] for the rigorous definitions). The projections of the orbits of that flow to B are called the billiard trajectories and correspond to free motions of particles inside B. Namely, the particle moves inside the set B with unit speed along a geodesic until it reaches one of the sets B_i (*collision*) where it reflects according to the law 'the angle of incidence is equal to the angle of reflection'.

The purpose of this paper is to establish the finiteness of topological entropy for a large class of semi-dispersing billiards, namely for non-degenerate semi-dispersing billiards, i.e. billiards on tables that satisfy a certain geometric non-degeneracy condition (see below). It was shown in [**6**] that this non-degeneracy condition implies the existence of local uniform estimates on the number of collisions. Estimates like that play an important role in various questions about billiards. For example, they appear as conditions for Sinai–Chernov's formulas for metric entropy of semi-dispersing billiards [**8**, **16**].

Note that Sinai–Chernov's formulas imply the finiteness of the metric entropy of non-degenerate semi-dispersing billiards in $\mathbb{R}^n$ or $\mathbb{T}^n$ with respect to the Liouville measure. However, little is known about the topological entropy of general semi-dispersing billiards. Most of the results known to the authors of this paper are proven only for two-dimensional semi-dispersing billiards (the connection between the topological entropy and the number of periodic points [**7**] and the results of [**12**]). The only result about the topological entropy of billiards of arbitrary dimension, that we are aware of, is the fact that the topological entropy of polygonal and polyhedral billiards is zero (proved for the two-dimensional case in [**11**], the proof of the general case is also outlined in [**11**], the rigorous proof can be found in [**10**]; see also [**8**] for the similar result about metric entropy).

In this paper we prove that the topological entropy of compact non-degenerate semi-dispersing billiards is finite. Moreover, our results are true not only for billiards in $\mathbb{R}^n$ or $\mathbb{T}^n$ but for billiards on any manifolds of non-positive sectional curvature.

In [**17**] exponential estimates on the number of periodic points for the first return map to the boundary, for billiards in $\mathbb{R}^k$, and the number of periodic trajectories for the flow, for non-degenerate billiards in $\mathbb{R}^2$, were proven. In §4 we prove the analogs of those results for billiards on arbitrary manifolds of non-positive curvature.

In §5 we prove some estimates for the topological entropy of Lorentz gas. In particular, we prove the existence of a limit of topological entropy of the Lorentz gas flow when the radius of the scatterer approaches zero.

The following non-degeneracy condition for semi-dispersing billiards was introduced in [**6**].

Definition 1.1 A billiard table B is *non-degenerate* in a subset $U \subset M$ (with constant $C > 0$), if for any $I \subset \{1, \ldots, n\}$ and for any $y \in (U \bigcap B) \backslash (\bigcap_{j \in I} B_j)$,

$$\max_{k \in I} \frac{\operatorname{dist}(y, B_k)}{\operatorname{dist}(y, \bigcap_{j \in I} B_j)} \geq C,$$

whenever $\bigcap_{j \in I} B_j$ is non-empty.

Roughly speaking, it means that if a point is d-close to all the walls from I then it is d/C-close to their intersection.

We will say that B is *non-degenerate* if there exist $\delta > 0$ and $C > 0$ such that B is non-degenerate, with constant C, in any δ-ball.

The following estimate on the number of collisions in non-degenerate semi-dispersing billiards was proven in [**6**].

PROPOSITION 1.1. *For any non-degenerate semi-dispersing billiard there exists a constant P such that, for every t, every trajectory of the billiard flow makes no more than* $P(t+1)$ *collisions with the boundary in the time interval* $[0, t]$.

Recall that there is a standard way to introduce a distance function in the tangent bundle TM to M (sometimes this distance function is called Sasaki metric). We will denote this distance function by $d_{TM}(\cdot, \cdot)$. Now we can use the distance $d_{TM}(\cdot, \cdot)$ to define the topological entropy $h_{\text{top}}(f)$ of any transformation f of any subset of TM (for a rigorous definition of the topological entropy of transformations of a non-compact space see, for example, [**13**]).

Definition 1.2. The topological entropy of the time-one map T^1 of the billiard flow will be called the topological entropy of the billiard.

Notice that the straightforward definition of the topological entropy of the billiard as the topological entropy of the whole billiard flow is meaningless, because, due to the discontinuity of the flow, the topological entropy of the whole billiard flow is always infinite.

We apply methods of singular Riemannian geometry (see [**2**, **4**, **9**]) to prove some estimates on the topological entropy of non-degenerate semi-dispersing billiards. In particular we will prove the following.

THEOREM 1. *The topological entropy of a compact non-degenerate semi-dispersing billiard on any manifold of non-positive sectional curvature is finite.*

Let us call a point $x \in \widetilde{TB}$ *$\mathbb{Z}$-regular* if $T^i(x)$ belongs to the interior of TB for all $i \in \mathbb{Z}$. For example, almost all points of $\widetilde{TB}$ are $\mathbb{Z}$-regular with respect to the Liouville measure. Clearly the restriction of the time-one map T^1 to the set $TB_{\mathbb{Z}}$ of $\mathbb{Z}$-regular points in B is continuous, and its topological entropy is less or equal to the topological entropy of T^1 on B. Thus, Theorem 1 together with Pesin and Pitskel's [**13**] results

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concerning the variational principle for the continuous maps of non-compact spaces, yields the following.

COROLLARY 1.1. *Metric entropy, of a compact non-degenerate semi-dispersing billiard on any manifold of non-positive bounded sectional curvature, with respect to any* T^1*-invariant probability measure* μ *such that* $\mu(TB_{\mathbb{Z}}) = 1$*, is finite. In particular, metric entropy is finite for any measure which is invariant with respect to the whole flow* T^t.

2. *Outline of the proof of Theorem 1 for simply connected M*

In order to keep the main ideas of the proof of Theorem 1 more transparent we will first prove Theorem 1 for simply connected M. At the end of §3 we will show how to adapt our arguments to the general case. Therefore, from now till the end of §3, we assume M to be simply connected.

Before we begin the proof, let us introduce the following.

Definition 2.1. We will say that two trajectories Γ_1 and Γ_2 are of the same *combinatorial class* if they collide with the same sequence of walls.

Additionally, if Γ_1 and Γ_2 have the same length $l \in \mathbb{R}$ and for each $t = 1, 2, \ldots, [l]$ they experience the same number of collisions by the time t, we will say that Γ_1 and Γ_2 are of the same *strict combinatorial class*.

For every piecewise smooth curve γ in M denote by $\dot{\gamma}(t)$ the right derivative of γ at the point $\gamma(t)$. For every $l \in \mathbb{N}$ and $\epsilon > 0$ we will construct an ϵ-net $A^l(\epsilon) \subset \widetilde{TB}$ for the distance $d_l(x, y) = \max_{0 \le i \le l} d_{TM}(T^i x, T^i y)$ and estimate the number of its elements. The construction of the ϵ-net $A^l(\epsilon)$ is based on the following lemma which will be proven in the next section.

LEMMA 2.1. *For every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that if* Γ_1, Γ_2 *are*
1. *of the same strict combinatorial class;*
2. *have equal length* $l \in \mathbb{N}$*;*
3. $d_M(\Gamma_1(0), \Gamma_2(0)) < \delta$ *and* $d_M(\Gamma_1(l), \Gamma_2(l)) < \delta$*;*

then $d_l(\dot{\Gamma}_1(0), \dot{\Gamma}_2(0)) \le \epsilon$.

Let us show how to construct $A^l(\epsilon)$ using Lemma 2.1.

Consider an arbitrary δ-cover Δ of the billiard B. Let C be an arbitrary strict combinatorial class of trajectories. For each pair of sets $U, V \in \Delta$ consider a billiard trajectory $\Gamma_{U,V}$ of class C such that $\Gamma_{U,V}(0) \in U$ and $\Gamma_{U,V}(l) \in V$ (provided such a trajectory exists) and set $A^l_C(\epsilon) = \{\dot{\Gamma}_{U,V}(0) \mid U, V \in \Delta\}$. One has $\mathrm{Card}(A^l_C) \le \mathrm{Card}(\Delta)^2 \le K$, where K is a positive constant that depends only on the billiard B and the number ϵ (clearly it depends only on B and δ, but δ is determined by ϵ). Now remark that since our billiard is non-degenerate, according to Proposition 1.1 the number of collisions that may occur in time l is not greater than $P(l + 1)$. Therefore, there is no more than $n^{P(l+1)+l}$ different strict combinatorial classes of trajectories that contain trajectories of length l. We take $A^l(\epsilon) = \bigcup A^l_C(\epsilon)$, where the union is taken over all strict combinatorial classes C. Clearly, $A^l(\epsilon)$ is an ϵ-net with respect to the metric d_l on TM, and

$$\mathrm{Card}(A^l(\epsilon)) \le Kn^{P(l+1)+l} \le Kn^{(P+1)(l+1)}$$

and, therefore,

$$h_{\text{top}}(T^1, \epsilon) \le \lim_{\epsilon \to 0} \overline{\lim_{l \to \infty}} \frac{\ln(\text{Card}(A^l(\epsilon)))}{l} \le \lim_{l \to \infty} \frac{\ln(K n^{(P+1)(l+1)})}{l} = (P+1)\ln(n).$$

Therefore, $h_{\text{top}}(T^1) \le (P+1)\ln(n)$.

3. *Proof of Lemma 2.1 and the general case of Theorem 1*

To prove Lemma 2.1 we apply the methods of singular Riemannian geometry similar to the way we did in [**6**].

First of all we have to recall the construction of a singular Riemannian space corresponding to a given billiard trajectory Γ which starts at the point X_0, ends at X_{j+1} and has collision points $X_1, \dots, X_j$. We construct a singular Riemannian space $\bar{M}$ in the following way: take $j+1$ isometric copies M_i, $i = 0, \dots, j$, of M and, for all $i = 0, \dots, j-1$, glue together M_i and M_{i+1} by the set B_k, which contains X_{i+1}. Notice that by construction, for each $i = 0, \dots, j-1$ there is a canonical isometric embedding $E_i : M \to \bar{M}$, which is an isometry between M and M_i and maps the subsets B_k, $k = 1, \dots, n$, in M into the subsets B_k in M_i.

The curve $G(\Gamma) = \bigcup_{i=0}^{j} E_i(X_i X_{i+1}) \in \bar{M}$ is a geodesic in $\bar{M}$ corresponding to the trajectory Γ in M and it has the same length in $\bar{M}$ as Γ in M. (Here and in the rest of the paper, we denote the piece of geodesic in M connecting points A and B by AB.)

Notice that if two trajectories have the same combinatorial class C then the singular Riemannian spaces corresponding to them are naturally isometric. We will denote this space by M_C.

It follows immediately from the construction of M_C, the fact that M is simply connected, and Reshetnyak's gluing theorem ([**14**], also see Theorem 6.1 in [**4**]) that M_C is a singular space of non-positive curvature.

Let Γ_1, Γ_2 be as in Lemma 2.1. Consider the geodesics $G(\Gamma_1)(t)$ and $G(\Gamma_2)(t)$, where t is the time parameter along $G(\Gamma_1)$ and $G(\Gamma_2)$.

Since M_C is a space of non-positive curvature, the function $D(t) = d_{M_C}(G(\Gamma_1)(t), G(\Gamma_2)(t))$ is convex (see [**3**, Theorem 14]). Therefore, for any $\delta > 0$, the fact that $D(0) < \delta$ and $D(l) < \delta$ implies that $D(t) < \delta$ for all $0 \le t \le l$.

Notice that the distance between the points $G(\Gamma_1)(t)$ and $G(\Gamma_2)(t)$ in M_C is bigger or equal to the distance between the points $\Gamma_1(t)$ and $\Gamma_2(t)$ in M.

Thus, we immediately have the following.

LEMMA 3.1. *For any $\delta > 0$, any semi-dispersing billiard B, any real number t_0, if two billiard trajectories $\Gamma_1(t)$ and $\Gamma_2(t)$ have*

1. *the same combinatorial class;*
2. *the same length t_0;*
3. *$d_M(\Gamma_1(0), \Gamma_2(0)) < \delta$ and $d_M(\Gamma_1(t_0), \Gamma_2(t_0)) < \delta$;*

then $d_M(\Gamma_1(t), \Gamma_2(t)) < \delta$ for all $0 \le t \le t_0$.

Now, to finish the proof of Lemma 2.1 it will be enough to prove the following.

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LEMMA 3.2. *Let $\Gamma_1(t)$ and $\Gamma_2(t)$, $0 \le t \le 1$, be two trajectories of the same combinatorial class C. Then for every ϵ there exists $\delta_C > 0$ such that if $d_M(\Gamma_1(0), \Gamma_2(0)) < \delta_C$ and $d_M(\Gamma_1(1), \Gamma_2(1)) < \delta_C$ then $d_{TM}(\dot\Gamma_1(0), \dot\Gamma_2(0)) < \epsilon$.*

Suppose that Lemma 3.2 is proven. Due to Proposition 1.1 there exist no more than n^{2P} different combinatorial classes containing a trajectory of length one. For a fixed ϵ let δ be equal to the minimum of all δ_C over all possible combinatorial classes C.

It follows from Lemma 3.1 that if Γ_1 and Γ_2 are as in Lemma 2.1 then we can apply Lemma 3.2 to each of the pairs of segments of $\Gamma_1(t)$, $i \le t \le i+1$, and $\Gamma_2(t)$, $i \le t \le i+1$, for all $i = 0, \ldots, l-1$. This immediately implies Lemma 2.1.

Let us prove Lemma 3.2.

Proof. Let us introduce some notation. Let $x \in M$, and let T be a linear transformation of T_xM. Let v be any vector, tangent to M, not necessary at point x. By $T(v)$ we will denote the vector obtained by the following procedure: first we translate v parallel along a geodesic to x (since M is assumed to be simply connected there is a unique geodesic joining any two points of M), and then apply the transformation T at x. Notice that if $T = \mathrm{Id}(x)$, the identity map in T_xM, then $T(v)$ is the result of the parallel translation of v to x.

For an arbitrary trajectory Γ of length one and the combinatorial class C we will use expanded notation $\Gamma((t_1, \gamma_1), \ldots, (t_m, \gamma_m))$, where each pair (t_k, γ_k), $t_k \in [0, 1]$, $\gamma_k \in M$, $k = 1, \ldots, m$, are the time and the coordinate of the kth collision. Let $\gamma_0 = \Gamma(0)$ and $\Gamma_0 = \dot\Gamma(0)$ and let $\gamma_{m+1} = \Gamma(1)$. Denote, for $k = 1, \ldots, m$, $\dot\Gamma_k = \dot\Gamma(t_k)$, i.e. the velocity vector at the time t_k.

Let $S(\gamma_k)$ be the reflection in $T_{\gamma_k}M$ with respect to the hyperplane tangent to $\partial B_{i(k)}$ at the point γ_k, where $\partial B_{i(k)}$ is the boundary of the wall containing γ_k.

Then the billiard motion law for the trajectory Γ can be written as

$$\dot\Gamma_k = (\mathrm{Id}(\gamma_k) S(\gamma_{k+1}) \ldots S(\gamma_{k+l}))\dot\Gamma_{k+l}, \tag{1}$$

for any $k, l = 0, \ldots m$, such that $k + l \le m$.

Now, denote by γ the uniform pointwise limit of the sequence of trajectories $\Gamma^n((t_1^n, \gamma_1^n), \ldots, (t_m^n, \gamma_m^n))$ of combinatorial class C. Let $(t_k, \gamma_k) \in [0, 1] \times \partial B_{i(k)}$ be an accumulation point of the sequence (t_k^n, γ_k^n). Also, let $\gamma_0 = \gamma(0)$, $t_0 = 0$, $t_{m+1} = 1$, and $\gamma_{m+1} = \gamma(1)$. (Notice that the times t_k and the points γ_k are defined non-uniquely, except for $k = 0$ and $k = m + 1$, i.e. there might be more than one accumulation point for the sequences (t_k^n, γ_k^n) for $k = 1, \ldots, m$.)

Obviously, there exists $0 \le k \le m$ such that the points γ_k and γ_{k+1} do not coincide. Then, denote by $\dot\gamma_k$ the vector, tangent at γ_k to the geodesic connecting γ_k and γ_{k+1}. Thus γ_k is defined for some $0 \le k \le m$. Let us define $\dot\gamma_k$ for all the other k. Namely, we put

$$\dot\gamma_l = \mathrm{Id}(\gamma_l) S(\gamma_{l+1}) \ldots S(\gamma_k)(\dot\gamma_k) \tag{2}$$

for all $0 \le l < k$ and

$$\dot\gamma_l = S(\gamma_l) S(\gamma_{l-1}) \ldots S(\gamma_{k+1})(\dot\gamma_k) \tag{3}$$

for all $m \ge l > k$.

Consider the closure $\bar{C}$ in the metric of uniform pointwise convergence: $d_{\bar{C}}(\Gamma_1, \Gamma_2) = \max_{t\in[0,1]} d_M(\Gamma_1(t), \Gamma_2(t))$, of the set of all the trajectories of the combinatorial class C and of length one. (Notice that, due to Lemma 3.1, $d_{\bar{C}}(\Gamma_1, \Gamma_2) = \max\{d_M(\Gamma_1(0), \Gamma_2(0)), d_M(\Gamma_1(1), \Gamma_2(1))\}$.) We claim that if $\gamma^n \in \bar{C}$ converges to $\gamma \in C$, then $\dot{\gamma}_0^n$ converges to $\dot{\gamma}_0$. Fix some choice of (t_k, γ_k), for $k = 1, \dots, m$. Let k be such that $\gamma_k \neq \gamma_{k+1}$. Let $\tilde{\gamma}^n$ be the part of γ^n connecting γ_k^n and γ_{k+1}^n, and let $\tilde{\gamma}$ be the part of γ connecting γ_k and γ_{k+1}. Then $\tilde{\gamma}^n$, $n = 1, 2, \dots$, and $\tilde{\gamma}$ are geodesic in M, such that $\tilde{\gamma}^n$ converges uniformly to $\tilde{\gamma}$. Therefore, $\dot{\gamma}_k^n$ converges to $\dot{\gamma}_k$ and then, due to the relations (2) and (3), $\dot{\gamma}_0^n$ converges to $\dot{\gamma}_0$. Thus, we have established that the map

$$L : \bar{C} \to TM : L(\gamma) = \dot{\gamma}_0$$

is continuous. It immediately implies that the map

$$F : \bar{C} \times \bar{C} \to \mathbb{R} : F(\gamma_1, \gamma_2) = d_{TM}(L(\gamma_1), L(\gamma_2))$$

is also continuous.

Let us now introduce the map

$$\Omega : \bar{C} \to B \times B; \quad \Omega(\gamma) = (\gamma(0), \gamma(1)),$$

which is bicontinuous and injective (both properties are due to the fact that M_C has non-positive curvature; see, for example, [**3**, Theorem 14]). It shows that, since B is compact, $\bar{C}$ is a compact set. On the other hand, function F is identically equal to zero on the diagonal $\{(\gamma, \gamma) \mid \gamma \in \bar{C}\} \subset \bar{C} \times \bar{C}$. It means that for every $\epsilon > 0$ there exists $\delta_C > 0$ such that $d_{\bar{C}}(\gamma_1, \gamma_2) \leq \delta_C$ implies $F(\gamma_1, \gamma_2) \leq \epsilon$. This proves Lemma 3.2 and, thus, finishes the proof of Theorem 1 for simply connected M. □

Now, let us show how to modify the proof to include the case when M is not simply connected.

Denote by $H(t)$ the number of different homotopy classes that can be represented by the curves which intersect the compact set B and have length less or equal to t.

We will say that two billiard trajectories Γ_1 and Γ_2 of length $l \in \mathbb{R}$ are of the same *homotopic combinatorial class* (respectively, the same *strict homotopic combinatorial class*) if:

1. Γ_1 and Γ_2 are of the same combinatorial class (respectively, the same strict combinatorial class);
2. $d_M(\Gamma_1(0), \Gamma_2(0)) < r_0$ and $d_M(\Gamma_1(l), \Gamma_2(l)) < r_0$, where r_0 is the minimum of the injectivity radius of M over all points of B;
3. the closed curve formed by Γ_1, Γ_2 and the two shortest geodesics connecting $\Gamma_1(0)$ with $\Gamma_2(0)$, and $\Gamma_1(l)$ with $\Gamma_2(l)$, is homotopically trivial.

(Notice that unlike the relation of being from the same combinatorial class (strict combinatorial class) the relation of being from the same homotopic combinatorial class (strict homotopic combinatorial class) is not an equivalency relation of the set of trajectories, because it does not possess the transitivity property.)

Now, Lemma 3.1 is true if we substitute in its statement 'homotopic combinatorial class' instead of 'combinatorial class' and add the condition that $\delta < r_0$. The proof is

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essentially the same. We consider the universal cover $\tilde{M}$ of M and the billiard $\tilde{B}$ in the pre-image of B under the covering map. Notice that the set $\tilde{B}$ in M is bounded by all the pre-images of the sets B_i in $\tilde{M}$, and each connected component of the pre-image is considered as a separate wall of the billiard $\tilde{B}$. The condition that Γ_1 and Γ_2 have the same homotopic combinatorial class guarantees that if their lifts $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ in $\tilde{B}$ are such that $d_{\tilde{M}}(\tilde{\Gamma}_1(0), \tilde{\Gamma}_2(0)) < \delta$ then $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have the same combinatorial class in $\tilde{B}$ and $d_{\tilde{M}}(\tilde{\Gamma}_1(t_0), \tilde{\Gamma}_2(t_0)) < \delta$. Then, exactly as before, we construct the singular space $M_{\tilde{C}}$ using the billiard $\tilde{B}$ and the combinatorial class $\tilde{C}$ of $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ in $\tilde{B}$, and show that $d_{\tilde{M}}(\tilde{\Gamma}_1(t), \tilde{\Gamma}_2(t)) < \delta$ for all $t \in [0, t_0]$. This immediately implies that $d_M(\Gamma_1(t), \Gamma_2(t)) < \delta$ for all $t \in [0, t_0]$.

Lemma 3.2 is true if we substitute in its statement 'homotopic combinatorial class' instead of 'combinatorial class'. The proof is again very similar to the proof of Lemma 3.2 for the simply connected case. Let us outline it. Let compact connected set $B' \subset \tilde{B}$ be such that B' covers B, that is, every point of B has a pre-image in B' and, moreover, every geodesic connecting two points of B has a lift that is contained in B'. Let $\tilde{C}'$ be an arbitrary combinatorial class in $\tilde{B}$ that contains pre-images of some trajectories of class C in B, and such that those pre-images start at some points of B'. Consider the set C' of all trajectories of $\tilde{B}$ that have length one, start at some point of B', and belong to the class $\tilde{C}'$. Consider the closure $\bar{C}'$ of C' in the metric of the uniform convergence for the curves on $\tilde{M}$. The functions L' and F' on the sets $\bar{C}'$ and $\bar{C}' \times \bar{C}'$ are defined exactly as the functions L and F were defined for the sets $\bar{C}$ and $\bar{C} \times \bar{C}$. The map Ω' is defined similar to the map Ω and maps $\bar{C}'$ into a compact set $B' \times B'(1)$, where $B'(1)$ is the set of points in $\tilde{M}$ which are at the distance less or equal to one from the set B'. In this way, we establish the compactness of $\bar{C}'$ and, thus, the uniform continuity of F'. It means that for every $\epsilon > 0$ there exists $\delta_{\tilde{C}} > 0$ such that $d_{\bar{C}'}(\gamma_1, \gamma_2) \leq \delta_{\tilde{C}}$ implies $F'(\gamma_1, \gamma_2) \leq \epsilon$.

Choose, $\delta'_C = \min \delta_{\tilde{C}}$, where the minimum is taken over all the classes $\tilde{C}$ in $\tilde{B}$ that contain the pre-images of the trajectories of length one of class C in B. Let $\delta_C = \min\{\delta'_C, r_0\}$.

Then, if Γ_1 and Γ_2 have the same homotopic combinatorial class, are such that $d_M(\Gamma_1(0), \Gamma_2(0)) < \delta_C$, $d_M(\Gamma_1(1), \Gamma_2(1)) < \delta_C$, and if their lifts, $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$, in $\tilde{B}$ are chosen so that $d_{\tilde{M}}(\tilde{\Gamma}_1(0), \tilde{\Gamma}_2(0)) < \delta_C$, then $\tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ have the same combinatorial class in $\tilde{B}$ and satisfy the condition $d_{\tilde{M}}(\tilde{\Gamma}_1(1), \tilde{\Gamma}_2(1)) < \delta_C$. Due to the choice of δ_C we see that it implies the statement of the non-simply connected version of Lemma 3.2.

Exactly as before the modified versions of Lemmas 3.1 and 3.2 imply Lemma 2.1 with 'strict combinatorial class' being changed to 'strict homotopic combinatorial class'.

Now, the modified version of Lemma 2.1 can be used to construct an ϵ-net $A^l(\epsilon)$. The construction is exactly the same as the construction of $A^l(\epsilon)$ in §2, except that instead of picking for each pair of U and V a single trajectory $\Gamma_{U,V}$ we pick as many trajectories $\Gamma^i_{U,V}$ as we can in order to satisfy the following conditions:

1. all the trajectories $\Gamma^i_{U,V}$ satisfy the conditions $\Gamma^i_{U,V}(0) \in U$ and $\Gamma^i_{U,V}(l) \in V$;
2. all the trajectories $\Gamma^i_{U,V}$ have the same strict combinatorial class C;
3. no two trajectories among $\Gamma^i_{U,V}$ have the same strict homotopic combinatorial class;

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4. every trajectory Γ of the strict combinatorial class C that satisfies the conditions $\Gamma(0) \in U$ and $\Gamma(l) \in V$ has the same strict homotopic combinatorial class with at least one of the trajectories $\Gamma^i_{U,V}$.

Clearly, for each pair of U and V one can choose at most $H(2l+2\delta)$ different trajectories $\Gamma^i_{U,V}$. In this way, for each strict combinatorial class C we construct a set $A^l_C(\epsilon)$ that consists of no more than $K'H(2l+2\delta)$ different points (here, as before, K' is a constant that depends only on B and ϵ). The union $A^l(\epsilon) = \bigcup A^l_C(\epsilon)$ over all the strict combinatorial classes of trajectories of length l is an ϵ-net with respect to the metric d_l on TM. Thus,

$$\begin{aligned} h_{\text{top}}(T^1, \epsilon) &\leq \lim_{\epsilon \to 0} \overline{\lim_{l\to\infty}} \frac{\ln(\text{Card}(A^l(\epsilon)))}{l} \leq \lim_{l\to\infty} \frac{\ln(K' n^{(P+1)(l+1)}) H(2l+2\delta)}{l} \\ &= (P+1)\ln(n) + \overline{\lim_{l\to\infty}} \frac{\ln(H(2l+2\delta))}{l} \\ &= (P+1)\ln(n) + 2\,\overline{\lim_{l\to\infty}} \frac{(H(l))}{l}. \end{aligned}$$

4. *Estimates on the number of periodic points and trajectories*

Here we will use our methods to prove some results about periodic points and trajectories of semi-dispersing billiards. The similar results for billiards in $\mathbb{R}^k$ (for periodic points) and $\mathbb{R}^2$ (for periodic trajectories) were proven in [**17**]. The advantage of our method is that the use of singular Riemannian geometry allows us to include the billiards on manifolds of variable non-positive curvature and at the same time to avoid a variational calculation used in [**17**].

We say that a periodic trajectory $\Gamma(t)$, $0 \leq t \leq l$, is of class C if we can choose the starting point $\Gamma(0)$ so that $\Gamma(0) \in B_{i_1}$, and then Γ collides with B_i, $i = i_2, \ldots, i_j$, corresponding to the class C, and eventually $\Gamma(0) = \Gamma(l)$. Also, we will call a curve $\nu(t)$ a periodic pseudo-trajectory of class C if it is a closed curve that consists of pieces of geodesics on M that connect some point $x_1 \in B_{i_1}$ with some point $x_2 \in B_{i_2}$, the point $x_2 \in B_{i_2}$ with some point $x_3 \in B_{i_3}, \ldots,$ the point $x_j \in B_{i_j}$ with the point $x_1 \in B_{i_1}$, and at each point x_k, $k = 1, \ldots, j$, the tangent vector to $\nu(t)$ changes according to the billiard rule with respect to B_{i_k}. (The difference with the usual trajectories is that a geodesic segment of a pseudo-trajectory between x_k and x_{k+1} may intersect some of the bodies B_i, $i = 1, \ldots, n$.) Notice that if Γ is any periodic trajectory than any periodic pseudo-trajectory close enough to Γ is a periodic trajectory.

Our main result on periodic trajectories is the following.

THEOREM 2. *Let B be a semi-dispersion billiard on a simply connected manifold M of non-positive sectional curvature. Let C be some combinatorial class of trajectories. (Notice that, unlike in our previous results, here we do not require B to be compact or non-degenerate.)*

Then, the periodic trajectories of class C all have the same length and form a parallel family in the following sense. Any two periodic trajectories Γ_1 and Γ_2 of class C can be joined by a continuous curve Γ_t, $1 \leq t \leq 2$, of periodic pseudo-trajectories of type C so that:

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1. *the surface* Σ_k, $k = 1, \dots, j$, *formed by the pieces of trajectories* Γ_t, $1 \le t \le 2$, *between the kth and* $(k+1)$*st collisions is a piece of* $\mathbb{R}^2$ *isometrically embedded into* M*;*
2. *the intersection* I_k *of the boundary of* B_{i_k} *with the trajectories from the family* Γ_t, $1 \le t \le 2$, *are isometrically embedded intervals of a straight line that connect the point* $\Gamma_1 \bigcap B_{i_k}$ *with the point* $\Gamma_2 \bigcap B_{i_k}$*;*
3. *inside of each flat surface* Σ_k, $k = 1, \dots, j$, *the pieces of trajectories from* Γ_t, $1 \le t \le 2$, *are parallel to each other.*

We immediately have the following.

COROLLARY 4.1. *For* M *and* C *as in Theorem 2:*
1. *if the curvature of* M *is strictly negative, then* C *contains no more than one periodic trajectory;*
2. *if for some periodic trajectory* Γ *of class* C *at least one of the sets* B_i, $i = i_1, i_2, \dots, i_j$, *is strictly convex at the point* $\Gamma \bigcap B_{i_k}$, *then* Γ *is the only periodic trajectory in its class.*

Let us prove Theorem 2.

Proof. Let Γ_1 and Γ_2 be two periodic trajectories of class C. Let $x = \Gamma_1(0)$ and $y = \Gamma_2(0)$, and $x' = E_j(\Gamma_1(0))$ and $y' = E_j(\Gamma_2(0))$. Extend the geodesics $G(\Gamma_1)$ and $G(\Gamma_2)$ a little beyond the points x, x' and y, y' correspondingly, to geodesics γ_1 and γ_2, in such a way that γ_1, γ_2 belong to B_{i_1}, prior to the points x and y, and to $E_j(B_{i_1})$, after the points x' and y' (i.e. so to say extend the geodesics 'into the walls'). Let $q \in \gamma_1$, $q \in B_{i_1}$, $q \ne x$, and let $p \in \gamma_2$, $p \in B_{i_1}$, $p \ne y$.

Then, since Γ_1 and Γ_2 are periodic trajectories, $q' = E_j(q) \in \gamma_1$, and $p' = E_j(p) \in \gamma_2$ (provided that q and p are chosen close enough to the x and y, respectively). Therefore,

$$\angle(qxy) = \angle(q'x'y') \quad \text{and} \quad \angle(xyp) = \angle(x'y'p'). \tag{4}$$

Thus, the sum of the angles of the geodesic quadrangle $xx'y'y$ is equal to 2π. Therefore, since M_C has non-positive curvature, the defects of the triangles $xx'y'$ and $xy'y$ are both equal to zero. Also, we see that

$$\angle(xy'y) + \angle(xy'x') = \angle(yy'x'). \tag{5}$$

Consider the triangles $XX'Y'$ and $XY'Y$ on $\mathbb{R}^2$ which have a common side XY', and $|XY| = d_{M_C}(x, y)$, $|XY'| = d_{M_C}(x, y')$, $|YY'| = d_{M_C}(y, y')$, $|XX'| = d_{M_C}(x, x')$, $|X'Y'| = d_{M_C}(x', y')$. Since the defects of $xx'y'$ and $xy'y$ are both equal to zero, triangles $xx'y'$ and $xy'y$ and triangles $XX'Y'$ and $XY'Y$ have equal corresponding angles. Due to equations (4) and (5) we see that $\angle(XYY') + \angle(YY'X') = \pi$. This, together with the fact that $|XY| = |X'Y'|$, shows that $XYY'X'$ is a parallelogram. Therefore, $d_{M_C}(x, x') = d_{M_C}(y, y')$. Denote this length by l.

Consider geodesics $g(t) = xy$, $g'(t) = x'y'$, $t \in [0, d_{M_C}(x, y)]$, connecting x with y, and correspondingly x' with y'. Since M_C has non-positive curvature the function $f(t) = d_{M_C}(g(t), g'(t))$ is convex, and since $f(0) = f(d_{M_C}(x, y))$, we see that $f(t)$ is a constant function. Denote this constant by l.

Consider a ruled surface S_1 (correspondingly, S_2) formed by the geodesics connecting y' with the points of the geodesic xy (correspondingly, x with the points of the geodesic $x'y'$). S_1 and S_2 have non-positive curvature with respect to the metric inherited from M_C (the result of Aleksandrov [**1**]; see also [**4**, Theorem 9.1]).

For piecewise smooth surfaces there is a well-defined concept of integral curvature measure, which has many properties similar to the integral curvature for smooth manifolds (see a review [**15**]), in particular, it satisfies the Gauss–Bonnet formula (see [**15**, Theorem 5.3.2]). Applying it to the surfaces S_1 and S_2, we conclude that their integral curvature is equal to zero everywhere (their defects are equal to zero, and their curvature measures are non-positive), and, therefore, S_1 is isometric to triangle XYY', and S_2 is isometric to triangle $XX'Y'$. Denote the isometries by F_1 and F_2, correspondingly.

Let $S = S_1 \bigcup S_2$, let $F : S \to XYY'X'$ be defined by $F|_{S_1} = F_1$, $F|_{S_2} = F_2$. Being a result of gluing of S_1 and S_2, the surface S also has non-positive curvature.

Let $M \in XY$ and $M' \in X'Y'$ be such that $MX = M'X'$. Consider $\alpha = F^{-1}(MM') \in S$. The curve α connects $m = F^{-1}(M) \in xy$ and $m' = F^{-1}(M') \in x'y'$, and its length is equal to the length of MM', and thus is equal to l. Therefore, α is the geodesic in M_C connecting m and m'.

Thus, we can describe S in the following way: surface S is formed by the geodesics G_t in M_C connecting $g_1(t)$ with $g_2(t)$, for $0 \le t \le d_{M_C}(x, y)$. Clearly, S is a piecewise smooth surface. Namely, it consists of the smooth pieces $L_k = S \bigcap M_k$, $k = 1, \dots, j$, which are glued together in the following way: L_k is glued with L_{k+1} along their common boundary C_{k+1}, where C_k, $k = 1, \dots, j$, are the curves of intersection of S with the boundaries of $E_k(B_{i_k})$.

The integral curvature measure at the interior points of L_k, $k = 1, \dots, j$, is equal to the smooth measure multiplied by the Gaussian curvature of S, and the integral curvature measure at the points of C_k is equal to the length measure on C_k multiplied by $(k_1 + k_2)$, where k_1 and k_2 are the oriented curvatures of the curve C_k in L_{k-1} and L_k, respectively.

We already know that S has zero integral curvature measure at all points. From the description above of the integral curvature measure on L_k and C_k it immediately follows that each piece L_k, $k = 1, \dots, j$, is flat inside and L_k is glued with L_{k+1} along a piece of C_k of a straight line. The other pieces of the boundary of L_k, $k = 1, \dots, j$, are all pieces of straight lines, since they are geodesics and belong to the boundary of a flat surface L_k.

Rescale the parameter t on the curves $g_1(t)$, $g_2(t)$ (and, thus, on the family G_t) so that t would vary from 1 to 2. Let $\Gamma_t = E^{-1}(G_t)$, where E^{-1} is the map $M_C \to M$ such that $E^{-1}(x) = E_k^{-1}(x)$, for $x \in M_k$, $k = 1, \dots, j$. We see that all G_t are periodic pseudo-trajectories of the same length. This finishes the proof of Theorem 2 (with $I_k = E_k^{-1}(C_k)$ and $\Sigma_k = E_k^{-1}(L_k)$). □

Let us call two periodic trajectories equivalent if they are parallel (in the sense explained in Theorem 2) and let us call two periodic points for the first return map to the boundary equivalent if the corresponding periodic billiard trajectories are equivalent.

Denote by P_k ($\tilde{P}_k$), $k \in \mathbb{N}$, the number of (equivalence classes of) periodic points of period k for the first return map to the boundary of the billiard B, and by P^t ($\tilde{P}^t$), $t \in \mathbb{R}^+$, the number of (equivalence classes of) periodic trajectories of the billiard flow of length less than or equal to t. Let for every $m \in \mathbb{N}$, $x \in \mathbb{R}^+$,

$$\theta(m,x) = \begin{cases} 0, & x < 2 \\ m(m-1), & 2 \le x < 3 \\ m(m-1)^{x-1}(m-2), & x \ge 3. \end{cases}$$

Theorem 2 implies the following.

COROLLARY 4.2. *Let B be a semi-dispersing billiard on a simply connected manifold M of non-positive sectional curvature. Then:*

1. *if the curvature of M is strictly negative then*

$$\tilde{P}_k = P_k \le \theta(n,k);$$

2. *if all the sets B_i, $i = 1, \dots, n$, are strictly convex then*

$$\tilde{P}_k = P_k \le \theta(n,k);$$

3. *otherwise, either*

$$\tilde{P}_k = P_k \le \theta(n,k) \quad \text{or} \quad \tilde{P}_k \le \theta(n,k), \quad P_k = \infty.$$

For $M = \mathbb{R}^k$, Corollary 4.2 was proven in [**17**].

Theorem 2 together with Proposition 1.1 implies the following.

COROLLARY 4.3. *Let B be a non-degenerate semi-dispersing billiard on a simply connected manifold M of bounded non-positive sectional curvature. (Notice that here again we no not require B to be compact, but we do require the non-degeneracy of B.) Then:*

1. *if the curvature of M is strictly negative then*

$$\tilde{P}^t = P^t \le \theta(n, P(t+1));$$

2. *if all the sets B_i, $i = 1, \dots, n$, are strictly convex then*

$$\tilde{P}^t = P^t \le \theta(n, P(t+1));$$

3. *otherwise, either*

$$\tilde{P}^t = P^t \le \theta(n, P(t+1)) \quad \text{or} \quad \tilde{P}^t \le \theta(n, P(t+1)), \quad P^t = \infty,$$

where P is the constant from Proposition 1.1.

In [**17**] Corollary 4.3 was proven for $M = \mathbb{R}^2$.

Remark. Notice that our estimates for the number of periodic points as well as for the topological entropy are applicable to billiards in polygons or polyhedras. However, for those billiards much finer results are known. In fact, the topological entropy of billiards in polygons or polyhedras is equal to zero [**10**, **11**], and the number of periodic points grows subexponentially [**11**].

5. *Topological entropy of Lorentz gas*

The Lorentz gas model is a billiard on $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$ with one wall which is a ball of radius $1/2 > r > 0$. In [**7**] it was proven that the first return map to the boundary of the Lorentz gas billiard has infinite topological entropy, and that the metric entropy of the Lorentz gas billiard with respect to the Liouville measure converges to zero, when $r \to 0$. In contrast to these results, we prove the following.

THEOREM 3. *Denote by $h_r(k)$ the topological entropy of the Lorentz gas billiard described above. Then:*

1. *$h_r(k)$ is finite;*
2. *there exist* $\lim_{r\to 0} h_r(k) = h_0(k)$, *and* $\infty > h_0(k) > 0$;
3. $h_0(k) \leq h_0(k+1)$ and $h_0(k) \geq \ln(2k-1)$.

The first statement of Theorem 3 follows immediately from Theorem 1. Moreover, for a fixed k, it is easy to see that $h_r(k)$ are uniformly bounded over r. Let us denote some upper bound by $Q(k)$.

To prove the second statement let us first introduce some notations. We denote by $B_r^{(m)}$ the ball of radius r centered at the point $(m) = (m_1, \ldots, m_k)$ on $\mathbb{R}^k$, $m_i \in \mathbb{Z}$, $i = 1, \ldots, k$. Denote by $F_r(n, k)$ the number of different combinatorial classes of billiard trajectories of length at most n, $n \in \mathbb{R}$, in $\tilde{B} = \mathbb{R}^k \backslash (\bigcup_{(m)\in\mathbb{Z}^k} \operatorname{Int} B_r^{(m)})$ starting at the ball $B_r^{(0)}$, where $(0) = (0, \ldots, 0)$. Then, exactly as in the proof of Theorem 1, we see that

$$h_r(k) = \overline{\lim_{n\to\infty}} \frac{\ln F_r(n,k)}{n}.$$

A curve γ in $\mathbb{R}^k$ that satisfies the following properties is called an r-pseudo-trajectory:

1. γ belongs to $\tilde{B}$, i.e. γ does not intersect the interiors of the balls $B_r^{(m)}$;
2. γ consists of several straight edges, with vertices that belong to the balls $B_r^{(m)}$.

In other words, γ is 'almost a billiard trajectory', except that it does not have to satisfy the 'angle of incidence is equal to the angle of reflection' law. Denote by $\Pi_r(n, k)$ the number of different combinatorial classes of r-pseudo-trajectories of length at most n that start at $B_r^{(0)}$. Then, clearly, $\Pi_r(n, k) \geq F_r(n, k)$. On the other hand, the shortest r-pseudo-trajectory in a given class C must be a billiard trajectory of length at most n (provided that C contains any r-pseudo-trajectories of length at most n). Thus, $\Pi_r(n, k) = F_r(n, k)$.

Now, we will show that $\Pi_r(n, k)$ is 'almost a decreasing function of r'. Let γ be an r-pseudo-trajectory of length l and with vertices $x_i \in B_r^{(m)_i}$, $i = 0, \ldots, p$, where $(m)_0 = (0)$. Then, since the distance between any two balls $B_r^{(m)_i}$ and $B_r^{(m)_{i+1}}$ is at least $1 - 2r$, we see that $p \geq l/(1 - 2r)$. Without loss of generality we may assume that $r \leq 1/4$. Then $p \geq 2l$. For a fixed $r' < r$, let y_i be the intersection of the boundary of the ball $B_{r'}^{(m)_i}$ with the interval of a straight line connecting x_i and $(m)_i$. Then the broken straight line γ' with vertices y_i, $i = 0, \ldots, p$, is an r'-pseudo-trajectory. Indeed, if γ' intersects with the interior of some ball $B_{r'}^{(m)}$, then γ intersects with the interior of the ball $B_r^{(m)}$, which contradicts the fact that γ is an r-pseudo-trajectory. Moreover, the length of γ' is at most $l + 2(r - r')p \leq l(1 + 4(r - r'))$. Thus,

$$\Pi_r(n, k) \leq \Pi_{r'}(n(1 + 4(r - r')), k).$$

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Thus, we see that

$$\begin{aligned} h_r(k) &= \overline{\lim_{n\to\infty}} \frac{\ln \Pi_r(n,k)}{n} \le \overline{\lim_{n\to\infty}} \frac{\ln \Pi_{r'}(n(1+4(r-r')),k)}{n} \\ &= (1+4(r-r'))\ \overline{\lim_{n\to\infty}} \frac{\ln \Pi_{r'}(n(1+4(r-r')),k)}{n(1+4(r-r'))} \\ &= (1+4(r-r'))h_{r'}(k). \end{aligned}$$

Thus,

$$h_r(k) - h_{r'}(k) \le 4(r-r')h_{r'}(k) \le 4(r-r')Q(k), \quad \text{if } r' < r. \tag{6}$$

Equation (6) immediately implies the second statement of Theorem 3, i.e. that there exists a limit $\lim_{r\to 0} h_r(k)$.

Moreover, equation (6) also implies that

$$h_0(k) = \lim_{r\to 0} h_r(k) = \lim_{r\to 0}\left(\overline{\lim_{n\to\infty}} \frac{\ln \Pi_r(n,k)}{n}\right) = \overline{\lim_{n\to\infty}} \frac{\ln(\lim_{r\to 0}\Pi_r(n,k))}{n}. \tag{7}$$

Denote by $\Pi_0(n,k) = \lim_{r\to 0}\Pi_r(n,k)$. Then it is easy to see that $\Pi_0(n,k)$ is equal to the number of different broken lines in $\mathbb{R}^k$ which start at (0), have length at most n, have vertices at the integer points, and such that:

1. the edges do not intersect any integer points, except for the vertices;
2. no three pairwise distinct consecutive vertices belong to one straight line, i.e. no edge is a continuation of the previous one.

Then, clearly, $\Pi_0(n,k+1) \ge \Pi_0(n,k)$. Thus, equation (7) implies that

$$h_0(k+1) = \overline{\lim_{n\to\infty}} \frac{\ln \Pi_0(n,k+1)}{n} \ge \overline{\lim_{n\to\infty}} \frac{\ln \Pi_0(n,k)}{n} = h_0(k).$$

Notice that, for integer n, the number $\Pi_0(n,k)$ can be estimated from below by the number of the broken lines such that each edge has length one. Therefore, $\Pi_0(n,k) \ge (2k-1)^n$. (Each of the edges can be parallel to one of the coordinate directions, and the only direction that is inadmissible is the direction that extends the previous edge.) Thus, $h_0(k) \ge \ln(2k-1)$.

Theorem 3 is proven.

Also, the geometric description of the numbers $\Pi_0(n,k)$ is useful if we want to obtain numerical estimates for the numbers $h_0(k)$. In particular, a computer aided calculation shows that $h_0(2) = 1.526\ldots$.

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