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COUNTEREXAMPLES TO CONJECTURES FOR UNIFORMLY OPTIMALLY RELIABLE GRAPHS

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In [7], several conjectures are listed about uniformly most reliable graphs, and, to date, no counterexamples have been found. These include the conjectures that an optimal reliable graph has degrees that are almost regular, has maximum girth, and has minimum diameter. In this article, we consider simple graphs and present one counterexample and another possible counterexample of these conjectures: maximum girth (i.e., maximize the length of the shortest circuit of the graph G) and minimum diameter (i.e., minimize the maximum possible distance between any pair of vertices).

1. INTRODUCTION

The graph $G(n,e)$ is simple with given n (number of nodes) and given e (number of edges). The all-terminal reliability of a graph G, $\mathrm{Rel}(G)$, is the probability that G remain connected after a fixed time period T. We assume that edges are independent random variables that survive this fixed time period T with probability p $(0 < p < 1)$ and vertices do not fail in this model. Colbourn's work [4] is an excellent reference on many of the combinatorial questions regarding network (all-terminal) reliability.

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Background material on graph theory and combinatorics can be obtained from [5] and [8]. In [2, p. 348], Boesch conjectures that uniformly most reliable graphs always exist and Myrvold [6] presents an infinite family of counterexamples to this conjecture. In Section 2, we give a counterexample to the conjecture about the minimum diameter of the graph G, and in Section 3, we provide another possible counterexample to the second conjecture about the maximum girth of the graph G.

2. COUNTEREXAMPLE TO CONJECTURE 1

Conjecture 1: A uniformly most reliable graph must have minimum diameter among the simple graphs with the same pair (n,e).

First, we recall that a standard formula for a reliability of a graph $G(n,e)$ with given n and e is

$$\mathrm{Rel}(p) = \sum_{i=0}^{e} N_i p^i (1-p)^{e-i} = 1 - \sum_{i=0}^{e} C_i p^{e-i}(1-p)^i, \tag{2.1}$$

where $N_i(G)$ or N_i denotes the number of i-edge connected spanning subgraphs of G, and the total number of cutsets (i.e., the number of sets of edges of size i whose removal disconnects G) is denoted by $C_i(G)$. Clearly, $C_i(G) = \binom{e}{i} - N_{e-i}$. The common size of the minimal cutsets (sometimes denoted by edge connectivity) and the number of minimal cutsets of G are denoted by $\lambda(G)$ and $C_\lambda(G)$, respectively. For any two graphs (G_1,G_2) with the same pair (n,e), we use the symbol $\mathrm{Rel}(G_1) > \mathrm{Rel}(G_2)$ to denote that G_1 is uniformly more reliable than G_2 $(0<p<1)$. In [2], Boesch showed that if G_1 and G_2 are two (n,e) graphs and if $\lambda(G_1) > \lambda(G_2)$, or $\lambda(G_1) = \lambda(G_2)$ and $C_\lambda(G_1) < C_\lambda(G_2)$, then $\mathrm{Rel}(G_1) > \mathrm{Rel}(G_2)$ for p close to 1. In [6], if $N_i(G_1) = N_i(G_2)$ for $i = 0,1,\ldots,k$ and $N_{k+1}(G_1) > N_{k+1}(G_2)$, then $\mathrm{Rel}(G_1) > \mathrm{Rel}(G_2)$ for $0 < p < \epsilon$ for some $\epsilon > 0$. Also, if $C_i(G_1) = C_i(G_2)$ for $i = 0,1,2,\ldots,k$ and $C_{k+1}(G_1) < C_{k+1}(G_2)$, then $\mathrm{Rel}(G_1) > \mathrm{Rel}(G_2)$ for $1-\epsilon < p < 1$ for some $\epsilon > 0$.

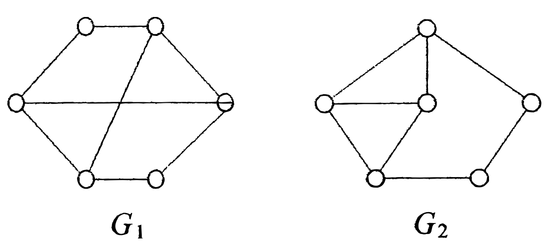

FIGURE 1. Two graphs, each with six nodes and eight edges.

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In Figure 1, there are two given graphs, G_1 and G_2; both graphs have the same number of vertices and links (i.e., $n = 6$, $e = 8$). Using (2.1) or the Maple Package, we obtain for the reliability polynomial for G_1 and G_2 respectively

$$\mathrm{Rel}(G_1, p) = 36p^5(1-p)^3 + 26p^6(1-p)^2 + 8p^7(1-p) + p^8, \tag{2.2}$$

$$\mathrm{Rel}(G_2, p) = 32p^5(1-p)^3 + 24p^6(1-p)^2 + 8p^7(1-p) + p^8. \tag{2.3}$$

We know from [2] that G_1 is a uniformly most reliable graph and note that $\mathrm{Rel}(G_1, p) > \mathrm{Rel}(G_2, p)$ for all $0 < p < 1$. However, the diameter of G_1 is 3 and the diameter of G_2 is 2, which contradicts Conjecture 1.

3. MAXIMUM GIRTH DOES NOT NECESSARILY IMPLY GREATER RELIABILITY

Conjecture 2: A uniformly most reliable graph must have maximum girth among the simple graphs with the same pair (n, e).

In this section, we present a possible counterexample to this second conjecture by looking at the two graphs in Figures 2 and 3 with the same pair (n, e) and showing that the one (G) with smaller girth is more reliable. If, as we believe but have not proven, it turns out that G is optimal, then this becomes a counterexample to Conjecture 2. Both graphs G and H have 30 nodes and 37 edges and their reliability polynomials are given in (3.1) and (3.2), respectively.

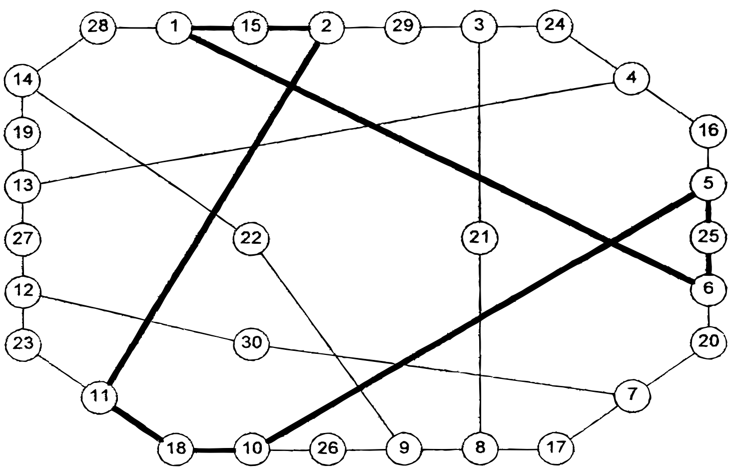

FIGURE 2. Graph G, with 30 nodes, 37 edges, and girth 9.

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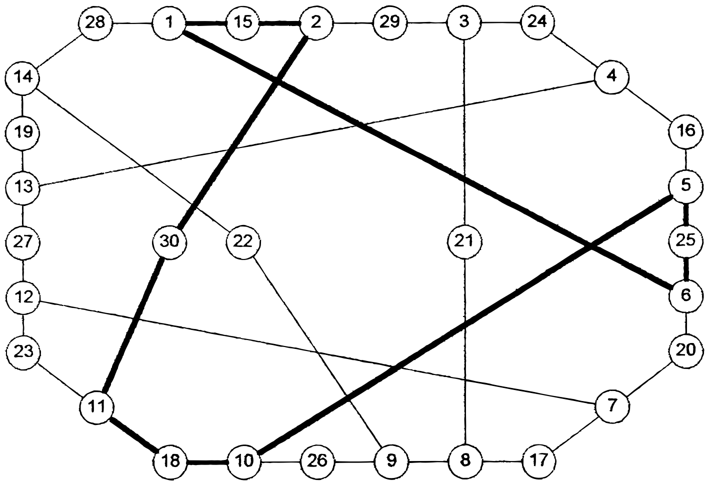

FIGURE 3. Graph H, also with 30 nodes and 37 vertices, but girth 10.

$$\begin{aligned}\mathrm{Rel}(G,p) = {} & 4{,}392{,}960p^{29}(1-p)^8 + 3{,}066{,}272p^{30}(1-p)^7 \\ & + 1{,}156{,}780p^{31}(1-p)^6 + 295{,}517p^{32}(1-p)^5 \\ & + 54{,}177p^{33}(1-p)^4 + 7138p^{34}(1-p)^3 \\ & + 650p^{35}(1-p)^2 + 37p^{36}(1-p) + p^{37}, \end{aligned} \tag{3.1}$$

$$\begin{aligned}\mathrm{Rel}(H,p) = {} & 4{,}386{,}976p^{29}(1-p)^8 + 3{,}062{,}748p^{30}(1-p)^7 \\ & + 1{,}155{,}986p^{31}(1-p)^6 + 295{,}429p^{32}(1-p)^5 \\ & + 54{,}173p^{33}(1-p)^4 + 7138p^{34}(1-p)^3 \\ & + 650p^{35}(1-p)^2 + 37p^{36}(1-p) + p^{37}. \end{aligned} \tag{3.2}$$

Again, we can see that $\mathrm{Rel}(G,p) > \mathrm{Rel}(H,p)$ for all $0 < p < 1$; however, the girth of G is 9 and the girth of H is 10 (the solid bold lines in each figure represent the cycle whose length is minimal and, hence, is the girth), which indicates that maximum girth does not necessarily imply greater reliability.

If (as we believe but have not proved) the graph G in Figure 2 is indeed optimal, then this becomes a counterexample to Conjecture 2.

4. CONCLUSION

The problem of finding the optimally reliable graph for given n and e is highly intractable. Although this field already has vast literature, the main problems of finding the optimal graph and proving optimality have not been solved. There may be more conjectures that still have no counterexamples or proofs. In [1] the present authors present seven new infinite families of graphs that are conjectured to be uniformly optimally reliable. This is a challenging problem with a potential for applications in statistics; we would like to see more probabilists and statisticians interested in this area.

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