Some open problems in concurrences

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An open problem

In the list of 12 *open* problems posed on pages 132-133 of a recent book [1] Problem 4 states, or rather conjectures:

If G_A, G_B and G_C are the orthogonal projections of the centroid G of triangle ABC on the sidelines BC, CA and AB, respectively, and if AG_A, BG_B and CG_C are concurrent, then ABC is isosceles. (*)

The problem is attributed to Temistocle Bîrsan. Needless to say, the converse is trivial and follows from symmetry, i.e., if ABC is isosceles, then AG_A, BG_B and CG_C are concurrent.

In this note, we provide a proof of the problem in (*), and we place it in a more general context.

A general context

Let ABC be a triangle. For any point P in the plane of ABC, we let P_A, P_B and P_C be the orthogonal projections of P on the sides BC, CA and AB, respectively. If AP_A, BP_B and CP_C are concurrent, then we denote the point of concurrence by P^*. Otherwise, we say that P^* does not exist or is not defined. Thus (*) asks for a proof that if G^* exists, then ABC is isosceles.

Let a, b, c, A, B and C denote the side lengths and angles of ABC in the standard manner, and let s denote its semi-perimeter, i.e., $s = \frac{1}{2}(a + b + c)$. Let O, I, G and H denote the circumcentre, incentre, centroid, and orthocentre, respectively. It is immediate that

$$H^* = H, O^* = G. \tag{1}$$

It also follows from Ceva's theorem that I^* exists since

$$\frac{BI_A}{I_AC}\frac{CI_B}{I_BA}\frac{AI_C}{I_CB} = \frac{s-b}{s-c}\frac{s-c}{s-a}\frac{s-a}{s-b} = 1.$$

The point of concurrence of AI_A, BI_B and CI_C is the point known as the *Gergonne* point. Denoting this point by R, we have

$$I^* = R. \tag{2}$$

In view of (1) and (2), it is now natural to ask whether G^* exists and what it is if it does. This is a reasonable context in which the problem posed in (*) above fits well.

A proof

Let M be the midpoint of BC, and let D be the orthogonal projection of A on the side BC, as shown in Figure 1. Note that the figure describes the case when $c > b$ and when C is acute. However, the calculations are still

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valid when C is obtuse (in both cases when G_A lies inside or outside the line segment BC). Also the formulas in (3) (below) still hold when $b \geqslant c$.

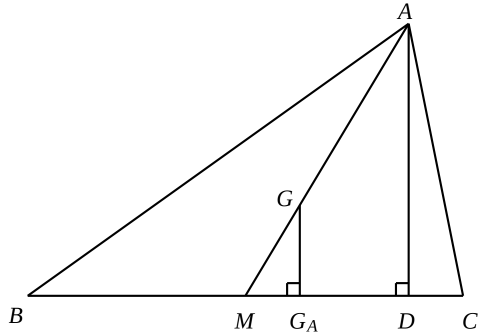

FIGURE 1

Referring to Figure 1, we see that

$$\frac{MG_A}{MD} = \frac{MG}{MA} = \frac{1}{3}.$$

and therefore

$$\begin{aligned} BG_A &= BM + MG_A = \frac{a}{2} + \frac{MD}{3} \\ &= \frac{a}{2} + \frac{BD - BM}{3} = \frac{a}{2} + \frac{1}{3}\left(c\cos B - \frac{a}{2}\right) \\ &= \frac{a}{2} + \frac{1}{3}\left(\frac{c\left(a^2 + c^2 - b^2\right)}{2ac} - \frac{a}{2}\right) = \frac{a}{2} + \frac{c^2 - b^2}{6a}. \end{aligned}$$

A formula for G_AC is obtained using $G_AC = BC - BG_A$. Similarly for the other sides. Thus we have

$$BG_A = \frac{a}{2} + \frac{c^2 - b^2}{6a},\ CG_B = \frac{b}{2} + \frac{a^2 - c^2}{6b},\ AG_C = \frac{c}{2} + \frac{b^2 - a^2}{6c},$$

$$G_AC = \frac{a}{2} - \frac{c^2 - b^2}{6a},\ G_BA = \frac{b}{2} - \frac{a^2 - c^2}{6b},\ G_CB = \frac{c}{2} - \frac{b^2 - a^2}{6c}. \quad (3)$$

If AG_A, BG_B and CG_C are concurrent, then it follows from Ceva's theorem that

$$\left(G_AC\right)\left(G_BA\right)\left(G_CB\right) - \left(BG_A\right)\left(CG_B\right)\left(AG_C\right) = 0.$$

Using the above formulas, and gathering terms in $b^2 - c^2$, etc., the left hand side simplifies to

$$\frac{bc\left(b^2 - c^2\right)}{24a} + \frac{ca\left(c^2 - a^2\right)}{24b} + \frac{ab\left(a^2 - b^2\right)}{24c} + \frac{\left(b^2 - c^2\right)\left(c^2 - a^2\right)\left(a^2 - b^2\right)}{216abc}$$

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$$= -\frac{(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)}{24abc} + \frac{(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)}{216abc}$$

$$= -\frac{(b^2 - c^2)(c^2 - a^2)(a^2 - b^2)}{27abc}.$$

If this expression vanishes, then $b = \pm c$, $c = \pm a$ or $a = \pm b$; but the negative values cannot apply, so at least two sides of the triangle are equal, and it is isosceles (or equilateral).

A historical note

As a referee kindly remarked, the points P for which P^* exists were studied as early as 1878, and they were found to form what is now known as the Darboux cubic. The points P which are equal to Q^* for some Q had been considered even earlier, and they were found to form another cubic, known now as the Lucas cubic. These cubics meet at nine points: A, B, C, H, the reflection of H in O, and four other points. Diagrams, equations and properties of these curves can be found on-line in [2]. These issues are also elaborated on in [3].

Examples: The Gergonne and Nagel points

In view of (1) and (2), it is natural to wonder whether R^* exists, where R is the Gergonne point. It turns out that, like G^*, R^* exists if, and only if, ABC is isosceles. The same holds for the Nagel point, where, the Nagel point is the point of concurrence of the cevians that join the vertices to the points where the respective excircles touch the opposite sides.

To prove these, and possibly other similar, statements, one first proves that if P is a point with barycentric coordinates (α, β, γ), $\alpha + \beta + \gamma = 1$, then

$$BP_A = \left(\gamma + \frac{\alpha}{2}\right)a + \frac{\alpha(c^2 - b^2)}{2a} = \frac{\alpha(a^2 - b^2 + c^2)}{2a} + a\gamma.$$

Similarly,

$$P_AC = \left(\beta + \frac{\alpha}{2}\right)a + \frac{\alpha(b^2 - c^2)}{2a} = \frac{\alpha(a^2 + b^2 - c^2)}{2a} + a\beta.$$

Also, the condition $\alpha + \beta + \gamma = 1$ is not needed since we will be interested in the ratio BP_A / P_AC. When this condition is not observed, we write $\alpha : \beta : \gamma$ for (α, β, γ). For $P = R$, we use the representation

$$\alpha : \beta : \gamma = (s - b)(s - c) : (s - c)(s - a) : (s - a)(s - b) \quad (4)$$

and the Ravi substitution $a = y + z$, $b = z + x$, $c = x + y$ to obtain

$$\frac{BP_A}{P_AC} = \frac{y^2(xy + yz + 3zx + z^2)}{z^2(xz + zy + 3yx + y^2)}. \quad (5)$$

Note that $x = \frac{1}{2}(b + c - a)$, and thus $x, y, z > 0$. Using *Maple*, the Ceva

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condition then simplifies to

$$(x - y)(y - z)(z - x)J(x, y, z) = 0,$$

where

$$J(x, y, z) = x^2y^2z^2(xy^2 + x^2y + yz^2 + y^2z + zx^2 + z^2x + 4xyz). \quad (6)$$

Since J is never 0, it follows that $x = y$, $y = z$ or $z = x$. Hence $a = b$, $b = c$ or $c = a$, and ABC is isosceles, as claimed.

A similar proof applies to the Nagel point. Here, (4) would be replaced by the barycentric coordinates of the Nagel point, namely

$$\alpha : \beta : \gamma = s - a : s - b : s - c,$$

and (5) by

$$\frac{BP_A}{P_AC} = \frac{xy + yz - zx + z^2}{xz + zy - yx + y^2},$$

and (6) by

$$J(x, y, z) = (x + y + z)^3(xy^2 + x^2y + yz^2 + y^2z + zx^2 + z^2x).$$

The same argument then applies.

In view of the dependence, albeit mild, of the proofs above on *Maple*, it would be interesting to come up with proofs that are more elegant, more geometric, and less computational.

A dynamical system

Let ABC be an isosceles triangle, with $AB = AC$ say, let AD be its altitude from A, and let H be its orthocentre. For any point P on the line AD, let E and F be the orthogonal projections of P on AB and AC, respectively, and let P^* be the point (on AD) where BF and CE intersect.

Starting with a point P on AD, we define the sequence $P^{[n]}$ for $n \geqslant 0$ by

$$P^{[0]} = P,\ P^{[n+1]} = \left(P^{[n]}\right)^*.$$

We will show that this sequence, with some controllable exceptions, converges to A or H according as ABC is acute or obtuse. To do so, we place ABC in the cartesian (x, y)-plane so that

$$B = (-1,0),\ C = (1,0),\ D = (0,0) \text{ and } A = (0, a) \text{ where } a > 0,$$

as shown in Figure 2. Let $P = (0, t)$ and let t_n, $n \geqslant 0$, be the y-coordinate of $P^{[n]}$. Thus $t = t_0$.

The equation of AB is $y = a + ax$ and the equation of PE is $y = t - \dfrac{x}{a}$. Solving these equations, we obtain

$$E = \left(\frac{a(t - a)}{a^2 + 1}, \frac{a(at + 1)}{a^2 + 1}\right).$$

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Thus the equation of EC is

$$y = (x - 1)\frac{a(1 + ta)}{ta - 2a^2 - 1}.$$

Therefore

$$t_{n+1} = \frac{-a^2 t_n - a}{at_n - \left(2a^2 + 1\right)}. \tag{7}$$

It is easy to check that

$$t_n = a \Leftrightarrow t_{n+1} = a,\ t_n = \frac{1}{n} \Leftrightarrow t_{n+1} = \frac{1}{n}.$$

It is also obvious that

$$t_0 = a \Leftrightarrow P = A,\ t_0 = \frac{1}{a} \Leftrightarrow P = H.$$

We exclude these trivial values of t_0, and we assume that

$$t_0 \neq a,\ t_0 \neq \frac{1}{a}.$$

This guarantees that t_n will never be a or $\frac{1}{a}$ for any n.

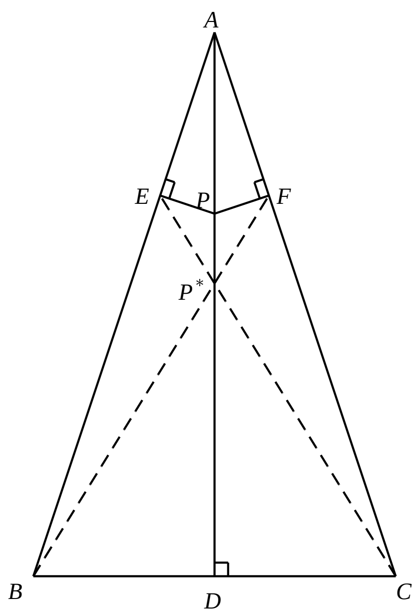

FIGURE 2

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We now try to diagonalise the difference equation (7) in the same way as we do the system

$$x' = -a^2x - ay,\ y' = ax - (2a^2 + 1)y \tag{8}$$

of differential equations. Since the characteristic equation of this system is

$$\chi(\lambda) = \det\begin{bmatrix} -a - \lambda & a \\ -a & -(2a^2 + 1) - \lambda \end{bmatrix} = (\lambda + a^2 + 1)(\lambda + 2a^2),$$

and since

$$a^2 + 1 = 2a^2 \Leftrightarrow a = 1,$$

we split our investigation into the two cases $a = 1$ and $a \neq 1$.

Case 1: $a = 1$, i.e., *ABC* is right-angled.

In this case, (7) takes the form

$$t_{n+1} = \frac{-t_n - 1}{t_n - 3}, \tag{9}$$

and the corresponding system (8) takes the form

$$x' = -x - y,\ y' = x - 3y. \tag{10}$$

To avoid the possibility that $t_n = 3$ for some n, we first check that if $k, n \in \mathbb{N}$, then

$$t_n = 1 + \frac{2}{k} \Leftrightarrow t_{n-1} = 1 + \frac{2}{k+1}.$$

It follows that if $n \geqslant 0$, then

$$t_n = 3 \Leftrightarrow t_0 = 1 + \frac{2}{n+1}.$$

Thus if t_0 is of the form $1 + \dfrac{2}{n+1}$ for some $n \geqslant 0$, then $t_n = 3$, and t_n is infinite. One may now stop here, or go on, seeing that $t_{n+1} = -1$, and $t_{n+2} = 0$. By induction, one can prove that

$$t_{n+2} = \frac{k-2}{k} \text{ for all } k \geqslant 2.$$

Thus t_n converges to 1, i.e., the point converges to the vertex A.

If t_0 is not of the form $1 + \dfrac{2}{n+1}$ for some $n \geqslant 0$, then we solve (10) obtaining the substitution

$$X = x,\ Y = y - 1 \tag{11}$$

and the triangular form

$$X' = -2X + Y,\ Y' = 2Y.$$

We use (11) to make the substitution

$$T_n = \frac{t_n}{t_n - 1},\ t_n = \frac{T_n}{T_n - 1},$$

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thus transforming (9) into the simple form

$$T_{n+1} = T_n - \tfrac{1}{2}.$$

It is obvious that $T_n \to -\infty$ as $n \to \infty$, and therefore $t_n \to 1$, i.e., our point converges to A.

Case 2: $a \neq 1$.

In this case, we try to avoid the possibility $t_n = \dfrac{2a^2 + 1}{a}$. Setting

$$k = \frac{2a^2 + 1}{a} \text{ and } q_n = \frac{a^2 - k^{n+1}}{a\left(1 - k^{n+1}\right)} \text{ for } n = 0, 1, 2, \ldots , \quad (12)$$

we can easily check that

$$t_n = q_s \Leftrightarrow t_{n+1} = q_{s-1}.$$

It follows that

$$t_n = q_0 = \frac{a^2 + 1}{a} \Leftrightarrow t_0 = q_n.$$

Thus we assume that t_0 is not an element of the sequence q_n given in (12). Diagonalising the system (8), we obtain the substitution

$$X = x - ay,\ Y = -ax + y,$$

and the diagonal form

$$X' = -2a^2 X,\ Y' = -\left(a^2 + 1\right) Y.$$

We use this and make the substitution

$$T_n = \frac{t_n - a}{-at_n + 1},\ t_n = \frac{T_n + a}{aT_n + 1},$$

and we obtain the geometric sequence

$$T_{n+1} = \frac{2a^2 T_n}{a^2 + 1}.$$

Thus $|T_n| \to 0$ if $a < 1$, and $|T_n| \to \infty$ if $a > 1$. But

$$T_n \to 0 \Leftrightarrow t_n \to a,\ |T_n| \to \infty \Leftrightarrow t_n \to \frac{1}{a}.$$

Since $a < 1 \Leftrightarrow A$ is obtuse and $a > 1 \Leftrightarrow A$ is acute, it follows that the sequence $P^{[n]}$ converges to H or A according as A is acute or obtuse. This is true of course under the assumption that the initial point is not a, $\frac{1}{a}$, or any element of the sequence given in (12). As mentioned earlier, if $t_0 = a$, i.e., $P = A$, then $P^{[n]} = A$ for all n; if $t_0 = \frac{1}{a}$, i.e., $P = H$, then $P^{[n]} = H$ for all n; if $t_0 = q_n$ for some n, then $P^{[n]}$ is the point at infinity, where we have chosen to terminate the iteration. Of course we can also stipulate, in view of (7), that if $t_n = \infty$, then $t_{n+1} = -a$, and the sequence would not terminate.

Remarks

1. It can be seen that if A is acute, and if the point P we started with is not any of the exceptional points and lies below A, then the convergence of $P^{[n]}$ to H is monotone. Examining the modes of convergence in other cases may turn out to be a nice project for undergraduate students and a gentle introduction to dynamical systems.
2. In dealing with the difference equation (7), it is natural to write (7) in the form

$$f(z) = \frac{-a^2z - a}{az - (2a^2 + 1)}$$

and to let z range in the extended complex plane $\mathbb{C} \cup \{\infty\}$. We then stipulate that $f((2a^2 + 1)/a) = \infty$ and $f(\infty) = -a$. However, we chose to isolate the possibility $t_n = \infty$ because of the geometric context in which (7) arose. Difference equations of the form $t_{n+1} = (at_n + b)/(cd_n + d)$, where $ad - bc \neq 0$ are treated in [4, §1.2., pp. 3-5], where they are referred to as Möbius transformations, and in [5, §2.6, Type II, pp. 93-94], where they are referred to as Riccati equations. Our treatment using the diagonalisation of the associated system of linear difference equations does not seem to have appeared anywhere else.

A dual problem

Let ABC be a triangle, and let P be an interior point. We have denoted by P_A, P_B, P_C the orthogonal projections of P on the sidelines BC, CA, AB and we have considered the problem of when the cevians AP_A, BP_B, CP_C are concurrent. When they are, we have denoted the point of concurrence by P^*. It is natural to consider the dual problem. Thus we denote by $P_{(A)}$, $P_{(B)}$, $P_{(C)}$ the points where the cevians through P meet the respective sides, and we consider the problem of when the perpendiculars erected from $P_{(A)}$, $P_{(B)}$, $P_{(C)}$ to the sides BC, CA, AB are concurrent. When they are, we denote the point of concurrence by $P^{(*)}$.

Let G, C, I, H, R, N denote, as usual, the centroid, circumcentre, incentre, orthocentre, Gergonne point, and Nagel point of ABC, respectively. It is trivial to see that $G^{(*)} = C$, $H^{(*)} = H$, and that $C^{(*)}$, $R^{(*)}$, $N^{(*)}$ exist. It is also shown in [6, Problem 7.46, pp. 173, 181] that $I^{(*)}$ exists. The questions that were asked about P^* can now be asked about $P^{(*)}$, and this may turn out to lead to interesting results.

Acknowledgement

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The answers to the *Nemo* page from November on probability and statistics were:

1.	Shakespeare	Antony and Cleopatra	Act 2, Scene 3
2.	JV Cunningham	Meditation on statistical method	
3.	Les Murray	The Statistics of Good	
4.	Frank Herbert	Dune (Chronicles 2)	
5.	Maurice G Kendall	Hiawatha designs an experiment	
6.	Ian McEwan	Sweet Tooth	Chapter 16

Congratulations to Henry Ricardo for identifying all these quotations. This month we revisit arithmetic. The quotations are to be identified by reference to author and work. Solutions are invited to the Editor by 31st May 2015.

1. Your ignorance so thick,
   You did not know your own arithmetic.
   We flung the graphs about your flying feet;
   We measured your diameter -
   Merely a line
   Of zeros prefaced by an integer.

2. CHORUS: If that be so, thy state of health is poor;
   But thine arithmetic is quite correct.

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