A TERMINAL COURSE IN MATHEMATICS

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For the past four years we have given, at the University of Alberta, a course entitled "The Nature of Mathematics". This course is open to first and second year students in arts and science and in education, and is designed primarily for those who will be taking only a single course in mathematics at the university. The only prerequisite for the course is high school mathematics and the course is not prerequisite for any other course.

The raison d'être for the course is our belief that the standard first year mathematics material consisting of trigonometry, analytic geometry and techniques of calculus leaves those students who take no further mathematics with a rather narrow and distorted view of the subject. The special techniques are soon forgotten and no understanding of what mathematics really is remains. While it is clear that no single course can give a student a deep understanding of mathematics it nevertheless seems desirable and feasible that a student taking only a single course should be given at least a glimpse of many aspects of the subject and some appreciation of the fact that mathematics is a living and rapidly expanding field of human activity whose importance stems not only from its role as handmaiden to the sciences but also from its position as an art. H.L. Mencken has defined an intelligent person as one who can discuss at least one single topic for at least five minutes. One objective of our course is to make students intelligent in this sense.

Our course consists of three lectures a week for both semesters. Most of the assignments consist of required reading in quite a few reference books. We proceed to give an outline of the course, to list some of the reference books and to give some sample examination questions.

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The course begins with a study of some topics from Topology. The purpose of starting here is to emphasize at the outset that mathematics does not deal exclusively with the subject matter of high school mathematics. Starting with some unicursal problems we go on to Euler's polyhedral formula and its application to a determination of the regular polyhedra. Next come coloring theorems and a variety of fixed point theorems. Jordan's curve theorem, orientability, one-sidedness, braids and knots and related topics are briefly discussed.

We then return to some arithmetic. Here we consider the basic laws of arithmetic, and calculation in bases other than decimal. Mathematical induction is discussed and applied and is contrasted with scientific induction. Various arithmetic functions are considered as well as some questions concerning the distribution of primes. Diophantine problems are considered and the present state of various important problems of number theory is mentioned. At this stage the concepts of group and field are presented and the field of integers with addition and multiplication modulo a prime is considered. Next comes a discussion of binomial coefficients and a study of certain combinatorial problems with emphasis on the concept of recurrence relation. It is again revealed that many natural problems lie beyond the scope of present day methods. The combinatorial problems lead to a discussion of large numbers and these in turn to a discussion of infinity. Here the concept of (1-1) correspondence is developed and the countability of certain sets and non-countability of others is established.

At all stages in the course students are invited to raise questions arising out of their outside reading or otherwise. We have found that such discussions form an important part of the lectures. Such questions lead, for example, to a discussion of how many mathematicians there are and what sort of work they do. Other questions bring out the relation between mathematics and other fields of intellectual activity. The dependence of the physical sciences on mathematics is common knowledge, but the ever increasing impact of mathematics on the biological and behavioral science is news to most students. Some attempts at relating mathematics to ethics and aesthetics are discussed. Questions also lead to a discussion of the various views concerning the basic nature of mathematics. The classical, logicalist, intuitionist and formalist viewpoints are considered with emphasis on the last. Also, as a result of questions raised by students, many miscellaneous questions on the history of mathematics, computing machinery, special devices for rapid calculations, paradoxes, mathematical recreations and other topics are discussed.

The second term begins with a review of elementary Euclidean geometry, or rather what should be a review. Many interesting results of elementary geometry, which were at one time well established as part of high school courses, are now quite new to university students. The discussion at this stage is on an intuitive basis but quite a few relatively novel geometrical topics are treated. For example we prove that for every pair of polygons A and B having the same area one can dissect A by means of a finite number of straight line cuts and reassemble the pieces to form B. Students are encouraged to find bounds for the number of cuts required in special cases. Dehn's analysis of the corresponding situation in three dimensions is mentioned. Definitions and basic properties of the conic sections and quadric surfaces are considered in this section.

We go on to a discussion of extremal problems, mainly of a geometrical nature. Many of the extremal problems in elementary calculus texts, and others as well, are tackled by synthetic devices. Geodesics on various surfaces are discussed and Steiner's argument for the isoperimetric problem is given and criticized. Extremal properties of some curves and surfaces are stated without proof. At this stage the basic principles of analytic geometry are developed and then used to prove some geometrical results. A discussion of three-dimensional analytic geometry leads to a discussion of the meaning of, and some results from, n-dimensional analytic geometry. This is followed by a discussion of n-dimensional geometry from a synthetic - combinatorial viewpoint. For example, the generalization of the Euler polyhedral formula is given and the numbers of vertices, edges, faces, etc. of an n-dimensional hypercube and simplex are obtained. Next we deal with an axiomatic approach to geometry. Some deductions are made from the axioms of incidence of finite projective geometry and various models for elliptic and hyperbolic geometry are presented.

The last part of the course is an introduction to the calculus. Some of the problems leading to the development of the calculus are mentioned and then the concepts of differentiation and integration are discussed. The functions here are restricted to polynomials. Applications to extremal problems and problems on rates of change, as well as problems on areas, volumes, motion on a line and calculation of work done in certain processes, are considered.

The course is by no means rigid and its content varies from year to year. In fact as mentioned earlier the content of a fair portion of the course is determined by the questions raised by the students, so that the above outline can only be considered as an approximation to the topics which may be covered in any one year.

The text, or rather principal reference book, has been R. Courant and H. Robbins, "What is Mathematics", Oxford (London, 1943). It is clear from the preceding outline that the course does not follow this book very closely. Most students find this book quite difficult but we feel that it is better to water down a good book rather than build up a weak one. We have found that some of the new books intended for similar courses are too shallow. However the new edition of M. Richardson, "Fundamentals of Mathematics", MacMillan (N.Y., 1957), has just come to our attention and as we are favorably impressed with it this book will probably be the principal reference book the next time the course is given. As for the books listed below, they represent of course a wide range of interest and depth. Students are strongly urged to spend at least three hours a week reading parts of these books, and are asked to report on their reading in one way or another.

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