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Relation between Euler–Bernoulli equation and contemporary knowledge in robotics

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SUMMARY
The motivation for this work is the state of modern structural mechanisms that are characterized by growing complexity and ever-increasing demands for rapid and accurate motion. These contradictory requirements are often achieved according to easier and easier structures characterized by flexibility segments. In most of cases, the elasticity of structures appears as an obstacle for a precise and rapid control of motion. The aim of this paper is to explore ways of implementation of structural properties of elasticity with the application of high fidelity models during synthesis and analysis of complex mechanisms. Precisely, the aim is to explore the possibility of using Euler–Bernoulli equation, if not in its original form, then to the same extent with the use of modern knowledge in robotics (based on the knowledge of classical mechanics), and to examine the affordability and confirmation of the method through simulation results for a typical robotic configuration.

KEYWORDS: Robot; Modeling; Elastic deformation; Link; Coupling; Dynamics; Kinematics; Trajectory planning.

Nomenclature

DOF	degree of freedom
t (s)	time
$\mathrm{d}t = 0.0001$ (s)	sample time
$T = 2$ (s)	whole period time
$p_s = [xyz\psi\wp\varphi]^{\mathrm{T}}$	Cartesian (external) coordinates
$\phi = [\rho_{1,1}\rho_{1,2}\rho_{1,3}\ \rho_{1,4}\dots\rho_{1,n}]^{\mathrm{T}}$	vector of internal coordinates
$x_{i,j}, y_{i,j}, z_{i,j}$	local coordinate frame, which is set in the base of considered mode
x_j, y_j, z_j	local coordinate frame, which is set in the base of the considered link
x, y, z	basic coordinate frame, which is set in the root of the considered robotic system
$j = 1, 2, 3, \dots, n_i$	serial number of the mode of considered link
$i = 1, 2, 3, \dots, m$	ordinal number of the link
$k = n_1 + n_2 + \dots + n_m$	whole number of the modes in considered robotics configuration
$M_{i,j} \in R^1$ (Nm)	load moment for the mode tip
$\varepsilon_{i,j} \in R^1$ (Nm)	bending moment for the mode tip
$\varepsilon_j \in R^{n_i}$ (Nm)	bending moments vector for each mode tip considered link
$\varepsilon = [\varepsilon_{1,1}\varepsilon_{1,2}\dots\varepsilon_{1,n_1}\ \varepsilon_{2,1}\varepsilon_{2,2}\dots\varepsilon_{2,n_2}\ \dots\dots\varepsilon_{m,n_m}]^{\mathrm{T}}$	vector of bending moments
$\varepsilon_m = [\varepsilon_{1,1}\varepsilon_{2,1}\varepsilon_{3,1}\ \dots\dots\varepsilon_{m,1}]^{\mathrm{T}}$	
$\varsigma \in R^1$ (Nm)	elasticity moment of the gear
$\hat{\#}_{i,j}$	quantities that are related to an arbitrary point of the elastic line of the mode, for example, $\hat{M}_{i,j}, \hat{x}_{i,j}, \hat{\varepsilon}_{i,j}$
$\#_{i,j}$	quantities that are not designated by "^" are defined for the mode tip, for example, $M_{i,j}, x_{i,j}, \varepsilon_{i,j}$
$\#_j$	quantities that characterize link
$\#^o$	quantities that define desired value
$\bar{\theta}_j \in R^1$ (rad)	rotation angle of the motor shaft after the reducer
$\vartheta_{i,j} \in R^1$ (rad)	bending angle of the considered mode
$\omega_{i,j} \in R^1$ (rad)	rotation angle of the considered mode tip (see ref. [22])
$\xi_j \in R^1$ (rad)	deflection angle of the gear
$\beta_{i,j} \in R^1$ (Nm^2)	flexural rigidity
$\eta_{i,j} \in R^1$ (s)	factor that characterizes part of damping in whole flexural characteristics
$H \in R^{k\times k}$	inertial matrix
$h \in R^k$	centrifugal, gravitational, Coriolis vector
$F_{uk} \in R^{6\times 1}$ [N(Nm)]	external contact force
$m_e = 1$ (kg)	equivalent mass
$b_e = 30$ [N/(m/s)]	equivalent damping
$k_{a1} = 10^4$ (N/m)	equivalent rigidity
$\mu = 0.1$	friction coefficient
J	Jacobian matrix mapping the effect of the external contact force
$T_{sti,j} \in R^1$ (m)	stationary part of flexible deformation caused by stationary moments that vary continuously over time
$T_{toi,j} \in R^1$ (m)	oscillatory part of flexible deformation

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$a_{i,j} \in R^1$ (m)	usually normal distance between j th and $j+1$ th joints
$a_{1,1} = l_{1,1}$(m), $a_{1,2} = l_{1,2}$ (m)	
$\alpha_{i,j} \in R^1(^o)$	angle between the axes z_{j-1} and z_j about axis x_j.
$\alpha_{1,1} = 0(^o)$, $\alpha_{1,2} = 0(^o)$,	
$d_{i,j} \in R^1$ (m)	distance between normal l_{j-1} and l_j along axis of jth joint
$d_{1,1} = 0$ (m), $d_{1,2} = 0$ (m)	
$R_j = 0.272\ (\Omega)$	rotor circuit resistance
u_j (V)	voltage
i_j (A)	rotor current
$C_{Ej} = 6.1$ [V/(rad/s)]	proportionality constants of the electromotive force
$C_{Mj} = 6.1$ (Nm/A)	proportionality constants of the moment
$B_{uj} = 0$ [Nm/(rad/s)]	coefficient of viscous friction
$I_j = 4.52$ (kgm^2)	inertia moments of the rotor and reducer
$S_j = 0.0446$	expression defining the reducer geometry
$\diamond \in R^{k\times k}$	matrix characterizing the mutual influence of the bending moments modes of all the links
$\diamond_m \in R^{m\times m}$	characterizes the influence of the bending moment of each mode on the motor dynamics
$\Theta \in R^{k\times k}$	matrix characterizing the robot configuration
$l_{i,j} \in R^1$ (m) $l_{1,1} = l_{1,2} = 0.3$ (m)	length of each mode
$c_{i,j} x b_{i,j}$	cross-section of rectangular shape
$c_{1,1} x b_{1,1} = 0.017 \times 0.018$ (m^2)	
$c_{1,2} x b_{1,2} = 0.0072 \times 0.0075$ (m^2)	
$r_{i,j} \in R^1$ (m)	flexure
$\Re \in R^1$ (m)	spatially distance
λ	trajectory mark
$m_b = 1, \in R^1$ (kg), $J_b = 0.00125$ (kgm^2)	mass in link base
$m = 2$ (kg), $J = 0.0025$ (kgm^2)	mass in link tip
$m_{eli,j}$ (kg), $J_{elzzi,j}$ (kgm^2)	mass, moment of inertia of the whole mode
$m_{el1,1} = 0.2479$ (kg), $J_{elzz1,1} = 0.0056$ (kgm^2)	
$m_{el1,2} = 0.0437$ (kg), $J_{elzz1,2} = 0.0010$ (kgm^2)	
$\hat{m}_{eli,j}$ (kg), $\hat{J}_{elzz\,i,j}$ (kgm^2)	equivalent mass, equivalent moment of inertia of the mode to the considered position on the flexible line according to the Strutt and Rayleigh (see ref. [22])
$\bar{w}_{i,j}$ (kg/m), $\bar{w}_{1,1} = 0.8262$ (kg/m), $\bar{w}_{1,2} = 0.1458$ (kg/m)	mass per unit of length,
$I_{momi,j}$ (m^4)	inertia moments of the cross-section of mode
$I_{mom1,1} = 0.8262 \times 10^{-8}$ (m^4)	
$I_{mom1,2} = 0.0253 \times 10^{-8}$ (m^4)	
E_{km} (Nm)	kinetic energy
E_p (Nm)	potential energy
Φ (Nm/s)	dissipative energy
ϕ	generalized coordinate
g (m/s^2)	gravity acceleration
$C \in R^{6\times 6}$	matrix of rigidity
$B \in R^{6\times 6}$	matrix of damping
$u \in R^1$ (V)	control signal
$C_{si,j} \in R^1$ (kg/s^2)	characteristics of stiffness of the mode considered link
$C_{s1,1} = 6.3617 \times 10^4$ (kg/s^2),	
$C_{s1,2} = 1.949 \times 10^3$ (kg/s^2)	
$B_{si,j} \in R^1$ (kg/s)	characteristics of damping of the mode considered link
$B_{s1,1} = 10$ (kg/s),	
$B_{s1,2} = 100$ (kg/s)	
$C_\xi = 1.8143 \times 10^3 \in R^1$ (Nm/rad)	characteristics of stiffness of the gear
$B_\xi = 10 \in R^1$ [Nm/(rad/s)]	characteristics of damping of the gear
$I_O = 0.7854 \times 10^{-11}$ (m^4)	polar moment of inertia, which we obtain depending on diameter and thickness of cross-section joints
$a_{uv} = 0.03$ (m)	length on which occurred deflection joints
$E_l = 69.3 \cdot 10^9$ (N/m^2)	module of elasticity for aluminium
$\delta\bar{\theta}(t_o) = 0$ (rad), $\delta\dot{\bar{\theta}}(t_o) = 0$ (rad/s)	initial exceptions of angles turning powers
$K_{lp} = 900 \times 10^6$, $K_{lv} = 60 \times 10^3$	position, velocity control gains for movements stabilization

1. Introduction

A mathematical model of a mechanism with one DOF, with one elastic gear that was defined by Spong (see ref. [1]) in 1987. On the basis of the same principle, the elasticity of gears is introduced in the mathematical model in this paper, as well in papers of refs. [2–7]. However, when the introduction of link flexibility in the mathematical model is concerned, it is necessary to point out some essential problems in this domain.

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With the aim to exploit the experience of previous research, Meirovitch theory was first analyzed. Meirovitch proposed "modal technique," more than 40 years ago, exactly in 1967. The author elaborated a particular application of the Euler–Bernoulli equation supposing that elastic deformation was a quantity defined in advance with respect to amplitude and frequency and, formed in this way, it was included into a dynamics model. Not finding any other solutions, many researchers in robotics (see refs. [8–16] etc.) applied the solution (see ref. [17]) in the description of the real dynamics of the robot system elastic deformations, or they used many ways to modify the solutions from ref. [17].

Having not found agreement with Meirovitch and his followers, the definition of elastic deformation was made taking into account the first research studies, i.e., the original form of Euler–Bernoulli equation.

Euler–Bernoulli equation was written in 1750. It was written by Bernoulli, a physicist, and Euler, a mathematician, his longtime friend and colleague. They did not even dream about the robotics and the knowledge we have now on disposal. Although it was made more than 250 years ago, Euler–Bernoulli equation is still usable and it can be connected logically with the contemporary knowledge from the robotics.

In this paper, Euler–Bernoulli equation is formed but "assumed modes technique" is not used in contrast to contemporaries who deal with this issue as well. The elastic deformation cannot be defined in advance (with both amplitude and frequency) and put in the system completely inversely. That means that the elastic deformation amplitude and its frequency change depending on the moments (perturbation, inertial moments, Coriolis, centrifugal moments, gravity moments as well as coupling moments between the present modes, and the play of the external forces). It, of course, depends on the mechanism configuration, weight, length of the segments of the reference trajectory choice, dynamic characteristics of the motor movements, etc.

This has been elaborated in detail in the work, but it is not the only essential problem existing in the pertinent literature.

In the previous papers, (see refs. [8–16] etc.) the general solution of the motion of an elastic robotic system has been obtained by considering flexural deformations as transversal oscillations that can be determined by the method of particular integrals of Bernoulli.

It is taken into consideration that any elastic deformation can be presented by superimposing Bernoulli's particular solutions of the oscillatory character and stationary solution of the forced character (see refs. [2–7]).

The motion equation at any point of considered mode (see refs. [18, 19]) follows directly from the Euler–Bernoulli equation for the preset boundary conditions.

Nowadays, taking into consideration significantly improved knowledge in the robotics (classical mechanics), Euler–Bernoulli equation can not be used anymore in its original form, as a purpose of synthesis and analysis of elastic robotic systems. Therefore, with respect to Euler and Bernoulli, it is necessary to further improve the equation. It is the only way for not losing information of complexity of movement dynamics of every mode within a segment (and broader within the total robotic configuration). Thus, it very important to connect original Euler–Bernoulli equation and modern robotic knowledge on the principles of classical mechanics. The foundations of classical mechanics are particularly emphasized because synthesis and analysis of kinematics and dynamics of robotic configurations in stiff and elastic elements are based on them. The elasticity of segments on the principles of classical mechanics is implemented in this paper.

This research has theoretical and practical significance. The purpose is to define as realistically as possible both kinematic and dynamic model of the mechanism with stiff and elastic elements that will describe the real system very well.

In refs. [5, 6], the general form of the mathematical model of the robotic system with elastic segments (Euler–Bernoulli equation) is given for the first time. This paper has the aim to make known this topic to the scientific community through modeling one "simple" example as well as to discover new phenomena in this field.

The future work should be directed to the implementation of gears elasticity and the flexibility of links on any model of a rigid robot and also on the model of a reconfigurable rigid robot as given in ref [20] or any other type of mechanism. The mechanism would be modeled to contain elastic elements and generate vibrations, which are used for conveying particulate and granular materials in ref. [21].

The procedure of defining the dynamic model of the system under the influence of external force with all elements of coupling is completely presented as well as with dynamic effects of present moments defined in Section 2. The kinematic model of the system is created (direct and inverse kinematics) in Section 3. Section 4 analyzes a simulation example for a dynamic movement of a multiple DOF elastic robotic pair with elastic gear and flexible link in the presence of the second mode and external force. Section 5 gives some concluding remarks.

2. Dynamics

"Original form of Euler–Bernoulli equation"of the elastic line of beam bending has the following form:

$$\beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2} + \hat{m}_{el1,1} \cdot \frac{\mathrm{d}^2 \hat{y}_{1,1}}{\mathrm{d}t^2} \cdot (x_{1,1} - \hat{x}_{1,1}) = 0, \qquad (1)$$

where $\hat{M}_{1,1} = \hat{m}_{el1,1} \cdot \frac{\mathrm{d}^2 \hat{y}_{1,1}}{\mathrm{d}t^2} \cdot (x_{1,1} - \hat{x}_{1,1})$ (Nm) is the load moment, in these source equations encompassing only inertia, $\hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot \frac{\partial^2 \hat{y}_{1,1}}{\partial \hat{x}_{1,1}^2}$ (Nm) is bending moment (see Fig. 1).

Equation (1) is defined under the assumption that the bending moment is opposed only by the proper inertial moment. Besides, it is supposed by definition that the motion in Eq. (1) is caused by an external force $F_{1,1}$, suddenly added and then removed. Euler–Bernoulli Eq. (1) is expanded in refs. [5, 6] from several aspects in order to be applicable in a broader analysis of elasticity of robot mechanisms. The load moment is composed of all moment acting on the first mode of the link and these are perturbation moments, inertial moments (single and coupled moments), centrifugal, gravitational, Coriolis moments (single and coupled), coupled bending moments of the other modes, as well as the external force (which can be defined as static or dynamic force), which is via Jacobian matrix transferred to the motion of the first mode that come out directly from the motion dynamics of elastic bodies.

They become more complex. This means that all these moments participate in generation of bending moment, i.e., in forming elastic deformation as well as in forming the elasticity line of the first mode. In that case, the model of elastic line of the elastic link's first mode has Euler–Bernoulli equation

$$\hat{H}_{1,j}\frac{\mathrm{d}^2\hat{y}_{1,j}}{\mathrm{d}t^2}+\hat{h}_{1,1}+j_{1,1}^{\mathrm{T}}F_{uk}+\Diamond_{1,j}\cdot\varepsilon_1+\beta_{1,1}\cdot\frac{\partial^2(\hat{y}_{1,1}+\eta_{1,1}\cdot\dot{\hat{y}}_{1,1})}{\partial\,\hat{x}_{1,1}^2}=0. \quad (2)$$

Let us consider a robotic system with m links, where by the first link contains n_1 modes, the second link n_2 modes, ... the m th link contains n_m modes. The model of the elastic line of this complex elastic robotic system is given in the matrix form by the following Euler–Bernoulli equation:

$$\hat{H}\cdot\frac{\mathrm{d}^2\hat{y}}{\mathrm{d}t^2}+\hat{h}+j_e^{\mathrm{T}}\cdot F_{uk}+\Diamond\cdot\Theta\cdot\varepsilon+\hat{\varepsilon}=0. \quad (3)$$

Detailed explanation of all components of Eqs. (2) and (3) can be find in ref. [5, 6]. Robotics researchers are especially interested in the first mode's tip motion.

The equation of the motion of the moments involved at any point of elastic line of the first mode, including the point of the first mode's tip, can be defined from Euler–Bernoulli Eq. (2). The equation of motion of all moments at the first mode's tip for the given boundary conditions can be defined by the following equation:

$$H_{1,j}\frac{\mathrm{d}^2y_{1,j}}{\mathrm{d}t^2}+h_{1,1}+j_{e1,1}^{\mathrm{T}}\cdot F_{uk}^{\mathrm{T}}+\Diamond_{1,j}\cdot\varepsilon_{1,j}+\varepsilon_{1,1}=0\Bigg|_{\substack{\sum F=0(\sum M=0)\\ \text{at the point of}\\ \text{first mode tip}}}. \quad (4)$$

Equation (4) is interesting because it allows one to calculate the position of the first mode's tip. If we know the position of each mode's tip, we can always calculate the position of the link's tip too and eventually the position of the robot's tip.

The equation of motion of all the moments at the point of each mode's tip of any link can be defined from Euler–Bernoulli Eq. (3) by setting boundary conditions. The vector equation of motion of all the moments involved for each mode's tip of any link is

$$H\frac{\mathrm{d}^2y}{\mathrm{d}t^2}+h+j_e^{\mathrm{T}}\cdot F_{uk}+\Diamond\cdot\Theta\cdot\varepsilon+\varepsilon=0\Bigg|_{\substack{\sum F=0(\sum M=0)\\ \text{at the tip of}\\ \text{any mode of the}\\ \text{link considered}}}. \quad (5)$$

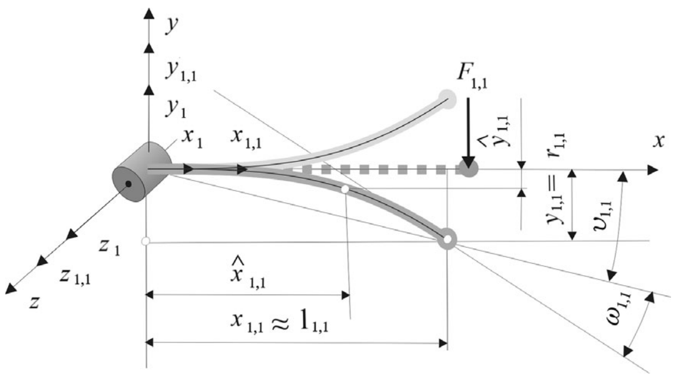

Fig. 1. Idealized motion of elastic body according to Bernoulli.

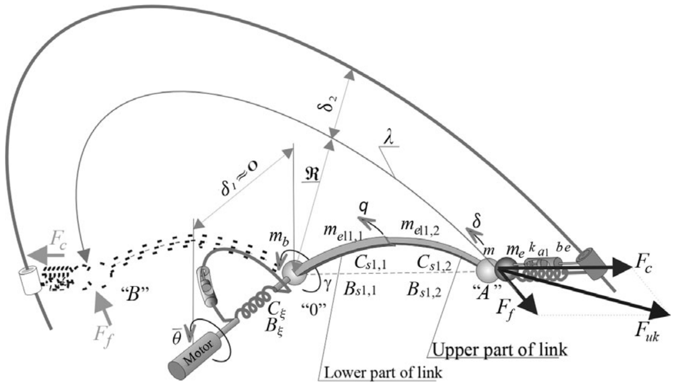

Fig. 2. Robotic mechanism.

The mathematical model of all m motors can be written in a vector form as

$$u=R\cdot i+C_E\cdot\dot{\bar{\theta}}$$

$$C_M\cdot i=I\cdot\ddot{\bar{\theta}}+B_u\cdot\dot{\bar{\theta}}-S\cdot(\Diamond_m\cdot\varepsilon+\varepsilon_m)\Bigg|_{\substack{\sum M=0\\ \text{about the rotation axis}\\ \text{of the each motors}}}. \quad (6)$$

Example. Let us analyze the behavior of the robotic pair consisting of elastic gear and flexible link in the presence of the second mode, as depicted in Fig. 2.

The link has two modes (the lower one and the upper one) and each of them is considered as a mode of rectangular cross-section $c_{i,j}xb_{i,j}$.

The presence of the second mode is introduced into the analysis of the robotic pair behavior. The relations between the important angles are defined in Fig. 3.

$$q=\bar{\theta}+\xi+\vartheta_{1,1},\quad \gamma=\bar{\theta}+\xi,\quad \delta=\vartheta_{1,2}+\omega_{1,1},\quad \omega_{1,j}=\frac{\vartheta_{1,j}}{2}. \quad (7)$$

The dynamic model (both the model of flexible line and model of the motion of each mode's tip) is defined according to classical principles but with the previously introduced new DH parameters, using Lagrange's equations.

The following quantities are adopted: q, δ, γ, and $\bar{\theta}$ as generalized coordinates (see Fig. 3).

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Small bending angles should be taken into consideration and it should be adopted that $l_{si,j} = l_{i,j}$ and if $\tan\vartheta_{i,j} = \frac{r_{i,j}}{l_{i,j}}$, then $\vartheta_{i,j} \approx \tan\vartheta_{i,j}$ and $r_{i,j} = l_{i,j} \cdot \vartheta_{i,j}$, $i = 1$, $j = 1, 2$

$$\vartheta_{i,j} = \frac{r_{i,j}}{l_{i,j}}. \tag{8}$$

The magnitude $r_{i,j}$ is the maximal deflection, i.e., the deflection at the each mode's tip.

The bending angles are expressed in Fig. 3 via generalized coordinates, Eqs. (7) and (8)

$$r_{1,1} = l_{1,1} \cdot (q - \gamma), r_{1,2} = l_{1,2} \cdot \left(\delta - \frac{q-\gamma}{2}\right), \tag{9}$$

$$\hat{m}_{el1,1} = \frac{33}{140} \cdot \bar{w}_{1,1} \cdot (x_{1,1} - \hat{x}_{1,1}),$$

$$\hat{m}_{el1,2} = \frac{33}{140} \cdot \bar{w}_{1,2} \cdot (x_{1,2} - \hat{x}_{1,2}), \tag{10}$$

$$\hat{J}_{elzz1,1} = \hat{m}_{el1,1} \cdot \left(\frac{x_{1,1} - \hat{x}_{1,1}}{2}\right)^2,$$

$$\hat{J}_{elzz1,2} = \hat{m}_{el1,2} \cdot \left(\frac{x_{1,2} - \hat{x}_{1,2}}{2}\right)^2. \tag{11}$$

Equation (10) sources from ref. [22]. Kinetic and potential energies of the mechanism presented in Fig. 2 are denoted as $\hat{\hat{E}}_{km}$ and $\hat{\hat{E}}_p$. All the angles in the expression for kinetic and potential energies characterizing flexibility of the links should also be expressed via generalized coordinates.

The bending moment is expressed at any point of mode in the form $\hat{\varepsilon}_{i,j} = \beta_{i,j} \cdot \frac{\partial^2(\hat{y}_{i,j} + \eta_{i,j} \cdot \dot{\hat{y}}_{i,j})}{\partial \hat{x}_{i,j}^2}$.

So that total potential energy is

$$\hat{\hat{E}}_p = E_{po} + E_{pels} + E_{pel\xi}. \tag{12}$$

So that total dissipative energy is

$$\Phi = \Phi_{els} + \Phi_{el\xi}. \tag{13}$$

Potential, dissipative energy as a result of elasticity of the first and second link is $E_{pels} = \frac{1}{2} \cdot C_{s1,1} \cdot r_{1,1}^2 + \frac{1}{2} \cdot C_{s1,2} \cdot r_{1,2}^2$, $\Phi_{els} = \frac{1}{2} \cdot B_{s1,1} \cdot \dot{r}_{1,1}^2 + \frac{1}{2} \cdot B_{s1,2} \cdot \dot{r}_{1,2}^2$ on the top of each link.

To bring previous expressions into the adequate form, or in other words, in the form dependent on generalized coordinates, multiplication and division of the same expressions are introduced with $l_{1,1}^2$, $l_{1,2}^2$, etc. $E_{pels} = \frac{1}{2} \cdot C_{s1,1} \cdot \frac{r_{1,1}^2}{l_{1,1}^2} \cdot l_{1,1}^2 + \frac{1}{2} \cdot C_{s1,2} \cdot \frac{r_{1,2}^2}{l_{1,2}^2} \cdot l_{1,2}^2$, $\Phi_{els} = \frac{1}{2} \cdot B_{s1,1} \cdot \frac{\dot{r}_{1,1}^2}{l_{1,1}^2} \cdot l_{1,1}^2 + \frac{1}{2} \cdot B_{s1,2} \cdot \frac{\dot{r}_{1,2}^2}{l_{1,2}^2} \cdot l_{1,2}^2$. By applying expression (7)–(9) in previous equations, potential energy of elastic link is needed:

$$E_{pels} = \frac{1}{2} C_{s1,1} (q-\gamma)^2 l_{1,1}^2 + \frac{1}{2} C_{s1,2} \left(\delta - \frac{q-\gamma}{2}\right)^2 l_{1,2}^2, \tag{14}$$

$$\Phi_{els} = \frac{1}{2} B_{s1,1} (\dot{q}-\dot{\gamma})^2 l_{1,1}^2 + \frac{1}{2} B_{s1,2} \left(\dot{\delta} - \frac{\dot{q}-\dot{\gamma}}{2}\right)^2 l_{1,2}^2. \tag{15}$$

Potential, dissipative energy of elastic joint is $E_{pel\xi} = \frac{1}{2} \cdot C_\xi \cdot \xi^2$ and $\Phi_{el\xi} = \frac{1}{2} \cdot B_\xi \cdot \dot{\xi}^2$. If it is expressed depending on the generalized coordinate (7) than it is

$$E_{pel\xi} = \frac{1}{2} \cdot C_\xi \cdot (\gamma - \bar{\theta})^2, \tag{16}$$

$$\Phi_{el\xi} = \frac{1}{2} \cdot B_\xi \cdot (\dot{\gamma} - \dot{\bar{\theta}})^2. \tag{17}$$

Let us define the equation of a flexible line of the first mode. The expressions $\hat{\hat{E}}_{km}$ and $\hat{\hat{E}}_p$ should be defined for the full length of the second mode and for any point of the first mode

$$l_{1,2} = x_{1,2}, m_{el1,2} = \frac{33}{140} \cdot \bar{w}_{1,2} \cdot l_{1,2},$$

$$J_{elzz1,2} = m_{el1,2} \cdot \left(\frac{l_{1,2}}{2}\right)^2, \tag{18}$$

and receive expressions $\hat{E}_{kmel\,q}$, $\hat{E}_{pel\,q}$, E_{pels}, $E_{pel\,\xi}$, Φ_{els}, $\Phi_{els\,\xi}$.

The original form of Euler–Bernoulli equation is the motion equation, but defined for every point of the considered mode. Just under defined conditions (18), the load torque $\hat{M}_{1,1}$ is defined using the Lagrangian equation, for every point of the considered mode. This means that the Lagrangian equation is used as an aid in order to form the elastic line of the first mode, or extended forms of Euler–Bernoulli equation. Instead of Lagrangian equation, any other motion equation could have been used.

By applying the Lagrange's equation with respect to the first generalized coordinate q, the load moment is obtained

$$\hat{M}_{1,1} = [\,\hat{H}_{1,1}\ \hat{H}_{1,2}\ \hat{H}_{1,3}\ 0\,] \cdot \ddot{\phi} + \hat{h}_1 + \hat{J}_{1,1} \cdot F_{uk\,x} + \hat{J}_{2,1} \cdot F_{uk\,y} - \frac{1}{2} \cdot C_{s_{1,2}} \cdot l_{1,2} \cdot r_{1,2} - \frac{1}{2} \cdot B_{s_{1,2}} \cdot l_{1,2} \cdot \dot{r}_{1,2},$$

which represents the sum of all moments that cause flexible deformation of the first mode and which is opposed by bending moment

$$\hat{\varepsilon}_{1,1} = \beta_{1,1} \cdot \frac{\partial^2(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}_{1,1}^2}.$$

The magnitude $\hat{M}_{1,1}$ includes the external force F_{uk} that across the Jacobi matrix $\hat{J}$ maps on the direction of the first generalized coordinate. This is just the procedure for obtaining Euler–Bernoulli equation by which the motion of any point on the flexible line of the first mode is performed

$$[\hat{H}_{1,1}\ \ \hat{H}_{1,2}\ \ \hat{H}_{1,3}\ \ 0] \cdot \ddot{\phi} + \hat{h}_1 + \hat{J}_{1,1} \cdot F_{uk\,x} + \hat{J}_{2,1} \cdot F_{uk\,y} - \frac{1}{2} \cdot C_{s_{1,2}} \cdot l_{1,2} \cdot r_{1,2} - \frac{1}{2} \cdot B_{s_{1,2}} \cdot l_{1,2} \cdot \dot{r}_{1,2} + \beta_{1,1} \cdot \frac{\partial^2(\hat{y}_{1,1} + \eta_{1,1} \cdot \dot{\hat{y}}_{1,1})}{\partial \hat{x}_{1,1}^2} = 0. \tag{19}$$

F_{uk} (N) is the dynamic force of the contact (in this case).

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The component of the entire external force in the radial direction (see Fig. 3) is $F_c = (m_e \cdot \ddot{\Re} + b_e \cdot \dot{\Re} + \cdot F_c^o + k_{a1} \cdot \Delta\Re)$, whereas the friction force is $F_f = -\mu \frac{\dot{p}_s}{|\dot{p}_s|} \cdot F_c$, as in ref [23]. The friction coefficient is μ. The velocity of the robot tip is $\dot{p}_s$.

$\Re$ is the distance from the point "0" to the trajectory, marked with λ on Fig. 2, and $\Delta\Re = l - \Re$, $l = l_{1,1} + l_{1,2}$. m_e (kg) is the equivalent mass, b_e (N/(m/s)) is the equivalent damping, k_{a1} (N/m) is the equivalent rigidity.

$\varsigma = C_\xi \cdot \xi + B_\xi \cdot \dot{\xi}$ is the elasticity moment of gear and $\varepsilon_{i,j} = (C_{s_{i,j}} \cdot r_{i,j} + B_{s_{i,j}} \cdot \dot{r}_{i,j}) \cdot l_{i,j}$ is the bending moment of each mode's tip motion

$$\begin{aligned}
&\phi = [q \quad \delta \quad \gamma \quad \bar{\theta}]^{\mathrm{T}},\\
&\hat{H}_{1,1} = \hat{m}_{el1,1} \cdot (x_{1,1} - \hat{x}_{1,1})^2 + (m + m_{el1,2}) \cdot l_{1,1}^2\\
&\qquad + (m + m_{el1,2}) \cdot l_{1,2}^2 + 2 \cdot (m + m_{el1,2}) \cdot l_{1,1} \cdot l_{1,2}\\
&\qquad \cdot \cos\delta + \frac{9}{4} \cdot \hat{J}_{elzz1,1} + \frac{9}{16} \cdot (J_{zz} + J_{elzz1,2}),\\
&\hat{H}_{1,2} = \ldots, \hat{H}_{1,3} = \ldots, \hat{h}_1 = \ldots.
\end{aligned}$$

Now it is seen how many elements should be used in order to extend Euler–Bernoulli equation with the aim to get real information about the dynamics of motion of each point on the elastic line of the considered mode.

There is a full analogy between Eqs. (2) and (19).

In an analogous way, the equation of flexible line of the second mode should be defined.

The expressions $\hat{\hat{E}}_{km}$ and $\hat{\hat{E}}_p$ need to be defined for the full length of the first mode and for any point of the second mode

$$\begin{aligned}
&l_{1,1} = x_{1,1}, m_{el1,1} = \frac{33}{140} \cdot \bar{w}_{1,1} \cdot l_{1,1},\\
&J_{elzz1,1} = m_{el1,1} \cdot \left(\frac{l_{1,1}}{2}\right)^2.
\end{aligned} \tag{20}$$

By applying Lagrange's equation with respect to the second generalized coordinate δ using the expressions $\hat{E}_{kmel\,\delta}$, $\hat{E}_{pel\,\delta}$, E_{pels}, $E_{pel\,\xi}$, Φ_{els}, and $\Phi_{els\,\xi}$, the load moment is obtained

$$\begin{aligned}
\hat{M}_{1,2} = [\,\hat{H}_{2,1} \quad \hat{H}_{2,2} \quad \hat{H}_{2,3} \quad 0\,] \cdot \ddot{\phi} + \hat{h}_2 + \hat{J}_{1,2} \cdot F_{uk\,x}\\
+ \hat{J}_{2,2} \cdot F_{uk\,y},
\end{aligned}$$

which represents the sum of all moments that cause flexible deformation of the second mode and which is opposed to the bending moment $\hat{\varepsilon}_{1,2} = \beta_{1,2} \cdot \frac{\partial^2(\hat{y}_{1,2} + \eta_{1,2} \cdot \dot{\hat{y}}_{1,2})}{\partial\,\hat{x}_{1,2}^2}$. The magnitude $\hat{M}_{1,2}$ includes also external force F_{uk}. This is just the procedure for obtaining Euler–Bernoulli equation of flexible line of the second mode. This is Euler–Bernoulli equation by which the motion of any point on the flexible line of the second mode could be defined

$$\begin{aligned}
&[\hat{H}_{2,1} \quad \hat{H}_{2,2} \quad \hat{H}_{2,3} \quad 0] \cdot \ddot{\phi} + \hat{h}_2 + \hat{J}_{1,2} \cdot F_{uk\,x}\\
&\quad + \hat{J}_{2,2} \cdot F_{uk\,y} + \beta_{1,2} \cdot \frac{\partial^2(\hat{y}_{1,2} + \eta_{1,2} \cdot \dot{\hat{y}}_{1,2})}{\partial\,\hat{x}_{1,2}^2} = 0.
\end{aligned} \tag{21}$$

There is a full analogy between the Eqs. (2) and (21). Robotics researchers are especially interested in the mode tip's motion. At this point, perturbation moments, inertial moments (single and coupled ones of other modes), centrifugal, gravitational, Coriolis moments (single and coupled), coupled bending moments of other modes, as well as the external force, the effect of the latter motion of the considered link being transferred through the Jacobian matrix.

The equation of motion of moments involved at any point of elastic line of the first mode, including the point of the first mode's tip, can be defined in the following way.

In order to form motion equations, Lagrangian equations are also used but only under boundary conditions, for the considered point of type modes, $l_{1,1} = x_{1,1}$ and $l_{1,2} = x_{1,2}$. Following this idea, the mathematical model of the observed mechanism in the classical form is formed.

The expressions $\hat{\hat{E}}_{km}$ and $\hat{\hat{E}}_p$ need to be defined for the full length of the first mode $l_{1,1} = x_{1,1}$ and for the full length of the second mode $l_{1,2} = x_{1,2}$. The expressions E_{kmel} and E_{pel} are derived in this way. The equation of the motion of tip point of considered elastic line of the first mode is obtained by applying Lagrange's equation with respect to the first generalized coordinate q and using the expressions E_{kmel}, E_{pel}, E_{pels}, $E_{pel\,\xi}$, Φ_{els}, and $\Phi_{els\,\xi}$:

$$\begin{aligned}
&[H_{1,1} \quad H_{1,2} \quad H_{1,3} \quad 0] \cdot \ddot{\phi} + h_1 + J_{e1,1} \cdot F_{ukx}\\
&\quad + J_{e2,1} \cdot F_{uky} - \frac{1}{2} \cdot C_{s_{1,2}} \cdot l_{1,2} \cdot r_{1,2} - \frac{1}{2} \cdot B_{s_{1,2}} \cdot l_{1,2} \cdot \dot{r}_{1,2}\\
&\quad + C_{s1,1} \cdot r_{1,1} \cdot l_{1,1} + B_{s1,1} \cdot \dot{r}_{1,1} \cdot l_{1,1} = 0.
\end{aligned} \tag{22}$$

Following the same procedure by applying Lagrange's equation with respect to the second generalized coordinates, δ obtains the equation of motion at the tip point of the considered elastic line's second mode.

$$\begin{aligned}
&[\,H_{2,1} \quad H_{2,2} \quad H_{2,3} \quad 0\,] \cdot \ddot{\phi} + h_2 + J_{e1,2} \cdot F_{uk\,x}\\
&\quad + J_{e2,2} \cdot F_{uk\,y} + C_{s_{1,2}} \cdot l_{1,2} \cdot r_{1,2} + B_{s_{1,2}} \cdot l_{1,2} \cdot \dot{r}_{1,2} = 0.
\end{aligned} \tag{23}$$

Lagrange's equation with respect to the third generalized coordinate γ, equation of motion is defined

$$\begin{aligned}
&[H_{3,1} \quad H_{3,2} \quad H_{3,3} \quad 0] \cdot \ddot{\phi} - C_{s1,1} \cdot r_{1,1} \cdot l_{1,1}\\
&\quad - B_{s1,1} \cdot \dot{r}_{1,1} \cdot l_{1,1} + \frac{1}{2} \cdot C_{s_{1,2}} \cdot l_{1,2} \cdot r_{1,2}\\
&\quad + \frac{1}{2} \cdot B_{s_{1,2}} \cdot l_{1,2} \cdot \dot{r}_{1,2} + B_\xi \cdot \dot{\xi} + C_\xi \cdot \xi = 0.
\end{aligned} \tag{24}$$

There is a full analogy between Eqs. (4) and (22)–(24). By applying Lagrange's equation with respect to the fourth generalized coordinates, $\bar{\theta}$ obtains the equation of the motor's

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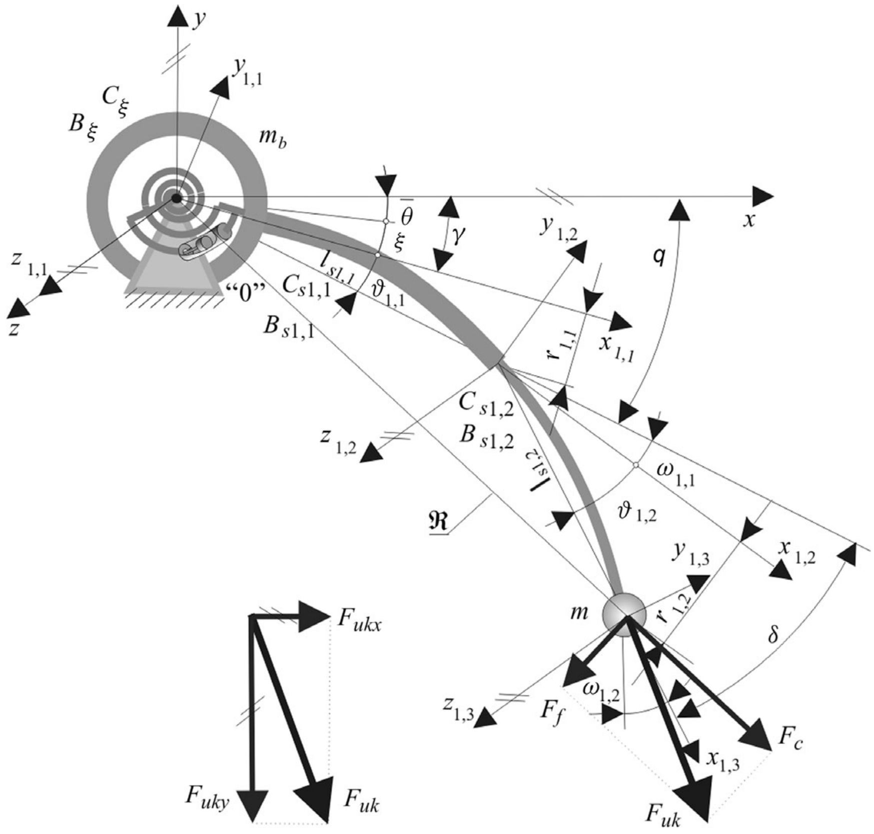

Fig. 3. Planar geometry of the mechanism in the vertical plane.

motion

$$u = R \cdot i + C_E \cdot \dot{\bar\theta},$$
$$C_M \cdot i = I \cdot \ddot{\bar\theta} + B \cdot \dot{\bar\theta} - S \cdot (B_\xi \cdot \dot\xi + C_\xi \cdot \xi). \quad (25)$$

Equations (22)–(25) that will be written in the matrix form obtain the mathematical model of the system

$$U = H \cdot \ddot\phi + h + C \cdot \phi + B \cdot \dot\phi + J^{\mathrm{T}} \cdot F_{uk}. \quad (26)$$

By Eq. (26) motions q, δ, γ, and $\bar\theta$ can be defined and through them the angle of deflection, as well as the bending angle for the each mode's tip, but the motions of particular points on the flexible line of present modes can not be defined. Equation (19) can not be equated to Eq. (22), because they are equations of different types. Equation (19) is the equation of flexible lines (Euler–Bernoulli equation) of the first mode, while Eq. (22) is the equation of motion at the point of the first mode's tip. Equation (21) is Euler–Bernoulli equation of the second mode, while Eq. (23) is the equation of motion at the point of the second mode's tip. Other equations of the model (24) and (25) are also equations of motion at a certain point. The system (26) consists of equations of the same type. Through them the motion of the robot's tip can be analyzed.

$H \in R^{4\times4}$, $h \in R^4$, $C \in R^{4\times4}$, $B \in R^{4\times4}$, and $J \in R^{4\times4}$. Control is denoted by $U = [0\ 0\ 0\ u]^{\mathrm{T}}$. Control via local feedbacks of motor motion with respect to position and velocity was applied

$$u = K_{lp} \cdot (\bar\theta^o - \bar\theta) + K_{lv} \cdot (\dot{\bar\theta}^o - \dot{\bar\theta}). \quad (27)$$

3. Kinematics

First, the solution of Euler–Bernoulli Eq. (1) original form can be analyzed. A general solution of motion, i.e., the form of transversal oscillations of flexible beams can be found in the method of particular integrals of Bernoulli, i.e.,

$$\hat y_{to1,1}(\hat x_{1,1}, t) = \hat X_{1,1}(\hat x_{1,1}) \cdot \hat T_{to1,1}(t). \quad (28)$$

Besides, it is supposed that according to the definition, the motion in Eq. (1) is caused by an external force $F_{1,1}$, added and then removed with the solution (28) of Bernoulli and it satisfies these assumptions.

By superimposing the particular solutions (28), any transversal oscillation can be presented in the following form:

$$\hat y_{to1}(\hat x_{1,j}, t) = \sum_{j=1}^{\infty} \hat X_{1,j}(\hat x_{1,j}) \cdot \hat T_{to1,j}(t). \quad (29)$$

Bernoulli wrote Eq. (29) based on "vision." Euler and Bernoulli did not define the mathematical model of a link with an infinite number of modes, but Bernoulli defined the motion solution (shape of an elastic line) of such a link, which is presented in Eq. (29). Euler and Bernoulli left the task of a link modeling with an infinite number of modes to their successors (see ref. [5, 6]).

The equation of Bernoulli (29) (see Fig. 4) defines a geometrical position of any spot on the elastic body line $\hat y_{to1}$ in direction y_1-axis, and in a direction of x_1-axis it would be a $\hat x_{to1}$ coordinate that is also a geometrical size and it can be presented in an analog way as well as the size

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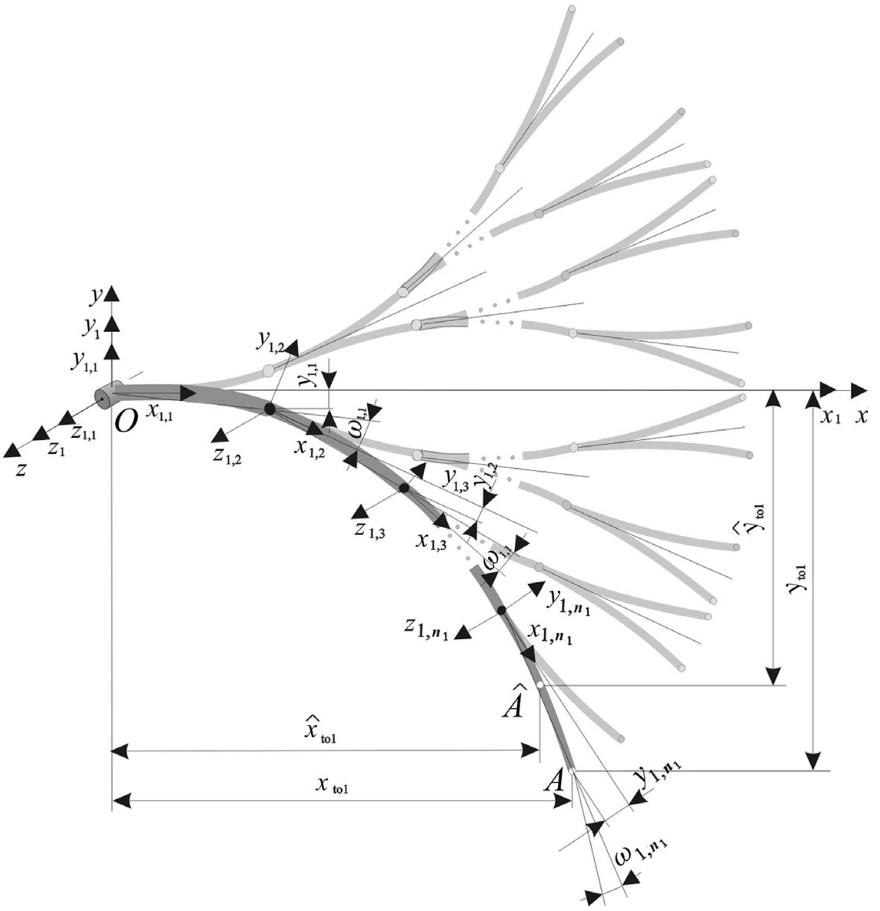

Fig. 4. Possible positions of the tip of elastic line with infinite number of modes.

$\hat{y}_{to1}$. The position of a tip of a presented body with indefinite number of modes is defined by coordinates x_{to1}, y_{to1} in x_1, y_1 level. It is supposed that all movements are made in $x_1 - y_1$ level, and a coordinate is $z_1 = 0$ in this case. Equation (29) is actually the solution of dynamics of the presented body's movement during the time. However, in order to calculate the coordinates $\hat{x}_{to1}$, $\hat{y}_{to1}$ in some specific moment of time (as is seen from Fig. 4), it is necessary (except from angles $\omega_{1,1}$, $\omega_{1,2}$, $\omega_{1,3} \ldots \omega_{1,j}$) to know sizes of elastic deformations of all modes $y_{to1,1}$, $y_{to1,2}$, $y_{to1,3} \ldots y_{to1,j} \ldots$ defined in a space of local coordination system $x_{i,j}$, $y_{i,j}$, $z_{i,j}$.

Generally, coordinates x_{to1}, y_{to1} are the total of elastic deformations, but precisely, in geometrical terms, it is the total of projected elastic deformations on axes x_1 and y_1, respectively.

Equation (29) has a significance as elastic deformation for each mode for Meirovitch (ref. [17]) and his followers (refs. [8–16]). Elastic deformation defined in this way as in (29) is entered in the total dynamic system model. This is essentially different interpretation of the Eq. (29) in comparison with the interpretation of the same equation in this manuscript.

In this paper, as explained above, Eq. (29) has completely new meaning. Equation (29) is a solution of dynamic models, i.e., form of elastic lines in space of Cartesian coordinates.

The motion of the considered robotic system mode is far more complex (see ref. [5, 6]) than motion of the body presented in Fig. 1. This means that the equations that describe the robotic system (its elements) must be also more complex than Eq. (1), formulated by Euler and Bernoulli. This fact is overlooked, and the original equations are widely used in the literature to describe the robotic system motion. This is very inadequate because valuable pieces of information about the complexity of the elastic robotic system's motion are thus lost.

Hence, we should emphasize the necessity of expanding the source equations for the purpose of modeling robotic systems and this should be done as in ref. [5, 6]. By superimposing the particular solutions of oscillatory character and stationary solution of forced character, position, and orientation of any elastic deformation can be presented in the following basic form (solution of Eq. (2))

$$\begin{aligned} \hat{y}_{1,1} &= \hat{a}_{1,1}(\hat{x}_{1,1}, \hat{T}_{st1,1}, \hat{T}_{to1,1}, \bar{\theta}, \alpha, t), \\ \hat{\psi}_{1,1} &= \hat{d}_{1,1}(\hat{x}_{1,1}, \hat{T}_{st1,1}, \hat{T}_{to1,1}, \bar{\theta}, \alpha, t). \end{aligned} \tag{30}$$

The solution of the system Eq. (3) and dynamic motor motion, i.e., the form of its elastic line, can be obtained in the presence of dynamics (angle) of rotation of each motor, as well as by taking into account the robotic configuration.

$$\begin{aligned} \hat{y} &= \hat{a}(\hat{x}_{i,j},\ \hat{T}_{sti,j},\ \hat{T}_{toi,j},\ \bar{\theta},\ \alpha,\ t), \\ \hat{x} &= \hat{b}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t), \\ \hat{z} &= \hat{c}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{sti,j}, \bar{\theta}, \alpha, t), \\ \hat{\psi} &= \hat{d}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t), \\ \hat{\xi} &= \hat{e}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t), \\ \hat{\varphi} &= \hat{f}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t). \end{aligned} \tag{31}$$

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Thus, the position and orientation of each point of the elastic line is defined in the space of Cartesian coordinates. It should be pointed out that the form of elastic line comes out directly from dynamics of system's motion. The motion of the mode's tip, its position and orientation, is defined by the sum of stationary and oscillatory motion (solution of Eq. (4))

$$\begin{aligned} y_{1,1} &= a_{1,1}(x_{1,1},\ T_{st1,1},\ T_{to1,1},\ \bar{\theta},\ \alpha,\ t), \\ \psi_{1,1} &= d_{1,1}(x_{1,1}, T_{st1,1}, T_{to1,1}, \bar{\theta}, \alpha, t). \end{aligned} \tag{32}$$

The robot tip's motion is defined by the sum of the stationary and oscillatory motion of each mode's tip plus the dynamics of motion of the motor powering each link, as well as by included robot configuration (solution of Eqs. (5) and (6))

$$\begin{aligned} y &= a(x_{i,j},\ T_{sti,j},\ T_{toi,j},\ \bar{\theta},\ \alpha,\ t), \\ x &= b(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t), \\ z &= c(x_{i,j}, T_{sti,j}, T_{sti,j}, \bar{\theta}, \alpha, t), \\ \psi &= d(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t), \\ \xi &= e(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t), \\ \varphi &= f(x_{i,j}, T_{sti,j}, T_{toi,j}, \bar{\theta}, \alpha, t). \end{aligned} \tag{33}$$

From Eq. (33), the motion of each mode's tip and link's tip can be calculated and finally of the robot tip's motion.

Example. In order to define the shape and position of elastic line of the first and second mode link from Fig. 2, during the realization of robot's task in the space of Cartesian coordinates, it is necessary to find the solution Eqs. (19), (21), (24), and (25). The general solution of the dynamics movement of the observed model is given

$$\begin{aligned} \hat{y} &= \hat{a}(\hat{x}_{i,j},\ \hat{T}_{sti,j},\ \hat{T}_{toi,j},\ \bar{\theta},\ \alpha,\ t), \\ \hat{x} &= \hat{b}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t), \\ \hat{\psi} &= \hat{d}(\hat{x}_{i,j}, \hat{T}_{sti,j}, \hat{T}_{toi,j}, \bar{\theta}, \alpha, t). \end{aligned} \tag{34}$$

Furthermore, the position and orientation of any point on the link elastic line during the robot's task realization need to be defined. Especially interesting point for robotics' experts is the position and orientation of the observed mechanism's tip. If its position is defined in every selected moment, then the trajectory of the tip's movement of the observed elastic mechanism is defined. First, the limit conditions for the selected point should be defined, and these are conditions (18) and (20). Then, Eqs. (22)–(25) will be valid and according to these conditions, all generalized coordinates could be calculated. According to Eqs. (7)–(9), all elastic system deformations could be defined. According to the analogy with robot systems, these values can be named "internal coordinates." A geometric link between these characteristics (internal coordinates) and the space of Cartesian coordinates (external coordinates) has been defined by using the transformation matrix, or so-called "direct kinematics" in the robotics.

The rotation matrix that describes the change of position (Cartesian coordinates) and orientation (Euler angles) of the tip of every mode of segment has the form

$$T_{ei}^{\,i-1} = \begin{bmatrix} \cos\rho_{i,j} & -\sin\rho_{i,j}\cos\alpha_{i,j} & \sin\rho_{i,j}\sin\alpha_{i,j} & l_{i,j}\cdot\cos\rho_{i,j} \\ \sin\rho_{i,j} & \cos\rho_{i,j}\cos\alpha_{i,j} & -\cos\rho_{i,j}\sin\alpha_{i,j} & l_{i,j}\cdot\sin\rho_{i,j} \\ 0 & \sin\alpha_{i,j} & \cos\alpha_{i,j} & d_{i,j} \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{35}$$

$\rho_{i,j}$, $l_{i,j}$, $\alpha_{i,j}$, and $d_{i,j}$ are the new DH parameters that also encompass the rigidity characteristics, see Fig. 3 and Eq. (7) ($\alpha_{i,j} = 0^o$ and $d_{i,j} = 0$ (m)):

$$T_{e1}^{\,0} = \begin{bmatrix} \cos q & -\sin q & 0 & l_{1,1}\cdot\cos q \\ \sin q & \cos q & 0 & l_{1,1}\cdot\sin q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{36}$$

The matrix rotation (36) describes position change and orientation of the top of the first mode's tip of the link

$$T_{e2}^{\,1} = \begin{bmatrix} \cos\delta & -\sin\delta & 0 & l_{1,2}\cdot\cos\delta \\ \sin\delta & \cos\delta & 0 & l_{1,2}\cdot\sin\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

The matrix rotation (37) describes position and orientation change of the top of the second mode's tip of the link.

For thirty matrix rotations adopted $l_{1,3} = 0$:

$$T_{e3}^{\,2} = \begin{bmatrix} \cos\omega_{1,2} & -\sin\omega_{1,2} & 0 & 0 \\ \sin\omega_{1,2} & \cos\omega_{1,2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{38}$$

The matrix rotation (38) describes orientation change of the top of the second mode's tip of the link in a local coordinate frame $x_{1,3}, y_{1,3}, z_{1,3}$.

The overall transformation matrix describes the change of the position and orientation of segment tip in a coordinate frame x, y, z :

$$T_{e3}^{\,0} = T_{e1}^{\,0}\cdot T_{e2}^{\,1}\cdot T_{e3}^{\,2}. \tag{39}$$

The Jacobi matrix for a manipulator with elastic joints and links maps the velocity vector of external coordinates $\dot{p}_s$ into the velocity vector of internal coordinates$\dot{\phi}$:

$$\dot{\phi} = J^{-1}(\phi)\cdot\dot{p}_s, \tag{40}$$

where $\dot{p}_s = [\dot{x}\quad \dot{y}\quad \dot{z}\quad \dot{\psi}\quad \dot{\wp}\quad \dot{\varphi}]^{\mathrm{T}}$ defines the velocity of a given point of the robotic system in Cartesian coordinates, whereas $\dot{\phi} = [\dot{\rho}_{1,1}\quad \dot{\rho}_{1,2}\quad \dot{\rho}_{1,3}\quad \dot{\rho}_{1,4}\quad \ldots\quad \dot{\rho}_{1,n}]^{\mathrm{T}}$ defines the velocity vector of internal coordinates. In this example,

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see Fig. 3 and Eq. (40) have the form

$$\begin{bmatrix} \dot{q} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} l_{1,2} \cdot \sin(q+\delta) + l_{1,1} \cdot \sin\delta & l_{1,2} \cdot \sin(q+\delta) \\ l_{1,2} \cdot \cos(q+\delta) & l_{1,2} \cdot \cos(q+\delta) + l_{1,1} \cdot \cos q \end{bmatrix}^{-1} \cdot \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}. \quad (41)$$

Elements of the Jacobian are only functions of the elements of the homogenous transformation matrix T_{e3}^{0}.

It is clear that each branched chain in the complex mechanism has its finite transformation matrix, as well as its Jacobi matrix.

The term "kinematics" is commonly used in the terms of rigid robot systems because, in this case, mechanisms geometry defines the position and orientation of the robot tip.

However, when both joint and link elasticity are present, then elastic deformation values, which are by their nature dynamic values, take part in the definition of the position and orientation of every point on the robot system's elastic line.

And for that reason, in order to keep the familiar terminology, in future, that solution of "direct kinematics" will be implied, "inverse kinematics" in elastic robot systems means the presence of elastic deformations.

It is clear now that Eq. (33), generally, and in the mentioned example, Eqs. (39) and (41), serve for the calculation of the robot tip's movement during the robot task realization and that it is based on motor rotating angles, elastic deformation values and all other kinematics and dynamic robot mechanism characteristics (such as its geometry, configuration, weight disposal, motor characteristics, reference trajectory choice as well as many other important characteristics that influence the robot system movement dynamics). In robotics, this procedure is called the solution of "direct kinematics" and "inverse kinematics."

That way we present the analogue

	between	
the "Original form of the Euler–Bernoulli equation solutions" that were defined by Bernoulli by Eq. (28) (i.e., Euler–Bernoulli equation solutions Eq. (31) or the form of equation of motion solutions of robot tip Eq. (33))	and	the procedure of the "direct kinematics" Eq. (39) and "inverse kinematics" solutions Eq. (41) in Robotics

In this way, the complete analogy between Eqs. (28), (39), and (41) is established. The analogue between the "Original form of the Euler–Bernoulli equation" and its solution and modern knowledge from Robotics is presented in this way.

4. Simulation Example for Elastic Mechanism

Tip of robot started from the position "A" and moves directly to the point "B" in the predicted time of $T = 2$ (s), (see Fig. 2). The trapezoidal profile of velocity together with time of acceleration and deceleration from $0.2 \cdot T$ is adopted.

The characteristics of stiffness and damping of gear in the real and reference regimes are not the same nor are stiffness and damping characteristics of the link $C_{\xi 1} = 0.99 \cdot C_{\xi 1}^{o}$, $B_{\xi 1} = 0.99 \cdot B_{\xi 1}^{o}$, $C_{s1,1} = 0.99 \cdot C_{s1,1}^{o}$, $B_{s1,1} = 0.99 \cdot B_{s1,1}^{o}$, $C_{s1,2} = 0.99 \cdot C_{s1,2}^{o}$, $B_{s1,2} = 0.99 \cdot B_{s1,2}^{o}$.

The only disturbance in the system is the ignorance of the rigidity characteristics and damping. The elastic deformation is a quantity that is at least partly encompassed by the reference trajectory.

The first detailed presentation of the procedure for creating reference trajectory is given in ref. [24].

As may be seen from Fig. 5, during its motion in the direction from the point "A" to the point "B" the robot tip tracks properly the reference trajectory in the Cartesian coordinate's space.

The presence of oscillations during robot's task performance is evident (if the change in velocity is taken into consideration with respect to the reference in the x-direction).

The position control law for controlling local feedback has been applied, so the tracking of reference external forces F_c^o, F_f^o depends directly on the deviation of the position from the reference, see Fig. 6. In the same figure, the total action of mechanism dynamics $M_{1,1}$ on "the first" (q) and $M_{1,2}$ on "the second" (δ) DOF is presented. Elastic deformations, joint deflection angle ξ, bending angle of the lower link part (the first mode) $\vartheta_{1,1}$, and bending angle of the upper link part (the second mode) $\vartheta_{1,2}$ are presented in Fig. 7. As the rigidity of the second mode is about 10 times smaller than the rigidity of the first one, it is logical that the bending angle of the second mode is about 10 times bigger.

5. Conclusions

The paper describes the methods of expanding Euler–Bernoulli equation with multiple points of view. Elastic deformation (moment of load) not only to build a perturbation and inertial moments, but there is the influence of gravitational, centrifugal, Coriolis torques (single and coupled), bending moments of other modes (which due to the coupling effect affect the motion of the considered mode), and moments that are caused by the action of external forces. Due to the strong coupling, there is diversity in the structure of the extended form of Euler–Bernoulli equation of each mode. The stiffness matrix is a full matrix as well as the damping matrix not only in the Euler–Bernoulli equation but also in equations of a motor. Damping is an integral part of the characteristics of elasticity of real systems and is naturally included in Euler–Bernoulli equation. All of these features and this whole discussion are not just related to Euler–Bernoulli equation, but also to motion equation for any point (and the top point) of the elastic line. This is the case because the motion equation follows directly from Euler–Bernoulli equation defining boundary conditions.

It is concluded that the definition of kinematic models is of particular importance. The dynamics of mechanism just over

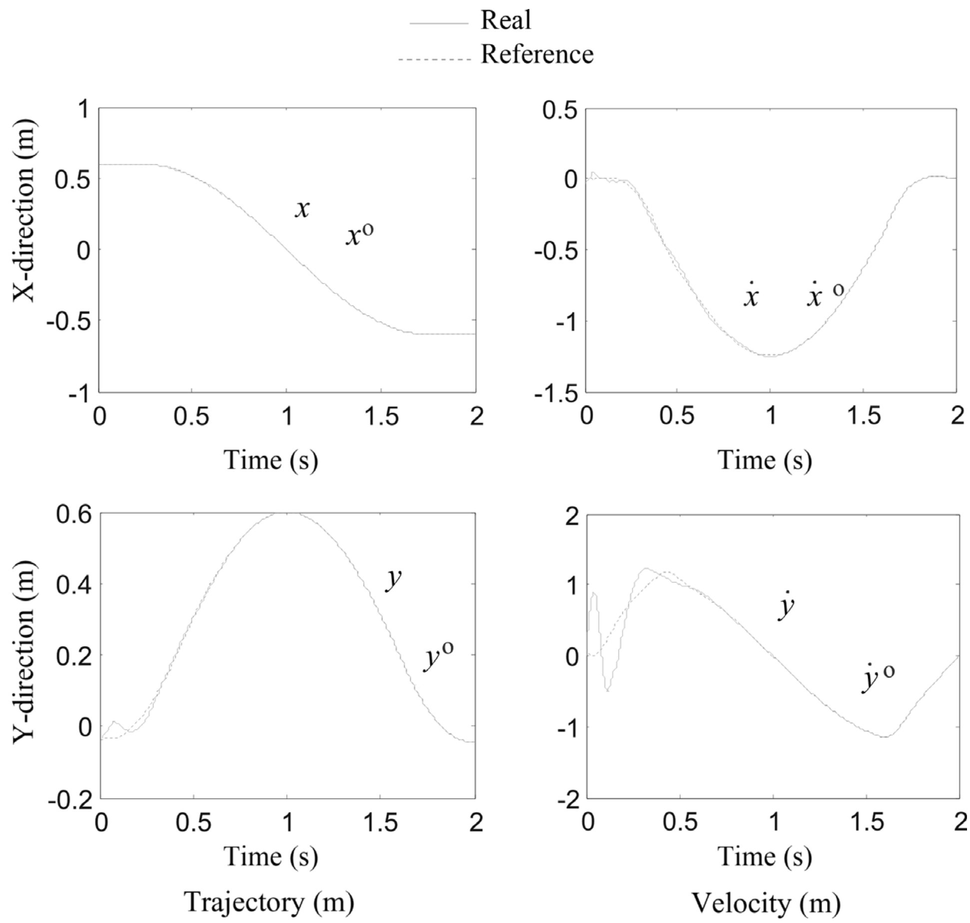

Fig. 5. Deviation of real robot tip position (velocity) from the reference.

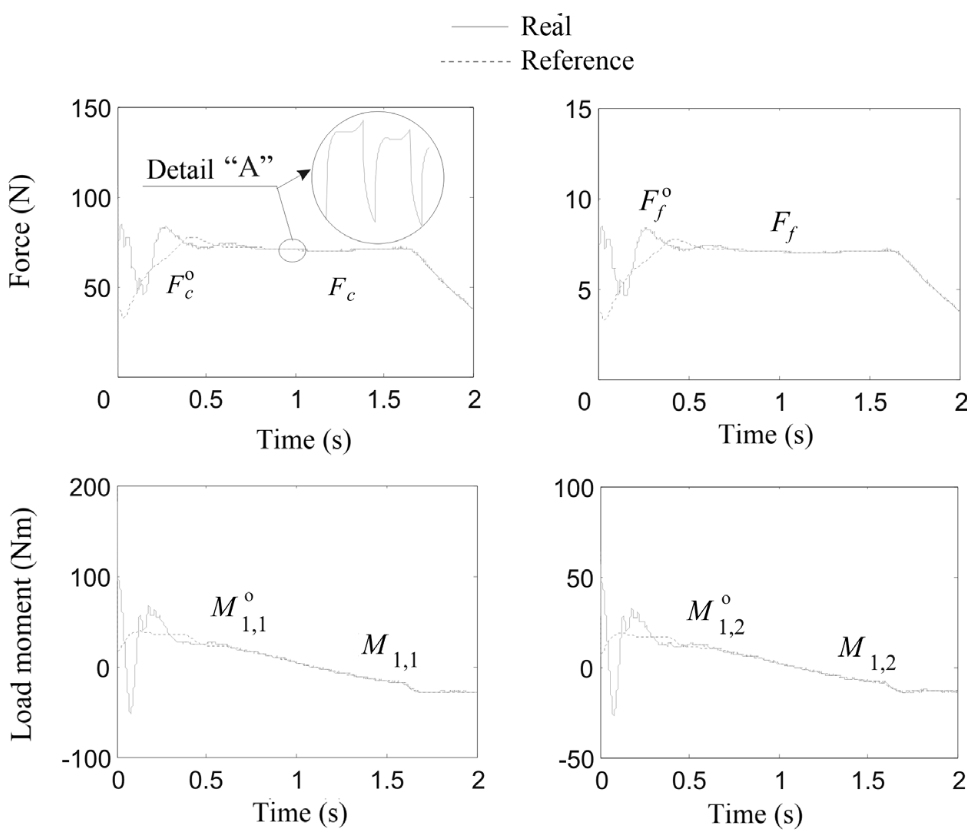

Fig. 6. External force and load moment.

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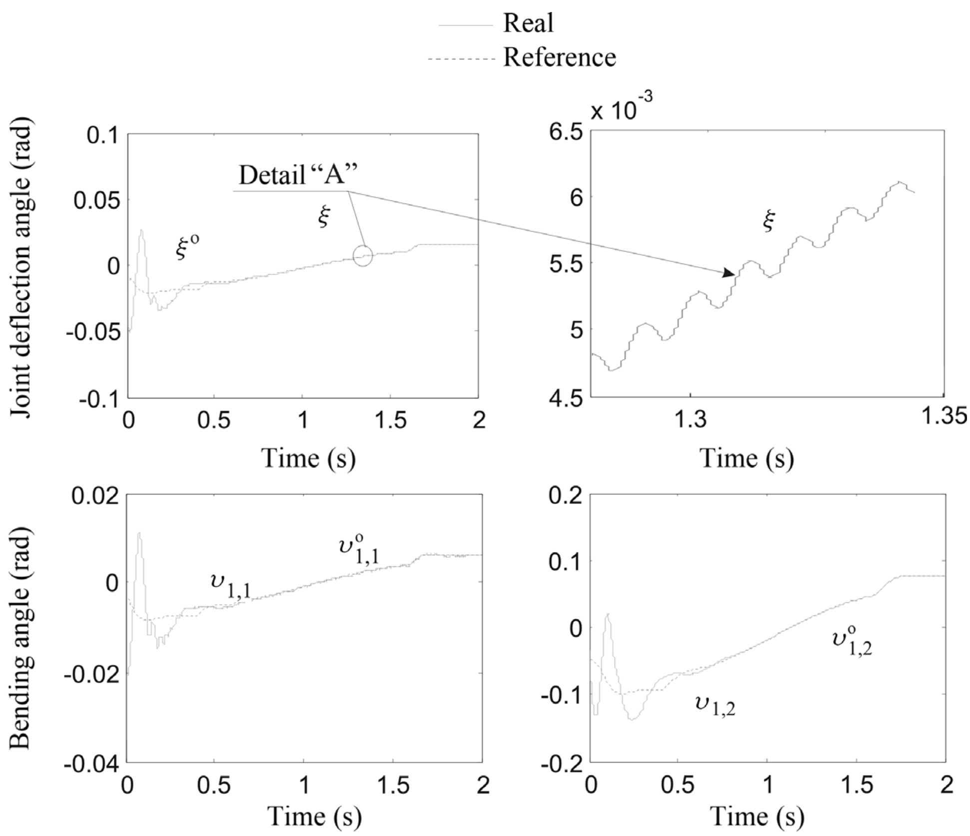

Fig. 7. Elastic deformations.

the sizes of elastic deformation is included into its definition, resulting from the dynamics of the motion of mechanism. This makes possible the process of defining new Denavit–Hartenberg parameters, a new form of matrix transformations and Jacobi matrix and the reference trajectory.

Analysis of simulation results of the selected mechanism confirms the possibility of using the extended form of Euler–Bernoulli equation, which realistically depicts the dynamics of motion of mechanism.

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