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Determination of singularity contours for five-bar planar parallel manipulators

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SUMMARY

From a design point of view, it is crucial to predict singular configurations of a manipulator in terms of inputs in order to improve the dexterity and workspace of a manipulator. In this paper, we present a simple, yet a systematic appoach to obtain singularity contours for a class of five-bar planar parallel manipulators which are based on five rigid links and five single degree of freedom joints – revolute and prismatic joints. The determinants of the manipulator Jacobian matrices are evaluated in terms of joint inputs for a specified set of geometric parameters, and the contours of the determinants at 0.0 plane which are the singularity contours in joint space are generated for the three types of singularities reported in the literature. The proposed approach/algorithm is simple and systematic, and the resulting equations are easy to solve on a computer. The singularity contours for all the class are presented in order to demonstrate the method. It is concluded that the proposed method is useful in trajectory planning and design of five-bar planar parallel manipulators in order to improve their dexterity and workspace.

KEYWORDS: Parallel manipulators; Five-bar; Singularity contours; Revolute and prismatic joints.

1. INTRODUCTION

Robot manipulators are anthropomorphic open loop mechanisms which usually have a longer reach, larger workspace, and more dexterity than closed chain mechanisms. They also have some disadvantages; the cantilever-type of structure for open manipulators naturally has low stiffness and therefore has undesirable dynamic characteristics, especially at high speed and heavy loading conditions. Mainly due to these disadvantages, many robotics tasks based on the high level of accuracy in positioning of the manipulators' end point are not generally realisable by conventional serially connected robot manipulators. One approach to achieve the required level of accuracy is to consider alternative structural designs which are more rigid than serial manipulators. Parallel connection is an alternative type to the serial mechanisms where closed-loop kinematics chains are included into a robot manipulator so that the output can be produced by in parallel-actuated closed-loop kinematics chains. Parallel manipulators have the inherent advantages of better load carrying capability and more precise positioning of the payload. The main disadvantages are difficulties in trajectory planning and control of such systems mainly due to singular configurations which are sometimes referred to as "special" or "critical" configurations.[1,2] The determination of these configurations will help us to understand the range of motion of a manipulator.

This paper presents the problem of determination of singular configurations of a class of five bar planar parallel manipulator. Singular configurations of a mechanism are undesirable due to the fact that the degree of freedom of the mechanism changes instantaneously. While the singular positions of serially connected manipulators result in the loss of one or more degrees of freedom,[3–5] those of parallel manipulators result in either a gain or a loss of one or more degrees of freedom.[6–10] At singular configurations, the determinant of the manipulator Jacobian becomes zero. The singularities of several mechanisms have been obtained. However, singularity analysis of parallel manipulators and closed-loop mechanisms is still an active topic of research. Gosselin and Angeles,[6] introduced the concept of two Jacobian matrices, which relate input velocities to output velocities, for parallel manipulator having equal number of inputs and outputs. The singularity of each matrix corresponds to loss or gain of degree of freedom and singularity of both occurs only when the mechanism is architecturally singular. Later, Daniali et al.,[8] have claimed that the singularity of both matrices is not necessarily architecture-dependent; depending on Jacobian formulation any of the three types of singularities described in reference [6] can occur. Sefrioui and Gosselin,[3] have reported on the quadratic nature of the singularity curves of planar three-degree-of-freedom parallel manipulators, the roots of the determinants of the manipulator's Jacobian are used to obtain a graphical representation of these curves in manipulator's Cartesian workspace. Merlet,[11] has explored on the singular configurations of a six-degree of freedom parallel manipulator using a geometric method rather than finding the roots of the determinants of the manipulator Jacobian matrix. Zlatanov et al.,[12] have presented a generalised approach to determine the singular configurations of any mechanism with arbitrary chains and an equal number of inputs and outputs. They use a velocity equation including the velocities of active and passive joints in order to determine singular configurations. Basu et al.,[9] have proposed two methods (algebraic and geometric) to deter-

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mine singular configurations of platform-type multi-loop spatial mechanisms containing spherical joints on the platform.

The approach proposed in this study consists of (i) obtaining analytical expressions for the position of the output point of a class of five-bar planar parallel manipulators, (ii) generating the two Jacobian matrices relating input velocity vector to the velocity vector of the output point, (iii) evaluating and plotting the determinants of the Jacobians in terms of the input positions for a given set of geometric parameters, and (iv) obtaining the contours of the determinants at 0.0 plane. These contours indicate the singular configurations of the manipulators in terms of joint inputs. This approach is simple, yet systematic and does not necessitate to determine the explicit expressions for the roots of the determinants in order to plot the singularity contours/loci. A computer program prepared in MATLAB is used to obtain the singularity contours. The proposed method is useful in trajectory planning and design of five-bar planar parallel manipulators in order to improve their dexterity and workspace.

2.1 Five-bar manipulators with revolute and prismatic joints

The number of the potential five-bar planar parallel manipulators made up of revolute and prismatic joints is thirty-two, which can be further reduced to six by considering some constraints described in reference [13]. They are RRRRR, RRRRP, PRRRP, RPRPR, RRRPR and RPRRP parallel manipulators, which are depicted in Figure 1. Their two joints are active and the rest are passive joints. If a five-bar mechanism consists of a single closed-loop and it has two active joints and three passive joints, such a mechanism is controllable and can be employed to position a point in a two dimensional space.[14] In reference [13], the workspace of these parallel manipulators are given, but their singular configurations have not been described. It is important to know their singularities before they can be used.

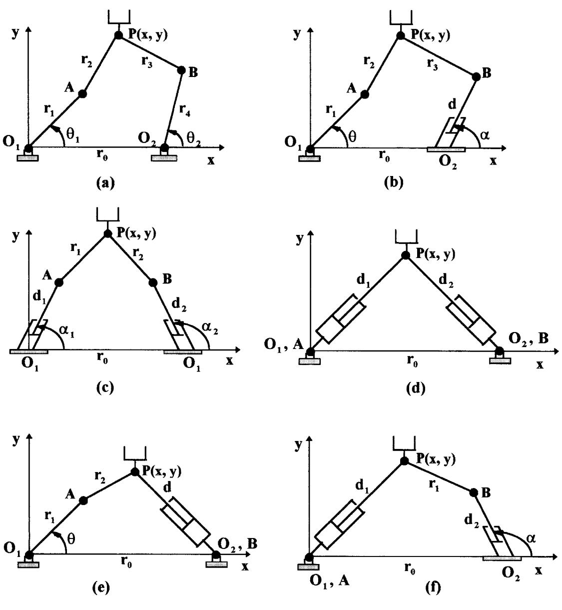

Fig. 1. A class of five-bar planar parallel manipulators.

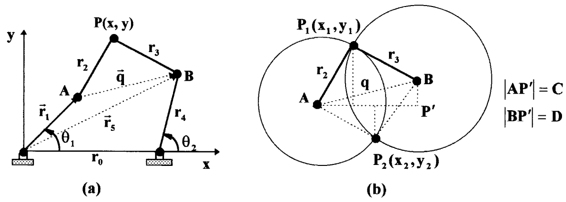

Fig. 2. Geometric representation of forward kinematics solutions.

2.2 Forward Kinematics Equations

Consider the RRRRR parallel manipulator shown in Figure 2(a). For the provided joint inputs θ_1 and θ_2, and the specified link lengths $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$, the analytical expressions for the coordinates of the output point **P** where the end effector is assembled to is obtained. The coordinates of **A** and **B** which are the **x** and **y** components of the $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_5$ vectors can be considered as the coordinates of the centers of two circles of radii $\mathbf{r}_2$ and $\mathbf{r}_3$. Please note that the center of two circles are expressed as functions of the inputs provided by the actuators fixed to the ground. It is well-known that the intersection of the two circles gives a maximum of two solutions which are the possible locations of point **P**. Referring to Figure 2(b), the analytical expressions for these two solutions are obtained using the following algorithm:

$$\begin{aligned}\vec{\mathbf{r}}_1 &= \mathbf{r}_1 \cos\theta_1\,\vec{\mathbf{i}} + \mathbf{r}_1 \sin\theta_1\,\vec{\mathbf{j}},\\ \vec{\mathbf{r}}_5 &= (\mathbf{r}_0 + \mathbf{r}_4 \cos\theta_2)\,\vec{\mathbf{i}} + \mathbf{r}_4 \sin\theta_2\,\vec{\mathbf{j}},\end{aligned} \tag{1}$$

$$\vec{\mathbf{q}} = \vec{\mathbf{r}}_5 - \vec{\mathbf{r}}_1 = \mathbf{C}\,\vec{\mathbf{i}} + \mathbf{D}\,\vec{\mathbf{j}} \tag{2}$$

$$\mathbf{C} = \mathbf{r}_{5x} - \mathbf{r}_{1x},\quad \mathbf{D} = \mathbf{r}_{5y} - \mathbf{r}_{1y},\quad \mathbf{q} = \sqrt{\mathbf{C}^2 + \mathbf{D}^2}, \tag{3}$$

$$\mathbf{Q} = \frac{\mathbf{r}_2^2 + \mathbf{q}^2 - \mathbf{r}_3^2}{2\mathbf{q}^2}, \tag{4}$$

and the coordinates of $\mathbf{P}_1$ and $\mathbf{P}_2$ are:

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{x}_A + \mathbf{QC} - \mathbf{D}\sqrt{\frac{\mathbf{r}_2^2}{\mathbf{q}^2} - \mathbf{Q}^2},\\ \mathbf{y}_1 &= \mathbf{y}_A + \mathbf{QD} + \mathbf{C}\sqrt{\frac{\mathbf{r}_2^2}{\mathbf{q}^2} - \mathbf{Q}^2},\end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{x}_2 &= \mathbf{x}_A + \mathbf{QC} + \mathbf{D}\sqrt{\frac{\mathbf{r}_2^2}{\mathbf{q}^2} - \mathbf{Q}^2},\\ \mathbf{y}_2 &= \mathbf{y}_A + \mathbf{QD} - \mathbf{C}\sqrt{\frac{\mathbf{r}_2^2}{\mathbf{q}^2} - \mathbf{Q}^2},\end{aligned} \tag{6}$$

So, it is now possible to determine the position of the output point for given joint inputs. Depending on the link lengths, $(\mathbf{x}_1, \mathbf{y}_1)$ and $(\mathbf{x}_2, \mathbf{y}_2)$ can have real and imaginary values. If they are imaginary, some kinematics constraints are not satisfied; the mechanism can not be assembled in those configurations. In the next section, link lengths not causing this problem are used in determining singularity contours.

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3. SINGULARITY ANALYSIS BASED ON JACOBIAN MATRIX

Gosselin and Angeles,[6] have proposed a method based on Jacobian matrix in order to evaluate the singularities of closed-loop mechanisms. In their method, it is assumed that the mechanism has an **m**-dimensional input vector Θ, which represents the set of the position of actuated joints, and an **m**-dimensional output vector **X**, which represents the Cartesian coordinates of the output point. The nonlinear kinematics equations, either forward or inverse kinematics equations, describing the input-output position relationship are expressed as;

$$\mathbf{F}(\Theta, \mathbf{X}) = \mathbf{0} \tag{7}$$

where **F** is two dimensional for the problem at hand and is a function of inputs and the outputs. Taking the first time derivative of Eq. 7 leads to the relationship between the input and the output velocity vectors as follows;

$$\frac{\partial \mathbf{F}}{\partial \Theta}\dot{\Theta} + \frac{\partial \mathbf{F}}{\partial \mathbf{X}}\dot{\mathbf{X}} = \mathbf{0} \Rightarrow \mathbf{A}\dot{\Theta} + \mathbf{B}\dot{\mathbf{X}} = \mathbf{0} \tag{8}$$

where **A** and **B** are configuration dependent $\mathbf{2}\times\mathbf{2}$ Jacobian matrices. The singular positions are the singular values of the **A** and **B** matrices. Based on the singularities of **A** and **B** matrices, Gosselin and Angeles,[6] have reported on three kinds of singularities for closed-loop kinematic chains. The first kind of the singularity is faced when $\mathbf{det[A]} = 0$ and $\mathbf{det[B]} \neq \mathbf{0}$. This implies that the output point **P** loses one or more degrees of freedom – regardless of the values of $\dot{\theta}_1$ and $\dot{\theta}_2$, the values of $\dot{\mathbf{x}}$ and $\dot{\mathbf{y}}$ are zero. The second kind happens when $\mathbf{det[A]} \neq 0$ and $\mathbf{det[B]} = 0$. This implies that the output point gains one or more degrees of freedom; the output is movable even when all the actuated joints are locked. The third kind is encountered when $\mathbf{det[A]} = 0$ and $\mathbf{det[B]} = 0$, simultaneously. The third kind of singularity corresponds to configurations where the manipulator can undergo finite motions when its actuators are locked or where a finite input does not produce output motion. So, the singularities occur whenever **A, B** or both become singular.

4. Singularity Contours

First of all, the forward kinematics equations for the manipulator under consideration is obtained from Eq. 5, and next their first derivative is taken and the resulting equations are expressed in the form of Eq. 8. Then, **A** and **B** matrices and their determinants are obtained for a range of joint inputs, and the specified link lengths $\mathbf{r}_i$. Finally, the contours for the determinants at 0.0 plane, which denote the loci of singular configurations, are plotted as function of joint inputs. This same procedure is applied for the six planar parallel manipulators considered in this paper.

4.1 RRRRR manipulator

The forward kinematics equations for the RRRRR manipulator shown in Figure 2(a) in the form of Eq. 7 are:

$$\begin{aligned}
&\mathbf{r_1}\cos\theta_1 + \frac{\mathbf{1}}{\mathbf{2}}[\mathbf{r_0} + \mathbf{r_1}(\cos\theta_2 - \cos\theta_1)] \\
&\qquad - \mathbf{r_1}(\sin\theta_2 - \sin\theta_1)\sqrt{\frac{\mathbf{r_2^2}}{\mathbf{q^2}} - \frac{\mathbf{1}}{\mathbf{4}}} - \mathbf{x} = \mathbf{0} \\
&\mathbf{r_1}\sin\theta_1 + \frac{\mathbf{r_1}}{\mathbf{2}}(\sin\theta_2 - \sin\theta_1) \\
&\qquad + [\mathbf{r_0} + \mathbf{r_1}(\cos\theta_2 - \cos\theta_1)]\sqrt{\frac{\mathbf{r_2^2}}{\mathbf{q^2}} - \frac{\mathbf{1}}{\mathbf{4}}} - \mathbf{y} = \mathbf{0}
\end{aligned} \tag{9}$$

where

$$\mathbf{q^2} = \mathbf{r_0^2} + \mathbf{2r_0r_1}(\cos\theta_2 - \cos\theta_1) + \mathbf{2r_1^2}[\mathbf{1} - \cos(\theta_1 - \theta_2)]$$

Note that for a parallel RRRRR manipulator $\mathbf{r_1} = \mathbf{r_4}$ and $\mathbf{r_2} = \mathbf{r_3}$. Taking the first derivative of Eq. 9 and then expressing the resulting equations in the form of Eq. 8 yields the following expressions:

$$\left.\begin{aligned}
\mathbf{a_{11}}\dot{\theta}_1 + \mathbf{a_{12}}\dot{\theta}_2 + \mathbf{b_{11}}\dot{\mathbf{x}} + \mathbf{b_{12}}\dot{\mathbf{y}} = \mathbf{0} \\
\mathbf{a_{21}}\dot{\theta}_1 + \mathbf{a_{22}}\dot{\theta}_2 + \mathbf{b_{21}}\dot{\mathbf{x}} + \mathbf{b_{22}}\dot{\mathbf{y}} = \mathbf{0}
\end{aligned}\right\} \Rightarrow \mathbf{A}\dot{\Theta} + \mathbf{B}\dot{\mathbf{X}} = \mathbf{0} \tag{10}$$

where

$$\begin{aligned}
\mathbf{a_{11}} &= \mathbf{0.5Kr_1q^4}\sin\theta_1 - (\mathbf{2Er_0} + \mathbf{q^4K^2})\mathbf{r_1}\cos\theta_1 \\
&\quad - \mathbf{2Er_1^2}\sin(\theta_1 - \theta_2) \\
\mathbf{a_{12}} &= \mathbf{0.5Kr_1q^4}\sin\theta_2 + (\mathbf{2Er_0} + \mathbf{q^4K^2})\mathbf{r_1}\cos\theta_2 \\
&\quad + \mathbf{2Er_1^2}\sin(\theta_1 - \theta_2) \\
\mathbf{a_{21}} &= -\mathbf{0.5Kr_1q^4}\cos\theta_1 - \mathbf{q^4K^2r_1}\sin\theta_1 + \mathbf{2Fr_0r_1}\cos\theta_1 \\
&\quad + \mathbf{2Fr_1^2}\sin(\theta_1 - \theta_2) \\
\mathbf{a_{22}} &= \mathbf{K^2r_1q^4}\sin\theta_2 - (\mathbf{2Fr_0} + \mathbf{0.5q^4K})\,\mathbf{r_1}\cos\theta_2 \\
&\quad - \mathbf{2FG_1^2}\sin(\theta_1 - \theta_2)
\end{aligned}$$

and

$$\mathbf{b_{11}} = \mathbf{Kq^2},\ \mathbf{b_{12}} = \mathbf{0},\ \mathbf{b_{21}} = \mathbf{0},\ \mathbf{b_{22}} = \mathbf{Kq^2}$$

and

$$\mathbf{K} = \sqrt{\frac{\mathbf{r_2^2}}{\mathbf{q^2}} - \frac{\mathbf{1}}{\mathbf{4}}},\ \mathbf{E} = \mathbf{0.5r_1r_2^2}\sin(\theta_2 - \theta_1),$$

$$\mathbf{F} = \mathbf{0.5r_2^2}[\mathbf{r_0} + \mathbf{r_1}(\cos\theta_2 - \cos\theta_1)]$$

The expressions for the determinants of the **A** and **B** Jacobian matrices are used to generate singularity contours for a given set of geometric parameters of the manipulator, and for $\mathbf{0} \leq \theta_1, \theta_2 \leq \mathbf{360°}$.

For $\mathbf{r_0} = \mathbf{r_1} = \mathbf{r_2} = \mathbf{10}$, **det**[**B**] can be zero for certain joint inputs, but **K** becomes imaginary. This follows that the kinematics constraints are not satisfied; the manipulator can not be assembled in those configurations. However, it is possible to find link lengths making **K** and $\mathbf{q}^2$ zero, but not making them smaller than zero for all the configurations. Such a study is out of scope of this study, and will be

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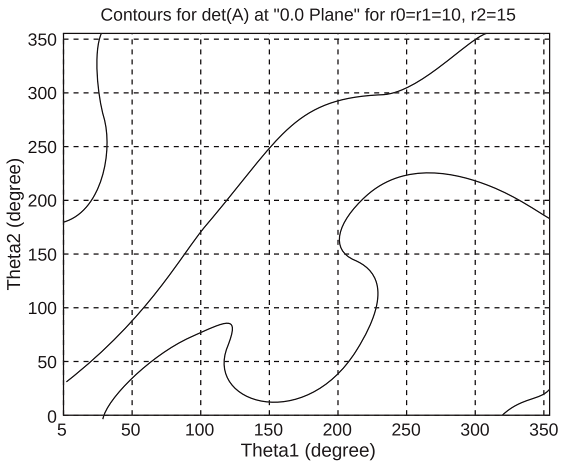

Fig. 3. Singularity contours (type I) for RRRRR manipulator for $\mathbf{0 \leq \theta_1, \theta_2 \leq 360°, r_0 = r_1 = 10,}$ and $\mathbf{r_2 = 15}$.

reported later. On the other hand, for $\mathbf{r_0 = r_1 = 10}$ and $\mathbf{r_2 = 15}$, **det**[**B**] does not become zero, and $\mathbf{K > 0}$, and $\mathbf{q^2 > 0}$. This implies that the output point does not gain a degree of freedom for the provided geometric parameters. But, **det**[**A**] becomes zero for certain joint inputs, and thus the output point lose one degree of freedom. The corresponding singularity contours are shown in Figure 3.

It is also possible to obtain the singularity contours for the third type provided that the kinematics constraints are satisfied and the manipulator can be assembled in all the configurations. The singularity contours for the third type of singularity consists of the intersection of singularity contours for type 1 and type 2. This explanation is also valid for the singularity contours for third type of singularity of the rest of the manipulators to be presented in the next subsections.

4.2 RRRRP manipulator

As depicted in Figure 1(b), the inputs are θ and **d**. The forward kinematics equations for the RRRRP manipulator in the form of Eq. 7 are given below:

$$\mathbf{r_1 \cos\theta + \frac{r_2^2 + q^2 - r_3^2}{2q^2}[r_0 + d\cos\alpha - r_1\cos\theta] - \left(\frac{d\sin\alpha - r_1\sin\theta}{2q^2}\right)\sqrt{4q^2r_2^2 - (r_2^2 + q^2 - r_3^2)^2} - x = 0}$$

$$\mathbf{r_1 \sin\theta + \frac{r_2^2 + q^2 - r_3^2}{2q^2}(d\sin\alpha - r_1\sin\theta) + \left(\frac{r_0 + d\cos\alpha - r_1\cos\theta}{2q^2}\right)\sqrt{4q^2r_2^2 - (r_2^2 + q^2 - r_3^2)^2} - y = 0} \tag{11}$$

where

$$\mathbf{q^2 = r_0^2 + r_1^2 + d^2 + 2r_0(d\cos\alpha - r_1\cos\theta) - 2r_1\, d\cos(\theta - \alpha)}$$

Taking the first derivative of Eq. 11 and then expressing the resulting equations in the form of Eq. 8 yields the following expressions:

$$\left.\begin{aligned}\mathbf{a_{11}\dot{\theta} + a_{12}\dot{d} + b_{11}\dot{x} + b_{12}\dot{y} = 0}\\ \mathbf{a_{21}\dot{\theta} + a_{22}\dot{d} + b_{21}\dot{x} + b_{22}\dot{y} = 0}\end{aligned}\right\} \Rightarrow \mathbf{A\dot{\Theta} + B\dot{X} = 0} \tag{12}$$

where

$$\mathbf{a_{11} = E[2r_0r_1\sin\theta + 2r_1 d\sin(\theta - \alpha)] + r_1K(r_2^2 - q^2 - r_3^2)\sin\theta + r_1K^2\cos\theta}$$
$$\mathbf{a_{12} = E[2d + 2r_0\cos\alpha - 2r_1\cos(\theta - \alpha)] + K(r_2^2 + q^2 - r_3^2)\cos\alpha - K^2\sin\alpha}$$
$$\mathbf{a_{21} = F[2r_0r_1\sin\theta + 2r_1 d\sin(\theta - \alpha)] - r_1K(r_2^2 - q^2 - r_3^2)\cos\theta + r_1K^2\sin\theta}$$
$$\mathbf{a_{22} = F[2d + 2r_0\cos\alpha - 2r_1\cos(\theta - \alpha)] + K(r_2^2 + q^2 - r_3^2)\sin\alpha + K^2\cos\alpha}$$

and

$$\mathbf{b_{11} = -2Kq^2,\ b_{12} = 0,\ b_{21} = 0,\ b_{22} = -2Kq^2}$$

where

$$\mathbf{K = \sqrt{4q^2r_2^2 - (r_2^2 + q^2 - r_3^2)^2}}$$
$$\mathbf{E = 2r_1K\cos\theta + K(r_0 + d\cos\alpha - r_1\cos\theta) - (d\sin\alpha - r_1\sin\theta)(r_2^2 - q^2 + r_3^2) - 2Kx}$$
$$\mathbf{F = 2r_1K\sin\theta + K(d\sin\alpha - r_1\sin\theta) + (r_0 + d\cos\alpha - r_1\cos\theta)(r_2^2 - q^2 + r_3^2) - 2Ky}$$

The determinants of the **A** and **B** matrices are used to generate singularity contours for various values of the geometric parameters of the manipulator, and for $\mathbf{0 \leq \theta \leq 360°}$, $\mathbf{5 \leq d \leq 15}$, $\mathbf{\alpha = 150°}$. For these values, **det[B]** is nonzero, and $\mathbf{K > 0}$, and $\mathbf{q^2 > 0}$, but **det[A]** becomes zero and thus the output point loses one degree of freedom for some joint inputs. The resulting singularity contours are shown in Figure 4.

4.3 PRRRP manipulator

As given in Figure 1(c), the inputs are $\mathbf{d_1}$ and $\mathbf{d_2}$. The forward kinematics equations for the PRRRP manipulator in the form of Eq. 7 are given below:

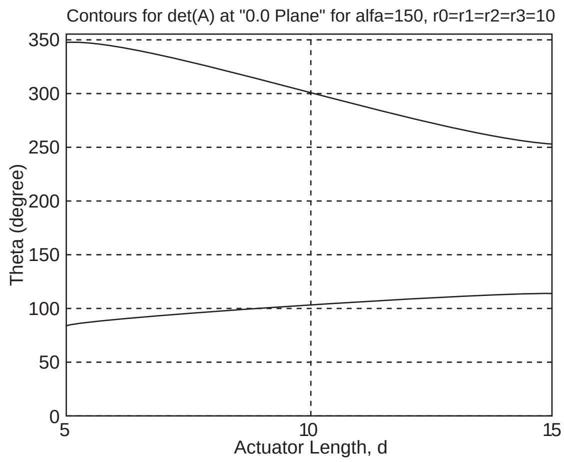

Fig. 4. Singularity contours (type I) for RRRRP manipulator for $\mathbf{0 \leq \theta \leq 360°, 5 \leq d \leq 15, \alpha = 150°, r_0 = r_1 = r_2 = r_3 = 10.}$

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$$\mathbf{d_1 \cos \alpha_1 + \frac{r_1^2+q^2-r_2^2}{2q^2}[r_0+d_2 \cos \alpha_2 - d_1 \cos \alpha_1]}$$

$$\mathbf{-\left(\frac{d_2 \sin \alpha_2 - d_1 \sin \alpha_1}{2q^2}\right)}$$

$$\mathbf{\sqrt{4q^2r_1^2-(r_1^2+q^2-r_2^2)^2}-x=0}$$

$$\mathbf{d_1 \sin \alpha_1 + \frac{r_1^2+q^2-r_2^2}{2q^2}[d_2 \sin \alpha_2 - d_1 \sin \alpha_1]}$$

$$\mathbf{+\left(\frac{r_0+d_2 \cos \alpha_2 - d_1 \cos \alpha_1}{2q^2}\right)}$$

$$\mathbf{\sqrt{4q^2r_1^2-(r_1^2+q^2-r_2^2)^2}-y=0} \tag{13}$$

where

$$\mathbf{q^2=r_0^2+d_1^2+d_2^2+2r_0(d_2 \cos \alpha_2 - d_1 \cos \alpha_1) - 2d_1d_2 \cos(\alpha_1-\alpha_2)}$$

Taking the first derivative of Eq.13 and then expressing the resulting equations in the form of Eq.8 yields the following expressions;

$$\left.\begin{matrix}\mathbf{a_{11}\dot{d}_1+a_{12}\dot{d}_2+b_{11}\dot{x}+b_{12}\dot{y}=0}\\ \mathbf{a_{21}\dot{d}_1+a_{22}\dot{d}_2+b_{21}\dot{x}+b_{22}\dot{y}=0}\end{matrix}\right\}\Rightarrow \mathbf{A\dot{\Theta}+B\dot{X}=0} \tag{14}$$

where

$$\mathbf{a_{11}=2E[d_1-r_0 \cos\alpha_1-d_2 \cos(\alpha_1-\alpha_2)] + K(q^2-r_1^2+r_2^2) \cos \alpha_1 + K^2 \sin \alpha_1}$$

$$\mathbf{a_{12}=2E[d_2+r_0 \cos\alpha_2-d_1 \cos(\alpha_1-\alpha_2)] + K(q^2+r_1^2-r_2^2) \cos \alpha_2 - K^2 \sin \alpha_2}$$

$$\mathbf{a_{21}=2F[d_1-r_0 \cos\alpha_1-d_2 \cos(\alpha_1-\alpha_2)] + K(q^2-r_1^2+r_2^2) \sin \alpha_1 - K^2 \cos \alpha_1}$$

$$\mathbf{a_{22}=2F[d_2+r_0 \cos\alpha_2-d_1 \cos(\alpha_1-\alpha_2)] + K(q^2+r_1^2-r_2^2) \sin \alpha_2 + K^2 \cos \alpha_2}$$

and

$$\mathbf{b_{11}=-2Kq^2,\quad b_{12}=0,\quad b_{21}=0,\quad b_{22}=-2Kq^2}$$

where

$$\mathbf{K=\sqrt{4q^2r_1^2-(r_1^2+q^2-r_2^2)^2}}$$

$$\mathbf{E=2d_1K \cos \alpha_1 + K(r_0+d_2 \cos \alpha_2 - d_1 \cos \alpha_1) - (d_2 \sin \alpha_2 - d_1 \sin \alpha_1)(r_1^2-q^2+r_2^2) - 2Kx}$$

$$\mathbf{F=2d_1K \sin \alpha_1 + K(d_2 \sin \alpha_2 - d_1 \sin \alpha_1) - (r_0+d_2 \cos \alpha_2 - d_1 \cos \alpha_1)(r_1^2-q^2+r_2^2) - 2Ky}$$

The determinants of the **A** and **B** matrices are used to generate singularity contours for various values of the geometric parameters of the manipulator, and for $\mathbf{5 \le d_1, d_2 \le 20}$, and $\mathbf{\alpha_1=70°}$, $\mathbf{\alpha_2=120°}$. For these values, **det[B]** is nonzero, and $\mathbf{K>0}$, and $\mathbf{q^2>0}$, but **det[A]** can become zero for some joint inputs and thus the output point loses one degree of freedom. The corresponding singularity contours are given in Figure 5.

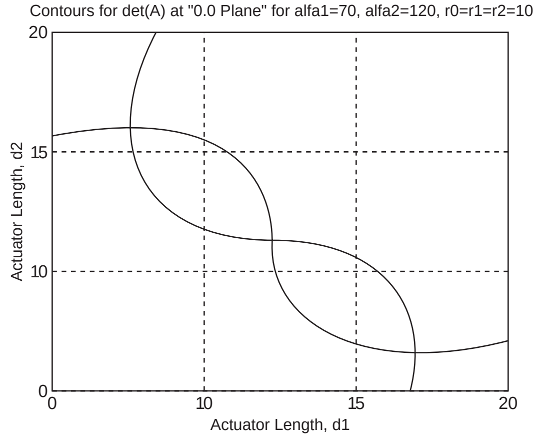

Fig. 5. Singularity contours (type I) for PRRRP manipulator for $\mathbf{5 \le d_1, d_2 \le 20}$ and $\mathbf{\alpha_1=70°}$, $\mathbf{\alpha_2=120°}$, $\mathbf{r_0=r_1=r_2=10}$.

4.4 RPRPR manipulator

As seen in Figure 1(d), the inputs are $\mathbf{d_1}$ and $\mathbf{d_2}$. Note that the centers **A** and **B** of the two circles are fixed to the ground. The forward kinematics equations for the RPRPR manipulator in the form of Eq.7 are given below;

$$\begin{matrix}\mathbf{\dfrac{d_1^2+r_0^2-d_2^2}{2r_0}-x=0}\\ \mathbf{\dfrac{1}{2r_0}\sqrt{4d_1^2r_0^2-(d_1^2+r_0^2-d_2^2)^2}-y=0}\end{matrix} \tag{15}$$

Taking the first derivative of Eq.15 and then expressing the resulting equations in the form of Eq.8 yields the following expressions;

$$\left.\begin{matrix}\mathbf{a_{11}\dot{d}_1+a_{12}\dot{d}_2+b_{11}\dot{x}+b_{12}\dot{y}=0}\\ \mathbf{a_{21}\dot{d}_1+a_{22}\dot{d}_2+b_{21}\dot{x}+b_{22}\dot{y}=0}\end{matrix}\right\}\Rightarrow \mathbf{A\dot{\Theta}+B\dot{X}=0} \tag{16}$$

where

$$\mathbf{a_{11}=d_1,\quad a_{12}=-d_2,\quad a_{21}=d_1(r_0^2-d_1^2+d_2^2),\quad a_{22}=d_2(r_0^2+d_1^2-d_2^2),}$$

$$\mathbf{b_{11}=-r_0,\quad b_{12}=0,\quad b_{21}=0,\quad b_{22}=-2r_0^2y}$$

Note that $\mathbf{det[A]=2r_0^2d_1d_2}$ and $\mathbf{det[B]=2r_0^3y}$. In order to make **det[A]** zero, either $\mathbf{d_1}$ or $\mathbf{d_2}$ must be zero, $\mathbf{r_0}$ is assumed to be nonzero for the sake of practical reasons. But, in reality both actuator inputs cannot be zero. They can only reach their limit positions. So, this type of singularity occurs at either the internal or external boundary of the workspace. On the other hand, in order to make **det[B]** zero, **y** must be zero. This implies that the output point **P** is on the **x** axis. This singularity is avoided if the value of $\mathbf{r_0}$ is chosen such that $\mathbf{(d_1)_{min}+(d_2)_{min}>r_0}$. This fact is in agreement with the results presented before,[7]. As given in Figure 6, when the actuator/joint inputs reach their limits, **det[B]** becomes zero. In such a configuration, the mechanism gains a degree of freedom; the output point **P** can have a nonzero velocity in

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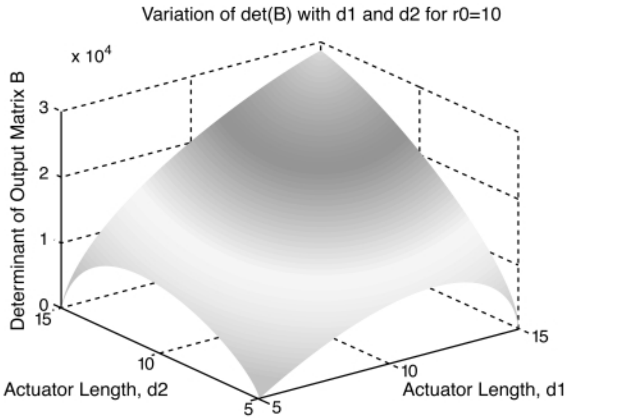

Fig. 6. Variation of det[B] with actuators lengths $\mathbf{d_1}$ **and** $\mathbf{d_2}$ for RPRPR manipulator for $\mathbf{5 \le d_1, d_2 \le 15}$ and $\mathbf{r_0 = 10}$.

the direction perpendicular to the x-axis even if the velocities ($\dot{\mathbf{d}}_1$ and $\dot{\mathbf{d}}_2$) of the actuators are zero.

4.5 RRRPR manipulator

The inputs are θ and $\mathbf{d}$, as given in Figure 1(e). The forward kinematics equations for the RRRPR manipulator in the form of Eq.7 are given below;

$$\begin{aligned} &\mathbf{r_1 \cos\theta + \frac{r_2^2 + q^2 - d^2}{2q^2}(r_0 - r_1 \cos\theta)} \\ &\quad \mathbf{- \left(\frac{r_1 \sin\theta}{2q^2}\right)\sqrt{4q^2r_2^2 - (r_2^2 + q^2 - d^2)^2} - x = 0} \\ &\mathbf{r_1 \sin\theta - \left(\frac{r_2^2 + q^2 - d^2}{2q^2}\right) r_1 \sin\theta} \\ &\quad \mathbf{+ \left(\frac{r_0 - r_1 \cos\theta}{2q^2}\right)\sqrt{4q^2r_2^2 - (r_2^2 + q^2 - d^2)^2} - y = 0} \end{aligned} \tag{17}$$

where

$$\mathbf{q^2 = r_0^2 + r_1^2 - 2r_0r_1 \cos\theta}$$

Taking the first derivative of Eq.17 and then expressing the resulting equations in the form of Eq.8 yields the following expressions;

$$\left.\begin{aligned} \mathbf{a_{11}\dot{\theta} + a_{12}\dot{d} + b_{11}\dot{x} + b_{12}\dot{y}} &= 0 \\ \mathbf{a_{21}\dot{\theta} + a_{22}\dot{d} + b_{21}\dot{x} + b_{22}\dot{y}} &= 0 \end{aligned}\right\} \Rightarrow \mathbf{A\dot{\Theta} + B\dot{X} = 0} \tag{18}$$

where

$$\begin{aligned} &\mathbf{a_{11} = 2Er_0r_1 \sin\theta + (r_2^2 - q^2 - d^2)r_1K \sin\theta + r_1K^2 \cos\theta} \\ &\mathbf{a_{12} = 2r_1d(q^2 - d^2 + r_2^2) \sin\theta - 2dK(r_0 - r_1 \cos\theta)} \\ &\mathbf{a_{21} = 2Fr_0r_1 \sin\theta + (q^2 - r_2^2 + d^2)r_1K \cos\theta + r_1K^2 \sin\theta} \\ &\mathbf{a_{22} = 2Kdr_1 \sin\theta + 2d(r_0 - r_1 \cos\theta)(r_2^2 + q^2 - d^2)} \end{aligned}$$

$$\mathbf{b_{11} = -2Kq^2, \quad b_{12} = 0, \quad b_{21} = 0, \quad b_{22} = -2Kq^2}$$

where

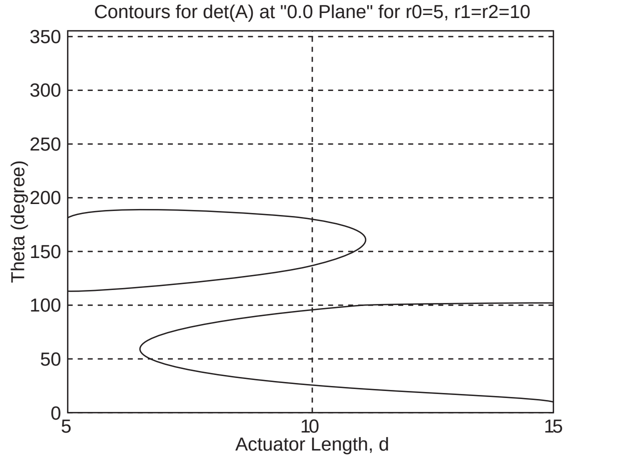

Fig. 7. Singularity contours (type I) for RRRPR manipulator for $\mathbf{0 \le \theta \le 360°}$, $\mathbf{5 \le d \le 15}$, $\mathbf{r_0 = 5}$ and $\mathbf{r_1 = r_2 = 10}$.

$$\mathbf{K = \sqrt{4q^2r_2^2 - (r_2^2 + q^2 - d^2)^2}}$$

$$\begin{aligned} &\mathbf{E = 2r_1K \cos\theta + K(r_0 - r_1 \cos\theta)} \\ &\quad \mathbf{+ r_1 \sin\theta(r_2^2 - q^2 + d^2) - 2Kx} \\ &\mathbf{F = r_1K \sin\theta + (r_0 - r_1 \cos\theta)(r_2^2 - q^2 + d^2) - 2Ky} \end{aligned}$$

For a set of geometric parameters of the manipulator, and for $\mathbf{0 \le \theta \le 360°}$, $\mathbf{5 \le d \le 15}$, **det[B]** is nonzero, $\mathbf{K > 0}$, and $\mathbf{q^2 > 0}$, but **det[A]** becomes zero for some joint inputs. This implies that the output point loses one degree of freedom in those configurations. The resulting singularity contours are given in Figure 7.

4.6 RPRRP manipulator

As shown in Figure 1(f), the inputs are $\mathbf{d_1}$ and $\mathbf{d_2}$. The forward kinematics equations for the RPRRP manipulator in the form of Eq.7 are given below;

$$\begin{aligned} &\mathbf{\frac{d_1^2 + q^2 - r_1^2}{2q^2}(r_0 + d_2 \cos\alpha)} \\ &\quad \mathbf{- \left(\frac{d_2 \sin\alpha}{2q^2}\right)\sqrt{4q^2d_1^2 - (d_1^2 + q^2 - r_1^2)^2} - x = 0} \\ &\mathbf{\frac{d_1^2 + q^2 - r_1^2}{2q^2} d_2 \sin\alpha} \\ &\quad \mathbf{+ \left(\frac{r_0 + d_2 \cos\alpha}{2q^2}\right)\sqrt{4q^2d_1^2 - (d_1^2 + q^2 - r_1^2)^2} - y = 0} \end{aligned} \tag{19}$$

where

$$\mathbf{q^2 = r_0^2 + d_2^2 + 2r_0d_2 \cos\alpha}$$

Taking the first derivative of Eq.19 and then expressing the resulting equations in the form of Eq.8 yields the following expressions;

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$$\left.\begin{array}{l}\mathbf{a_{11}\dot{d}_1 + a_{12}\dot{d}_2 + b_{11}\dot{x} + b_{12}\dot{y} = 0} \\ \mathbf{a_{21}\dot{d}_1 + a_{22}\dot{d}_2 + b_{21}\dot{x} + b_{22}\dot{y} = 0}\end{array}\right\} \Rightarrow \mathbf{A\dot{\Theta} + B\dot{X} = 0} \qquad (20)$$

where

$$\mathbf{a_{11} = 2Kd_1(r_0 + d_2 \cos \alpha) - 2d_1d_2(r_1^2 + q^2 - d_1^2) \sin \alpha}$$
$$\mathbf{a_{12} = 2E(d_2 + r_0 \cos \alpha) + K \cos \alpha(q^2 + d_1^2 - r_1^2) - K^2 \sin \alpha}$$
$$\mathbf{a_{21} = 2Kd_1d_2 \sin \alpha - 2d_1(r_0 + d_2 \cos \alpha)(r_1^2 + q^2 - d_1^2)}$$
$$\mathbf{a_{22} = 2F(d_2 + r_0 \cos \alpha) + K \sin \alpha(q^2 + d_1^2 - r_1^2) + K^2 \cos \alpha}$$
$$\mathbf{b_{11} = -2Kq^2, \quad b_{12} = 0, \quad b_{21} = 0, \quad b_{22} = -2Kq^2}$$

where

$$\mathbf{K} = \sqrt{\mathbf{4q^2d_1^2 - (d_1^2 + q^2 - r_1^2)^2}}$$
$$\mathbf{E = K(r_0 + d_2 \cos \alpha) - d_2 \sin \alpha(r_1^2 - q^2 + d_1^2) - 2Kx}$$
$$\mathbf{F = Kd_2 \sin \alpha + (r_0 + d_2 \cos \alpha)(r_1^2 - q^2 + d_1^2) - 2Ky}$$

For various values of the geometric parameters of the manipulator, and for $\mathbf{5 \leq d_1, d_2 \leq 15}$, $\mathbf{det[B]}$ is nonzero, and $\mathbf{K > 0}$, and $\mathbf{q^2 > 0}$, but $\mathbf{det[A]}$ becomes zero for some joint inputs and thus the output point loses one degree of freedom in those configurations. The corresponding singularity contours are given in Figure 8.

5. Conclusions

Based on the determinants of the Jacobian matrices, a method is proposed to generate the singularity contours of a class of five-bar planar parallel manipulators in terms of input positions. The singularity contours for all the class are presented in order to demonstrate the method. With such a tool, the singularity configurations of a parallel manipulator under design can be predicted by changing the geometrical parameters of the manipulator, and thus a better optimization of the dexterity and workspace of the manipulator is obtainable. This approach contributes to previously published work from the point of view of being simple and systematic, and requiring small computation time.

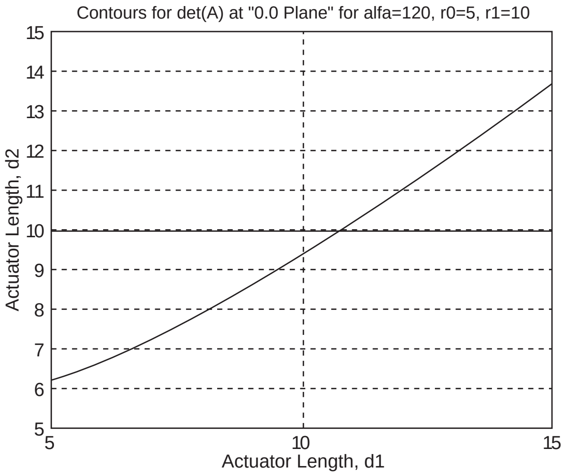

Fig. 8. Singularity contours (type I) for RPRRP manipulator for $\mathbf{5 \leq d_1, d_2 \leq 15}$, $\boldsymbol{\alpha = 120°}$, $\mathbf{r_0 = 5}$ and $\mathbf{r_1 = 10}$.

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