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EULER–KRONECKER CONSTANTS FOR CYCLOTOMIC FIELDS

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Abstract

The Euler–Mascheroni constant $\gamma = 0.5772\ldots$ is the $K = \mathbb{Q}$ example of an Euler–Kronecker constant γ_K of a number field K. In this note, we consider the size of the $\gamma_q = \gamma_{K_q}$ for cyclotomic fields $K_q := \mathbb{Q}(\zeta_q)$. Assuming the Elliott–Halberstam Conjecture (EH), we prove uniformly in Q that

$$\frac{1}{Q}\sum_{Q<q\leq 2Q} |\gamma_q - \log q| = o(\log Q).$$

In other words, under EH, the $\gamma_q/\log q$ in these ranges converge to the one point distribution at 1. This theorem refines and extends a previous result of Ford, Luca and Moree for prime q. The proof of this result is a straightforward modification of earlier work of Fouvry under the assumption of EH.

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1. Introduction

For a number field K, the *Euler–Kronecker constant* γ_K is given by

$$\gamma_K := \lim_{s\to 1^+}\Big(\frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s-1}\Big),$$

where $\zeta_K(s)$ is the Dedekind zeta-function for K. The Euler–Mascheroni constant $\gamma = 0.5772\ldots$ is the $K = \mathbb{Q}$ case, where $\zeta_{\mathbb{Q}}(s) = \zeta(s)$ is the Riemann zeta-function. We consider the constants $\gamma_q = \gamma_{K_q}$ for cyclotomic fields $K_q := \mathbb{Q}(\zeta_q)$, where $q \in \mathbb{Z}^+$ and ζ_q is a primitive qth root of unity.

The recent interest in the distribution of the γ_q is inspired by work of Ihara [4, 5]. He proposed, for every $\varepsilon > 0$, that there is a $Q(\varepsilon)$ for which

$$(c_1 - \varepsilon)\log q \leq \gamma_q \leq (c_2 + \varepsilon)\log q$$

for every integer $q \geq Q(\epsilon)$, where $0 < c_1 \leq c_2 < 2$ are absolute constants. This conjecture was disproved by Ford *et al.* in [2] assuming a strong form of the

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Hardy–Littlewood k-tuple conjecture. However, assuming the Elliott–Halberstam conjecture (see [1]), these same authors also proved that the conjecture holds for almost all primes q, with $c_1 = c_2 = 1$. We recall the Elliott–Halberstam Conjecture as formulated in terms of the Von Mangoldt function $\Lambda(n)$, the Chebyshev function $\psi(x)$ and Euler's totient function $\varphi(n)$.

ELLIOTT–HALBERSTAM CONJECTURE (EH). If we let

$$E(x;m,a) := \sum_{\substack{p\equiv a \pmod m \\ p\le x \text{ prime}}} \Lambda(p) - \frac{\psi(x)}{\varphi(m)},$$

then for every $\varepsilon > 0$ and $A > 0$, we have

$$\sum_{m\le x^{1-\varepsilon}} \max_{(a,m)=1} |E(x;m,a)| \ll_{A,\varepsilon} \frac{x}{(\log x)^A}.$$

Assuming EH, Ford *et al.* proved (see [2, Theorem 6(i)]), for every $\varepsilon > 0$, that

$$1-\varepsilon < \frac{\gamma_q}{\log q} < 1+\varepsilon$$

for almost all primes q (that is, the number of exceptional $q \le x$ is $o(\pi(x))$ as $x \to \infty$). Here we extend and refine this result to all integers q.

THEOREM 1.1. *Under EH, for $Q \to +\infty$, we have*

$$\frac{1}{Q}\sum_{Q<q\le 2Q} |\gamma_q - \log q| = o(\log Q),$$

where the sum is over integers q.

REMARK 1.2. Theorem 1.1 shows that EH implies that the distribution of $\gamma_q/\log q$ in $[Q,2Q]$ converges to the one point distribution supported on 1.

To prove Theorem 1.1, we use the work of Fouvry [3] that allowed him to unconditionally prove that

$$\frac{1}{Q}\sum_{Q<q\le 2Q} \gamma_q = \log Q + O(\log\log Q).$$

Our conditional result is a point-wise refinement of Fouvry's asymptotic formula under EH.

2. Proof of Theorem 1.1

For brevity, we shall assume that the reader is familiar with Fouvry's paper [3]. The key formula is (see (3) of [3]) the following expression for γ_q in terms of logarithmic derivatives of Dirichlet L-functions:

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$$\gamma_q = \gamma + \sum_{1<q^*|q} \sum_{\chi^* \bmod q^*} \frac{L'(1,\chi^*)}{L(1,\chi^*)}. \tag{2.1}$$

Here the inner sum runs over the primitive Dirichlet characters χ^* modulo q^*.

We follow the strategy and notation in [3], which makes use of the modified Chebyshev function

$$\psi(x;q,a) := \sum_{\substack{n\le x \\ n\equiv a \ (\bmod\ q)}} \Lambda(n),$$

and the integral

$$\Phi_{\chi^*}(x) := \frac{1}{x-1}\int_1^x \Big(\sum_{n\le t} \frac{\Lambda(n)}{n}\chi^*(n)\Big)\,dt.$$

However, we replace the sums $\Gamma_i(Q)$ and $\Gamma_{1,j}(Q)$ defined in [3] with the pointwise terms $\gamma_i(q)$ and $\gamma_{1,j}(q)$. Following the approach in [3], which is based on (2.1), we have

$$\gamma_q = \gamma + A(q) + B(q) - \gamma_2(q) - \gamma_3(q) - (\gamma_{1,1}(q) + \gamma_{1,2}(q) + \gamma_{1,3}(q)),$$

where

$$A(q) = \sum_{q^*|q} \sum_{\chi^* \bmod q^*} \frac{L'}{L}(1,\chi^*) + \Phi_{\chi^*}(x),$$

$$B(q) = \sum_{\substack{\chi \bmod q \\ \chi\neq\chi_0}} \Phi_\chi(x) - \sum_{q^*|q} \sum_{\chi^* \bmod q^*} \Phi_{\chi^*}(x),$$

$$\gamma_2(q) = \frac{1}{x-1}\int_1^x \frac{\varphi(q)\psi(t;q,1) - \psi(t)}{t}\,dt,$$

$$\gamma_3(q) = \frac{1}{x-1}\int_1^x \sum_{\substack{n\le t \\ (n,q)\neq 1}} \frac{\Lambda(n)}{n}\,dt,$$

$$\gamma_{1,1}(q) = \frac{1}{x-1}\int_1^x \int_1^{\min(q,t)} \Big(\frac{\varphi(q)\psi(u;q,1) - \psi(u)}{u^2}\,du\Big)\,dt,$$

$$\gamma_{1,2}(q) = \frac{1}{x-1}\int_1^x \int_{\min(q,t)}^{\min(x_1,t)} \Big(\frac{\varphi(q)\psi(u;q,1) - \psi(u)}{u^2}\,du\Big)\,dt,$$

$$\gamma_{1,3}(q) = \frac{1}{x-1}\int_1^x \int_{\min(x_1,t)}^{t} \Big(\frac{\varphi(q)\psi(u;q,1) - \psi(u)}{u^2}\,du\Big)\,dt.$$

To complete the proof, for $\varepsilon > 0$, we let $x := q^{100}$ and $x_1 := q^{1+\varepsilon}$. Apart from $\gamma_{1,1}(q)$, which gives the $-\log q$ terms in Theorem 1.1, we shall show that these summands are all small.

Estimation of $A(q)$: By Proposition 1 and Remark (i) of [3],

$$\sum_{q=Q}^{2Q} |A(q)| = O(Q).$$

Estimation of $B(q)$: For $B(q)$, by (26) and Lemma 3 of [3], we simplify

$$\begin{aligned}
B(q) &= -\frac{1}{x-1}\int_1^x \sum_{q^*|q} \sum_{\chi^* \bmod q^*} \sum_{\substack{n\le t\\(n,q)>1}} \frac{\Lambda(n)\chi^*(n)}{n}\,dt\\
&= -\frac{1}{x-1}\int_1^x \sum_{q^*|q} \sum_{\chi^* \bmod q^*} \sum_{\substack{p^\nu\le t\\p|q}} \frac{\log p\cdot\chi^*(p^\nu)}{p^\nu}\,dt\\
&= -\frac{1}{x-1}\int_1^x \sum_{q^*|q} \sum_{\substack{p^\nu\le t\\p|q\\p\nmid q^*}} \sum_{d|(p^\nu-1,q^*)} \frac{\log p}{p^\nu}\cdot\varphi(d)\mu\Big(\frac{q^*}{d}\Big)\,dt\\
&= -\frac{1}{x-1}\int_1^x \sum_{\substack{p^\nu\le t\\p|q}} \sum_{d|p^\nu-1} \frac{\log p}{p^\nu}\cdot\varphi(d)\sum_{\substack{q^*|q\\d|q^*\\p\nmid q^*}} \mu\Big(\frac{q^*}{d}\Big)\,dt.
\end{aligned}$$

We note that the innermost sum

$$\sum_{\substack{q^*|q\\d|q^*\\p\nmid q^*}} \mu\Big(\frac{q^*}{d}\Big)$$

is always 0 or 1, so we conclude that $B(q)\le 0$ for any q. Proposition 2 of [3] gives

$$\sum_{q=Q}^{2Q} B(q) = O(Q),$$

and so we have

$$\sum_{q=Q}^{2Q} |B(q)| = O(Q).$$

Estimation of $\gamma_2(q)$: By Lemma 8 of [3], uniformly in Q with $u\ge 1$, we have

$$\sum_{q=Q}^{2Q} \psi(u;q,1) \ll u.$$

Therefore,

$$\sum_{q=Q}^{2Q} |\varphi(q)\psi(t;q,1)-\psi(t)| = O(Qt),$$

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and so we conclude that

$$\sum_{q=Q}^{2Q} |\gamma_2(q)| = O(Q).$$

Estimation of $\gamma_3(q)$: By definition, γ_3 is positive, so by (36) of [3],

$$\sum_{q=Q}^{2Q} |\gamma_3(q)| = O(Q).$$

Estimation of $\gamma_{1,1}(q)$: Since $\psi(u; q, 1) = 0$ for $u < q$, we have

$$\gamma_{1,1}(q) = -\frac{1}{x-1} \int_1^x \Big(\int_1^{\min(q,t)} \frac{\psi(u)}{u^2}\, du \Big)\, dt.$$

Dividing both sides of (41) of [3] by Q,

$$\gamma_{1,1}(q) = -\log q + O(1).$$

Estimation of $\gamma_{1,2}(q)$: By the same proof as (42) of [3], we have

$$\sum_{q=Q}^{2Q} |\gamma_{1,2}(q)| \ll \varepsilon Q \log Q.$$

Summing the above estimates, we conclude unconditionally that

$$\frac{1}{Q} \sum_{q=Q}^{2Q} |\gamma_q - \log q| = \frac{1}{Q} \sum_{q=Q}^{2Q} |\gamma_{1,3}(q)| + O(\varepsilon \log Q).$$

Estimation of $\gamma_{1,3}(q)$: If we assume Conjecture EH holds, then we have (as in Lemma 7 of [3]) that

$$\sum_{\substack{q \le 2Q \\ (q,a)=1}} \varphi(q) \Big| \psi(x; q, a) - \frac{\psi(x)}{\varphi(q)} \Big| = O_A(Qx(\log x)^{-A+2}).$$

Therefore,

$$\frac{1}{Q} \sum_{q=Q}^{2Q} |\gamma_{1,3}(q)| = O_{\epsilon,A}(\log^{-A} Q).$$

By combining these estimates, we obtain the main result

$$\frac{1}{Q} \sum_{q=Q}^{2Q} |\gamma_q - \log q| = o(\log Q).$$

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