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DISTRIBUTIONS OF JUMPS IN A CONTINUOUS-STATE BRANCHING PROCESS WITH IMMIGRATION

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Abstract

We study the distributional properties of jumps in a continuous-state branching process with immigration. In particular, a representation is given for the distribution of the first jump time of the process with jump size in a given Borel set. From this result we derive a characterization for the distribution of the local maximal jump of the process. The equivalence of this distribution and the total Lévy measure is then studied. For the continuous-state branching process without immigration, we also study similar problems for its global maximal jump.

Keywords: Branching process; continuous state; immigration; maximal jump; jump time; jump size

2010 Mathematics Subject Classification: Primary 60J80; 60H20

1. Introduction

A continuous-state branching process (CB-process) is a nonnegative Markov process describing the random evolution of a population in an isolated environment. The *branching property* means that if $X = (X_t : t \geq 0)$ and $Y = (Y_t : t \geq 0)$ are two independent CB-processes with the same transition semigroup, then $X + Y = (X_t + Y_t : t \geq 0)$ is also a CB-process with that transition semigroup. A continuous-state branching process with immigration (CBI-process) is a generalization of the CB-process which considers the possibility of input of immigrants during the evolution of the population. The transition semigroup of the CBI-process is uniquely determined by its branching mechanism Φ and immigration mechanism Ψ, both are functions on the nonnegative half-line. We refer the reader to Kawazu and Watanabe (1971) and Lamperti (1967a), (1967b) for early work on CB- and CBI-processes as biological models. See also Duquesne and Le Gall (2002), Kyprianou (2014), and Li (2011) for up-to-date treatments of those processes. We also mention that the CBI-process has been used widely in mathematical finance as models of interest rate, asset price, and so on. A special form of the process is known in the financial world as the Cox–Ingersoll–Ross model; see, e.g. Brigo and Mercurio (2006) and Lamberton and Lapeyre (1996).

The CBI-process is a Feller process, so it has a càdlàg realization $X = (X_t : t \geq 0)$. Let $\Delta X_s := X_s - X_{s-}$ (≥ 0) denote the size of the jump of X at time $s > 0$. In this work, we are interested in distributional properties of jumps of the CBI-process. In particular, we shall give a representation of the distribution of the first occurrence time τ_A of its jump with jump size in some given Borel set $A \subset (0, \infty)$. From this result we derive a characterization for the distribution of the local maximal jump $\max_{0<s\leq t} \Delta X_s$ for any $t > 0$. Under suitable

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assumptions, we prove that this distribution and the total Lévy measure of the process are equivalent. For the CB-process, we also study similar problems for the global maximal jump $\sup_{0<s<\infty} \Delta X_s$. The tool of stochastic equations of the CBI-process established in Dawson and Li (2006) and Fu and Li (2010) plays a key role in the proof of our main result. The results obtained in this work are of clear interest in applications of the CB- and CBI-processes as biological and financial models.

The paper is organized as follows. In Section 2, some basic facts on CB- and CBI-processes are reviewed. In Section 3 we give the characterization of the distribution of the jump time τ_A for $A \subset (0, \infty)$. In Section 4 we establish a number of distributional properties of the local and global maximal jumps of the process.

2. CB- and CBI-processes

In this section we review several basic facts on CB- and CBI-processes for the convenience of the reader. Let us fix a *branching mechanism* Φ, which is a function on $\mathbb{R}_+ := [0, \infty)$ with the representation

$$\Phi(z) = \alpha z + \beta z^2 + \int_{(0,\infty)} \pi_0(\mathrm{d}\theta)(\mathrm{e}^{-z\theta} - 1 + z\theta), \qquad z \geq 0, \tag{2.1}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ are two constants, and π_0 is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \pi_0(\mathrm{d}\theta)(\theta \wedge \theta^2) < \infty. \tag{2.2}$$

A *CB-process* with branching mechanism Φ is a nonnegative Markov process with transition semigroup $(P_t)_{t\geq 0}$ defined by

$$\int_{\mathbb{R}_+} \mathrm{e}^{-\lambda y} P_t(x, \mathrm{d}y) = \mathrm{e}^{-x v_t(\lambda)}, \qquad \lambda \geq 0,$$

where $t \mapsto v_t(\lambda)$ is the unique nonnegative solution of

$$v_t(\lambda) = \lambda - \int_0^t \Phi(v_s(\lambda))\, \mathrm{d}s, \qquad t \geq 0,$$

or, in the equivalent differential form,

$$\frac{\mathrm{d}}{\mathrm{d}t} v_t(\lambda) = -\Phi(v_t(\lambda)), \qquad v_0(\lambda) = \lambda.$$

Under the integrability condition (2.2), the CB-process started from any deterministic initial value has finite expectation. This, in particular, allows us to compensate large jumps of the process generated by the branching mechanism; see the stochastic integral equation (3.1).

We say that the CB-process is *subcritical* if $\alpha > 0$, *critical* if $\alpha = 0$, and *supercritical* if $\alpha < 0$. In view of (2.1), we have

$$\Phi'(z) = \alpha + 2\beta z + \int_0^\infty \pi_0(\mathrm{d}\theta)\theta(1 - \mathrm{e}^{-z\theta}),$$

which is increasing in $z \geq 0$. Then Φ is a convex function. Consequently, the limit $\Phi(\infty) := \lim_{z\to\infty} \Phi(z)$ exists in $[-\infty, 0] \cup \{\infty\}$. The limit $\Phi'(\infty) := \lim_{z\to\infty} \Phi'(z)$ exists in $(-\infty, \infty]$. In fact, we have

$$\Phi'(\infty) = \alpha + 2\beta \cdot \infty + \int_0^\infty \theta \pi_0(\mathrm{d}\theta)$$

with $0\cdot\infty = 0$ by convention. Observe that $\Phi(\infty) \in [-\infty, 0]$ if and only if $\Phi'(\infty) \in (-\infty, 0]$, and $\Phi(\infty) = \infty$ if and only if $\Phi'(\infty) \in (0, \infty]$. For $\lambda \geq 0$ let

$$\Phi^{-1}(\lambda) = \inf\{z \geq 0\colon \Phi(z) > \lambda\}.$$

Of course, we have $\Phi^{-1}(\lambda) = \infty$ for all $\lambda \geq 0$ if $\Phi(\infty) \in [-\infty, 0]$. If $\Phi(\infty) = \infty$ then $\Phi^{-1}\colon [0, \infty) \to [\Phi^{-1}(0), \infty)$ is the inverse of the restriction of Φ to $[\Phi^{-1}(0), \infty)$.

The CBI-process generalizes the CB-process given above. Let Ψ be an *immigration mechanism*, which is a function on $\mathbb{R}_+$ with representation

$$\Psi(z) = \gamma z + \int_{(0,\infty)} \pi_1(\mathrm{d}\theta)(1 - \mathrm{e}^{-z\theta}), \qquad z \geq 0, \tag{2.3}$$

where $\gamma \in \mathbb{R}_+$ and π_1 is a σ-finite measure on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \pi_1(\mathrm{d}\theta)(1 \wedge \theta) < \infty.$$

A nonnegative Markov process is called a *CBI-process* with branching mechanism Φ and immigration mechanism Ψ if it has transition semigroup $(Q_t)_{t\geq 0}$ given by

$$\int_{\mathbb{R}_+} \exp\{-\lambda y\} Q_t(x, \mathrm{d}y) = \exp\bigg\{-x v_t(\lambda) - \int_0^t \Psi(v_s(\lambda))\,\mathrm{d}s\bigg\}, \qquad \lambda \geq 0. \tag{2.4}$$

This reduces to a CB-process when $\Psi \equiv 0$. We refer the reader to Kawazu and Watanabe (1971) for a discussion of CB- and CBI-processes with more general branching and immigration mechanisms.

From (2.4), we see that $(Q_t)_{t\geq 0}$ is a Feller semigroup, so the CBI-process has a Hunt process realization; see, e.g. Chung (1982, p. 75). Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x)$ be such a realization. Then the sample path $\{X_t\colon t \geq 0\}$ is $\mathbb{P}_x$-almost surely càdàg for every $x \geq 0$. Let $\mathbb{E}_x$ denote the expectation with respect to the probability measure $\mathbb{P}_x$.

Proposition 2.1. *For $t \geq 0$, $x \geq 0$, and $\lambda \geq 0$, we have*

$$\mathbb{E}_x\bigg[\exp\bigg\{-\lambda \int_0^t X_s\,\mathrm{d}s\bigg\}\bigg] = \exp\bigg\{-x u_t(\lambda) - \int_0^t \Psi(u_s(\lambda))\,\mathrm{d}s\bigg\}, \tag{2.5}$$

where $t \mapsto u_t(\lambda)$ is the unique nonnegative solution of

$$u_t(\lambda) = t\lambda - \int_0^t \Phi(u_s(\lambda))\,\mathrm{d}s, \qquad t \geq 0, \tag{2.6}$$

or, in the equivalent differential form,

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t(\lambda) = \lambda - \Phi(u_t(\lambda)), \qquad u_0(\lambda) = 0. \tag{2.7}$$

Proof. As special cases of Li (2011, Theorem 9.16), we have (2.5) with $t \mapsto u_t(\lambda)$ being the unique nonnegative solution of (2.6), which is equivalent to its differential form (2.7). □

Proposition 2.2. *For $\lambda > 0$, the mapping $t \mapsto u_t(\lambda)$ is strictly increasing and* $\lim_{t\to\infty} u_t(\lambda) = \Phi^{-1}(\lambda)$.

Proof. Consider a Hunt realization X of the CB-process with branching mechanism Φ. By Proposition 2.1, we have

$$\mathbb{E}_x\left[\exp\left\{-\lambda\int_0^t X_s\,\mathrm{d}s\right\}\right] = \exp\{-x u_t(\lambda)\}. \tag{2.8}$$

As observed in the proof of Li (2011, Proposition 3.1), we have $\mathbb{P}_x(X_t > 0) > 0$ for $x > 0$ and $t \geq 0$. By (2.8), we see that $t \mapsto u_t(\lambda)$ is strictly increasing, so $(\partial/\partial t)u_t(\lambda) > 0$ for all $\lambda > 0$. Let $u_\infty(\lambda) = \lim_{t\to\infty} u_t(\lambda) \in (0, \infty]$. In the $\Phi(\infty) \in [-\infty, 0]$ case, we have $\Phi(z) \leq 0$ for all $z \geq 0$. Then $(\partial/\partial t)u_t(\lambda) \geq \lambda$ and $u_\infty(\lambda) = \infty$. In the $\Phi(\infty) = \infty$ case, we note that $\Phi(u_t(\lambda)) = \lambda - (\partial/\partial t)u_t(\lambda) < \lambda$, and, hence, $u_t(\lambda) < \Phi^{-1}(\lambda)$, implying $u_\infty(\lambda) \leq \Phi^{-1}(\lambda) < \infty$. It follows that

$$0 = \lim_{t\to\infty}\frac{\partial}{\partial t}u_t(\lambda) = \lambda - \lim_{t\to\infty}\Phi(u_t(\lambda)) = \lambda - \Phi(u_\infty(\lambda)).$$

Then, we have $u_\infty(\lambda) = \Phi^{-1}(\lambda)$. □

Corollary 2.1. *Let $X = (\Omega, \mathscr{F}, \mathscr{F}_t, X_t, \mathbb{P}_x)$ be a Hunt realization of the CB-process with branching mechanism Φ. Then for $x > 0$ and $\lambda > 0$, we have*

$$\mathbb{E}_x\left[\exp\left\{-\lambda\int_0^\infty X_s\,\mathrm{d}s\right\}\right] = \exp\{-x\Phi^{-1}(\lambda)\}. \tag{2.9}$$

Note that (2.9) can also be derived from the theory of Lévy processes; see, e.g. Kyprianou (2014, Corollary 12.10).

3. Distributional properties of jump times

Let Φ and Ψ be the branching and immigration mechanisms with representations (2.1) and (2.3), respectively. Suppose that on a suitable filtered probability space $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbb{P})$ satisfying the usual hypotheses, we have a standard $\mathscr{G}_t$-Brownian motion $(B_t : t \geq 0)$, a $\mathscr{G}_t$-Poisson point process $(p_t : t \geq 0)$ on $(0, \infty)^2$ with characteristic measure $\pi_0(\mathrm{d}z)\,\mathrm{d}y$, and a $\mathscr{G}_t$-Poisson point process $(q_t : t \geq 0)$ on $(0, \infty)$ with characteristic measure $\pi_1(\mathrm{d}z)$. Suppose that $(B_t : t \geq 0)$, $(p_t : t \geq 0)$, and $(q_t : t \geq 0)$ are independent. Let $N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}y)$ denote the Poisson random measure on $(0, \infty)^3$ associated with $(p_t : t \geq 0)$, and $\tilde{N}_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}y)$ the compensated measure of $N_0(\mathrm{d}s, \mathrm{d}z, \mathrm{d}y)$. Let $N_1(\mathrm{d}s, \mathrm{d}z)$ denote the Poisson random measure on $(0, \infty)^2$ associated with $(p_t : t \geq 0)$. By the results of Dawson and Li (2006) and Fu and Li (2010), for any $\mathscr{G}_0$-measurable nonnegative random variable X_0 there is a unique nonnegative strong solution $X = (X_t : t \geq 0)$ of the stochastic equation

$$\begin{aligned} X_t = X_0 &+ \int_0^t \sqrt{2\beta X_s}\,\mathrm{d}B_s + \int_{(0,t]}\int_{(0,\infty)}\int_{(0,X_{s-}]} z\tilde{N}_0(\mathrm{d}s,\, \mathrm{d}z,\, \mathrm{d}y) + \int_0^t (\gamma - \alpha X_s)\,\mathrm{d}s \\ &+ \int_{(0,t]}\int_{(0,\infty)} zN_1(\mathrm{d}s,\, \mathrm{d}z). \end{aligned} \tag{3.1}$$

It was also proved in Dawson and Li (2006) and Fu and Li (2010) that X is a CBI-process with branching mechanism Φ and immigration mechanism Ψ. For $x \geq 0$, let $\mathbb{P}_x$ denote the conditional law of X given $X_0 = x$.

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In the sequel, we provide some results on the distributional properties of the first jump time of the CBI-process with jump size in some given sets. To present the results, let us introduce some notation. For any Borel set $A \subset (0, \infty)$ with $\pi_0(A) + \pi_1(A) < \infty$, we define

$$\Psi_A(z) = \Psi(z) - \int_A \pi_1(\mathrm{d}\theta)(1 - \mathrm{e}^{-z\theta}) \tag{3.2}$$

and

$$\Phi_A(z) = \Phi(z) + \int_A \pi_0(\mathrm{d}\theta)(1 - \mathrm{e}^{-z\theta}). \tag{3.3}$$

Then Φ_A is also a branching mechanism and Ψ_A an immigration mechanism. For example, we have

$$\Phi_A(z) = \alpha_A z + \beta z^2 + \int_{(0,\infty)\setminus A} \pi_0(\mathrm{d}\theta)(\mathrm{e}^{-z\theta} - 1 + z\theta), \tag{3.4}$$

where $\alpha_A = \alpha + \int_A \theta\pi_0(\mathrm{d}\theta)$.

Proposition 3.1. *Suppose that $A \subset (0, \infty)$ is a Borel set with $\pi_0(A) + \pi_1(A) < \infty$. For $t > 0$ let $J_t(A) := \mathrm{card}\{s \in (0, t]\colon \Delta X_s = X_s - X_{s-} \in A\}$. Then, for any $x \geq 0$, we have $\mathbb{P}_x(J_t(A) < \infty) = 1$.*

Proof. Let N_0^A and $N_0^{A^c}$ be the restrictions of N_0 to $(0, \infty) \times A \times (0, \infty)$ and $(0, \infty) \times ((0, \infty) \setminus A) \times (0, \infty)$, respectively. Similarly, let N_1^A and $N_1^{A^c}$ be the restrictions of N_1 to $(0, \infty) \times A$ and $(0, \infty) \times ((0, \infty) \setminus A)$, respectively. Then we can express (3.1) as

$$\begin{aligned} X_t = X_0 &+ \int_0^t \sqrt{2\beta X_s}\,\mathrm{d}B_s + \int_{(0,t]}\int_{(0,\infty)\setminus A}\int_{(0,X_{s-}]} z\tilde{N}_0^{A^c}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}y) \\ &+ \int_0^t (\gamma - \alpha_A X_s)\,\mathrm{d}s + \int_{(0,t]}\int_{(0,\infty)\setminus A} zN_1^{A^c}(\mathrm{d}s, \mathrm{d}z) \\ &+ \int_{(0,t]}\int_A\int_{(0,X_{s-}]} zN_0^A(\mathrm{d}s, \mathrm{d}z, \mathrm{d}y) + \int_{(0,t]}\int_A zN_1^A(\mathrm{d}s, \mathrm{d}z). \end{aligned}$$

Note that the last two terms on the right-hand side of the above equation never jump simultaneously, so we have

$$J_t(A) = \int_{(0,t]}\int_A\int_{(0,X_{s-}]} N_0^A(\mathrm{d}s, \mathrm{d}z, \mathrm{d}y) + \int_{(0,t]}\int_A N_1^A(\mathrm{d}s, \mathrm{d}z).$$

For any $k \geq 1$ let

$$J_t(k, A) = \int_{(0,t]}\int_A\int_{(0,k]} N_0^A(\mathrm{d}s, \mathrm{d}z, \mathrm{d}y) + \int_{(0,t]}\int_A N_1^A(\mathrm{d}s, \mathrm{d}z).$$

It follows that

$$\mathbb{E}_x[J_t(k, A)] = kt\pi_0(A) + t\pi_1(A) < \infty,$$

and so $\mathbb{P}_x(J_t(k, A) < \infty) = 1$. Since $s \mapsto X_t$ is càdlàg, we have $\sup_{0<s\leq t} X_s < \infty$. Note also that $J_t(A) \leq J_t(k, A)$ on the event $\sup_{0<s\leq t} X_s < k$. It follows that

$$\begin{aligned} \mathbb{P}_x(J_t(A) = \infty) &\leq \sum_{k=1}^{\infty} \mathbb{P}_x\Big(\{J_t(A) = \infty\} \cap \Big\{\sup_{0<s\leq t} X_s < k\Big\}\Big) \\ &= \sum_{k=1}^{\infty} \mathbb{P}_x\Big(\{J_t(A) = J_t(k, A) = \infty\} \cap \Big\{\sup_{0<s\leq t} X_s < k\Big\}\Big) \end{aligned}$$

$$\leq \sum_{k=1}^{\infty} \mathbb{P}_x(J_t(k, A) = \infty)$$
$$= 0.$$

Then $\mathbb{P}_x(J_t(A) < \infty) = 1$. □

Theorem 3.1. *Suppose that $A \subset (0, \infty)$ is a Borel set with $\pi_0(A) + \pi_1(A) < \infty$. Let $\tau_A = \min\{s > 0\colon \Delta X_s = X_s - X_{s-} \in A\}$, which is well defined by the result of Proposition 3.1. Then, for any $x \geq 0$ and $t \geq 0$, we have*

$$\mathbb{P}_x(\tau_A > t) = \exp\bigg\{-t\pi_1(A) - x u_t^A(\pi_0(A)) - \int_0^t \Psi_A(u_s^A(\pi_0(A)))\,\mathrm{d}s\bigg\}, \tag{3.5}$$

where $u_t^A(\lambda)$ is the unique nonnegative solution of

$$\frac{\mathrm{d}}{\mathrm{d}t} u_t^A(\lambda) = \lambda - \Phi_A(u_t^A(\lambda)), \qquad u_0^A(\lambda) = 0. \tag{3.6}$$

Proof. We shall use the notation introduced in the proof of Proposition 3.1. Let $(X_t^A : t \geq 0)$ be the solution of

$$X_t^A = X_0 + \int_0^t \sqrt{2\beta X_s^A}\,\mathrm{d}B_s + \int_{(0,t]} \int_{(0,\infty)\setminus A} \int_{(0, X_{s-}^A]} z \tilde{N}_0^{A^c}(\mathrm{d}s,\, \mathrm{d}z,\, \mathrm{d}y)$$
$$+ \int_0^t (\gamma - \alpha_A X_s^A)\,\mathrm{d}s + \int_{(0,t]} \int_{(0,\infty)\setminus A} z N_1^{A^c}(\mathrm{d}s,\, \mathrm{d}z). \tag{3.7}$$

Then $(X_t^A : t \geq 0)$ is a CBI-process with branching mechanism Φ_A and immigration mechanism Ψ_A. By Dawson and Li (2012, Theorem 2.2), we have $X_t^A \leq X_t$ for all $t \geq 0$. (Intuitively, we can obtain $(X_t^A : t \geq 0)$ by removing from $(X_t : t \geq 0)$ all masses produced by jumps of sizes in the set A.) We claim that, up to a null set,

$$\{\tau_A > t\} = \bigg\{\int_{(0,t]} \int_A \int_{(0, X_{s-}^A]} N_0^A(\mathrm{d}s,\, \mathrm{d}z,\, \mathrm{d}y) + \int_{(0,t]} \int_A N_1^A(\mathrm{d}s,\, \mathrm{d}z) = 0\bigg\}. \tag{3.8}$$

Indeed, since $X_s = X_s^A$ for $0 \leq s < \tau_A$, we have

$$\{\tau_A > t\} = \{\tau_A > t\} \cap \bigg\{\int_{(0,t]} \int_A \int_{(0, X_{s-}]} N_0^A(\mathrm{d}s,\, \mathrm{d}z,\, \mathrm{d}y) + \int_{(0,t]} \int_A N_1^A(\mathrm{d}s,\, \mathrm{d}z) = 0\bigg\}$$
$$\subset \bigg\{\int_{(0,t]} \int_A \int_{(0, X_{s-}^A]} N_0^A(\mathrm{d}s,\, \mathrm{d}z,\, \mathrm{d}y) + \int_{(0,t]} \int_A N_1^A(\mathrm{d}s,\, \mathrm{d}z) = 0\bigg\}.$$

Since $\Delta X_{\tau_A} \in A$ when $\tau_A \leq t$, we have

$$\{\tau_A \leq t\} \subset \{\tau_A \leq t\} \cap \bigg\{\int_{\{\tau_A\}} \int_A \int_{(0, X_{s-}]} N_0^A(\mathrm{d}s,\, \mathrm{d}z,\, \mathrm{d}y) + \int_{\{\tau_A\}} \int_A N_1^A(\mathrm{d}s,\, \mathrm{d}z) > 0\bigg\}$$
$$= \{\tau_A \leq t\} \cap \bigg\{\int_{\{\tau_A\}} \int_A \int_{(0, X_{s-}^A]} N_0^A(\mathrm{d}s,\, \mathrm{d}z,\, \mathrm{d}y) + \int_{\{\tau_A\}} \int_A N_1^A(\mathrm{d}s,\, \mathrm{d}z) > 0\bigg\}$$
$$\subset \bigg\{\int_{(0,t]} \int_A \int_{(0, X_{s-}^A]} N_0^A(\mathrm{d}s,\, \mathrm{d}z,\, \mathrm{d}y) + \int_{(0,t]} \int_A N_1^A(\mathrm{d}s,\, \mathrm{d}z) > 0\bigg\}.$$

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Then (3.8) holds. Since $(X_t^A : t \geq 0)$ is a strong solution of (3.7), it is progressively measurable with respect to the filtration generated by B, $N_0^{A^c}$, and $N_1^{A^c}$, which is independent of N_0^A and N_1^A. Then we have

$$\mathbb{P}_x(\tau_A > t) = \exp\{-t\pi_1(A)\}\mathbb{E}_x\left[\exp\left\{-\pi_0(A)\int_0^t X_s^A \,\mathrm{d}s\right\}\right].$$

Finally, we obtain (3.5) by Proposition 2.1. □

Corollary 3.1. (i) *If A and B are Borel subsets of $(0,\infty)$ such that $A \subset B$ and $\pi_0(B) < \infty$, then $u_t^A(\pi_0(A)) \leq u_t^B(\pi_0(B))$ for $t \geq 0$.*

(ii) *If $A \subset (0,\infty)$ is a Borel set satisfying $\pi_0(A) + \pi_1(A) = 0$, then $\mathbb{P}_x(\tau_A = \infty) = 1$ for $x \geq 0$.*

(iii) *If $\Psi \neq 0$ and $A \subset (0,\infty)$ is a Borel set satisfying $\pi_0(A) + \pi_1(A) > 0$, then $\mathbb{P}_x(\tau_A < \infty) = 1$ for $x \geq 0$.*

Proof. (i) By applying Theorem 3.1 to the special case $\Psi \equiv 0$, we have

$$\exp\{-xu_t^A(\pi_0(A))\} = \mathbb{P}_x(\tau_A > t) \geq \mathbb{P}_x(\tau_B > t) = \exp\{-xu_t^B(\pi_0(B))\}.$$

Taking any $x > 0$, we obtain the result.

(ii) By Theorem 3.1, we have $\mathbb{P}_x(\tau_A > t) = 1$ for every $t \geq 0$. Then $\mathbb{P}_x(\tau_A = \infty) = \lim_{t\to\infty} \mathbb{P}_x(\tau_A > t) = 1$.

(iii) By choosing a smaller set if it is necessary, we may assume that $0 < \pi_0(A) + \pi_1(A) < \infty$. If $\pi_1(A) > 0$ then $t\pi_1(A) \to \infty$ as $t \to \infty$. In the $\pi_1(A) = 0$ case, we must have $\pi_0(A) > 0$, so $s \mapsto u_s^A(\pi_0(A))$ is strictly increasing by Proposition 2.2. Since $\Psi \neq 0$, one can see that

$$\lim_{t\to\infty} \int_0^t \Psi_A(u_s^A(\pi_0(A)))\,\mathrm{d}s = \infty.$$

In view of (3.5), we have $\mathbb{P}_x(\tau_A = \infty) = \lim_{t\to\infty} \mathbb{P}_x(\tau_A > t) = 0$ in both cases. □

Corollary 3.2. *Suppose that $\Psi \equiv 0$. Then, for any $x \geq 0$ and Borel set $A \subset (0,\infty)$ satisfying $0 < \pi_0(A) < \infty$, we have*

$$\mathbb{P}_x(\tau_A = \infty) = \exp\{-x\Phi_A^{-1}(\pi_0(A))\}.$$

Proof. By applying Theorem 3.1 to the special case $\Psi \equiv 0$, we have

$$\mathbb{P}_x(\tau_A = \infty) = \lim_{t\to\infty} \mathbb{P}_x(\tau_A > t) = \lim_{t\to\infty} \exp\{-xu_t^A(\pi_0(A))\}.$$

Then the result follows by Proposition 2.2. □

4. Local and global maximal jumps

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \mathbb{P}_x)$ be a Hunt realization of the CBI-process with branching mechanism Φ and immigration mechanism Ψ given by (2.1) and (2.3), respectively. In this section we shall give some characterizations of the local and global maximal jumps of the process.

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Theorem 4.1. *Suppose that* $r \geq 0$ *and* $\pi_0(r, \infty) + \pi_1(r, \infty) < \infty$. *Then, for any* $x \geq 0$ *and* $t > 0$, *we have*

$$\mathbb{P}_x\Big(\max_{s\in(0,t]} \Delta X_t \leq r\Big) = \exp\bigg\{-t\pi_1(r, \infty) - xu_t^r(\pi_0(r, \infty)) - \int_0^t \Psi_{(r,\infty)}(u_s^r(\pi_0(r, \infty)))\,\mathrm{d}s\bigg\},$$

where $u_t^r(\lambda)$ *is the unique nonnegative solution of*

$$\frac{\mathrm{d}}{\mathrm{d}t}u_t^r(\lambda) = \lambda - \Phi_{(r,\infty)}(u_t^r(\lambda)), \qquad u_0^r(\lambda) = 0.$$

Proof. Since $\mathbb{P}_x(\max_{s\in(0,t]} \Delta X_t \leq r) = \mathbb{P}_x(\tau_{(r,\infty)} > t)$, the result follows by Theorem 3.1. □

Corollary 4.1. *Suppose that* $\Psi \neq 0$. *Then* $\mathbb{P}_x(\sup_{s\in(0,\infty)} \Delta X_s = \sup(\pi_0 + \pi_1)) = 1$ *for any* $x \geq 0$, *where* $\sup(\pi_0 + \pi_1) = \sup \operatorname{sup}(\pi_0 + \pi_1)$.

Proof. Since $(\pi_0+\pi_1)(\sup(\pi_0+\pi_1), \infty) = 0$, for any $t > 0$, we have $\mathbb{P}_x(\sup_{s\in(0,t]} \Delta X_s \leq \sup(\pi_0 + \pi_1)) = 1$ by Theorem 4.1. Then

$$\mathbb{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \leq \sup(\pi_0 + \pi_1)\Big) = \lim_{t\to\infty} \mathbb{P}_x\Big(\sup_{s\in(0,t]} \Delta X_s \leq \sup(\pi_0 + \pi_1)\Big) = 1.$$

For any $z < \sup(\pi_0 + \pi_1)$, we have $(\pi_0 + \pi_1)[z, \sup(\pi_0 + \pi_1)] > 0$. By Corollary 3.1(iii),

$$\mathbb{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \in [z, \sup(\pi_0 + \pi_1)]\Big) \geq \mathbb{P}_x(\tau_{[z,\sup(\pi_0+\pi_1)]} < \infty) = 1.$$

Since $z < \sup(\pi_0 + \pi_1)$ was arbitrary, it follows that $\mathbb{P}_x(\sup \Delta X = \sup(\pi_0 + \pi_1)) = 1$. □

Corollary 4.2. *Suppose that* $\Psi \equiv 0$. *Then, for any* $x \geq 0$ *and* $r \geq 0$ *satisfying* $0 < \pi_0(r, \infty) < \infty$, *we have*

$$\mathbb{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \leq r\Big) = \exp\{-x\Phi^{-1}_{(r,\infty)}(\pi_0(r, \infty))\}.$$

Proof. This follows by Theorem 4.1 and Proposition 2.2. □

Corollary 4.3. *Suppose that* $\Psi \equiv 0$ *and let* $\sup(\pi_0) = \sup \operatorname{sup}(\pi_0)$. *Then, for any* $x \geq 0$, *we have*

$$\mathbb{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s = \sup(\pi_0)\Big) = 1 - \exp\{-x\Phi^{-1}_{\{\sup(\pi_0)\}}(\pi_0(\{\sup(\pi_0)\}))\}$$

with $\Phi_{\{0\}} = \Phi_{\{\infty\}} = \Phi$ *and* $\pi_0(\{0\}) = \pi_0(\{\infty\}) = 0$ *by convention.*

Proof. By the proof of Corollary 4.1, we have $\mathbb{P}_x(\sup_{s\in(0,\infty)} \Delta X_s \leq \sup(\pi_0)) = 1$. For any $z < \sup(\pi_0)$, we have $\pi_0(z, \sup(\pi_0)] > 0$. By Corollary 4.2, it follows that

$$\mathbb{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \in (z, \sup(\pi_0)]\Big) = 1 - \exp\{-x\Phi^{-1}_{(z,\sup(\pi_0)]}(\pi_0(z, \sup(\pi_0)])\}.$$

Then we obtain the desired result by letting $z \to \sup(\pi_0)$. □

Corollary 4.4. *Suppose that* $\alpha > 0$ *and* $\Psi \equiv 0$. *If the measure* π_0 *has unbounded support then, for any* $x > 0$, *we have, as* $r \to \infty$,

$$\mathbb{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s > r\Big) = 1 - \exp\{-x\Phi^{-1}_{(r,\infty)}(\pi_0(r, \infty))\} \sim \frac{x}{\alpha}\pi_0(r, \infty).$$

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Proof. By (3.4), we see by dominated convergence that $(\partial/\partial z)\Phi_{(r,\infty)}(0) = \alpha_{(r,\infty)}$. It follows that $(\partial/\partial z)\Phi^{-1}_{(r,\infty)}(0) = 1/\alpha_{(r,\infty)}$. Then, as $r \to \infty$,

$$\Phi^{-1}_{(r,\infty)}(\pi_0(r,\infty)) \sim \frac{\pi_0(r,\infty)}{\alpha_{(r,\infty)}} \sim \frac{\pi_0(r,\infty)}{\alpha},$$

and the desired result follows from Corollary 4.2. □

We remark that a special form of Corollary 4.2 has been obtained by Bertoin (2011). In the next theorem we establish the equivalence of the distribution of the local maximal jump of the CBI-process and the *total Lévy measure* $\pi_0 + \pi_1$. In view of Theorem 4.1, we may have $\mathbb{P}_x(\max_{s\in(0,t]} \Delta X_s = 0) > 0$, so we discuss the absolute continuity only on the set $(0, \infty)$.

Theorem 4.2. *Suppose that $x + \gamma > 0$. Then, for any $t > 0$, the measure $\pi_0 + \pi_1$ and the distribution $\mathbb{P}_x(\max_{s\in(0,t]} \Delta X_s \in \cdot)|_{(0,\infty)}$ are equivalent.*

Proof. Recall that Ψ and Ψ_A are defined by (2.3) and (3.2), respectively. If $A \subset (0, \infty)$ is a Borel set with $\pi_0(A) + \pi_1(A) = 0$, by Theorem 3.1, we have

$$\mathbb{P}_x\Big(\max_{s\in(0,t]} \Delta X_s \in A\Big) \le \mathbb{P}_x(\tau_A \le t) = 1 - \mathbb{P}_x(\tau_A > t) = 0.$$

Then $\mathbb{P}_x(\max_{s\in(0,t]} \Delta X \in \cdot)|_{(0,\infty)}$ is absolutely continuous with respect to $\pi_0 + \pi_1$. To prove the absolute continuity of $\pi_0+\pi_1$ with respect to $\mathbb{P}_x(\max_{s\in(0,t]} \sup \Delta X \in \cdot)|_{(0,\infty)}$, we consider a Borel set $A \subset (0, \infty)$ and a constant $r > 0$. Since

$$\begin{aligned}\Big\{\max_{s\in(0,t]} \Delta X_s \in A\Big\} &\supset \Big\{\max_{s\in(0,t]} \Delta X_s \in A \cap [r,\infty)\Big\}\\ &\supset \{\tau_{A\cap[r,\infty)} \le t\} \cap \{\tau_{[r,\infty)\setminus A} > t\}\\ &= \{\tau_{[r,\infty)} \le t\} \setminus \{\tau_{[r,\infty)\setminus A} \le t\},\end{aligned}$$

we have

$$\mathbb{P}_x\Big(\max_{s\in(0,t]} \Delta X \in A\Big) \ge \mathbb{P}_x(\tau_{[r,\infty)} \le t) - \mathbb{P}_x(\tau_{[r,\infty)\setminus A} \le t).$$

Suppose that $\mathbb{P}_x(\sup_{s\in(0,t]} \Delta X \in A) = 0$. Then $\mathbb{P}_x(\tau_{[r,\infty)} \le t) = \mathbb{P}_x(\tau_{[r,\infty)\setminus A} \le t)$, so the result of Theorem 3.1 implies that

$$\begin{aligned}&t\pi_1[r,\infty) + xu_t^{[r,\infty)}(\pi_0[r,\infty)) + \int_0^t \Psi_{[r,\infty)}(u_s^{[r,\infty)}(\pi_0[r,\infty)))\,\mathrm{d}s\\ &\quad = t\pi_1([r,\infty)\setminus A) + xu_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))\\ &\qquad + \int_0^t \Psi_{[r,\infty)\setminus A}(u_s^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A)))\,\mathrm{d}s.\end{aligned} \tag{4.1}$$

By Corollary 3.1(i), we have

$$u_t^{[r,\infty)}(\pi_0[r,\infty)) \ge u_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A)), \qquad t \ge 0. \tag{4.2}$$

Then (4.1) implies that

$$\begin{aligned}&t\pi_1[r,\infty) + xu_t^{[r,\infty)}(\pi_0[r,\infty)) + \int_0^t \Psi_{[r,\infty)}(u_s^{[r,\infty)}(\pi_0[r,\infty)))\,\mathrm{d}s\\ &\quad \le t\pi_1([r,\infty)\setminus A) + xu_t^{[r,\infty)}(\pi_0[r,\infty)) + \int_0^t \Psi_{[r,\infty)\setminus A}(u_s^{[r,\infty)}(\pi_0[r,\infty)))\,\mathrm{d}s.\end{aligned}$$

By reorganizing the terms in the above inequality, we obtain

$$t\pi_1([r,\infty)\cap A)\le\int_0^t \mathrm{d}s\int_{[r,\infty)\cap A}(1-\exp\{-\theta u_s^{[r,\infty)}(\pi_0[r,\infty))\})\pi_1(\mathrm{d}\theta).$$

It follows that $\pi_1([r,\infty)\cap A)=0$. Since $r>0$ was arbitrary in the above, we have proved that $\pi_1(A)=0$. Using (4.1) and (4.2), we have

$$\begin{aligned}&xu_t^{[r,\infty)}(\pi_0[r,\infty))+\gamma\int_0^t u_s^{[r,\infty)}(\pi_0[r,\infty))\,\mathrm{d}s\\&\qquad\le xu_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))+\gamma\int_0^t u_s^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))\,\mathrm{d}s,\end{aligned}$$

and so using (4.2) again, we obtain

$$u_t^{[r,\infty)}(\pi_0[r,\infty))=u_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))=:a(r,t).$$

It follows that

$$\frac{\partial u_t^{[r,\infty)}(\pi_0[r,\infty))}{\partial t}=\frac{\partial u_t^{[r,\infty)\setminus A}(\pi_0([r,\infty)\setminus A))}{\partial t}.$$

Then we can use (3.6) to see that

$$\pi_0([r,\infty))-\Phi_{[r,\infty)}(a(r,t))=\pi_0([r,\infty)\setminus A)-\Phi_{[r,\infty)\setminus A}(a(r,t)),$$

and, hence,

$$\Phi_{[r,\infty)}(a(r,t))=\Phi_{[r,\infty)\setminus A}(a(r,t))+\pi_0([r,\infty)\cap A).$$

However, by (3.3), we should have

$$\Phi_{[r,\infty)}(a(r,t))=\Phi_{[r,\infty)\setminus A}(a(r,t))+\int_{[r,\infty)\cap A}\pi_0(\mathrm{d}\theta)(1-\exp\{-a(r,t)\theta\}).$$

It follows that $\pi_0([r,\infty)\cap A)=0$, implying that $\pi_0(A)=0$. This completes the proof. □

The conclusion of Theorem 4.2 does not necessarily hold in the $x=\gamma=0$ case. As a counterexample, consider the case where $X_0=0$, $\pi_0=\delta_1$, and $\pi_1=\delta_2$. In this case, we have $\tau_{\{2\}}\le t$ when $\tau_{\{1\}}\le t$, otherwise $X_s=0$ for all $s\in[0,t]$. It follows that

$$\mathbb{P}_0\Big(\max_{s\in(0,t]}\Delta X_s=1\Big)=0.$$

Then π_0 is not absolutely continuous with respect to $\mathbb{P}_0(\max_{s\in(0,t]}\Delta X_s\in\cdot)$.

For critical and subcritical branching CB-processes without immigration, we may also discuss the absolute continuity of the distribution of its global maximal jump. Such a result is presented in the following theorem.

Theorem 4.3. *Suppose that $\alpha\ge 0$ and $\Psi\equiv 0$. Then, for any $x>0$, the Lévy measure π_0 and the distribution $\mathbb{P}_x(\sup_{s\in(0,\infty)}\Delta X_s\in\cdot)|_{(0,\infty)}$ are equivalent.*

Proof. Since $\Psi = 0$ and $\alpha \geq 0$, we have $X_t \to 0$ almost surely as $t \to \infty$. If $A \subset (0, \infty)$ is a Borel set so that $\pi_0(A) = 0$, by Corollary 3.1(ii), we have

$$\mathbb{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \in A\Big) \leq \mathbb{P}_x(\tau_A < \infty) = 1 - \mathbb{P}_x(\tau_A = \infty) = 0.$$

Then $\mathbb{P}_x(\sup_{s\in(0,\infty)} \Delta X_s \in \cdot)|_{(0,\infty)}$ is absolutely continuous with respect to π_0. Now suppose that $A \subset (0, \infty)$ is a Borel set with $\pi_0(A) > 0$. For any $r > 0$, one can see, as in the proof of Theorem 4.2, that

$$\mathbb{P}_x\Big(\sup_{s\in(0,\infty)} \Delta X_s \in A\Big) \geq \mathbb{P}_x(\tau_{[r,\infty)} < \infty) - \mathbb{P}_x(\tau_{[r,\infty)\setminus A} < \infty).$$

If $\mathbb{P}_x(\sup_{s\in(0,\infty)} \Delta X_s \in A) = 0$, we have $\mathbb{P}_x(\tau_{[r,\infty)} < \infty) = \mathbb{P}_x(\tau_{[r,\infty)\setminus A} < \infty)$, so Corollary 3.2 implies that

$$\Phi^{-1}_{[r,\infty)}(\pi_0[r,\infty)) = \Phi^{-1}_{[r,\infty)\setminus A}(\pi_0([r,\infty) \setminus A)) =: a(r).$$

It follows that

$$\begin{aligned}
\Phi_{[r,\infty)}(a(r)) &= \pi_0[r,\infty) \\
&= \pi_0([r,\infty) \setminus A) + \pi_0(A \cap [r,\infty)) \\
&= \Phi_{[r,\infty)\setminus A} \circ \Phi^{-1}_{[r,\infty)\setminus A}(\pi_0([r,\infty) \setminus A)) + \pi_0(A \cap [r,\infty)) \\
&= \Phi_{[r,\infty)\setminus A}(a(r)) + \pi_0(A \cap [r,\infty)).
\end{aligned}$$

Then, as in the proof of Theorem 4.2, we must have $\pi_0(A \cap [r,\infty)) = 0$. This contradicts $\pi_0(A) > 0$ since $r > 0$ was arbitrary. It then follows that $\mathbb{P}_x(\sup_{s\in(0,\infty)} \Delta X_s \in A) > 0$. □

In the above theorem, we consider only the critical and subcritical cases. The supercritical case is more subtle since in that case we may have $\sup_{s\in(0,\infty)} \Delta X_s = \sup(\pi_0)$ with strictly positive probability by Corollary 4.3. We leave the consideration of the details to the interested reader.

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