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Differential forms on universal K3 surfaces

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Abstract

We give a vanishing and classification result for holomorphic differential forms on smooth projective models of the moduli spaces of pointed K3 surfaces. We prove that there is no nonzero holomorphic k-form for 0 < k < 10 and for even k > 19. In the remaining cases, we give an isomorphism between the space of holomorphic k-forms with that of vector-valued modular forms ($10 \le k \le 18$) or scalar-valued cusp forms (odd $k \ge 19$) for the modular group. These results are in fact proved in the generality of lattice-polarisation.

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1. Introduction

Let $\mathcal{F}_{g,n}$ be the moduli space of *n*-pointed K3 surfaces of genus g > 2, i.e., primitively polarised of degree 2g - 2. It is a quasi-projective variety of dimension 19 + 2n with a natural morphism $\mathcal{F}_{g,n} \to \mathcal{F}_g$ to the moduli space \mathcal{F}_g of K3 surfaces of genus g, which is generically a $K3^n$ -fibration. In this paper we study holomorphic differential k-forms on a smooth projective model of $\mathcal{F}_{g,n}$. They do not depend on the choice of a smooth projective model, and thus are fundamental birational invariants of $\mathcal{F}_{g,n}$. We prove a vanishing result for about half of the values of the degree k, and for the remaining degrees give a correspondence with modular forms on the period domain.

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Our main result is stated as follows.

THEOREM 1.1. Let $\overline{\mathcal{F}}_{g,n}$ be a smooth projective model of $\mathcal{F}_{g,n}$ with g > 2. Then we have a natural isomorphism:

$$H^{0}(\bar{\mathcal{F}}_{g,n},\Omega^{k}) \simeq \begin{cases} 0 & 0 < k \le 9\\ M_{\wedge^{k},k}(\Gamma_{g}) & 10 \le k \le 18\\ 0 & k > 19, \ k \in 2\mathbb{Z} \\ S_{19+m}(\Gamma_{g},\det) \otimes \mathbb{C}S_{n,m} & k = 19+2m, \ 0 \le m \le n \end{cases}$$
(1.1)

Here Γ_g is the modular group for K3 surfaces of genus g, which is defined as the kernel of $O^+(L_g) \rightarrow O(L_g^{\vee}/L_g)$ where $L_g = 2U \oplus 2E_8 \oplus \langle 2 - 2g \rangle$ is the period lattice of K3 surfaces of genus g. In the second case, $M_{\wedge^k,k}(\Gamma_g)$ stands for the space of vector-valued modular forms of weight (\wedge^k, k) for Γ_g (see [4]). In the last case, $S_{19+m}(\Gamma_g, \text{det})$ stands for the space of scalar-valued cusp forms of weight 19 + m and determinant character for Γ_g , and $S_{n,m}$ stands for the right quotient $\mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$, which is a left \mathfrak{S}_n -set. Theorem 1.1 is actually formulated and proved in the generality of lattice-polarisation (Theorem 2.6).

In the case of the top degree k = 19 + 2n, namely for canonical forms, the isomorphism (1·1) is proved in [2]. Theorem 1·1 is the extension of this result to all degrees k < 19 + 2n. The spaces in the right-hand side of (1·1) can also be geometrically explained as follows. In the case $k \le 18$, $M_{\wedge^k,k}(\Gamma_g)$ is identified with the space of holomorphic k-forms on a smooth projective model of \mathcal{F}_g , pulled back by $\mathcal{F}_{g,n} \to \mathcal{F}_g$. In the case k = 19 + 2m, $S_{19+m}(\Gamma_g, \det)$ is identified with the space of canonical forms on $\overline{\mathcal{F}}_{g,m}$, and the tensor product $S_{19+m}(\Gamma_g, \det) \otimes \mathbb{C}S_{n,m}$ is the direct sum of pullback of such canonical forms by various projections $\mathcal{F}_{g,n} \to \mathcal{F}_{g,m}$. Therefore Theorem 1·1 can be understood as a kind of classification result which says that except for canonical forms, there are essentially no new differential forms on the tower $(\mathcal{F}_{g,n})_n$ of moduli spaces. In fact, this is how the proof proceeds.

The space $S_l(\Gamma_g, \det)$ is nonzero for every sufficiently large l, so the space $H^0(\bar{\mathcal{F}}_{g,n}, \Omega^k)$ for odd $k \ge 19$ is typically nonzero (at least when k is large). On the other hand, it is not clear at present whether $M_{\wedge^k,k}(\Gamma_g) \ne 0$ or not in the range $10 \le k \le 18$. This is a subject of study in the theory of vector-valued orthogonal modular forms.

The isomorphism (1·1) in the case k = 19 + 2m is an \mathfrak{S}_n -equivariant isomorphism, where \mathfrak{S}_n acts on $H^0(\bar{\mathcal{F}}_{g,n}, \Omega^k)$ by its permutation action on $\mathcal{F}_{g,n}$, while it acts on $S_{19+m}(\Gamma_g, \det) \otimes \mathbb{C}S_{n,m}$ by its natural left action on $S_{n,m}$. Therefore, taking the \mathfrak{S}_n -invariant part, we obtain the following simpler result for the unordered pointed moduli space $\mathcal{F}_{g,n}/\mathfrak{S}_n$, which is birationally a $K3^{[n]}$ -fibration over \mathcal{F}_g .

COROLLARY 1.2. Let $\overline{\mathcal{F}_{g,n}/\mathfrak{S}_n}$ be a smooth projective model of $\mathcal{F}_{g,n}/\mathfrak{S}_n$. Then we have a natural isomorphism:

$$H^{0}(\overline{\mathcal{F}_{g,n}}/\mathfrak{S}_{n},\Omega^{k}) \simeq \begin{cases} 0 & 0 < k \le 9\\ M_{\wedge^{k},k}(\Gamma_{g}) & 10 \le k \le 18\\ 0 & k > 19, \ k \in 2\mathbb{Z}\\ S_{19+m}(\Gamma_{g}, \det) & k = 19 + 2m, \ 0 \le m \le n \end{cases}$$

Differential forms on universal K3 surfaces

The universal K3 surface $\mathcal{F}_{g,1}$ is an analogue of elliptic modular surfaces ([6]), and the moduli spaces $\mathcal{F}_{g,n}$ for general *n* are analogues of the so-called Kuga varieties over modular curves ([7]). Starting with the case of elliptic modular surfaces [6], holomorphic differential forms on the Kuga varieties have been described in terms of elliptic modular forms: [7] for canonical forms, and [1] for the case of lower degrees (somewhat implicitly). Theorem 1.1 can be regarded as a K3 version of these results.

As a final remark, in view of the analogy between universal K3 surfaces and elliptic modular surfaces, invoking the classical fact that elliptic modular surfaces have maximal Picard number ([6]) now raises the question if $H^{k,0}(\bar{\mathcal{F}}_{g,n}) \oplus H^{0,k}(\bar{\mathcal{F}}_{g,n})$ is a sub \mathbb{Q} -Hodge structure of $H^k(\bar{\mathcal{F}}_{g,n}, \mathbb{C})$. This is independent of the choice of a smooth projective model $\bar{\mathcal{F}}_{g,n}$.

The rest of this paper is devoted to the proof of Theorem 1.1. In Section 2.1 we compute a part of the holomorphic Leray spectral sequence associated to a certain type of $K3^n$ -fibration. This is the main step of the proof. In Section 2.2 we study differential forms on a compactification of such a fibration. In Section 2.3 we deduce (a generalised version of) Theorem 1.1 by combining the result of Section 2.2 with some results from [2–5]. Sometimes we drop the subscript X from the notation Ω_X^k when the variety X is clear from the context.

2. Proof

2.1. Holomorphic Leray spectral sequence

Let $\pi : X \to B$ be a smooth family of *K*3 surfaces over a smooth connected base *B*. In this subsection *X* and *B* may be analytic. We put the following assumption:

Condition $2 \cdot 1$. In a neighbourhood of every point of B, the period map is an embedding.

This is equivalent to the condition that the differential of the period map

$$T_b B \rightarrow \operatorname{Hom}(H^{2,0}(X_b), H^{1,1}(X_b))$$

is injective for every $b \in B$, where X_b is the fiber of π over b.

For a natural number n > 0 we denote by $X_n = X \times_B \cdots \times_B X$ the *n*-fold fiber product of *X* over *B*, and let $\pi_n \colon X_n \to B$ be the projection. We denote by Ω_{π_n} the relative cotangent bundle of π_n , and $\Omega_{\pi_n}^p = \wedge^p \Omega_{\pi_n}$ for $p \ge 0$ as usual.

PROPOSITION 2.2. Let $\pi : X \to B$ be a K3 fibration satisfying Condition 2.1. Then we have a natural isomorphism:

$$(\pi_n)_*\Omega_{X_n}^k \simeq \begin{cases} \Omega_B^k & k \le \dim B\\ 0 & k > \dim B, \ k \ne \dim B \mod 2\\ K_B \otimes (\pi_n)_*\Omega_{\pi_n}^{2m} & k = \dim B + 2m, \ 0 \le m \le n \end{cases}$$

This assertion amounts to a partial degeneration of the holomorphic Leray spectral sequence. Recall ([8, section 5.2]) that $\Omega_{X_n}^k$ has the holomorphic Leray filtration $L^{\bullet}\Omega_{X_n}^k$ defined by

$$L^l\Omega^k_{X_n} = \pi_n^*\Omega^l_B \wedge \Omega^{k-l}_{X_n},$$

whose graded quotients are naturally isomorphic to

$$\operatorname{Gr}_{L}^{l}\Omega_{X_{n}}^{k} = \pi_{n}^{*}\Omega_{B}^{l}\otimes\Omega_{\pi_{n}}^{k-l}$$

This filtration induces the holomorphic Leray spectral sequence

$$(E_r^{l,q}, d_r) \Rightarrow E_{\infty}^{l+q} = R^{l+q} (\pi_n)_* \Omega_{X_n}^k$$

which converges to the filtration

$$L^{l}R^{l+q}(\pi_{n})_{*}\Omega^{k}_{X_{n}} = \operatorname{Im}(R^{l+q}(\pi_{n})_{*}L^{l}\Omega^{k}_{X_{n}} \to R^{l+q}(\pi_{n})_{*}\Omega^{k}_{X_{n}}).$$

By [8, proposition 5.9], the E_1 page coincides with the collection of the Koszul complexes associated to the variation of Hodge structures for π_n :

$$(E_1^{l,q}, d_1) = (\mathcal{H}^{k-l,l+q} \otimes \Omega_B^l, \bar{\nabla}).$$
(2.1)

Here $\mathcal{H}^{*,*}$ are the Hodge bundles associated to the fibration $\pi_n \colon X_n \to B$, and

$$\bar{\nabla}:\mathcal{H}^{*,*}\otimes\Omega_B^*\to\mathcal{H}^{*-1,*+1}\otimes\Omega_B^{*+1}$$

are the differentials in the Koszul complexes (see [8, section $5 \cdot 1 \cdot 3$]). For degree reasons, the range of (l, q) in the E_1 page satisfies the inequalities

$$0 \le l \le \dim B$$
, $0 \le k - l \le 2n$, $0 \le l + q \le 2n$.

The first two can be unified:

$$\max(0, k - 2n) \le l \le \min(\dim B, k), \quad 0 \le l + q \le 2n.$$

$$(2.2)$$

We calculate the E_1 to E_2 pages on the edge line l + q = 0.

LEMMA 2.3. The following holds:

- (1) $E_1^{l,-l} = 0$ when $l \le \min(\dim B, k)$ with $l \ne k \mod 2$;
- (2) $E_2^{l,-l} = 0$ when $l < \min(\dim B, k)$;
- (3) For $l_0 = \min(\dim B, k)$ we have $E_1^{l_0, -l_0} = E_2^{l_0, -l_0} = \dots = E_{\infty}^{l_0, -l_0}$.

Proof. By (2.1), we have $E_1^{l,-l} = \mathcal{H}^{k-l,0} \otimes \Omega_B^l$. By the Künneth formula, the fiber of $\mathcal{H}^{k-l,0}$ over a point $b \in B$ is identified with

$$H^{k-l,0}(X_b^n) = \bigoplus_{(p_1,\cdots,p_n)} H^{p_1,0}(X_b) \otimes \cdots \otimes H^{p_n,0}(X_b),$$
(2.3)

where (p_1, \dots, p_n) ranges over all indices with $\sum_i p_i = k - l$ and $0 \le p_i \le 2$.

(1) When k - l is odd, every index (p_1, \dots, p_n) in (2·3) must contain a component $p_i = 1$. Since $H^{1,0}(X_b) = 0$, we see that $H^{k-l,0}(X_b^n) = 0$. Therefore $\mathcal{H}^{k-l,0} = 0$ when k - l is odd.

(3) Let $l_0 = \min(\dim B, k)$. By the range (2.2) of (l, q), we see that for every $r \ge 1$ the source of d_r that hits $E_r^{l_0, -l_0}$ is zero, and the target of d_r that starts from $E_r^{l_0, -l_0}$ is also zero. This proves our assertion.

(2) Let $l < \min(\dim B, k)$. In view of (1), we may assume that l = k - 2m for some m > 0. By (2·2), the source of d_1 that hits $E_1^{l,-l}$ is zero. We shall show that $d_1 : E_1^{l,-l} \to E_1^{l+1,-l}$ is injective. By (2.1), this morphism is identified with

$$\bar{\nabla}:\mathcal{H}^{2m,0}\otimes\Omega_B^l\to\mathcal{H}^{2m-1,1}\otimes\Omega_B^{l+1}.$$
(2.4)

By the Künneth formula as in (2.3), the fibers of the Hodge bundles $\mathcal{H}^{2m,0}$, $\mathcal{H}^{2m-1,1}$ over $b \in B$ are respectively identified with

$$H^{2m,0}(X_b^n) = \bigoplus_{|\sigma|=m} H^{2,0}(X_b)^{\otimes \sigma}, \qquad (2.5)$$

$$H^{2m-1,1}(X_b^n) = \bigoplus_{\substack{|\sigma'|=m-1 \ i \notin \sigma'}} \bigoplus_{i \notin \sigma'} H^{2,0}(X_b)^{\otimes \sigma'} \otimes H^{1,1}(X_b)$$

$$= \bigoplus_{\substack{|\sigma|=m}} \bigoplus_{i \in \sigma} H^{2,0}(X_b)^{\otimes \sigma-\{i\}} \otimes H^{1,1}(X_b).$$
(2.6)

In (2.5), σ ranges over all subsets of $\{1, \dots, n\}$ consisting of *m* elements, and $H^{2,0}(X_b)^{\otimes \sigma}$ stands for the tensor product of $H^{2,0}(X_b)$ for the *j*th factors X_b of X_b^n over all $j \in \sigma$. The notations σ', σ in (2.6) are similar, and $H^{1,1}(X_b)$ in (2.6) is the $H^{1,1}$ of the *i*th factor X_b of X_b^n .

Let us write $V = H^{2,0}(X_b)$ and $W = (T_b B)^{\vee}$ for simplicity. The homomorphism (2.4) over $b \in B$ is written as

$$\bigoplus_{|\sigma|=m} \left(V^{\otimes \sigma} \otimes \wedge^{l} W \to \bigoplus_{i \in \sigma} V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_{b}) \otimes \wedge^{l+1} W \right).$$
(2.7)

By [8, lemma 5.8], the (σ, i) -component

$$V^{\otimes \sigma} \otimes \wedge^{l} W \to V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W$$
(2.8)

factorises as

$$V^{\otimes \sigma} \otimes \wedge^{l} W \to V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b) \otimes W \otimes \wedge^{l} W$$
$$\to V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_b) \otimes \wedge^{l+1} W,$$

where the first map is induced by the adjunction $V \to H^{1,1}(X_b) \otimes W$ of the differential of the period map for the *i*th factor X_b , and the second map is induced by the wedge product $W \otimes \wedge^l W \to \wedge^{l+1} W$. By linear algebra, this composition can also be decomposed as

$$V^{\otimes \sigma} \otimes \wedge^{l} W \to V^{\otimes \sigma - \{i\}} \otimes V \otimes W^{\vee} \otimes \wedge^{l+1} W$$

$$\to V^{\otimes \sigma - \{i\}} \otimes H^{1,1}(X_{b}) \otimes \wedge^{l+1} W,$$
(2.9)

where the first map is induced by the adjunction $\wedge^{l}W \to W^{\vee} \otimes \wedge^{l+1}W$ of the wedge product, and the second map is induced by the adjunction $V \otimes W^{\vee} \to H^{1,1}(X_b)$ of the differential of the period map. By our initial Condition 2.1, the second map of (2.9) is injective. Moreover, since $l + 1 \leq \dim W$ by our assumption, the wedge product $\wedge^{l}W \times W \to \wedge^{l+1}W$ is nondegenerate, so its adjunction $\wedge^{l}W \to W^{\vee} \otimes \wedge^{l+1}W$ is injective. Thus the first map of (2.9) is also injective. It follows that (2.8) is injective. Since the map (2.7) is the direct sum of its (σ, i) -components, it is injective. This finishes the proof of Lemma 2.3.

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We can now complete the proof of Proposition $2 \cdot 2$.

Proof of Proposition 2.2. By Lemma 2.3 (2), we have $E_{\infty}^{l,-l} = 0$ when $l < l_0 = \min(\dim B, k)$. Together with Lemma 2.3 (3), we obtain

$$(\pi_n)_* \Omega_{X_n}^k = E_\infty^0 = E_\infty^{l_0, -l_0} = E_1^{l_0, -l_0}.$$

When $k \leq \dim B$, we have $l_0 = k$, and $E_1^{l_0, -l_0} = \Omega_B^k$ by (2.1). When $k > \dim B$, we have $l_0 = \dim B$, and $E_1^{l_0, -l_0} = \mathcal{H}^{k-\dim B, 0} \otimes K_B$ by (2.1). When $k - \dim B$ is odd, this vanishes by Lemma 2.3 (1).

In the case $k = \dim B + 2m$, the vector bundle $\mathcal{H}^{2m,0} \otimes K_B = (\pi_n)_* \Omega_{\pi_n}^{2m} \otimes K_B$ can be written more specifically as follows. For a subset σ of $\{1, \dots, n\}$ with cardinality $|\sigma| = m$, we denote by $X_{\sigma} \simeq X_m$ the fiber product of the *i*th factors $X \to B$ of $X_n \to B$ over all $i \in \sigma$. We denote by

$$X_n \xrightarrow{\pi_\sigma} X_\sigma \xrightarrow{\pi^\sigma} B$$

the natural projections. The Künneth formula (2.5) says that

$$(\pi_n)_*\Omega^{2m}_{\pi_n}\simeq \bigoplus_{|\sigma|=m}\pi^{\sigma}_*K_{\pi^{\sigma}}.$$

Combining this with the isomorphism

$$\pi^{\sigma}_* K_{X_{\sigma}} \simeq K_B \otimes \pi^{\sigma}_* K_{\pi^{\sigma}} \tag{2.10}$$

for each X_{σ} , we can rewrite the isomorphism in the last case of Proposition 2.2 as

$$(\pi_n)_*\Omega_{X_n}^{\dim B+2m} \simeq \bigoplus_{|\sigma|=m} \pi_*^{\sigma} K_{X_{\sigma}}.$$
(2.11)

2.2. Extension over compactification

Let $\pi: X \to B$ be a K3 fibration as in Section 2.1. We now assume that X, B are quasi-projective and π is a morphism of algebraic varieties. We take smooth projective compactifications of X_n, X_σ, B and denote them by $\bar{X}_n, \bar{X}_\sigma, \bar{B}$ respectively.

PROPOSITION 2.4. We have

$$H^{0}(\bar{X}_{n}, \Omega^{k}) \simeq \begin{cases} H^{0}(\bar{B}, \Omega^{k}) & k \leq \dim B \\ 0 & k > \dim B, \ k \neq \dim B \mod 2 \\ \oplus_{\sigma} H^{0}(\bar{X}_{\sigma}, K_{\bar{X}_{\sigma}}) & k = \dim B + 2m, \ 0 \leq m \leq n \end{cases}$$

In the last case, σ ranges over all subsets of $\{1, \dots, n\}$ with $|\sigma| = m$. The isomorphism in the first case is given by the pullback by $\pi_n \colon X_n \to B$, and the isomorphism in the last case is given by the direct sum of the pullbacks by $\pi_\sigma \colon X_n \to X_\sigma$ for all σ .

Proof. The assertion in the case $k > \dim B$ with $k \neq \dim B \mod 2$ follows directly from the second case of Proposition 2.2. Next we consider the case $k \leq \dim B$. We may assume that $\pi_n: X_n \to B$ extends to a surjective morphism $\bar{X}_n \to \bar{B}$. Let ω be a holomorphic *k*-form on \bar{X}_n . By the first case of Proposition 2.2, we have $\omega|_{X_n} = \pi_n^* \omega_B$ for a holomorphic

k-form ω_B on *B*. Since ω is holomorphic over \bar{X}_n , ω_B is holomorphic over \bar{B} as well by a standard property of holomorphic differential forms. (Otherwise ω must have pole at the divisors of \bar{X}_n dominating the divisors of \bar{B} where ω_B has pole.) Therefore the pullback $H^0(\bar{B}, \Omega^k) \to H^0(\bar{X}_n, \Omega^k)$ is surjective.

Finally, we consider the case $k = \dim B + 2m$, $0 \le m \le n$. Let ω be a holomorphic k-form on \bar{X}_n . By (2.11), we can uniquely write $\omega|_{X_n} = \sum_{\sigma} \pi_{\sigma}^* \omega_{\sigma}$ for some canonical forms ω_{σ} on X_{σ} .

Claim 2.5. For each σ , ω_{σ} is holomorphic over \bar{X}_{σ} .

Proof. We identify X_n with the fiber product $X_{\sigma} \times_B X_{\tau}$ where $\tau = \{1, \dots, n\} - \sigma$ is the complement of σ . We may assume that this fiber product diagram extends to a commutative diagram of surjective morphisms



between smooth projective models. We take an irreducible subvariety $\tilde{B} \subset \bar{X}_{\tau}$ such that $\tilde{B} \to \bar{B}$ is surjective and generically finite. Then $\pi_{\tau}^{-1}(\tilde{B}) \subset \bar{X}_n$ has a unique irreducible component dominating \tilde{B} . We take its desingularisation and denote it by Y. By construction $\pi_{\sigma}|_Y: Y \to \bar{X}_{\sigma}$ is dominant (and so surjective) and generically finite. On the other hand, for any $\sigma' \neq \sigma$ with $|\sigma'| = m$, the projection $\pi_{\sigma'}|_Y: Y \to X_{\sigma'}$ is not dominant. Indeed, such σ' contains at least one component $i \in \tau$, so if $Y \to X_{\sigma'}$ was dominant, then the *i*th projection $Y \to X$ would be also dominant, which is absurd because it factorises as $Y \to \tilde{B} \subset \bar{X}_{\tau} \to X$.

We pullback the differential form $\omega = \pi_{\sigma}^* \omega_{\sigma} + \sum_{\sigma' \neq \sigma} \pi_{\sigma'}^* \omega_{\sigma'}$ to *Y* and denote it by $\omega|_Y$. Since ω is holomorphic over \bar{X}_n , $\omega|_Y$ is holomorphic over *Y*. Since $\pi_{\sigma'}^* \omega_{\sigma'}|_Y$ is the pullback of the canonical form $\omega_{\sigma'}$ on $X_{\sigma'}$ by the non-dominant map $Y \to X_{\sigma'}$, it vanishes identically. Hence $\pi_{\sigma}^* \omega_{\sigma}|_Y = \omega|_Y$ is holomorphic over *Y*. Since $\pi_{\sigma}|_Y : Y \to \bar{X}_{\sigma}$ is surjective, this implies that ω_{σ} is holomorphic over \bar{X}_{σ} as before.

The above argument will be clear if we consider over the generic point η of B: we restrict ω to the fiber of $(X_{\eta})^n \to (X_{\eta})^{\tau}$ over the geometric point \tilde{B} of $(X_{\eta})^{\tau}$ over η .

By Claim 2.5, the pullback

$$(\pi_{\sigma}^*)_{\sigma} \colon \bigoplus_{|\sigma|=m} H^0(\bar{X}_{\sigma}, K_{\bar{X}_{\sigma}}) \to H^0(\bar{X}_n, \Omega^{\dim B+2m})$$

is surjective. It is also injective as implied by $(2 \cdot 11)$. This proves Proposition $2 \cdot 4$.

2.3. Universal K3 surface.

Now we prove Theorem 1.1, in the generality of lattice-polarisation. Let *L* be an even lattice of signature (2, *d*) which can be embedded as a primitive sublattice of the *K*3 lattice $3U \oplus 2E_8$. We denote by

$$\mathcal{D} = \{ \mathbb{C}\omega \in \mathbb{P}L_{\mathbb{C}} \mid (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0 \}^+$$

the Hermitian symmetric domain associated to L, where + means a connected component.

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Let $\pi: X \to B$ be a smooth projective family of K3 surfaces over a smooth quasiprojective connected base B. We say ([3]) that the family $\pi: X \to B$ is *lattice-polarised* with period lattice L if there exists a sub local system Λ of $R^2\pi_*\mathbb{Z}$ such that each fiber Λ_b is a hyperbolic sublattice of the Néron-Severi lattice $NS(X_b)$ and the fibers of the orthogonal complement Λ^{\perp} are isometric to L. Then we have a period map

$$\mathcal{P}: B \to \Gamma \backslash \mathcal{D}$$

for some finite-index subgroup Γ of $O^+(L)$. By Borel's extension theorem, \mathcal{P} is a morphism of algebraic varieties.

Let us put the assumption

$$\mathcal{P}$$
 is birational and $-\operatorname{id} \notin \Gamma$. (2.12)

For such a family $\pi: X \to B$, if we shrink *B* as necessary, then \mathcal{P} is an open immersion and Condition 2.1 is satisfied. For example, the universal *K*3 surface $\mathcal{F}_{g,1} \to \mathcal{F}_g$ for g > 2restricted over a Zariski open set of \mathcal{F}_g satisfies this assumption with $L = L_g$ and $\Gamma = \Gamma_g$ (see Section 1 for these notations).

As in Section 1, we denote by $M_{\wedge^k,k}(\Gamma)$ the space of vector-valued modular forms of weight (\wedge^k, k) for Γ , $S_l(\Gamma, \det)$ the space of scalar-valued cusp forms of weight l and character det for Γ , and $S_{n,m} = \mathfrak{S}_n/(\mathfrak{S}_m \times \mathfrak{S}_{n-m})$.

THEOREM 2.6. Let $\pi: X \to B$ be a lattice-polarised K3 family with period lattice L of signature (2, d) with $d \ge 3$ and monodromy group Γ satisfying (2.12). Then we have an \mathfrak{S}_n -equivariant isomorphism

$$H^{0}(\bar{X}_{n}, \Omega^{k}) \simeq \begin{cases} 0 & 0 < k < d/2 \\ M_{\wedge^{k}, k}(\Gamma) & d/2 \le k < d \\ 0 & k > d, \ k - d \notin 2\mathbb{Z} \\ S_{d+m}(\Gamma, \det) \otimes \mathbb{C}S_{n,m} & k = d + 2m, \ 0 \le m \le n \end{cases}$$

Proof. When $k \leq d$, we have $H^0(\bar{X}_n, \Omega^k) \simeq H^0(\bar{B}, \Omega^k)$ by Proposition 2.4. Then \bar{B} is a smooth projective model of the modular variety $\Gamma \setminus D$. By a theorem of Pommerening [5], the space $H^0(\bar{B}, \Omega^k)$ for k < d is isomorphic to the space of Γ -invariant holomorphic k-forms on D, which in turn is identified with the space $M_{\wedge^k,k}(\Gamma)$ of vector-valued modular forms of weight (\wedge^k, k) for Γ (see [4]). The vanishing of this space in 0 < k < d/2 is proved in [4, theorem 1.2] in the case when L has Witt index 2, and in [4, theorem 1.5 (1)] in the case when L has Witt index ≤ 1 .

The vanishing in the case k > d with $k \neq d \mod 2$ follows from Proposition 2.4. Finally, we consider the case k = d + 2m, $0 \le m \le n$. By Proposition 2.4, we have a natural \mathfrak{S}_n -equivariant isomorphism

$$H^0(ar{X}_n, \Omega^{d+2m}) \simeq igoplus_{|\sigma|=m} H^0(ar{X}_\sigma, K_{ar{X}_\sigma}),$$

where \mathfrak{S}_n permutes the subsets σ of $\{1, \dots, n\}$. Here note that the stabiliser of each σ acts on $H^0(\bar{X}_{\sigma}, K_{\bar{X}_{\sigma}})$ trivially by (2.10). Therefore, as an \mathfrak{S}_n -representation, the right-hand side can be written as

$$H^0(\bar{X}_m, K_{\bar{X}_m}) \otimes \left(\bigoplus_{|\sigma|=m} \mathbb{C}\sigma\right) \simeq H^0(\bar{X}_m, K_{\bar{X}_m}) \otimes \mathbb{C}S_{n,m}.$$

Finally, we have $H^0(\bar{X}_m, K_{\bar{X}_m}) \simeq S_{d+m}(\Gamma, \det)$ by [3, theorem 3.1].

Remark 2.7. The case $k \ge d$ of Theorem 2.6 holds also when d = 1, 2. We put the assumption $d \ge 3$ for the requirement of the Koecher principle from [5]. Therefore, in fact, only the case (d, k) = (2, 1) with Witt index 2 is not covered.

REFERENCES

- B. B. GORDON. Algebraic cycles and the Hodge structure of a Kuga fiber variety. *Trans. Amer. Math. Soc.* 336(2) (1993), 933–947.
- [2] S. MA. Mukai models and Borcherds products. To appear in Amer. J. Math. ArXiv:1909.03946.
- [3] S. MA. Kodaira dimension of universal holomorphic symplectic varieties. J. Inst. Math. Jussieu 21(6) (2022), 1849–1866.
- [4] S. MA. Vector-valued orthogonal modular forms. To appear in *Mem. Eur. Math. Soc.* ArXiv:2209.10135.
- [5] K. POMMERENING. Die Fortsetzbarkeit von Differentialformen auf arithmetischen Quotienten von hermiteschen symmetrischen Räumen. J. Reine Angew. Math. 356 (1985), 194–220.
- [6] T. SHIODA. On elliptic modular surfaces. J. Math. Soc. Japan 24 (1972), 20-59.
- [7] V. V. SHOKUROV. Holomorphic differential forms of higher degree on Kuga's modular varieties. *Math.* USSR-Sb. 30(1) (1976), 119–142.
- [8] C. VOISIN. Hodge Theory and Complex Algebraic Geometry II. (Cambridge University Press, 2003).