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WHY IS OPTIMAL GROWTH THEORY MUTE? RESTORING ITS RIGHTFUL VOICE

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Optimal growth theory as it stands today does not work. Using strictly concave utility functions systematically inflicts on the economy distortions that are either historically unobserved or unacceptable by society. Moreover, we show that the traditional approach is incompatible with competitive equilibrium: Any economy initially in such equilibrium will always veer away into unwanted trajectories if its investment is planned using a concave utility function. We then propose a rule for the optimal savings-investment rate based on competitive equilibrium that simultaneously generates three intertemporal optima for society. The rule always leads to reasonable time paths for all central economic variables, even under very different hypotheses about the future evolution of population and technical progress.

Keywords: Optimal Savings Rate, Calculus of Variations, Optimal Control

1. INTRODUCTION

This paper will argue that optimal growth theory, as it has been developed, has never been able to come up with a reasonable answer to the problem of determining how much a nation should save. We will show that the traditional approach, based on the systematic use of strictly concave utility functions, never delivered, and when the bold step of modifying the utility function to obtain a reasonable answer was taken, it unfailingly led to nonsensical values for other variables of central

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importance such as the growth rate of real income per person, the marginal product of capital, or the capital–output ratio.

Our profession should have taken note of those inadequacies long ago. They had been met already by the very originator of the theory, Ramsey (1928), who tried to put numbers on the theory, and whose disappointment when obtaining an "optimal" savings rate of 60% is almost palpable. Thirty years later, Goodwin (1961) obtained even worse results in all models he considered—but contrary to Ramsey, he set out to defend them in a dumbfounding way. Finally, King and Rebelo (1993) convincingly showed that it was an impossible task to replicate the observed development of an economy by assuming some form of the traditional model. They tried to modify in many ways not only the parameters of the models they were using, but eventually the very nature of the latter—to no avail. Their conclusion was unequivocal.

The central result of this paper is two-fold: First, we demonstrate that the concavity of the utility functions precludes any possibility of a sustained competitive equilibrium and, second, any economy initially in such equilibrium will always veer off from that situation into unwanted trajectories if it is governed by the standard model. We then propose the following solution to the problem of optimal growth: Optimal trajectories of the economy, and, first and foremost, the optimal savings rate, should be determined by the Euler equation resulting from competitive equilibrium. By saving and investing along the lines defined by such an equilibrium, society is able to reach simultaneously the following intertemporal optima, in addition to the minimization of production costs:

- maximization over an infinite time span of the sum of discounted consumption (welfare) flows,
- maximization of the total value of society's activity defined as the sum of consumption flows and the rate of increase of the value of capital,
- maximization of the total remuneration of labor.

This is demonstrated in Theorem 1. In addition, we show that for all parameters in the range of observed or predictable values, as well as for quite different hypotheses regarding the future evolution of population or technical progress, we are always led to very reasonable time paths for all central variables of the economy. We would even qualify those time paths as "welcome," since we show that if we approach a situation of competitive equilibrium, the capital–output ratio will decrease and the share of labor in national income will increase, thus offering an appreciable alternative to current, gloomy predictions.

We will proceed as follows. In Section 2, we review evidence of the nonapplicability of the traditional approach, from the Ramsey model to the King and Rebelo (1993) piece. Prompted by the latter study, in Section 3 of this paper, we extend our analysis to the implied initial growth rate as well as to the limiting value of the marginal productivity of capital, and show that as soon as adjustments are made to the utility function to obtain a reasonable initial savings rate, historically unobserved or unwanted values appear.

We then demonstrate (Section 4) that competitive equilibrium is unsustainable in the traditional model. We suppose that initially the economy is in a situation of competitive equilibrium and that from that point onward it can follow any of the following two possible kinds of paths:

(I) Investment is planned in such a way as to maximize intertemporally discounted utility flows, the utility function being the widely used affine transform of a strictly concave power function; this is the traditional approach.
(II) Investment is made in such a way as to conform to the Euler equation defining competitive equilibrium. This will be our suggested solution to the basic problem of optimal economic growth.

We will show that in the first scenario, although central variables have normal, historically observed *initial* values, *in all cases* their time paths run astray, and we explain analytically this behavior.

Section 5 provides our solution: We show that scenario II, while securing the intertemporal optima for society we mentioned earlier, *always* yields reasonable results for the following fundamental variables: the optimal savings rate, the implied growth rate of income per person, and the capital–output ratio; in addition, it secures the most welcome feature of an increasing share of the remuneration of labor in total income.

In Section 6, we take the natural step of checking the robustness of these results not only to changes in the values of the parameters of the model, but also to very different evolutions of population and technical progress. Indeed, we hold that a model for medium- and long-run horizons should take into account, in particular, the quasi-certainty of a non-exponential, *S*-shaped, evolution of population. We will show that despite significantly different hypotheses, the time paths of the central variables just mentioned remain within very reasonable, predictable ranges, thus conferring a welcome robustness to the model.

2. THREE ESSAYS THAT SHOULD HAVE BEEN ALARM BELLS: RAMSEY (1928), GOODWIN (1961), AND KING AND REBELO (1993)

2.1. Ramsey: The First Difficulties

The reader of Ramsey's beautifully written, highly original essay "A mathematical theory of saving" can almost feel the disappointment of its author when he wrote, "The rate of saving which the rule requires [60%] is greatly in excess of that which anyone would suggest," adding that the utility function he used was "put forward merely as an illustration" (pp. 548–549). Ramsey had tested his model just at one point of his utility function. In the full version of our paper, available in the supplementary material available online (see Supplementary Material section at end of article), we show that Ramsey would have obtained even worse results if he had calculated the optimal savings rate over the whole interval of the utility function he had considered; 60% was the minimum value over that interval; it reached 80% when consumption was 50% higher. We also showed that for his

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model to yield an acceptable savings rate, one should consider utility functions that nobody would ever imagine—and even less a whole society, as we will see. Indeed, the optimal savings rate s^* implied by the model can be determined as

$$s^* = \left[1 + \frac{CU'(C)}{B - U(C)}\right]^{-1}, \tag{1}$$

where B denotes the bliss level defined by Ramsey. Integrating this differential equation results in the utility function

$$U(C) = \kappa C^{1-1/s^*} + B, \tag{2}$$

where κ, the constant of integration, can be identified with any point in (C, U) space, for instance, any point chosen by Ramsey. We get

$$U(C) = (U_1 - B)\left(\frac{C}{C_1}\right)^{1-1/s^*} + B. \tag{3}$$

Using Ramsey's first point $(C_1, U_1) = (150, 2)$ as well as his $B = 8$ value, and setting $s^* = 0.1$, the resulting function $U(C) = (U_1 - 8)\,(C/150)^{-9} + 8$ is the only utility function going through (150, 2) and yielding, under the Ramsey rule, a constant optimal savings rate equal to 10%. The bad news is that this function makes no sense at all. To paraphrase Ramsey, its extreme properties are also "greatly in excess of that which anyone would suggest." Indeed, the curve is close to a vertical, almost immediately followed by a horizontal; the bliss level is practically attained at $C = 300$ already $[U(300) = 7.99]$ (see Figure 1). The marginal utility is $U'(C) = 0.36(C/150)^{-10}$; this implies that multiplying C by a factor λ divides the marginal utility by a factor λ^{10}. An example illustrates the oddity of such a construct. Consider any country whose real income per person, over a very long time span, was multiplied by $\lambda = 10^{9/10} \approx 7.943$. Thanks to the work of Johnston and Williamson (2013), we can estimate that such an increase took about 115 years to be achieved in the United States (on a time frame ending in 2012) and 150 years in the United Kingdom. Applying the above-mentioned utility function would mean that at the beginning of the 20th century the marginal utility of consumption in the United States was by $\lambda^{10} = 10^{(9/10)10} =$ *one billion* times higher than it is today, certainly an indefensible proposition.

One may think that choosing a larger optimal savings rate might improve the situation. That is not the case: A 20% savings rate entails a utility curve hardly distinguishable from the preceding one. In the same figure, we have also depicted the curve corresponding to the constant rate $s^* = 60\%$. It can be seen that Ramsey chose a utility function that seemed reasonable to him (and probably to most of his readers) which was very close to a function implying an optimal savings level equal to 60% at *all* its points.

We might also attribute the observed antinomy between what appears as a reasonable utility function and a reasonable optimal savings rate to the very model

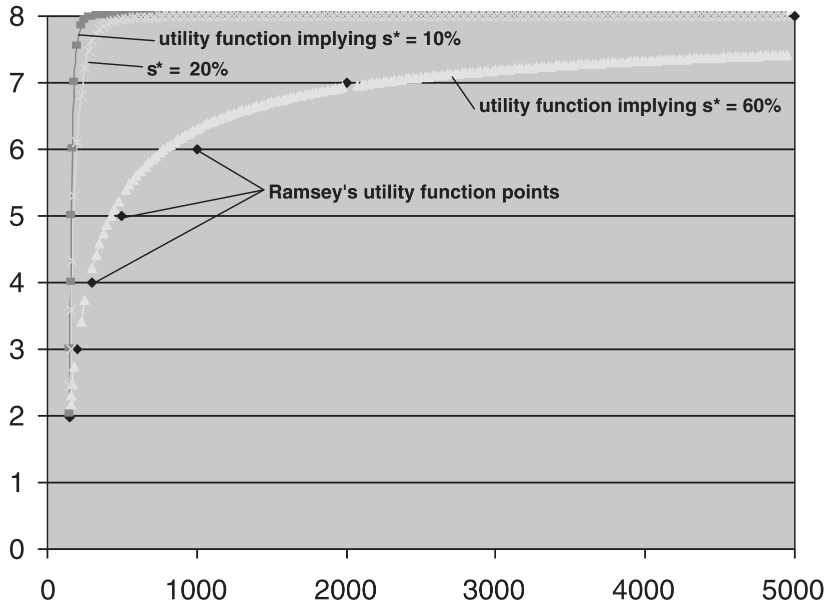

FIGURE 1. The dots depict Ramsey's utility function; they are close to a utility curve entailing a constant savings rate equal to 60% at all its points. For the savings rate to be equal to 10%, the utility function should correspond to the upper curve. The $s^* = 20\%$ curve is hardly distinguishable from the $s^* = 10\%$ line.

that Ramsey put forward (in which, for instance, the future utility flows are not discounted). This is not the case either. We will show that, time and again, for whatever model we might consider, not only such bland opposition is maintained, but it extends to unrealistic values of other variables of fundamental importance such as the marginal productivity of capital, the growth rate of income per person, or the capital–output ratio.

2.2. The Second Warning Bell: Goodwin (1961)

Context and results. The problem of the optimal savings rate came again to the forefront with the paper by Richard Goodwin entitled "The optimal growth path for an underdeveloped economy" (1961). Before describing Goodwin's models and results, it may be useful to consider the times at which the author was writing. Although his paper was published in 1961, its substance was originally presented to the Oxford–London–Cambridge Seminar on November 10, 1956. Those years were marked by the widely shared belief, even in countries like the United Kingdom or France, that planning was the answer to all possible economic woes, from shortages to inflation and unemployment. We therefore should hardly be surprised when Goodwin boldly wrote, "The planners may determine the

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marginal utility curve in any way or may accept any sort of directive about it" (p. 763)—a statement that would seem quite extraordinary today, to say the least, but that explains the reaction he would have when confronted by his results.

The author used three types of utility functions: The first was derived numerically through the UK marginal income-tax schedule, 1953–1954, for a married couple with two children, the second was $\ln(C - \bar{C})$, where $\bar{C}$ is a subsistence level, and the third was $\left[(C - \bar{C})^{1-\epsilon} - 1\right]/(1 - \epsilon)$. Production was supposed to be a linear function of capital.

His results should have been startling for anybody, including the author himself. In model I (corresponding to the first utility function), the optimal savings rate grew to 62% after 28 years, with an implied marginal savings rate of 79% at year 20. In model II, the optimal savings rate was 59% at year 24, with a marginal savings rate equal to 68% at year 12. Model III (where Goodwin chose $\epsilon = 0.2$) was even more disastrous, leading to an optimal savings rate equal to 83% at year 36, and marginal rates of at least 95% between years 28 and 32.

Goodwin's reaction. Contrary to Ramsey's natural reaction to such excessive savings rates, Goodwin found those numbers perfectly justifiable. Already after getting model I results, he explained them by the gains of productivity that might be bestowed onto future generations; those gains would be so big that they would justify huge sacrifices made by present generations; in his own words, "So great are the gains that we are fully justified in robbing the poor to give to the rich!" (p. 765). With such a conviction, it is not surprising that when all results of his three models were in, Goodwin wrote, "Some violent process of capital accumulation of the type illustrated is the ideal. The simplifications of the model give an unduly sharp outline of the ideal policy, but its general character is surely a sound guide to policy" (pp. 772–773).

It is difficult to gauge what the general feeling of the profession has been after the publication of those strong statements, but no doubt some members must have had serious reservations. It seems appropriate to mention that in a conference given in 2006, Robert Solow said that he vividly remembered having read Goodwin's paper just before or just after its publication, and to have been "very worried" about its excessive optimal savings rates.

2.3. The Paper That Should Have Been the Final Alarm Bell: King and Rebelo (1993)

Twenty years ago, King and Rebelo published an important, illuminating study on the transition paths for a neoclassical economy with intertemporally optimizing households. Basically, they worked with three utility functions: (i) $\log C$, (ii) a transform of the log function of the Stone–Geary type, and (iii) $(-1/9)(C^{-9} - 1)$. The production function was of the Cobb–Douglas type, with a one-third capital share and labor-augmenting progress. In a second part of their paper, they also considered a constant elasticity of substitution (CES) production function with an elasticity of substitution between 0.9 and 1.25, and finally introduced a large array of variants to the basic model. The authors' conclusion is unambiguous: "In exploring some plausible alterations of the basic model, we found that it was impossible to explain important components of economic growth in terms of transition dynamics without introducing some related implication that strongly contradicted historical experience" (p. 929).

3. THE ILL-FATED ROLE OF UTILITY FUNCTIONS

[Please see this section online, where it is shown that any power function of the type $U = (C^{\alpha} - 1)/\alpha$, with $-8.8 < \alpha < 0.8$, leads to at least one central variable displaying an unacceptable evolution; as to the negative exponential utility function $U = (-1/\beta)e^{-\beta C}$, $\beta > 0$, sometimes declared fit for service, it does not allow an equilibrium point any more.]

4. HOW THE STRICT CONCAVITY OF UTILITY FUNCTIONS MAKES COMPETITIVE EQUILIBRIUM UNSUSTAINABLE

We will now show that the traditional approach, in its attempt to optimize the evolution of an economy by positing a strictly concave utility function, is simply incompatible with competitive equilibrium. To do so, we will assume that an economy is initially in a state of competitive equilibrium. We will then suppose that *two different courses* can be pursued:

(I) Investment is planned in such a way as to maximize intertemporally discounted utility flows, the utility function being the widely used affine transform of a strictly concave power function; this is the traditional approach.

(II) Investment is made in such a way as to conform to the Euler equation defining competitive equilibrium. This will be our suggested solution to the basic problem of optimal economic growth.

We will also widen our hypothesis regarding the structure of the production process by allowing, in both scenarios, technical progress to be not only labor-augmenting, but also capital-augmenting. In the traditional literature on the neoclassical model, only labor-augmenting technical progress is allowed, apparently for the following reason: That restricting hypothesis is considered necessary for the growth rate of income per person to converge asymptotically toward the rate of labor-augmenting progress, the only exception applying in the Cobb–Douglas case. We have recently shown this assumption to be wrong by demonstrating a new property of general means of order p when p is negative—precisely the case where $0 < \sigma < 1$ [La Grandville (2011)] and we will check that, indeed, in both scenarios I and II the growth rate of income per person does converge toward the rate of labor-augmenting progress, although progress is capital-enhancing as well.

In each of scenarios I and II, we will depict the evolution of the economy represented by the following variables: the optimal savings rate, the growth rate of income per person, the marginal product of capital, and the capital–output ratio.

4.1. Initial Conditions as Determined from Optimal Time Paths

We suppose that at the initial time competitive equilibrium prevails in the economy. This implies that the capital stock is in such an amount that its marginal productivity is equal to the rate of interest. Total output (*net* of depreciation), denoted by Y_t, is given by a production function of CES form featuring both labor- and capital-augmenting technical progress; to this aim, we define factor-enhancing functions of time, G_t and H_t, such that their growth rates $\dot{G}_t/G \equiv g(t)$ and $\dot{H}_t/H_t \equiv h(t)$, respectively, are positive; G_0 and H_0 are normalized to 1. Labor is an exogenous increasing function of time L_t, with $L_0 = 1$. In a first step, we will consider that the functions G_t, H_t, and L_t are the exponentials $G_t = e^{gt}$, $H_t = e^{ht}$, and $L_t = e^{nt}$, respectively; in Section 6, to test the robustness of the model we suggest, we will suppose that those exponentials are replaced by S-shaped functions tending, with $t \to \infty$, toward horizontal asymptotes. The production function is the general mean of order p of the enhanced inputs $G_t K_t/K_0$ and $H_t L_t/L_0$:

$$Y_t = F(G_t K_t, H_t L_t) = Y_0\{\delta[G_t K_t/K_0]^p + (1-\delta)[H_t L_t/L_0]^p\}^{1/p}\ , \ \ p \neq 0, \quad \textbf{(4)}$$

where the order p is an increasing function of the elasticity of substitution σ: $p = 1 - 1/\sigma$. Note that p will always be negative because σ is supposed to be in the range where it has most often been observed, i.e., between 0 and 1. However, for comparison purposes, we will also give results corresponding to the $p = 0,\ \sigma = 1$ Cobb–Douglas case

$$Y_t = Y_0(G_t K_t/K_0)^\delta (H_t L_t/L_0)^{1-\delta}. \quad \textbf{(5)}$$

In the case $0 < \sigma < 1$, the fundamental competitive equilibrium equality $F_{K_t} = i$ leads to the following equation in K_t:

$$F_{K_t}(G_t K_t, H_t L_t) = Y_0\{\delta[G_t K_t/K_0]^p + (1-\delta)[H_t L_t/L_0]^p\}^{(1/p)\ -1}$$
$$.\,\delta {K_t}^{p-1}(G_t/K_0)^p = i, \quad p < 0,\ 0 < \sigma < 1, \quad \textbf{(6)}$$

which can be solved to yield the optimal time path K_t^*:

$$K_t^* = \frac{K_0}{L_0}\left(\frac{1-\delta}{\delta}\right)^{\sigma/(\sigma-1)} \frac{L_t H_t G_t^{-1}}{\left[i^{\sigma-1}\delta^{-\sigma}\,(Y_0/K_0)^{1-\sigma}\,G_t^{1-\sigma} - 1\right]^{\sigma/(\sigma-1)}},\ \ 0 < \sigma < 1. \quad \textbf{(7)}$$

K_0 and Y_0 are identified by setting $t = 0$ in (7); we obtain $K_0/Y_0 = \delta/i$. We now can normalize Y_0 to 1; thus, $K_0 = \delta/i$; finally, the optimal time path of capital is

$$K_t^* = \frac{\delta}{i}\left(\frac{1-\delta}{G_t^{1-\sigma} - \delta}\right)^{\sigma/(\sigma-1)} L_t H_t G_t^{-1}, \quad 0 < \sigma < 1. \quad \textbf{(8)}$$

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The optimal trajectory of output and income Y_t^* follows from substituting (8) into (4), using the same identifications. We obtain

$$\begin{aligned} Y_t^* &= L_t H_t \left[\delta \left(\frac{1-\delta}{G_t^{1-\sigma} - \delta} \right) + 1 - \delta \right]^{1/p} \\ &= L_t H_t \left(\frac{1 - \delta G_t^{\sigma-1}}{1-\delta} \right)^{\sigma/(1-\sigma)}, \ 0 < \sigma < 1. \end{aligned} \quad \textbf{(9)}$$

We can verify that $K_0 = \delta / i$ and $Y_0 = 1$.

An important observation is now in order. Note that when $\sigma \neq 1$ the time path K_t^* is defined *for all t* if and only if $\sigma < 1$.[1] Indeed, such is the condition for the denominator $G_t^{1-\sigma} - \delta$ in (8) and (9) to be positive for all t. Since G_t is larger than 1, as well as increasing and unbounded, if $\sigma > 1$ there always exists a time $\bar{t}$ from which $G_t^{1-\sigma} - \delta$ becomes zero and then negative. The economic reason for this is the following: We know that σ is a powerful engine of growth; this is due to its considerable enhancement of the marginal productivity of capital; but, it cannot become too powerful, because to maintain the equality $F_K = i$, capital should then increase extremely fast, entailing explosive growth: It can be verified that $\lim_{t \to \bar{t}} K_t^* = \infty$. It is also a good place to remember that, time and again, the empirical estimates of σ have been strictly lower than 1, and, on the other hand, that $\sigma > 1$ would make very little sense, since it would imply that any amount of output could be produced either without capital or without labor (indeed, in that case the isoquants cut the axes).

K_t^* and Y_t^*, given by (8) and (9), respectively, lead to the following optimal evolution of the capital–output ratio:

$$K_t^* / Y_t^* = \frac{\delta}{i} G_t^{-(1-\sigma)}, \quad 0 \le \sigma \le 1. \quad \textbf{(10)}$$

Innocuous as this last formula may seem, it carries a wealth of good news. The first is that, contrary to what we saw just before where *all* concave power utility functions made the capital-output ratio increase to absurd values, here the ratio *always diminishes*—no one would want an economy where the stock of capital increases more rapidly than its output when it is expected, on the contrary, that technological progress will enable to use relatively *less* capital for a given output. The second good news is that since the remuneration of capital is fixed at $F_{K_t^*} = i$, the share of capital in total income $F_{K_t^*} K_t^* / Y_t^* = i K_t^* / Y_t^* = \delta G_t^{-(1-\sigma)}$ will always diminish to the benefit of the share of labor, equal to $1 - \delta G_t e^{-(1-\sigma)}$.

There are now several ways to determine the optimal savings and investment rates. One of the simplest is to first evaluate the optimal growth rate of Y_t^*. Denoting the growth rates of G_t, H_t, and L_t by g_t, h_t, and n_t, respectively, we get

$$\dot{Y}_t^* / Y_t^* = n_t + h_t + \sigma \delta \left(G_t^{1-\sigma} - \delta \right)^{-1} g_t, \quad 0 \le \sigma \le 1. \quad \textbf{(11)}$$

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Applying (10), the growth rate of capital is $\dot{K}_t^*/K_t^* = \dot{Y}_t^*/Y_t^* - (1-\sigma)g_t$ and therefore, after simplifications,

$$\dot{K}_t^*/K_t^* = n_t + h_t + g_t\left[\frac{\sigma}{1-\delta G_t^{\sigma-1}} - 1\right], \quad 0 \leq \sigma \leq 1. \qquad \textbf{(12)}$$

The optimal savings rate s_t^* is equal to $\dot{K}_t^*/Y_t^* = (\dot{K}_t^*/K_t^*)\left(K_t^*/Y_t^*\right)$; so we have

$$s_t^* = \frac{\delta}{i}\left\{n_t + h_t + g_t\left[\frac{\sigma}{1-\delta G_t^{\sigma-1}} - 1\right]\right\} G_t^{-(1-\sigma)}, \quad 0 \leq \sigma \leq 1. \qquad \textbf{(13)}$$

We are now in a position to identify the optimal initial level of consumption $C_0^* = \left(1 - s_0^*\right) Y_0^*$; since $Y_0^* = 1$, we have

$$C_0^* = 1 - \frac{\delta}{i}\left\{n_0 + h_0 + g_0\left[\frac{\sigma}{1-\delta} - 1\right]\right\}, \quad 0 \leq \sigma \leq 1.^2 \qquad \textbf{(14)}$$

Hence, the optimal initial conditions K_0^* and C_0^* corresponding to $F_{K_t} = i$ are given by $K_0^* = \delta/i$ and equation (14). They define the common starting point shared by scenarios I and II.

Before describing the evolution of the economy in each of those settings, let us consider the values taken by the common initial savings rate s_0^* and the common initial growth rate of real income per person $\dot{y}_0^*/y_0^*$. Indeed, we need to ascertain, in particular, that intricate as formula (13) for s_t^* may look, it always yields very reasonable numbers already at time $t = 0$. We will take $\delta = 0.25$ and $n = 0.01$; the factor-enhancing growth rates g and h will be those measured by Sato (2006, p. 60) for the United States over the period 1909–1989: $g = 0.004$ and $h = 0.02$. Table 1 indicates s_0^* for σ in the range of 0.5–0.8 (most observed) and i between 0.04 and 0.06.

It can be seen that the initial savings and growth rates implied by competitive equilibrium are in a very reasonable range, historically observed. They stay in stark contrast to the results presented earlier, corresponding to all possible concave power utility functions. Consider, for instance, the case of the logarithmic utility

TABLE 1. The initial savings rate s_0^* implied by competitive equilibrium, as a function of the elasticity of substitution σ and the rate of interest i, in percent ($\delta = 0.25$; $n = 0.01$; $h = 0.02$; $g = 0.004$)

i \ σ	0.5	0.55	0.6	0.65	0.7	0.75	0.8
0.04	17.9	18.1	18.3	18.4	18.6	18.8	18.9
0.05	14.3	14.5	14.6	14.7	14.9	15.0	15.1
0.06	11.9	12.1	12.2	12.3	12.4	12.5	12.6

TABLE 2. The initial values of the growth rate of income per person $\dot{y}_0^*/y_0^*$ implied by competitive equilibrium, as a function of the elasticity of substitution σ, in percent ($\delta = 0.25$; $n = 0.01$; $h = 0.02$; $g = 0.004$)

σ	0.5	0.55	0.6	0.65	0.7	0.75	0.8
$\dot{y}_0^*/y_0^*$	2.07	2.07	2.08	2.09	2.09	2.10	2.11

function, corresponding to $\alpha = 0$ [second line in Table 2 in the full version (refer to the online version), with $i = 0.04$], and take $\sigma = 0.5$. It can be seen that the initial "optimal" savings rate necessary to put the economy on the stable branch in the phase diagram is 41%, implying also a never-observed *real* growth rate equal to 15%. If σ had been equal to 0.8, the results would have been even more disastrous: The initial savings rate would have climbed to 50% and the growth rate to 17%.

Thus, equipped with initial conditions corresponding to competitive equilibrium, we can describe what will happen to the economy if either scenario I or II is pursued; in scenario I, investment is planned on the basis of not just one, but *all* possible concave power utility functions. Its fateful consequences are laid out in Section 4.2; the inability of scenario I to maintain trajectories that would replicate competitive equilibrium is explained in Section 4.3.

Scenario II is our solution to the problem of optimal economic growth: investing in such a way that competitive equilibrium is maintained through time. We will show that it entails no less than five maximization objectives for society, apart from the minimization of production costs. We lay out the resulting, very reasonable time paths in Section 5. The robustness of these results is finally tested in Section 6 by considering quite different evolutions of population and technical progress.

4.2. Scenario I: Planning with Strictly Concave Utility Functions from an Initial Situation of Competitive Equilibrium: A Disaster in the Making

Given the above-defined initial conditions reflecting competitive equilibrium, we now maximize $\int_0^\infty U(C_t)e^{-it}dt$ under the constraint $C_t = F(K_t, t) - \dot{K}_t$, where $F(.)$ is defined by (4), and where $U(C) = (C^\alpha - 1)/\alpha$. Our functional is $\int_0^\infty U[F(K,t) - \dot{K}]e^{-it}dt = \int_0^\infty V(K, \dot{K}, t]dt$; the Euler equation

$$\frac{\partial V}{\partial K} - \frac{d}{dt}\frac{\partial V}{\partial \dot{K}} = 0$$

together with the constraint leads to the system of first-order nonlinear equations

$$\dot{C} = \frac{C}{1-\alpha}\left\{[\delta(e^{gt}iK/\delta)^p + (1-\delta)e^{p(n+h)t}]^{(1/p)-1}\,\delta^{1-p}i^p e^{pgt}K^{p-1} - i\right\}, \tag{15}$$

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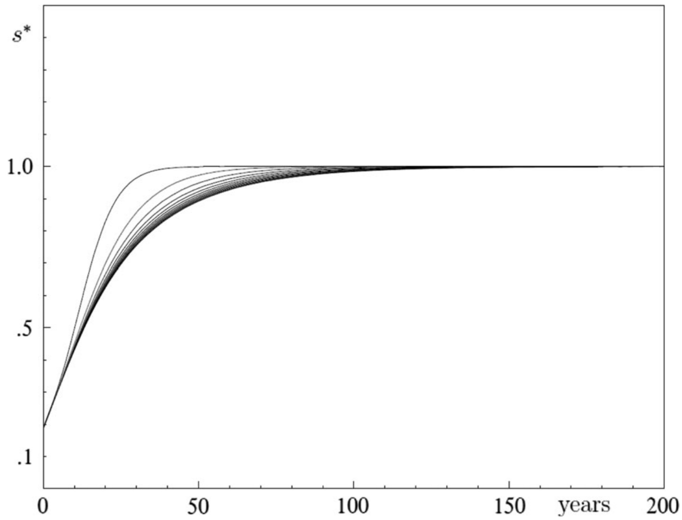

Figure 2. The inordinate behavior of the savings rate for 25 values of α in the utility function $(C^\alpha - 1)/\alpha$ ranging from $\alpha = 0.8$ (upper curve) to $\alpha = -8.8$, in steps of -0.4 (lower curve).

$$\dot{K} = [\delta(e^{gt} iK/\delta)^p + (1-\delta)e^{p(n+h)t}]^{1/p} - C. \tag{16}$$

The concavity of the integrand with respect to K and $\dot{K}$ and the transversality conditions (shown to be met at the end of this section—see online version) ensure that this system leads to a unique maximum, given the above-defined initial conditions K_0^* and C_0^*.

We started the tests of the utility function by using the parameter values mentioned above: $n = 0.01$, $\delta = 0.25$, $i = 0.04$, $\sigma = 0.8$, $h = 0.02$, and $g = 0.004$. Solving numerically the system (15)–(16) and plugging the solution K_t^* into (4) enables us to determine the evolution of the central variables of the economy, such as the optimal savings rate, consumption, the growth rates of income per person and of the capital stock, the marginal productivity of capital, and the capital-output ratio. We did this for 25 values of the parameter α of the utility function, ranging from 0.8 (upper curve in the left part of the diagram) to -8.8 in steps of -0.4 (see the corresponding diagrams in the supplementary material). None of these evolutions make sense. As an example, we give here the evolution of the optimal savings rate (see Figure 2). *For all values of* α, the savings rate becomes equal to or larger than 50% before 14 years. Another example: From an initial, reasonable value equal to $K_0^*/Y_0^* = \delta/i = 6.25$—corresponding to competitive equilibrium—the capital–output ratio increases and tends asymptotically toward an absurd value equal to 32 for *any* utility function. One would expect, of course, that technical progress enhancing capital would *reduce,* not increase, the need of fixed capital for one unit of net output. On the other hand, in the competitive equilibrium model we suggest hereafter, we will see that the capital–output ratio decreases, if ever slowly.

4.3. The Incompatibility of the Traditional Approach and Competitive Equilibrium: An Analytic Explanation

We illustrated numerically the fact that competitive equilibrium could not be sustained in the traditional model, this approach leading to definitely unwarranted time paths for variables of central importance: fast-declining consumption and overaccumulation of the capital stock.

The reason is simple: Only affine utility functions make the Euler equation compatible with competitive equilibrium; strictly concave functions always entail an Euler equation different from $F_K(K, L, t) = i(t)$. Indeed, maximizing $\int_0^\infty U(C_t) \exp(-\int_0^t i(z)dz)dt$ under the constraint $C_t = F(K_t, t) - \dot{K}_t$ leads to the Euler second-order differential equation

$$i(t) = F_K(K, t) + \dot{U}'(C)/U'(C) = F_K(K, t) + \left[U''(C)/U'(C)\right]\dot{C} \qquad \textbf{(17)}$$

where $C_t = F(K_t, t) - \dot{K}_t$. We can also explain why consumption will always decrease. For all parameters within observed or predictable ranges, the positive effect of technical progress on F_K is not sufficient to compensate the decreasing return on capital. Therefore, to maintain the right-hand side of (17) equal to i (constant here), $\dot{C}$ must be negative since $U''(C) < 0$. In the solution we propose, capital will increase in such a way that those two effects will exactly compensate each other.

5. A SUGGESTED SOLUTION

In the intertemporal optimization problem considered above, the only way to enforce $i(t) = F_K(K, L, t)$ is to have $U''(C) = 0$, i.e., $U(C) = aC + b$, where a and b are constants, a particular case being ours,[3] $U(C) = C$. It is a good place to remember that utility functions at the macroeconomic level were simple, direct transpositions of functions considered at the micro-level. We take the liberty of suggesting that before bending down into a concave curve the relationship between consumption and society's representation of welfare, we first take away from net national income the huge amount of expenditures that have simply no relationship with any present or future well-being.

We thus should be tending toward a measure of the quality of life that offers much less reason to be transformed into a concave function than what was the case previously. There are sound reasons not to introduce such transformations. Consider, for instance, medical discoveries that enhance both the length and the quality of life of a large part of the population, either in rich or poor countries. Would not we then conclude that those health services generate linear or even *convex*, rather than concave, utility flows? Also, contrary to what is assumed at the individual micro-level, *the very knowledge that not only some given person, but also the rest of society as well as all future generations are able to benefit from those discoveries* can hardly induce to penalize them with a transformation into some concave function. Exactly the same reasoning would apply to the all-important expenditures on education.

Consequently, in what follows we take the step of considering that C stands for *welfare* flows, F representing the output net of (a) physical and natural capital depreciation, and (b) all goods and services reducing welfare. In the same way, $\dot{K}$ is standing only for investment in goods and services improving society's present and future well-being.

We are thus led to maximize $W_0 = \int_0^\infty C_t \exp(-\int_0^t i(z)dz)dt$ under the constraint $C_t = F(K_t, L_t, t) - \dot{K}_t$. The Euler equation, of course, leads to the competitive equilibrium condition $i(t) = F_K(K, L, t)$ and, if $i(t)$ is constant, to equations (7)–(14) introduced in Section 4 to determine the optimal time paths and their initial values corresponding to such equilibrium.

We now want to show that all equations (7)–(14) will *always* yield reasonable initial values and future time paths for the following fundamental variables: the optimal savings rate, the implied growth rate of income per person, the capital–output ratio, and the share of labor in net income.

Before proceeding we should point out how appropriate the adjective "optimal" in this context is, since all time paths described hereafter correspond to no less than *five* simultaneous optima, in addition to the minimization of production costs.

5.1. The Intertemporal Optimality of Competitive Equilibrium: Its Multiple Facets

We will show how investing in such a way that the marginal productivity of capital stays equal to the rate of interest generates for society five benefits of considerable importance; those benefits may be very surprising in the sense that they can be—and most probably are—far removed from the initial objective of investors—which might simply have been the minimization of their production costs. We will prove the following results.

THEOREM 1. *Let the production function $F(K_t, L_t, t)$ be concave and homogeneous of degree one in K, L; technical progress may be labor- and capital-augmenting. If investment is carried out through time in such a way that the marginal productivity of capital is maintained equal to the rate of interest $i(t)$, and if capital is remunerated by $i(t)K(t)$, society simultaneously maximizes five magnitudes:*

(1) *the sum of the discounted consumption flows society can acquire from now to infinity $\int_0^\infty C_t \exp(-\int_0^t i(z)dz)dt$;*
(2) *the value of society's activity at any point of time t, defined by the consumption flow received at time t plus the rate of increase in the value of the capital stock at that time. In present value, this sum is equal to*

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$C_t \exp(-\int_0^t i(z)dz) + \frac{d}{dt}[\lambda(t)K(t)]$, where $\lambda(t)$ is the discounted price of capital;

(3) *the total value of society's activity over an infinite time span*

$$\int_0^\infty \left\{C_t \exp\left(-\int_0^t i(z)dz\right) dt + \frac{d}{dt}[\lambda(t)K(t)]\right\} dt;$$

(4) *the remuneration of labor at any point of time $F(K_t, L_t, t) - i(t)K(t)$;*

(5) *the total remuneration of labor over an infinite time span*

$$\int_0^\infty \exp\left(-\int_0^t i(z)dz\right)[F(K_t, L_t, t) - i(t)K(t)]dt.$$

Proof of (1). Maximizing $\int_0^\infty C_t \exp(-\int_0^t i(z)dz)dt$ under the constraint $C_t = F(K_t, L_t, t) - \dot{K}(t)$ amounts to maximizing

$$W_0 = \int_0^\infty \left[F(K_t, L_t, t) - \dot{K}(t)\right] \exp\left(-\int_0^t i(z)dz\right) dt, \tag{18}$$

denoted $\int_0^\infty \varphi(K, \dot{K}, t)dt$. Applying the Euler equation $\varphi_K - \frac{d}{dt}\varphi_{\dot{K}} = 0$ results in the condition

$$F_K(K_t, L_t, t) = i(t). \tag{19}$$

Due to the concavity of $\varphi(K, \dot{K}, t)$ in the variables K and $\dot{K}$, we may apply Takayama's theorem to ascertain that (19) is a necessary and sufficient condition for a global maximum of W, provided that the transversality conditions at infinity are met. Those conditions are $\lim_{t\to\infty} \partial\varphi/\partial\dot{K} = 0$ and $\lim_{t\to\infty} \varphi(K, \dot{K}, t) = 0$. The first condition can be immediately checked: It implies $\lim_{t\to\infty} \partial\varphi/\partial\dot{K} = \lim_{t\to\infty} \exp(-\int_0^t i(z)dz) = 0$, always verified. The second condition is met as long as $\int_0^\infty \varphi(K, \dot{K}, t)dt$ converges, a property obtained due to the fast convergence of the exponential $\exp(-\int_0^t i(z)dz)$. ■

Proof of (2). As defined above, the value of society's activity is measured by the Dorfmanian $D(K, \dot{K}, t)$[4]; using the constraint introduced before, it can be expressed as

$$D(K_t, \dot{K}_t, t) = C_t \exp\left(-\int_0^t i(z)dz\right) + \frac{d}{dt}[\lambda(t)K(t)] \tag{20}$$

$$= [F(K_t, L_t, t) - \dot{K}_t] \exp\left(-\int_0^t i(z)dz\right) + \lambda(t)\dot{K}_t + \dot{\lambda}(t)K_t, \tag{21}$$

where $\lambda(t)$ is the price of one unit of capital in use at time t, in present value. Setting the gradient of D with respect to K_t and $\dot{K}_t$ to 0 gives

$$\frac{\partial D}{\partial K_t} = F_{K_t}(K_t, L_t, t) \exp\left(-\int_0^t i(z)dz\right) + \dot{\lambda}(t) = 0 \tag{22}$$

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and

$$\frac{\partial D}{\partial \dot{K}_t} = -\exp\left(-\int_0^t i(z)dz\right) + \lambda(t) = 0; \tag{23}$$

eliminating $\lambda(t)$ yields $F_K(K_t, L_t, t) = i(t)$, which, together with the concavity of the Dorfmanian with respect to K and $\dot{K}$, gives a sufficient condition for a global maximum of D. (An alternate proof is available in the full version available as supplementary material.)

We still have to prove that $\lambda(t)$, given by equation (23) as equal to $\exp(-\int_0^t i(z)dz)$, is indeed the present value of one additional unit of capital used at time t. This will be true if only if at any time t the rate of increase of the *optimal* value of the functional W_t^* with respect to capital is equal to 1. We have, from

$$W_t = \int_t^{\infty} \left[F(K_\tau, L_\tau, \tau) - \dot{K}(\tau)\right] \exp\left(-\int_t^{\tau} i(z)dz\right) d\tau,$$

$$\begin{aligned}\frac{\partial W_t}{\partial K_t} &= \frac{\partial}{\partial K_t} \int_t^{\infty} [F(K_\tau, L_\tau, \tau) - \dot{K}_\tau] \exp\left(-\int_t^{\tau} i(z)dz\right) d\tau \\ &= \int_t^{\infty} F_{K_\tau}(K_\tau, L_\tau, \tau) \exp\left(-\int_t^{\tau} i(z)dz\right) d\tau.\end{aligned} \tag{24}$$

Replacing $F_K(K_\tau, L_\tau, \tau)$ in the last term of (24) with $i(\tau)$ gives

$$\frac{\partial W_t^*}{\partial K_t} = \int_t^{\infty} i(\tau) \exp\left(-\int_t^{\tau} i(z)dz\right) d\tau = \left[-\exp\left(-\int_t^{\tau} i(z)dz\right)\right]_t^{\infty} = 1; \tag{25}$$

hence, $\lambda(t) = \exp(-\int_0^t i(z)dz)$ is indeed the present value of $\partial W_t^*/\partial K_t$ and therefore the discounted price of one additional unit of capital set in use at time t, as was to be shown for the Dorfmanian to measure the value of society's activity. ■

Proof of (3). Maximizing at any point of time a function $f(t)$ will generate a maximum of the integral $\int_0^{\infty} f(t)dt$ as long as the integral converges, which is the case here. We can verify that $F_K(K_t, L_t, t) = i(t)$ optimizes the total value of society's activity over $t \in [0, \infty)$ by maximizing the indefinite integral of the Dorfmanian

$$\begin{aligned}\int_0^{\infty} D(K, \dot{K}, t)dt &= \int_0^{\infty} \{C_t \exp\left(-\int_0^t i(z)dz\right) dt + \frac{d}{dt}[\lambda(t)K(t)]\}dt \\ &= \int_0^{\infty} \{[F(K_t, L_t, t) - \dot{K}_t] \exp\left(-\int_0^t i(z)dz\right) + \lambda(t)\dot{K}_t + \dot{\lambda}(t)K_t\}dt.\end{aligned} \tag{26}$$

The Euler equation can be shown to be equal to

$$\frac{\partial D}{\partial K_t} - \frac{d}{dt}\frac{\partial D}{\partial \dot{K}_t} = \exp\left(-\int_0^t i(z)dz\right)[F_K(K_t, L_t, t) - i(t)] = 0, \tag{27}$$

leading to $F_K(K_t, L_t, t) = i(t)$. (For an alternate proof, see the full version online.) ■

Proof of (4). When maximizing the value of society's activity at any point of time, we have determined the value of $\lambda(t)$ as $\exp(-\int_0^t i(z)dz)$; substituting this value into the Dorfmanian expressed either as $D(K, \dot{K}, t)$ gives the Dorfmanian evaluated at its maximum value, denoted by D^*:

$$\begin{aligned} D^* &= C_t^* \exp\left(-\int_0^t i(z)dz\right) + \lambda(t)\dot{K}_t^* + \dot{\lambda}(t)K_t^* \\ &= C_t^* \exp\left(-\int_0^t i(z)dz\right) + \exp\left(-\int_0^t i(z)dz\right)\dot{K}_t^* \\ &\qquad - i(t)\exp\left(-\int_0^t i(z)dz\right)K_t^* \\ &= \exp\left(-\int_0^t i(z)dz\right)\left[C_t^* + \dot{K}_t^* - i(t)K_t^*\right] \\ &= \exp\left(-\int_0^t i(z)dz\right)\left[F(K_t^*, L_t, t) - i(t)K_t^*\right]. \end{aligned}$$

Since $i(t)K_t$ is the remuneration of capital, the bracketed term is the remuneration of labor, which has been maximized with D. ■

Proof of (5). The maximization of the total remuneration of labor over $[0, \infty)$, the integral $\int_0^\infty \exp(-\int_0^t i(z)dz)[F(K_t, L_t, t) - i(t)K(t)]dt$, follows immediately either from a differential or a variational argument as those used in the proof of (3). ■

Taken individually, any of those five outcomes of competitive equilibrium admittedly constitutes surprises. One of the most startling outcomes is that the equality $F_K(K_t, L_t, t) = i(t)$ not only maximizes intertemporally consumption as well as the value of society's activity, but also maximizes the remuneration of labor, and, additionally, that the last two quantities are equal.

We will now show that, surprising as this last equality may be, it perfectly squares with a basic principle of national accounting, namely that at any time t the total remuneration of factors must be equal to consumption plus investment. The Dorfmanian, denoted by D^* at its maximal value, has just been shown to be equal to the present value of the remuneration of labor; thus, the *current* value at time t of that remuneration is $D^* \exp(\int_0^t i(z)dz)$; on the other hand, the remuneration of capital is $i(t)K_t^*$. We must now verify that the sum of those factor payments is equal to consumption plus investment at their optimum values. We have indeed $i(t)K_t^* + [\exp(\int_0^t i(z)dz)]D^* = i(t)K_t^* + \exp(\int_0^t i(z)dz)[C_t^* \exp(-\int_0^t i(z)dz) + \frac{d}{dt}(\lambda_t^* K_t^*)] = i(t)K_t^* + e^{\int_0^t i(z)dz}[C_t^* e^{-\int_0^t i(z)dz} + e^{-\int_0^t i(z)dz}\dot{K}_t^* - i(t)e^{-\int_0^t i(z)dz}K_t^*] = C_t^* + \dot{K}_t^*$, as was to be ascertained.

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5.2. The Optimal Evolution of the Economy under Competitive Equilibrium

We now want to assess the values taken by central variables of the economy, namely the savings-investment rate, the growth rate of real income per person, and the capital–output ratio if we manage to save and invest in such a way as to maintain competitive equilibrium.

In a first approach, we assume constant growth rates for L_t, G_t, and H_t, denoted by n, g, and h, respectively. (In the next section, we will assume very different time paths for those variables.) We choose $n = 0.01$; for g, h, and σ, we took the estimates made by Sato for the US economy over an 80-year time span. Thus, $\sigma = 0.8$, $h = 0.02$, and $g = 0.004$ as a first series of values for those parameters.

The optimal time path of the savings rate. We are now in a position to undertake the comparative dynamics of the optimal savings rate, and answer, in particular, the nagging question asked by Frank Ramsey and certainly by anybody who would take up the subject of optimal growth: Will technical progress increase or decrease the optimal savings rate? We will now use our central equation (6) not only, as we did before, to determine the initial conditions prevailing in a competitive economy, but also to study its whole time path:

$$s_t^* = \frac{\delta}{i}\left\{n + h + g\left[\frac{\sigma}{1 - \delta e^{-g(1-\sigma)t}} - 1\right]\right\} e^{-g(1-\sigma)t}, \quad 0 \le \sigma \le 1. \qquad \textbf{(28)}$$

Examination of (28) immediately reveals that s_t^* is an increasing function of the elasticity of substitution σ and a decreasing function of the rate of interest. It will also decrease through time for any given value of the parameters. Those dependencies are very natural. For instance, the property $\partial s_t^*/\partial\sigma > 0$ is easily understood if we think of σ as a powerful engine of growth; the reason is that income per person, as a general mean of order p ($p = 1 - 1/\sigma$), is an increasing function of its order and therefore of σ, with an inflection point close to $p = 0$, i.e., when σ is in the observed range, considered here ($0.5 < \sigma < 0.8$).[5]

Also, it would be disastrous if the sacrifice made by society through time in the form of its savings rate were increasing or constant despite technical progress. Tables 3 and 4 present first results for the values of the parameters indicated above.

The good news is that the optimal savings rate is always in very reasonable ranges. From (28) it can be seen that its welcome decrease through time is solely due to the presence of capital-augmenting technical progress. (If g were equal to zero, the optimal savings rate would remain at the constant level $\frac{\delta}{i}(n + h)$.) Also, whatever values of g and h, a value $\sigma = 1$ would make s^* remain constant, at level $(\delta/i)\left[n + h + g(\frac{\delta}{1-\delta})\right]$. However, we should underline that, time and again, the elasticity of substitution has been observed to be *lower than,* not equal to, 1, and that, as we had observed in Section 3.1, $\sigma = 1$ constitutes the *upper limit* for which a competitive equilibrium can be sustained.

TABLE 3. The optimal savings rate $s^*(t, i)$ as a function of the rate of preference for the present, and as a slowly decreasing function of time; $\sigma = 0.8$; $n = .01$; $\delta = 0.25$; $g = .004$; $h = 0.02$

t \ i	0.04	0.045	0.05	0.055	0.06
0	18.9	16.8	15.1	13.8	12.6
30	18.4	16.4	14.8	13.4	12.3
60	18.0	16.0	14.4	13.1	12.0

TABLE 4. The optimal savings rate $s^*(t, i)$ as a function of the elasticity of substitution σ, and as a slowly decreasing function of time; $i = 0.04$; $n = .01$; $\delta = 0.25$; $g = .004$; $h = 0.02$

t \ σ	0.5	0.55	0.6	0.65	0.07	0.75	0.8
0	17.9	18.1	18.3	18.4	18.6	18.8	18.9
30	16.8	17.1	17.4	17.6	17.9	18.2	18.4
60	15.8	16.2	16.5	16.9	17.2	17.6	18.0

The positive dependency between s_t^* and the rate of labor-augmenting technical progress h is immediately established from (28), but an assessment of the effect of changing g on s_t^* cannot be easily made analytically due to the complexity of $\partial s_t^*/\partial g$. However, a clear pattern can be established with numerical representations. At any time t, whether an increase in g will lead to an increase in s_t^* will depend on the value of the elasticity of substitution. There will always exist a value $\bar{\sigma}$ for which $\partial s_t^*/\partial g = 0$. For instance, at $t = 0$, it can be seen from (28) that the coefficient multiplying g, equal to $\frac{\sigma}{1-\delta} - 1$, reduces to zero for $\sigma = 1 - \delta$; above this critical value, $\partial s_t^*/\partial g > 0$; and below, $\partial s_t^*/\partial g < 0$. In our example, this value is $\bar{\sigma} = 0.75$; as time progresses, the value of $\bar{\sigma}$ increases (for instance, with $t = 30$, $\bar{\sigma} = 0.82$). In any case, the changes in s_t^* impacted by an increase in g are minimal: for instance, doubling g from 0.004 to 0.008 makes s_{30}^* decrease from 18.4% to 18.1%.

5.3. The Optimal Growth Rate of Income Per Person

From (11), the optimal growth rate of income per person is now

$$\dot{y}_t^*/y_t^* = h + \sigma g \frac{\delta}{e^{g(1-\sigma)t} - \delta}, \quad \sigma \leq 1. \qquad \textbf{(29)}$$

It immediately appears that $\dot{y}_t^*/y_t^*$, an increasing function of the elasticity of substitution, is higher than h and very slowly decreases asymptotically toward h,

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TABLE 5. The optimal growth rate of income per person $r^*(t, i) = \dot{y}_t^*/y_t^*$ as a function of the elasticity of substitution; $n = .01$; $\delta = 0.25$; $g = .004$; $h = 0.02$

$t \backslash \sigma$	0.5	0.55	0.6	0.65	0.7	0.75	0.8
0	2.07	2.07	2.08	2.09	2.09	2.10	2.11
30	2.06	2.07	2.08	2.08	2.08	2.09	2.10
60	2.06	2.06	2.07	2.08	2.08	2.09	2.10

as illustrated in Table 5. (Notice once more that the ultimate growth rate of income per person may converge toward the rate of labor-augmenting technical progress even in the presence of capital-augmenting progress—this is due to the property of general means with negative order we mentioned earlier.)

5.4. The Optimal Time Path of the Capital–Output Ratio

In a reassuring way, the capital–output ratio K^*/Y^*, determined from (8) and (9) as

$$K^*/Y^* = \frac{\delta}{i} e^{-(1-\sigma)gt}, \quad \sigma \leq 1, \tag{30}$$

is a slowly decreasing function of time (see Table 6). It would be indeed bad news if this ratio were to stay constant (the case of $\sigma = 1$, with $K^*/Y^* = \delta/i$), meaning that society would have to match any growth rate of its standard of living with the same growth rate of fixed capital; it would be absurd news if, as seen above in the traditional approach (Section 3.2), from a competitive equilibrium value the capital–output ratio were to increase five-fold whatever the $\alpha < 1$ value in the utility function, despite the presence of capital-augmenting technical progress! Here the ratio's rate of decline is $(1 - \sigma)\, g$, depending positively on g and negatively on σ, which makes good economic sense.

TABLE 6. The capital–output ratio K^*/Y^* as a function of time and the rate of preference for the present; $n = .01$; $\delta = 1/4$; $\sigma = 0.8$; $h = .02$; $g = 0.004$

	$\sigma = 0.5$			$\sigma = 0.8$		
$t \backslash i$	0.04	0.05	0.06	0.04	0.05	0.06
0	6.25	5.00	4.17	6.25	5	4.17
30	5.89	4.71	3.92	6.10	4.88	4.07
60	5.54	4.43	3.70	5.96	4.76	3.97

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5.5. The Evolution of the Labor Share in Competitive Equilibrium

From Section 4, we know that the remuneration of labor, equal to the value of society's activity, is maximized at any point of time, and therefore intertemporally. But what is the evolution of the share of labor through time? From (30), we can determine the share of capital as $iK^*/Y^* = \delta e^{-(1-\sigma)gt}, \quad \sigma \leq 1$. Therefore, the share of labor, denoted by θ_t^*, is equal to

$$\theta_t^* = 1 - \delta e^{-(1-\sigma)gt}. \tag{31}$$

It can be seen that initially this share is independent of σ and g, and that it slowly increases asymptotically from $1 - \delta$ to 1. For instance, with $\delta = 0.25$, $\sigma = 0.5$, and $g = 0.04$, $\theta_0^* = 0.75$ and $\theta_{30}^* = 0.76$. In this 30-year time span, it can be calculated from (8) and (9) (and the same other parameters as in Section 5.4) that the total remuneration of labor, $Y - iK$, has been multiplied by 2.56, implying an increase of the wage rate equal to 2.1 % per year.

6. THE ROBUSTNESS OF THE OPTIMAL SAVINGS RATE

Modelling the economy in the very long run definitely requires that we do not suppose—as we did—that the population evolution will be exponential; we should rather use some S-shaped time-path, possibly converging toward a horizontal asymptote. Our aim now is to know how the optimal savings rate and the other central variables of the economy would react if not only the population $L(t)$ but also the factor enhancing functions $G(t)$ and $H(t)$ were to follow such S-shaped evolutions. To that effect, we supposed that the growth rates of these functions would be asymptotically decreasing toward 0. Due to space constraints, our methodology and the detailed results are available on the online document only. The results exactly square with what we expected: the optimal savings rate is lower than in the exponential case while, not surprisingly, the capital-output ratio remains in the same range and decreases through time. Such robustness of the model is due to two factors: first, while the S-shaped time-paths are highly different in the distant future from the exponential ones, they remain relatively close in the short and medium term; the second reason is the quick convergence of the indefinite integral we are maximizing.

7. CONCLUSION

Extending the concept of a concave utility function from micro-representations to macroeconomics was an intuitive, apparently defensible idea, but it led optimal growth theory into a blind alley, precluding any possibility of solving its central problem: simultaneously determining meaningful time paths for the optimal savings rate and for other central variables of an economy. Ever since Ramsey's first experiment, it has been repeatedly demonstrated that such a function, whatever extreme properties it was imparted with, led to at least one evolution of a fundamental variable that was either strongly contradictory to historical experience or simply unacceptable by society.

For our part, we have, on the one hand, confirmed the serious warning signs sent to our profession in the writings of Ramsey and Goodwin, and most forcibly by King and Rebelo; in fact, we confirmed what would be concluded by anyone who would care to solve numerically the differential equations implied by the theory. On the other hand, we offered an explanation to those dire results: the traditional approach prevents competitive equilibrium to be sustained. In particular, we showed that if the economy was initially in a state close to competitive equilibrium, any attempt to define an optimal investment time path along traditional lines inevitably led to a catastrophic evolution of the economy, marked by a permanent decrease in consumption accompanied by an inordinate accumulation of capital.

Our solution to the problem of optimal growth is then the following: first, rather than bending all consumption into a concave function as it has been done until now, we retain in consumption what can be considered as *welfare flows* for society. This approach leads in a natural way to the following objective, probably conforming to the desires of most individuals: maximizing the sum of discounted *welfare* flows (contrast this with the traditional approach: imposing a utility function on every individual, with the *certainty* that it will lead to unwanted time paths for the economy). Then, define with i the rate of preference of society for the present, which naturally incorporates a risk premium. We believe that it will definitely be easier to obtain a consensus on such a rate than on some utility function, even if society is completely unaware of the impracticability of such functions. That rate could be linked to historically observed real rates of return on capital. Then, as a rule, savings and investment decisions should conform to the equation of competitive equilibrium $i = F_K(K, L, t)$. This is the Euler equation for the maximization just defined; with the general, historically observed hypotheses of the neoclassical model, the equation will always have a solution $K_t^* = F_K^{-1}(i, L, t)$, leading to a meaningful savings-investment rate $s_t^* = \dot{K}_t^*/F(K_t^*, L_t, t)$.

This proposal offers three advantages:

(1) The time path K_t^* is optimal in more than one way: In addition to minimizing production costs, it maximizes intertemporally the following magnitudes: the sum of discounted consumption flows; the total value of society's activity, equal to the sum of consumption and the increase in the value of capital; and, finally, the total remuneration of labor, shown to be equal to this sum.

(2) K_t^* always leads to reasonable time paths of the economy. In addition, the optimal savings rate and the capital–output ratio—reflecting a sacrifice made by society—both exhibit the most welcome feature of being slowly decreasing over time.

(3) All implied time paths are extremely robust to variations in the parameters of the model, as well as to highly different predictions regarding the future

evolutions of population and technical progress. Even drastic predictions—for instance, assuming that the variables reflecting those evolutions will soon reach a plateau—are unable to make central variables of the economy deviate from reasonable, predictable ranges.

In his introduction to the 1975 edition of Adam Smith's magnum opus, William Letwin wrote, "Far from being a hymn in praise of anarchic greed, the 'Wealth of Nations' is a reasoned argument for justice, order, liberty and *prudent plenty*" (p. 7; our italics). It is definitely arguable that with optimal growth theory we are looking for rules enabling society to achieve this last objective. It is our hope that the numbers suggested here, based on competitive equilibrium with its associate optima, contribute to that rightful purpose.

SUPPLEMENTARY MATERIAL

The supplementary material for this article is available online at www.journals.cambridge.org/jid/10.1017/S1365100516000742.

NOTES

1. In the $\sigma = 1$ Cobb–Douglas case, formulas (7)–(9) have to be reworked from $F_K = i$, using this time (5) for $F(.)$. The results are $K_t^* = \frac{\delta}{i} L_t H_t G_t^{\delta/(1-\delta)}$, $Y_t^* = L_t H_t G_t^{\delta/(1-\delta)}$, and $K_t^*/Y_t^* = \delta/i$. As mentioned before, we give these results for complete reference only, because time and again σ has been observed as smaller than 1.

2. Note that formulas (10)–(13) apply directly in the $\sigma = 1$ case. One gets $K_t^*/Y_t^* = \delta/i$, $\dot{Y}_t^*/Y_t^* = \dot{K}_t^*/K_t^* = n_t + h_t + \frac{\delta}{1-\delta} g_t$, $s_t^* = \frac{\delta}{i}(n_t + h_t + \frac{\delta}{1-\delta} g_t)$, and $C_0^* = 1 - \frac{\delta}{i}(n_0 + h_0 + \frac{\delta}{1-\delta} g_0)$.

3. As a referee has pointed out, it is not the first time a linear objective has been used: Intriligator (1971) and Kamikigashi and Roy (2006) are examples, albeit in different contexts.

4. In his remarkable essay "An economic interpretation of optimal control", Robert Dorfman (1969) introduced a "modified Hamiltonian", as he called it (p. 822). To pay hommage to the memory of Robert Dorfman, we call this new Hamiltonian a "Dorfmanian".

5. In La Grandville (1989), we conjectured that the spectacular growth in East-Asian countries was due less to technical progress than a higher elasticity of substitution; see also Klump and La Grandville (2000). The conjecture was successfully tested by Yuhn (1991) in the case of South Korea. For the existence of a unique inflection point in the general mean, see the conjecture offered in La Grandville and Solow (2006); the proof is due to Thanh and Minh (2008).

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