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The Griffith formula and the Rice–Cherepanov integral for crack problems with unilateral conditions in nonsmooth domains

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As a paradigm for non-interpenetrating crack models, the Poisson equation in a nonsmooth domain in R^2 is considered. The geometrical domain has a cut (a crack) of variable length. At the crack faces, inequality type boundary conditions are prescribed. The behaviour of the energy functional is analysed with respect to the crack length changes. In particular, the derivative of the energy functional with respect to the crack length is obtained. The associated Griffith formula is derived, and properties of the solution are investigated. It is shown that the Rice–Cherepanov integral defined for the solutions of the unilateral problem defined in the nonsmooth domain is path-independent. Finally, a non-negative measure characterising interaction forces between the crack faces is constructed.

1 Introduction

In this paper the differentiability of the energy functional for an elliptic equation with respect to the crack length is shown. The method of proof is different from the proof in the linear case [1], since we cannot in general expect that the solution to the variational inequality for the displacement of an elastic membrane with unilateral conditions prescribed on the crack faces is differentiable with respect to the crack length. The method of the proof presented in the paper is general, and can also be applied to the energy functionals of the linear elasticity system with the non-interpenetration conditions prescribed on the crack faces (see Fig. 1 for an example in $2D$ elasticity).

In the case of a $2D$ elasticity system, the condition which is prescribed on the crack faces takes the form

$$[v_1n_1 + v_2n_2] \geqslant 0 \,,$$

where $v = (v_1, v_2)$ is the displacement field, $n = (n_1, n_2)$ is the normal vector, and $[v \cdot n]$ denotes the jump of the normal component of v across the crack. In this paper, we consider the scalar displacement u of an elastic membrane and, therefore, we prescribe

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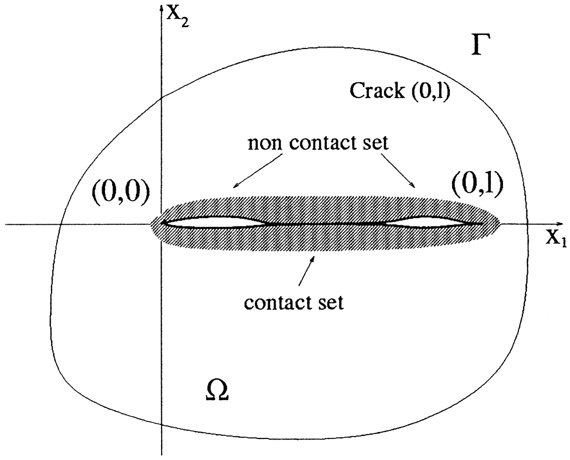

FIGURE 1. Partially ‘open’ crack $(0, l)$ in $2D$ elasticity.

the following unilateral condition on the crack faces:

$$[u] \geqslant 0,$$

which makes sense from the mathematical point of view. However, it is difficult physically to justify the condition for the membrane model. On the other hand, the simplified model for the membrane can be used for testing numerical methods, as well as for determination of the singularities of the displacement near the crack tips. The same method of analysis, as proposed in the paper for the membrane model, will be used for more realistic models in linear elasticity in a forthcoming paper.

In the linear case, i.e. for the homogeneous Neumann boundary conditions prescribed on the crack faces in the scalar case, or for the traction-free crack faces in elasticity, the results are well known. We refer the reader elsewhere [2] for the models currently used in the fracture mechanic, and for a review of the recent results [3] on the applications to crack propagation.

In the linear case, both the first and second order derivatives of the energy functionals with respect to the crack length are evaluated and used for numerical methods of analysis of crack propagation in solids. However, it seems that we cannot in general expect the second order differentiability of the energy functional with respect to the crack length in the case of the nonlinear problem in which unilateral conditions are prescribed on the crack faces, i.e. only the second order directional differentiability can be obtained. Indeed, from the local point of view, we expect the gradient of the solution to have an inverse square root singularity at the prescribed tips, but to be bounded at the edges of the contact set. We refer elsewhere [4] for the shape differentiability properties of solutions to variational inequalities in smooth domains.

1.1 Problem formulation

Let $D \subset R^2$ be a bounded domain with smooth boundary Γ, and $\Xi_{l+\delta}$ be the set $\{(x_1, x_2) |\ 0 < x_1 < l + \delta,\ x_2 = 0\}$. We assume that this set belongs to the domain D

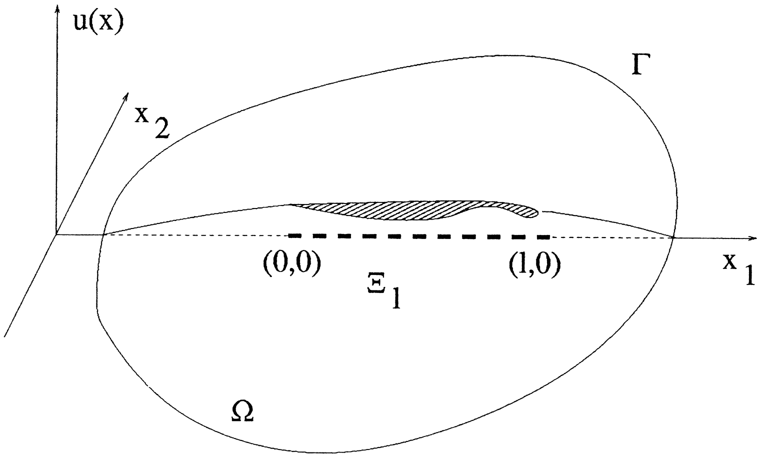

FIGURE 2. Loaded membrane with partially 'open' crack Ξ_l.

for all sufficiently small δ, and $l > 0$. The domains with cracks $\Xi_{l+\delta}$, Ξ_l are denoted by $\Omega_\delta = D \setminus \overline{\Xi}_{l+\delta}$, $\Omega = D \setminus \overline{\Xi}_l$, respectively. We consider an elastic membrane in the reference domain Ω with crack Ξ_l of the length l and with the unilateral condition prescribed on the crack for the displacement of the membrane (see Fig. 2).

Therefore, in the domain Ω, we consider the following boundary value problem for a function u, which satisfies

$$-\Delta u = f, \tag{1}$$

$$u = 0 \quad \text{on} \quad \Gamma, \quad [u] \geqslant 0 \quad \text{on} \quad \Xi_l. \tag{2}$$

Here $f \in C^1(\bar{D})$ is a given function, $[u] = u^+ - u^-$ is the jump of u across Ξ_l. The vector $n = (0, 1)$ is orthogonal to Ξ_l, and $u^\pm$ denote the traces of u on the crack faces, corresponding to the positive and negative directions of n. The problem formulation (1), (2) is not complete to ensure the uniqueness of the solution. In fact, to ensure non-interpenetralibity, we consider the minimisation of the functional

$$I(\phi) = \frac{1}{2}\int_\Omega |\nabla \phi|^2 - \int_\Omega f\phi$$

over the set of all admissible functions from the Sobolev space $H^1(\Omega)$. That is, introduce the sets

$$K_0 = \{w \in H^1(\Omega) | \; [w] \geqslant 0 \quad \text{on} \quad \Xi_l; \; w = 0 \quad \text{on} \quad \Gamma\},$$

$$K_\delta = \{w \in H^1(\Omega_\delta) | \; [w] \geqslant 0 \quad \text{on} \quad \Xi_{l+\delta}; \; w = 0 \quad \text{on} \quad \Gamma\}.$$

The function u is the solution of the variational inequality

$$u \in K_0 : \quad \int_\Omega \langle \nabla u, \nabla v - \nabla u \rangle \geqslant \int_\Omega f(v - u) \quad \forall v \in K_0. \tag{3}$$

In particular, u satisfies (1), (2). There are additional relations holding on Ξ_l, and we shall discuss them in the sequel.

For a small parameter δ, the family of perturbed problems defined in Ω_δ is considered.

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We want to find a function u^δ such that

$$-\Delta u^\delta = f, \tag{4}$$

$$u^\delta = 0 \quad \text{on} \quad \Gamma, \ [u^\delta] \geqslant 0 \quad \text{on} \quad \Xi_{l+\delta}. \tag{5}$$

Similar to (3), the function u^δ is the solution of the variational inequality

$$u^\delta \in K_\delta : \quad \int_{\Omega_\delta} \langle \nabla u^\delta, \nabla v - \nabla u^\delta \rangle \geqslant \int_{\Omega_\delta} f(v - u^\delta) \quad \forall v \in K_\delta. \tag{6}$$

The energy functional for problem (3) is defined by the formula

$$J(\Omega) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \int_\Omega f u, \tag{7}$$

and the energy functional for problem (6) is equal to

$$J(\Omega_\delta) = \frac{1}{2} \int_{\Omega_\delta} |\nabla u^\delta|^2 - \int_{\Omega_\delta} f u^\delta. \tag{8}$$

The aim of this paper is to find the derivative

$$\frac{dJ(\Omega_\delta)}{d\delta}\Big|_{\delta=0} = \lim_{\delta \to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \tag{9}$$

which describes the behaviour of the energy functional $J(\Omega)$ with respect to the variation of the crack length, and to analyse the Rice–Cherepanov integrals corresponding to problem (3).

The dependence of the energy functional on the crack length is important in the fracture mechanics. The derivative of the functional is often used to formulate fracture criteria. The formulae for derivatives of the energy functional with respect to the crack length are called the *Griffith formulae*. Invariant integrals over curves surrounding the crack tips are usually called the *Rice–Cherepanov integrals* (for the history of the question see Parton & Morozov [5] and Cherepanov [2]).

Concerning the background material used in the present paper, derivatives of the energy functional for the Poisson equation and for the linear elasticity equations with linear boundary conditions holding at Ξ_l have been studied extensively [6, 7] (see also [8, 9, 1]). The regularity of solutions in nonsmooth domains have been analysed at length [10, 11, 12, 13]. As for asymptotic properties of solutions in domains with cracks (with linear boundary conditions on Ξ_l), we refer the reader elsewhere [6, 14, 15]. Other aspects of elliptic problems in domains with nonsmooth boundaries can be found [16, 17, 18, 19, 20].

2 Preliminary statements and formulae

To find the derivative (9), the transformation of the domain Ω_δ onto the domain Ω is introduced. The transformation is constructed in the following way.

Let $\theta \in C_0^\infty(D)$ be any function such that $\theta = 1$ in a neighbourhood of the point $x_l = (l, 0)$. To simplify the arguments, the function θ is assumed to be equal to zero in a neighbourhood of the point $(0, 0)$. Consider the transformation of the independent

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variables

$$y_1 = x_1 - \delta\theta(x_1, x_2),$$
$$y_2 = x_2, \tag{10}$$

where $(x_1, x_2) \in \Omega_\delta, (y_1, y_2) \in \Omega$. The Jacobian q_δ of this transformation is equal to

$$\left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right| = 1 - \delta\theta_{x_1}.$$

For small δ, the Jacobian q_δ is positive, hence the transformation (10) is one-to-one. Therefore, in view of (10), we have $y = y(x, \delta), x = x(y, \delta)$.

Let $u^\delta(x)$ be the solution of (6), and $u^\delta(x) = u_\delta(y), x = x(y, \delta)$. We have the following formulae:

$$u^\delta_{x_1} = u_{\delta y_1}(1 - \delta\theta_{x_1}),$$
$$u^\delta_{x_2} = u_{\delta y_1}(-\delta\theta_{x_2}) + u_{\delta y_2}. \tag{11}$$

Consequently,

$$\int_{\Omega_\delta} |\nabla u^\delta|^2 dx = \int_\Omega \langle A_\delta \nabla u_\delta, \nabla u_\delta\rangle dy,$$

where $A_\delta = A_\delta(y)$ is the matrix such that

$$A_\delta(y) = \frac{1}{1 - \delta\theta_{x_1}} \begin{pmatrix} (1 - \delta\theta_{x_1})^2 + \delta^2\theta^2_{x_2} & -\delta\theta_{x_2} \\ -\delta\theta_{x_2} & 1 \end{pmatrix}, \quad \theta = \theta(x(y, \delta)).$$

Note that $A_0(y) = E$ is the identity matrix.

It is easy to find the derivative of $A_\delta(y)$ with respect to δ, namely,

$$A'(y) = \frac{dA_\delta(y)}{d\delta}|_{\delta=0} = \lim_{\delta\to 0} \frac{A_\delta(y) - A_0(y)}{\delta}.$$

We have

$$A'(y) = \begin{pmatrix} -\theta_{y_1}(y) & -\theta_{y_2}(y) \\ -\theta_{y_2}(y) & \theta_{y_1}(y) \end{pmatrix}. \tag{12}$$

By the change of variables, it follows that

$$\int_{\Omega_\delta} f u^\delta dx = \int_\Omega \frac{f(x(y, \delta))u_\delta(y)}{1 - \delta\theta_{x_1}} dy.$$

Denote

$$f^\delta(y) = \frac{f(x(y, \delta))}{1 - \delta\theta_{x_1}}$$

and find the derivative

$$f'(y) = \frac{df^\delta(y)}{d\delta}|_{\delta=0} = \lim_{\delta\to 0} \frac{f^\delta(y) - f^0(y)}{\delta}.$$

Assuming that y, δ are independent variables in (10), we have $x = x(y, \delta)$. Differentiation of (10) with respect to δ yields

$$0 = \frac{dx_1}{d\delta} - \theta - \delta\theta_{x_1}\frac{dx_1}{d\delta},$$

whence

$$\frac{dx_1}{d\delta} = \frac{\theta}{1 - \delta\theta_{x_1}}, \quad \frac{dx_2}{d\delta} = 0. \tag{13}$$

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Consequently, by (13),

$$\frac{\partial f(x(y,\delta))}{\partial \delta}|_{\delta=0} = f_{x_1}\frac{dx_1}{d\delta}|_{\delta=0} + f_{x_2}\frac{dx_2}{d\delta}|_{\delta=0} = f_{y_1}\theta. \tag{14}$$

Now we are in a position to find the derivative $f'(y)$. Indeed, by (14),

$$f'(y) = \lim_{\delta\to 0}\left(\frac{f(x(y,\delta))}{1-\delta\theta_{x_1}}\right) - f(y))\frac{1}{\delta} = \lim_{\delta\to 0}\frac{f(x(y,\delta)) - f(y)}{\delta} +$$

$$+\theta_{x_1}f(y)|_{\delta=0} = f_{y_1}\theta + \theta_{y_1}f = \frac{\partial}{\partial y_1}(\theta f),$$

i.e.

$$f'(y) = (\theta f)_{y_1}(y). \tag{15}$$

Since $f \in C^1(\bar{\Omega})$, we can see that as $\delta \to 0$

$$\frac{f^\delta(y) - f^0(y)}{\delta} \to f'(y) \quad \text{in} \quad L^\infty(\Omega). \tag{16}$$

Also, notice that, in addition to (12), as $\delta \to 0$

$$\frac{A_\delta(y) - A_0(y)}{\delta} \to A'(y) \quad \text{in} \quad L^\infty(\Omega). \tag{17}$$

In view of (10), let $x = x(y,\delta)$. Then $w^\delta(x) = w_\delta(y)$. The inclusion $w^\delta \in K_\delta$ implies $w_\delta \in K_0$, and, conversely, $w_\delta \in K_0$ implies $w^\delta \in K_\delta$. This means that the transformation (10) maps K_δ on K_0, and it is one-to-one. Now we shall prove an auxiliary statement which is used in the sequel.

Lemma 2.1 *Let u^δ be the solution of* (6), *$u^\delta(x) = u_\delta(y)$, and u be the solution of* (3). *Then*

$$\|u_\delta - u\|_{H^1(\Omega)} \to 0, \quad \delta \to 0. \tag{18}$$

Proof The function $u^\delta \in K_\delta$ is the solution of the variational inequality (6). We change the variables in (6) in accordance with (10). To this end, we write (11) as

$$\nabla_x u^\delta = \nabla_y u_\delta - \delta g D_1 u_\delta,$$

where $D_1 u_\delta = u_{\delta y_1}, g = \nabla_x \theta$, which transforms (6) into the inequality

$$\int_\Omega \langle \nabla u_\delta, \nabla\tilde{v} - \nabla u_\delta\rangle \frac{1}{q_\delta} \geqslant \int_\Omega \langle h_\delta, \nabla\tilde{v} - \nabla u_\delta\rangle + \int_\Omega f^\delta(\tilde{v} - u_\delta) + \tag{19}$$

$$+\delta\int_\Omega \langle \nabla u_\delta, gD_1\tilde{v} - gD_1u_\delta\rangle\frac{1}{q_\delta} + \delta^2\int_\Omega \langle gD_1u_\delta, gD_1\tilde{v} - gD_1u_\delta\rangle\frac{1}{q_\delta} \quad \forall\, \tilde{v} \in K_0.$$

Here

$$h_\delta = \frac{\delta g D_1 u_\delta}{q^\delta} \to 0 \quad \text{in} \quad [L^2(\Omega)]^2$$

as $\delta \to 0$. It is of importance that the inequality (19) holds for all $\tilde{v} \in K_0$. ¿From (6) it follows that

$$\|u^\delta\|_{H^1(\Omega_\delta)} \leqslant c$$

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uniformly in δ, consequently,

$$\|u_\delta\|_{H^1(\Omega)} \leqslant c \tag{20}$$

uniformly in δ.

We can substitute $\tilde{v} = u, v = u_\delta$ in (19), (3), respectively, and sum the relations. This implies

$$\int_\Omega \left\langle \nabla u - \nabla u_\delta, \nabla u - \frac{\nabla u_\delta}{q_\delta} \right\rangle \leqslant \int_\Omega (f^\delta - f)(u_\delta - u) + \int_\Omega \langle h_\delta, \nabla u_\delta - \nabla u\rangle + P(\delta, u, u_\delta, g). \tag{21}$$

By (19), (20), we have $P(\delta, u, u_\delta, g) \to 0$ as $\delta \to 0$. The inequality (21) can be written as

$$\|\nabla u - \nabla u_\delta\|_0^2 + \int_\Omega \left\langle \nabla u - \nabla u_\delta, \nabla u_\delta - \frac{\nabla u_\delta}{q_\delta} \right\rangle \leqslant \int_\Omega \langle h_\delta, \nabla u_\delta - \nabla u\rangle$$

$$+ \int_\Omega (f^\delta - f)(u_\delta - u) + P(\delta, u, u_\delta, g),$$

where $\|\cdot\|_0$ is the norm in $L^2(\Omega)$. Hence,

$$\frac{1}{2}\|\nabla u - \nabla u_\delta\|_0^2 \leqslant \|\nabla u_\delta - \frac{\nabla u_\delta}{q_\delta}\|_0^2 + \|h_\delta\|_0 \|\nabla u_\delta - \nabla u\|_0$$

$$+ \|f^\delta - f\|_0 \|u_\delta - u\|_0 + P(\delta, u, u_\delta, g). \tag{22}$$

It is easy to see that

$$\|\nabla u_\delta - \frac{\nabla u_\delta}{q_\delta}\|_0 \leqslant \delta \frac{\max_\Omega |\theta_x|}{\min_\Omega |q_\delta|} \|\nabla u_\delta\|_0 \to 0, \quad \delta \to 0.$$

In this case, the inequality (22) implies,

$$\|\nabla u_\delta - \nabla u\|_0 \to 0$$

as $\delta \to 0$ which completes the proof of Lemma 2.1. □

Remark 2.1 *Since f^δ is the smooth function, we have*

$$\|f^\delta - f\|_0 \leqslant c\delta$$

with a constant c being uniform with respect to δ. Taking into account the formulae for $h_\delta, P(\delta, u, u_\delta, g)$, it follows from (22) *that the result of Lemma 1 can be improved, namely, there exists a constant $c > 0$ such that*

$$\|u_\delta - u\|_{H^1(\Omega)} \leqslant c\delta.$$

3 The main results

To underline the dependence of the domain Ω on the crack length l we shall write Ω_l instead of Ω in some places of this section.

Let $J(\Omega_l)$ be defined by formula (7), and the function θ be chosen the same as that

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at the beginning of § 2. Our purpose is to establish the Griffith formula, which gives the derivative of the energy functional with respect to the crack length for problem (3).

Theorem 3.1 *The derivative of $J(\Omega_l)$ with respect to l is given by*

$$\frac{dJ(\Omega_l)}{dl} = -\frac{1}{2}\int_\Omega (\theta_{y_1}(u_{y_1}^2 - u_{y_2}^2) + 2\theta_{y_2}u_{y_1}u_{y_2}) - \int_\Omega (\theta f)_{y_1} u. \tag{23}$$

Proof Introduce the notation

$$\Pi(\Omega;\varphi) = \frac{1}{2}\int_\Omega |\nabla\varphi|^2 - \int_\Omega f\varphi,$$

$$\Pi_\delta(\Omega;\varphi) = \frac{1}{2}\int_\Omega \langle A_\delta\nabla\varphi, \nabla\varphi\rangle - \int_\Omega f^\delta\varphi,$$

$$\Pi(\Omega_\delta;\varphi) = \frac{1}{2}\int_{\Omega_\delta} |\nabla\varphi|^2 - \int_{\Omega_\delta} f\varphi.$$

The solution u of problem (3) satisfies the relation

$$\Pi(\Omega;u) = \min_{\varphi\in K_0} \Pi(\Omega;\varphi)$$

and the solution u^δ of problem (6) satisfies

$$\Pi(\Omega_\delta;u^\delta) = \min_{\varphi\in K_\delta} \Pi(\Omega_\delta;\varphi).$$

We have noted that transformation (10) establishes a one-to-one mapping between K_δ and K_0, hence

$$\min_{\varphi\in K_0} \Pi_\delta(\Omega;\varphi) = \min_{\varphi\in K_\delta} \Pi(\Omega_\delta;\varphi). \tag{24}$$

According to our notation,

$$J(\Omega) = \Pi(\Omega;u); \quad J(\Omega_\delta) = \Pi(\Omega_\delta;u^\delta),$$

where u and u^δ are the solutions of (3) and (6), respectively. Now we can find the limit (9). Indeed, by (24),

$$\frac{J(\Omega_\delta) - J(\Omega)}{\delta} = \frac{\Pi(\Omega_\delta;u^\delta) - \Pi(\Omega;u)}{\delta}$$

$$= \frac{\Pi_\delta(\Omega;u_\delta) - \Pi(\Omega;u)}{\delta} \leqslant \frac{\Pi_\delta(\Omega;u) - \Pi(\Omega;u)}{\delta}$$

and consequently,

$$\limsup_{\delta\to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} \leqslant \limsup_{\delta\to 0} \frac{\Pi_\delta(\Omega;u) - \Pi(\Omega;u)}{\delta}$$

$$= \frac{1}{2}\int_\Omega \langle A'\nabla u, \nabla u\rangle - \int_\Omega f'u. \tag{25}$$

On the other hand, by Lemma 1 and (16), (17), as $\delta \to 0$

$$\frac{\Pi_\delta(\Omega;u_\delta) - \Pi(\Omega;u_\delta)}{\delta} = \frac{1}{2}\int_\Omega \left\langle \frac{A_\delta - A_0}{\delta}\nabla u_\delta, \nabla u_\delta \right\rangle \tag{26}$$

$$\pm\frac{1}{2}\int_\Omega \langle A'\nabla u_\delta, \nabla u_\delta\rangle - \frac{1}{\delta}\int_\Omega (f^\delta - f)u_\delta = \frac{1}{2}\int_\Omega \left\langle \left(\frac{A_\delta - A_0}{\delta} - A'\right)\nabla u_\delta, \nabla u_\delta \right\rangle$$

$$+\frac{1}{2}\int_\Omega \langle A'\nabla u_\delta, \nabla u_\delta\rangle - \frac{1}{\delta}\int_\Omega (f^\delta - f)u_\delta \to \frac{1}{2}\int_\Omega \langle A'\nabla u, \nabla u\rangle - \int_\Omega f'u.$$

Hence

$$\liminf_{\delta\to 0}\frac{J(\Omega_\delta) - J(\Omega)}{\delta} \geqslant \liminf_{\delta\to 0}\frac{\Pi_\delta(\Omega; u_\delta) - \Pi(\Omega; u_\delta)}{\delta}$$

$$= \frac{1}{2}\int_\Omega \langle A'\nabla u, \nabla u\rangle - \int_\Omega f'u. \tag{27}$$

Comparing (25) and (27), we find

$$\lim_{\delta\to 0}\frac{J(\Omega_\delta) - J(\Omega)}{\delta} = \frac{1}{2}\int_\Omega \langle A'\nabla u, \nabla u\rangle - \int_\Omega f'u,$$

i.e.

$$\frac{dJ(\Omega_l)}{dl} = \frac{1}{2}\int_\Omega \langle A'\nabla u, \nabla u\rangle - \int_\Omega f'u. \tag{28}$$

By (12) and (15), a substitution of A' and f' in (28) implies the Griffith formula (23). The proof of Theorem 3.1 is complete. □

The solution u of problem (3) satisfies the following boundary conditions:

$$[u] \geqslant 0,\ [u_{y_2}] = 0,\ u_{y_2} \leqslant 0,\ u_{y_2}[u] = 0 \quad \text{on} \quad \Xi_l. \tag{29}$$

We do not provide detailed proof of (29). In fact, (29) can be obatained by substituting the proper test functions in variational inequality (3), and integrating by parts. The derivation of similar nonlinear boundary conditions of the inequality type is performed elsewhere [12] in the case of plate equations.

First, we have to prove that the right-hand side of (23) does not depend upon θ. It follows from Yakunina [21] that the solution of problem (3) has an additional regularity up to the crack faces. For any $x \in \Xi_l$ there exists a neighbourhood V of the point x such that

$$u \in H^2(V \setminus \Xi_l). \tag{30}$$

Consequently, $u \in H^{3/2}_{loc}(\Xi_l^\pm)$.

To prove that the right-hand side of (23) is independent of θ, we consider two functions θ_1, θ_2 with the required properties. Denote by Λ the difference between right-hand sides of (23) corresponding to θ_1, θ_2. Then

$$\Lambda = -\frac{1}{2}\int_\Omega (\theta_{y_1}(u_{y_1}^2 - u_{y_2}^2) + 2\theta_{y_2}u_{y_1}u_{y_2}) - \int_\Omega (\theta f)_{y_1} u, \tag{31}$$

where $\theta = \theta_1 - \theta_2$. Since θ_1, θ_2 are equal to 1 in some neighbourhoods of the point x_l, in (31) we integrate outside of a ball B_{x_l} centred at x_l. Integrating by parts in (31), we find

$$\Lambda = \int_{\Omega\setminus B_{x_l}} \theta u_{y_1}(\Delta u + f) + \int_{\Xi_l\setminus B_{x_l}} \theta[u_{y_2}u_{y_1}],$$

and by (1) and (29),

$$\Lambda = \int_{\Xi_l \setminus B_{x_l}} \theta [u_{y_2} u_{y_1}].$$

To prove $\Lambda = 0$, it suffices to establish that

$$u_{y_2}[u_{y_1}] = 0 \quad \text{a.e. on} \quad \Xi_l \cap \{\text{supp}\theta\}. \tag{32}$$

Here, by $\{\text{supp}\theta\}$ we denote the support of θ. Introduce the set

$$M = \{x \in \Xi_l \cap \{\text{supp}\theta\} | \ \ [u(x)] > 0\}.$$

The set M is open, and by (30), u is continuous up to Ξ_l. By (29), we have

$$u_{y_2} = 0 \quad \text{a.e. on} \quad M. \tag{33}$$

The complement of M is characterised by the condition

$$[u] = 0 \quad \text{on} \qquad (\Xi_l \cap \{\text{supp}\theta\}) \setminus M.$$

Hence (see Kinderlehrer & Stampacchia [22], Ch.2, Theorem A.1)

$$[u_{y_1}] = 0 \quad \text{a.e. on} \qquad (\Xi_l \cap \{\text{supp}\theta\}) \setminus M. \tag{34}$$

Consequently, by (33) and (34), we arrive at (32), which proves the independence of the right-hand side of (23) on θ.

Note that the independence of the right-hand side of (23) on θ follows by the existence of derivative (23). Indeed, since

$$\liminf_{\delta \to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta} = \limsup_{\delta \to 0} \frac{J(\Omega_\delta) - J(\Omega)}{\delta}$$

and both sides are independent of θ, we conclude that the derivative

$$\frac{dJ(\Omega_\delta)}{d\delta}\Big|_{\delta=0}$$

exists and does not depend upon θ.

The proved assertion means that the right-hand side of (23) actually depends upon the point x_l, and the right-hand side f of (1). This allows us to write (23) as the following Griffith formula:

$$\frac{dJ(\Omega_l)}{dl} = k(x_l, f), \tag{35}$$

where $\Omega_l = D \setminus \overline{\Xi}_l$, k is a functional depending on x_l, f. In particular, we have

$$J(\Omega_{l+\delta}) = J(\Omega_l) + k(x_l, f)\delta + \alpha(\delta)\delta,$$

where $\Omega_{l+\delta} = D \setminus \overline{\Xi}_{l+\delta}$ and $\alpha(\delta) \to 0$ as $\delta \to 0$.

Note that $k(x_l, f) = 0$, provided that the solution u is sufficiently smooth, which implies

$$\frac{dJ(\Omega_l)}{dl} = 0. \tag{36}$$

In particular, the equality (36) holds for $u \in H^2(\Omega_l)$. Indeed, in this case, we can extend Ξ_l beyond the points $(l, 0), (0, 0)$, so that the extension denoted by Ξ_* crosses the boundary Γ. As a result, the domain Ω is divided into two subdomains, Ω_1, Ω_2. By (23), (32) and

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(1), we have

$$\frac{dJ(\Omega_l)}{dl} = \sum_{i=1}^{2}\left\{-\frac{1}{2}\int_{\Omega_i}(\theta_{y_1}(u_{y_1}^2 - u_{y_2}^2) + 2\theta_{y_2}u_{y_1}u_{y_2}) - \int_{\Omega_i}(\theta f)_{y_1}u\right\}$$

$$= \sum_{i=1}^{2}\int_{\Omega_i}\theta u_{y_1}(\Delta u + f) + \int_{\Xi_*}\theta u_{y_2}[u_{y_1}] = 0. \tag{37}$$

In fact, to prove (37) we need a local regularity of the solution near the point x_l. The inclusion $u \in H^2(\Omega)$ provides the sufficient regularity to integrate by parts in (37).

An additional regularity near the point x_l can be shown in some particular cases. For example, assume that the solution u satisfies the condition

$$[u] = 0 \quad \text{on} \quad B_{x_l} \cap \Xi_l,$$

where B_{x_l} is a ball centred at x_l. In this case we can prove that the equation

$$-\Delta u = f \tag{38}$$

holds in B_{x_l} in the sense of distributions, consequently, $u \in H^3_{loc}(B_{x_l})$, and all the integrals in (37) are well defined. Hence, equality (36) follows from (1) and (32).

Let us prove that (38) holds in B_{x_l}, provided that $[u] = 0$ on $B_{x_l} \cap \Xi_l$. Consider a closed smooth curve such that it confines a bounded simply connected domain $Q \subset \Omega_l$ and contains $\Xi_l \cap B_{x_l}$ as a part of its boundary γ. It is well known that the conditions $u \in H^1(Q), \Delta u \in L^2(Q)$ imply $\frac{\partial u}{\partial n} \in H^{-1/2}(\gamma)$, where n is an external normal vector to the boundary γ. Moreover, the following Green formula holds:

$$(\Delta u, v)_Q - \left\langle \frac{\partial u}{\partial n}, v\right\rangle_{1/2,\gamma} = -\langle \nabla u, \nabla v\rangle_Q \quad \forall v \in H^1(Q), \tag{39}$$

where $\langle\cdot,\cdot\rangle_{1/2,\gamma}$ means the duality pairing between $H^{1/2}(\gamma)$ and $H^{-1/2}(\gamma)$, and the lower indices Q in $(\cdot,\cdot)_Q$, $\langle\cdot,\cdot\rangle_Q$, mean the integration over Q. It is clear that the domain Q can be chosen in different ways. In any case, one of the inclusions $\Xi_l^+ \cap B_{x_l} \subset \gamma$, $\Xi_l^- \cap B_{x_l} \subset \gamma$ holds. Let $\phi \in C_0^\infty(B_{x_l})$. We substitute $v = u + \phi$ into the variational inequality (3) as a test function. This implies

$$\int_{B_{x_l}\setminus\Xi_l}\langle \nabla u, \nabla\phi\rangle = \int_{B_{x_l}\setminus\Xi_l} f\phi,$$

whence, integrating by parts and taking into account (1) and (39), we obtain

$$\left\langle \frac{\partial u}{\partial n}, \phi\right\rangle_{1/2,\gamma}^{-} - \left\langle \frac{\partial u}{\partial n}, \phi\right\rangle_{1/2,\gamma}^{+} = 0 \quad \forall \phi \in C_0^\infty(B_{x_l}). \tag{40}$$

The signs $\pm$ correspond to $B_{x_l}^\pm$, respectively. Here $B_{x_l}^+ = B_{x_l} \cap \{x_2 > 0\}, B_{x_l}^- = B_{x_l} \cap \{x_2 < 0\}$. Identity (40) actually provides the precise meaning of the condition $[\frac{\partial u}{\partial n}] = 0$ on Ξ_l in (29). Since $u \in H^1(B_{x_l} \setminus \Xi_l)$ and $[u] = 0$ on $B_{x_l} \cap \Xi_l$, it follows that $u \in H^1(B_{x_l})$. For example, below we denote by $(\Delta u + f, \phi)$ the value of the distribution $\Delta u + f \in \mathscr{D}'(B_{x_l})$ evaluated on the test function $\phi \in C_0^\infty(B_{x_l})$. Consider the following equalities:

$$(\Delta u + f, \phi) = -\langle \nabla u, \nabla\phi\rangle_{B_{x_l}} + (f, \phi)_{B_{x_l}}$$

$$= -\langle \nabla u, \nabla\phi\rangle_{B_{x_l}^+} - \langle \nabla u, \nabla\phi\rangle_{B_{x_l}^-} + (f, \phi)_{B_{x_l}}. \tag{41}$$

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We can integrate by parts in the right-hand side of (41) which implies

$$(\Delta u + f, \phi) = \langle \frac{\partial u}{\partial n}, \phi \rangle^{+}_{1/2,\gamma} - \langle \frac{\partial u}{\partial n}, \phi \rangle^{-}_{1/2,\gamma} + (\Delta u + f, \phi)_{B^{+}_{x_l}} + (\Delta u + f, \phi)_{B^{-}_{x_l}}. \quad (42)$$

By (40) and (1), the right-hand side of (42) is equal to zero, which proves that (38) holds in B_{x_l} in the sense of distributions.

We can rewrite (23) in the form which does not contain θ. To this end, consider a ball $B_{x_l}(r)$ of radius r with a boundary $\Gamma(r)$, such that $\theta = 1$ on $B_{x_l}(r)$. Integration by parts in (23) implies

$$\frac{dJ(\Omega_l)}{dl} = \int_{\Omega \setminus B_{x_l}(r)} \theta u_{y_1}(\Delta u + f) + \int_{\Xi_l \setminus B_{x_l}(r)} \theta u_{y_2}[u_{y_1}]$$

$$+ \int_{B_{x_l}(r)\setminus \Xi_l} \theta f u_{y_1} + \frac{1}{2}\int_{\Gamma(r)} \theta(v_1(u^2_{y_1} - u^2_{y_2}) + 2v_2 u_{y_1} u_{y_2}),$$

where (v_1, v_2) is the unit external normal vector to $\Gamma(r)$. Hence, by (1) and (32),

$$\frac{dJ(\Omega_l)}{dl} = \int_{B_{x_l}(r)\setminus \Xi_l} f u_{y_1} + \frac{1}{2}\int_{\Gamma(r)} (v_1(u^2_{y_1} - u^2_{y_2}) + 2v_2 u_{y_1} u_{y_2}). \quad (43)$$

Now assume that $f = 0$ in some neighbourhood V of the point x_l. For small r, we have $B_{x_l}(r) \subset V$, and the formula (43) implies

$$\frac{dJ(\Omega_l)}{dl} = \frac{1}{2}\int_{\Gamma(r)} (v_1(u^2_{y_1} - u^2_{y_2}) + 2v_2 u_{y_1} u_{y_2}). \quad (44)$$

The right-hand side of (44) does not depend upon r, consequently we arrive at the following conclusion. Let u be the solution of the problem (3), and f be equal to zero in some neighbourhood of the point x_l. Then the integral

$$I = \int_{\Gamma(r)} (v_1(u^2_{y_1} - u^2_{y_2}) + 2v_2 u_{y_1} u_{y_2})$$

is independent of r for all sufficiently small r. Moreover, the above arguments show that the integral

$$I = \int_{C} (v_1(u^2_{y_1} - u^2_{y_2}) + 2v_2 u_{y_1} u_{y_2}) \quad (45)$$

does not depend upon C for any closed curve C surrounding the point x_l. In this case, $v = (v_1, v_2)$ is the normal unit vector to the curve C. A part of this curve may belong to Ξ_l. In this last case, we can integrate over Ξ^+ or Ξ^-, since, in view of (29) and (32), the jump $[u_{y_1} u_{y_2}]$ is equal to zero on Ξ_l. Here $\Xi^{\pm} = \Xi^{\pm}_l \cap C$ (see Fig. 3).

The integral of the form (45) is called the Rice–Cherepanov integral. We have to note that the statement obtained is proved for nonlinear boundary conditions (29). This statement is similar to the well-known result in the linear elasticity theory with linear boundary conditions presribed on Ξ_l [2]. Of course, the above independence takes place provided that f is equal to zero in the domain with the boundary C.

Finally, we construct a measure μ defined on Borel sets of Ξ_l. The measure characterises interaction forces between the crack faces. First, recall that the smallest σ-algebra containing all compact sets in Ξ_l is called the Borel σ-algebra. Any σ-additive real-valued function defined on the Borel σ-algebra, which is finite for all compact sets $B \subset \Xi_l$ is called a measure on Ξ_l.

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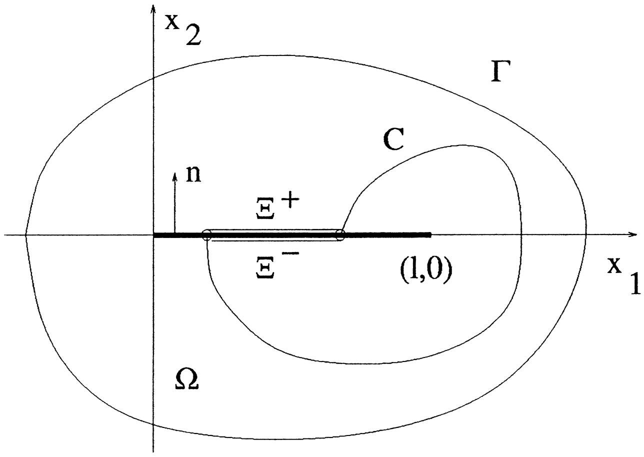

FIGURE 3. Piecewise smooth nonintersecting path C which begins and ends on the crack Ξ_l and surrounds the tip $(0, l)$.

Let $C_0(\Xi_l)$ be the space of continuous functions defined on Ξ_l and having compact support in Ξ_l. Convergence in $C_0(\Xi_l)$ is defined in the following way. We say that $\phi_m \to \phi$ in $C_0(\Xi_l)$ if the support of ϕ_m , $m = 1, 2, ...$, belongs to a fixed compact subset $B \subset \Xi_l$ and the sequence ϕ_m converges to ϕ uniformly on Ξ_l. In what follows, we may assume that Ξ_l is a C^1-curve in D without self-intersections such that the points $(0, 0), (l, 0)$ are the curve ends. In such a case, a solution of problem (3) obviously exists. Denote by $H^{1,0}(\Omega_l) \cap C_0(\Xi_l)$ the linear subset in $H^{1,0}(\Omega_l)$ which contains the functions with continuous and compactly supported traces on both crack faces $\Xi_l^{\pm}$. Here

$$H^{1,0}(\Omega_l) = \{u \in H^1(\Omega_l) | \; u = 0 \quad \text{on} \quad \Gamma\}.$$

Let u be the solution of problem (3). Then $u + \bar{u} \in K_0$ for every $\bar{u} \in K_0$. Consequently, from (3) it follows that

$$\int_{\Omega_l} \langle \nabla u, \nabla \bar{u} \rangle \geqslant \int_{\Omega_l} f \bar{u}. \tag{46}$$

Now we are in a position to prove the following statement.

Theorem 3.2 *On the σ-algebra of Borel sets of Ξ_l we can construct a non-negative measure μ such that the representation*

$$\int_{\Omega_l} \langle \nabla u, \nabla \bar{u} \rangle - \int_{\Omega_l} f \bar{u} = \int_{\Xi_l} [\bar{u}] d\mu \tag{47}$$

holds for all $\bar{u} \in H^{1,0}(\Omega_l) \cap C_0(\Xi_l)$.

Proof To simplify the formulae below, we present the proof for the case when Ξ_l is a segment of a straight line. Consider a linear space M of functions defined on Ξ_l:

$$M = \{\bar{u}^* | \; \bar{u}^* = [\bar{u}], \; \bar{u} \in H^{1,0}(\Omega_l) \cap C_0(\Xi_l)\}.$$

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Define a linear functional on M by the formula

$$L(\bar{u}^*) = \int_{\Omega_l} \langle \nabla u, \nabla \bar{u} \rangle - \int_{\Omega_l} f\bar{u}.$$

It is clear that the functional L is well defined. Its value at the point $\bar{u}^*$ does not depend upon the choice of $\bar{u}$. Indeed, if $\bar{u}_1^* = \bar{u}_2^*$ then by (46), taking into account that $-\bar{u}, \bar{u} \in K_0$ for $\bar{u} = \bar{u}_1 - \bar{u}_2$, it follows that $L(\bar{u}_1^*) = L(\bar{u}_2^*)$.

We show that the space $C_0^1(\Xi_l)$ of continuously differentiable functions on Ξ_l with compact supports is included in M. To prove this statement, we take an arbitrary function $\psi \in C_0^1(\Xi_l)$. Extend the curve Ξ_l beyond both tips $(0,0),(l,0)$, as the straight line, and define the smooth function ξ along the normal n to the extended curve $\tilde{\Xi}_l$:

$$\xi(\tilde{x}) = \begin{cases} 1, & \text{if } \tilde{x} = \bar{x} + \varepsilon n, \quad \bar{x} \in \tilde{\Xi}_l, \quad 0 \leqslant \varepsilon \leqslant \frac{\varepsilon_0}{2} \\ 0, & \text{if } \varepsilon > \varepsilon_0. \end{cases}$$

Since the extension $\tilde{\Xi}_l$ is smooth, the function $\xi(\tilde{x})$ is well-defined. Now it is possible to construct a function Φ in the domain $\{x_2 > 0\}$ assuming that the function ψ is extended to $\tilde{\Xi}_l$ by zero. Indeed, in the domain $\{x_2 > 0\}$, we define

$$\Phi(\tilde{x}) = \psi(\bar{x})\xi(\tilde{x}), \quad \text{if } \tilde{x} = \bar{x} + \varepsilon n, \quad \bar{x} \in \tilde{\Xi}_l, \ \varepsilon > 0.$$

In this case $\Phi(\bar{x}) = \psi(\bar{x})$, $\bar{x} \in \Xi_l$. One can see that the points $\tilde{x} = \bar{x} + \varepsilon n$ may not belong to Ω_l, in general. This should present no difficulty, since ψ vanishes outside of Ξ_l. Assuming that the function Φ is identically equal to zero for $\{x_2 < 0\}$, we obtain $\Phi \in H^{1,0}(\Omega_l) \cap C_0(\Xi_l)$ and, moreover,

$$[\Phi] = \psi \quad \text{on} \quad \Xi_l.$$

Hence, the inclusion $C_0^1(\Xi_l) \subset M$ follows. As a consequence, the positive functional L can be extended to a linear continuous form over the space $C_0(\Xi_l)$ [23]. On the other hand, any positive functional on $C_0(\Xi_l)$ is defined by a measure μ, and

$$L(\phi) = \int_{\Xi_l} \phi d\mu \quad \forall \phi \in C_0(\Xi_l).$$

By the definition of L, we obtain (47), which completes the proof of Theorem 3.2. □

If Ξ_l is a C^2-curve (and, in particular, Ξ_l is a segment of a straight line), it is possible to find the density of the measure μ. This means the existence of a locally integrable function p such that

$$\mu(B) = \int_B p d\Xi_l.$$

In fact, as indicated above, in such a case the function u has the second derivatives square integrable up to Ξ_l. Consequently, we can integrate in the left-hand side of (47), which implies

$$-\int_{\Xi_l} \frac{\partial u}{\partial n}[\bar{u}] d\Xi_l = \int_{\Xi_l} [\bar{u}] d\mu. \tag{48}$$

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It follows from (48) that the density of the measure μ is equal to

$$p = -\frac{\partial u}{\partial n}.$$

Using the regularity result (30) we have $p \in H^{1/2}_{loc}(\Xi_l)$.

4 Conclusion

The Griffith formula for the Poisson equation established in this paper is proved for the nonlinear boundary conditions. Also, the path independence of the Rice–Cherepanov integral is stated. Proof of independence of the Rice–Cherepanov integral uses an additional regularity of the solution up to the crack faces. The derivation of the Griffith formula is based on the regularity of a variational solution. It is of interest to establish the highest regularity of the solution in the vicinity of the crack tips. We hope this will allow us to extract some useful corollaries from the formulae obtained in the paper.

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