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TIME AVERAGES FOR THE LAPLACE GROUP

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Abstract The imaginary powers of the Laplace operator over the circle give a C_0 group of bounded linear operators on $L^p_\theta(0, 2\pi)$ $(1 < p < \infty)$. Whereas the group is unbounded on L^4, this paper shows that the L^4 long-time averages of each f in L^2 are bounded. This is a Fourier restriction phenomenon.

Keywords: imaginary powers of the Laplace operator; Riesz potentials; Fourier restriction phenomena

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1. Introduction

Let $\{R^t\}_{t\in\mathbb{R}}$ be a C_0 group of bounded linear operators, acting on $L^p_\theta(0, 2\pi)$ for some $1 < p < \infty$. We use θ to indicate the space variable. Define the long-time average:

$$A^{(p)} f = \left\{ \limsup_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|R^t f\|^p_{L^p_\theta} \,\mathrm{d}t \right\}^{1/p}. \tag{1.1}$$

If the group is bounded with $\|R^t\| \leqslant M$, then clearly it follows that $A^{(p)} f \leqslant M\|f\|_p$. Remarkably this inequality holds for some unbounded C_0 groups. Let Δ be the Laplace operator over the circle that satisfies $\Delta \mathrm{e}^{\mathrm{i}n\theta} = n^2 \mathrm{e}^{\mathrm{i}n\theta}$. Zygmund [**12**, Theorem 1] showed that the periodic Schrödinger group $\mathrm{e}^{\mathrm{i}t\Delta}$ has

$$\iint_{[0,2\pi]\times[0,2\pi]} |\mathrm{e}^{\mathrm{i}t\Delta} f(\theta)|^4 \,\mathrm{d}t\,\mathrm{d}\theta \leqslant 2\|f\|^4_{L^2_\theta}. \tag{1.2}$$

We obtain related estimates for the Laplace group $R^t = \Delta^{-\mathrm{i}t/2}$, where

$$(\Delta^{-\mathrm{i}t/2} f)(\theta) \sim \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{a_n}{|n|^{\mathrm{i}t}} \mathrm{e}^{\mathrm{i}n\theta} \tag{1.3}$$

for $f = \sum a_n \mathrm{e}^{\mathrm{i}n\theta} \in L^2_\theta$. For brevity we take $a_0 = 0$ throughout. The main result is the following theorem.

Theorem 1.1. *Let $f \sim \sum a_n \mathrm{e}^{\mathrm{i}n\theta} \in L^2_\theta$. Then the long-time averages of the Laplace group satisfy*

$$\|f\|^4_{L^2_\theta} \leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \|\Delta^{-\mathrm{i}t/2} f\|^4_{L^4_\theta} \,\mathrm{d}t \leqslant 4\|f\|^4_{L^2_\theta}. \tag{1.4}$$

This (quite surprising) result relies upon a smoothing effect of the time average, since $\Delta^{-\mathrm{i}t/2}$ is an unbounded group on L^4_θ, as we show in §§ 2 and 3, where we prove $L^p_\theta \to L^p_\theta$ operator bounds for $\Delta^{-\mathrm{i}t/2}$.

The proof of Theorem 1.1 is carried out for trigonometric polynomials by a combinatorial argument in § 4, and then extended in § 5 to the general L^2_θ case. We introduce a Banach space $B^4_t L^4_\theta$ of L^4_θ-valued Bohr almost-periodic functions in time such that $\Delta^{-\mathrm{i}t/2}$ is bounded as an operator $L^2_\theta \to B^4_t L^4_\theta$.

The group $\Delta^{-\mathrm{i}t/2}$ arises via the periodic Riesz potential kernel in several applications [**7**, § 19.3]. The spectral theory of operator groups on L^4_θ is treated in [**2**].

2. Upper bounds on the Laplace group

Proposition 2.1. *Let $p > 1$ and $r = \max(p, q)$, where $1/p + 1/q = 1$. Then, for $0 < \epsilon < 2/(r-2)$, there exist constants $c_p(\epsilon), C_p(\epsilon) > 0$ such that*

$$c_p(\epsilon)|t|^{(1/2)-(1/r)} \leqslant \|\Delta^{-\mathrm{i}t/2}\|_{L^p_\theta \to L^p_\theta} < C_p(\epsilon)(1+|t|)^{1-(2/r)+\epsilon} \quad (t \in \mathbb{R}). \tag{2.1}$$

In particular, for $p = 4$ the following holds:

$$c_4(\epsilon)|t|^{1/4} \leqslant \|\Delta^{-\mathrm{i}t/2}\|_{L^4_\theta \to L^4_\theta} < C_4(\epsilon)(1+|t|)^{(1/2)+\epsilon} \quad (t \in \mathbb{R}). \tag{2.2}$$

Proof of the upper bound. The strong form of Marcinkiewicz's Multiplier Theorem [**5**, § 8], applied to $\phi(y) = |y|^{-\mathrm{i}t}$, gives an upper bound for the operator norm. As ϕ has constant modulus 1, and as the variation over dyadic intervals $[2^k, 2^{k+1}]$ is uniformly bounded by $|t|$, this ϕ determines an L^p_θ multiplier $a_n \mapsto \phi(n)a_n$ for all $p > 1$, and we deduce

$$\|\Delta^{-\mathrm{i}t/2}\|_{L^p_\theta \to L^p_\theta} \leqslant C_p(1+|t|) \quad (t \in \mathbb{R}). \tag{2.3}$$

So the operators are bounded, with norms of at most linear growth in $|t|$. Let $r > 2$, suppose $0 < \epsilon < 2/(r-2)$, and $p = 2 + (2/\epsilon)$. Now we may apply Riesz–Thorin interpolation between L^2_θ and L^p_θ. Plugging in the exact value $\|\Delta^{-\mathrm{i}t/2}\|_{L^2_\theta \to L^2_\theta} = 1$ and the bound (2.3) gives

$$\|\Delta^{-\mathrm{i}t/2}\|_{L^r_\theta \to L^r_\theta} < C_r(\epsilon)(1+|t|)^{1-(2/r)+\epsilon} \quad (t \in \mathbb{R}), \tag{2.4}$$

where ϵ can be made arbitrarily small, at the cost of growth in $C_r(\epsilon)$. For $1 < q < 2$, since $\Delta^{-\mathrm{i}t/2}$ is self-adjoint, we have

$$\|\Delta^{-\mathrm{i}t/2}\|_{L^q_\theta \to L^q_\theta} < C_q(\epsilon)(1+|t|)^{-1+(2/q)+\epsilon} \quad (t \in \mathbb{R}) \tag{2.5}$$

by considering the dual exponent $r = q/(q-1)$. Thus the right-hand side of (2.1) is proven. □

3. Lower bound on the Laplace group

We shall use a test function related to the imaginary part of the periodic zeta function [**1**] to prove the left-hand inequality of Proposition 2.1. Let $s = \sigma + it$ be within strip

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$\Omega = \{s : 0 < \operatorname{Re} s < 1\}$. Define

$$m_s(\theta) = \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n^{1-s}}. \tag{3.1}$$

If $\sigma < 1/p$, then both m_σ and $m_{\sigma+\mathrm{i}t}$ are L^p_θ functions and the following holds:

$$m_s(\theta) = (\Delta^{-\mathrm{i}\tau/2} m_{s+\mathrm{i}\tau})(\theta) \quad (\tau \in \mathbb{R}). \tag{3.2}$$

The *Hurwitz generalized zeta function* [**10**, § 2] is initially defined by

$$\zeta(s,a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \quad (a > 0), \tag{3.3}$$

which is clearly analytic in the half-plane $\operatorname{Re} s > 1$ with pole at $s = 1$, and may be continued to exponents $s \in \Omega$ via the loop integral

$$\zeta(s,a) = \frac{\mathrm{e}^{-\mathrm{i}\pi s}\Gamma(1-s)}{2\pi\mathrm{i}} \int_C \frac{z^{s-1}\mathrm{e}^{-az}}{1-\mathrm{e}^{-z}}\,\mathrm{d}z \quad (a > 0), \tag{3.4}$$

where the path C encircles $\mathbb{R}^+$ anticlockwise, including only the pole at $z = 0$. From the Fourier representation of $\zeta(s,a)$, we derive

$$m_s(\theta) = \frac{\sin(\frac{1}{2}\pi s)\Gamma(s)}{(2\pi)^s}\left\{\zeta\left(s, \frac{\theta}{2\pi}\right) - \zeta\left(s, 1 - \frac{\theta}{2\pi}\right)\right\}. \tag{3.5}$$

Hence $m_s(\theta)$ may be continued to an entire function of s. Now making use of the loop integral representation, we obtain

$$m_s(\theta) = \frac{\sin(\frac{1}{2}\pi s)}{\pi^s} K^s\left(1 - \frac{\theta}{\pi}\right), \tag{3.6}$$

where

$$K^s(\beta) = \int_0^\infty x^{s-1}\frac{\sinh(\beta x)}{\sinh(x)}\,\mathrm{d}x \tag{3.7}$$

and, for $\beta \to 1^-$, we deduce the asymptotic behaviour

$$K^s(\beta) \sim \frac{\Gamma(s)}{(1-\beta)^s} = \Gamma(s)\left(\frac{\pi}{\theta}\right)^s. \tag{3.8}$$

We now consider the approximating integral for $m_s(\theta)$, which is a standard Mellin transform [**8**, p. 521] for $s \in \Omega$ with value

$$k_s(\theta) = \int_0^\infty \frac{\sin(\theta u)}{u^{1-s}}\,\mathrm{d}u = \frac{\sin(\frac{1}{2}\pi s)\Gamma(s)\operatorname{sgn}(\theta)}{|\theta|^s} \quad (-\pi < \theta < \pi). \tag{3.9}$$

Define $\Omega_p = \{s : 0 < \operatorname{Re} s < 1/p\}$. If $s \in \Omega_p$, then $k_s \in L^p_\theta(-\pi,\pi)$, and

$$\|k_{\sigma+\mathrm{i}t}\|_{L^p_\theta} = \frac{|\sin(\frac{1}{2}\pi(\sigma+\mathrm{i}t))\ \Gamma(\sigma+\mathrm{i}t)|}{(1-p\sigma)^{1/p}\pi^\sigma}. \tag{3.10}$$

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Given fixed $\sigma \in (0,1)$, as $|t| \to \infty$ it is well known from Stirling's formula [**9**, p. 58] that

$$|\Gamma(\sigma + \mathrm{i}t)| \sim \sqrt{2\pi}\mathrm{e}^{-(\pi/2)|t|}|t|^{\sigma-(1/2)}. \tag{3.11}$$

Thus we have the asymptotic behaviour

$$\frac{\|k_\sigma\|_{L^p_\theta}}{\|k_{\sigma+\mathrm{i}t}\|_{L^p_\theta}} \sim \frac{\Gamma(\sigma)|t|^{(1/2)-\sigma}}{\sqrt{2\pi}} \quad (|t| \to \infty). \tag{3.12}$$

For large $|t|$, this estimate allows us to obtain the lower bound

$$\|\Delta^{-\mathrm{i}t/2}\|_{L^p_\theta \to L^p_\theta} \geqslant \frac{1}{\sqrt{2\pi}}|t|^{(1/2)-(1/p)}. \tag{3.13}$$

4. Proof of Theorem 1.1 (trigonometric polynomials)

The left-hand side of Theorem 1.1 follows from Hölder's inequality, as

$$\|f\|_{L^2_\theta} = \|\Delta^{-\mathrm{i}t/2}f\|_{L^2_\theta} \leqslant \|\Delta^{-\mathrm{i}t/2}f\|_{L^4_\theta} \quad (t \in \mathbb{R}). \tag{4.1}$$

We now prove the right-hand side for trigonometric polynomials. Let $f = \sum_{-N}^{N} a_n \mathrm{e}^{\mathrm{i}n\theta}$ with $a_0 = 0$. The notation $\sum^N$ indicates finite sums of this form, and we sum over all indices subject to the stated conditions. Applying the operator $\Delta^{-\mathrm{i}t/2}$ to f gives

$$|\Delta^{-\mathrm{i}t/2}f(\theta)|^4 = \left|\sum^{N} \frac{a_n \mathrm{e}^{\mathrm{i}n\theta}}{|n|^{\mathrm{i}t}}\right|^4 = \sum_{n_1,n_2,n_3,n_4}^{N} a_{n_1}a_{n_2}\bar{a}_{n_3}\bar{a}_{n_4}\mathrm{e}^{\mathrm{i}\theta(n_1+n_2-n_3-n_4)}\left|\frac{n_3n_4}{n_1n_2}\right|^{\mathrm{i}t}. \tag{4.2}$$

Integrating with respect to θ, we obtain

$$\|\Delta^{-\mathrm{i}t/2}f\|^4_{L^4_\theta} = \int_0^{2\pi} |\Delta^{-\mathrm{i}t/2}f(\theta)|^4 \frac{\mathrm{d}\theta}{2\pi} = \sum_{n_1+n_2=n_3+n_4}^{N} a_{n_1}a_{n_2}\bar{a}_{n_3}\bar{a}_{n_4}\left|\frac{n_3n_4}{n_1n_2}\right|^{\mathrm{i}t}. \tag{4.3}$$

Now we may form the long-time average. Let

$$S = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sum_{n_1+n_2=n_3+n_4}^{N} a_{n_1}a_{n_2}\bar{a}_{n_3}\bar{a}_{n_4}\left|\frac{n_3n_4}{n_1n_2}\right|^{\mathrm{i}t}\,\mathrm{d}t. \tag{4.4}$$

Terms with $|n_1n_2| \neq |n_3n_4|$ vanish, hence we arrive at

$$S = \sum_{\substack{n_1+n_2=n_3+n_4\\ |n_1n_2|=|n_3n_4|}}^{N} a_{n_1}a_{n_2}\bar{a}_{n_3}\bar{a}_{n_4}. \tag{4.5}$$

Now separate the case $S^+ : n_1n_2 = n_3n_4$ from $S^- : n_1n_2 = -n_3n_4$ to give

$$S = \sum_{\substack{n_1+n_2=n_3+n_4\\ n_1n_2=n_3n_4}}^{N} a_{n_1}a_{n_2}\bar{a}_{n_3}\bar{a}_{n_4} + \sum_{\substack{n_1+n_2=n_3+n_4\\ n_1n_2=-n_3n_4}}^{N} a_{n_1}a_{n_2}\bar{a}_{n_3}\bar{a}_{n_4}. \tag{4.6}$$

The circular case

The S^+ sum reduces to the system

$$\begin{aligned} n_1^2 + n_2^2 &= n_3^2 + n_4^2 \\ n_1 + n_2 &= n_3 + n_4, \end{aligned} \tag{4.7}$$

which corresponds to the intersections of circles and lines at lattice points $\mathbb{Z}^2$. This Diophantine system is considered by Zygmund [**12**], and Bourgain [**4**, § 2], in the context of the Schrödinger group. We may evaluate S^+ precisely; all off-axis points (n_1, n_2) on the lattice $\{-N, \dots, N\} \times \{-N, \dots, N\}$ give contributions to the sum. Those with $n_1 \neq n_2$ generate two solutions (n_3, n_4) and (n_4, n_3), whereas those of form (n_1, n_1) give just one. Thus we obtain

$$S^+ = 2\biggl\{\sum^{N} |a_{n_1}|^2\biggr\}^2 - \sum^{N} |a_{n_1}|^4 \tag{4.8}$$

and deduce that

$$\|f\|^4_{L^2_\theta} \leqslant S^+ \leqslant 2\|f\|^4_{L^2_\theta}. \tag{4.9}$$

The hyperbolic case

The other term S^- is a sum over intersections of hyperbolae and parallel lines

$$\begin{aligned} n_3 + n_4 &= n_1 + n_2 \\ n_3 n_4 &= -n_1 n_2. \end{aligned} \tag{4.10}$$

This general Diophantine system may be solved by change of variables. Solutions are less clear than for S^+. The sum is over a more sparse set, as the only possible solutions are given by

$$n_3, n_4 = \tfrac{1}{2}\Bigl(n_1 + n_2 \pm \sqrt{(n_1 + 3n_2)^2 - 8n_2^2}\Bigr), \tag{4.11}$$

where n_3 and n_4 are integers. In general, S^- is non-empty, for instance $(n_1, n_2, n_3, n_4) = (2, 3, 6, -1)$ is an element for $N \geqslant 6$. All solutions may be generated using forms reminiscent of Pythagorean triples. Setting $X = n_1 + 3n_2$ and $Y = n_2$, we arrive at the following case of Pell's equation:

$$X^2 - 8Y^2 = k^2, \tag{4.12}$$

with k integral. Here we rely on the fact that $\mathbb{Q}(\sqrt{-2})$ is a Euclidean domain [**6**]. Now assume (X, Y, k) have no pairwise common factor. The only possible common divisor of $X + k$, $X - k$ is 2. Thus $X + k = 2P^2$, $X - k = 4Q^2$ give the general solution of (4.12) with (X, Y) positive:

$$(X, Y, k) = (P^2 + 2Q^2, PQ, P^2 - 2Q^2), \tag{4.13}$$

where $P, Q \neq 0$. These give the minimal solutions of S^- via

$$(n_1, n_2, n_3, n_4) = \Bigl(X - 3Y, Y, \tfrac{1}{2}(X - 2Y + k), \tfrac{1}{2}(X - 2Y - k)\Bigr). \tag{4.14}$$

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Putting $x = P - Q$ and $y = Q$ leads to the symmetric form

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{pmatrix} = \begin{pmatrix} P^2 + 2Q^2 - 3PQ \\ PQ \\ P^2 - PQ \\ 2Q^2 - PQ \end{pmatrix} = \begin{pmatrix} x(x-y) \\ y(x+y) \\ x(x+y) \\ -y(x-y) \end{pmatrix} \tag{4.15}$$

for non-zero $|x| \neq |y|$. These are the basic solutions, which we can scale to give the general solution in the integer lattice to (4.10). We must allow an additional factor p, where p is odd and square-free, giving the explicit expansion

$$S^- = 8 \operatorname{Re} \sum_{\substack{0<x<y \\ p \in \mathbb{P}}} a_{p(x^2-xy)} a_{p(y^2+xy)} \bar{a}_{p(x^2+xy)} \bar{a}_{p(y^2-xy)}, \tag{4.16}$$

where $\mathbb{P} = \{\pm 1, \pm 3, \pm 5, \pm 7, \pm 11, \pm 13, \pm 15, \dots\}$. The summation is over the valid range of coefficients, that is to say all subscripts must fall inside $\{-N, \dots, +N\}$, so that x and y must be less than $\sqrt{N/2p}$. We can now make the required estimate:

$$S^- \leqslant 2\|f\|^4_{L^2_\theta}. \tag{4.17}$$

Adding this bound to (4.9) gives the right-hand side of Theorem 1.1 for finite sums.

5. General L^2 case

We extend the previous result to the whole of L^2_θ, making use of the theory of vector-valued Bohr almost-periodic functions, from [**3**] and [**11**].

Definition 5.1. Let X be a Banach space, and $g : \mathbb{R} \to X$ be continuous. We say that $\tau \in \mathbb{R}$ is an *ϵ-almost period* of g if

$$\|g(t+\tau) - g(t)\|_X \leqslant \epsilon \quad (t \in \mathbb{R}). \tag{5.1}$$

The function g is *Bohr almost periodic* if for each $\epsilon > 0$, there exists $\lambda > 0$ such that each interval $(t, t+\lambda)$ contains at least one ϵ-almost period τ. Let $B^4_t X$ be the completion of the space of Bohr almost periodic X-valued functions for the norm

$$\|g\|^4_{B^4_t X} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \|g(t)\|^4_X \, \mathrm{d}t. \tag{5.2}$$

The Mean Value Theorem for almost periodic functions shows that this limit exists; the Uniqueness Theorem proves that this is indeed a valid norm.

Theorem 5.2. *The map $f \mapsto \Delta^{-\mathrm{i}t/2} f$ is bounded $L^2_\theta \to B^4_t L^4_\theta$ with norm at most 4.*

Proof. Given f as in Theorem 1.1, let $f_N = \sum_{|n|<N} a_n \mathrm{e}^{\mathrm{i}n\theta}$, so that, $f_N \to f$ in L^2_θ, as $N \to \infty$. Now let $F_N(t, \theta) = \sum_{|n|<N} a_n \mathrm{e}^{\mathrm{i}n\theta} |n|^{-it}$. These partial sums are almost periodic in t, with values in L^4_θ, and give a Cauchy sequence (F_N) in $B^4_t L^4_\theta$, the Banach space

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obtained by completing the space of finite sums $\sum_{m,n} b_n \mathrm{e}^{\mathrm{i}n\theta} r_m^{-\mathrm{i}t}$ with respect to the norm (5.2), where $X = L^4_\theta$. Let F be the limit of this sequence in $B^4_t L^4_\theta$. The Fourier coefficients depend continuously on the $B^4_t L^4_\theta$ norm, so that we can regard F as a function with

$$F(t, \theta) = \Delta^{-\mathrm{i}t/2} f(\theta), \tag{5.3}$$

as the interpretation in L^2_θ is unambiguous. Since

$$\|F_N\|_{B^4_t L^4_\theta} \to \|F\|_{B^4_t L^4_\theta} \quad \text{and} \quad \|f_N\|_{L^2_\theta} \to \|f\|_{L^2_\theta},$$

as $N \to \infty$, we deduce the general theorem from the finite sum case:

$$\|F\|^4_{B^4_t L^4_\theta} \leqslant 4 \|f\|^4_{L^2_\theta}. \tag{5.4}$$

□

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