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Resonant absorption and heating in a nonlinear dissipative resonance layer

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Abstract. The technique for analytical investigation of the Alfvén resonance, proposed by Dmitrienko [*J. Plasma Phys.* **57**, 311 (1997); **62**, 145 (1999)] is used to obtain the matching conditions and to calculate the rate of resonant absorption and heating in a nonlinear dissipative resonance layer. It is shown that one matching condition is the invariability condition for the coefficient of the singular part of the solution describing the MHD mode outside the resonance layer, and the other condition is a jump of the coefficient of a regular solution, the value of which depends on the ratio of the dissipative to nonlinear spatial scales. It is established that the ratio of the jump of the imaginary part of the regular solution to the coefficient of the singular solution is exactly equal to the rate of resonant heating, and this heating is calculated as a function of the ratio of the linear dissipative to nonlinear scales. The resonant absorption coefficient is calculated as well. It is shown that the nonlinearity leads to a decrease of the heating rate and the absorption coefficient compared with those predicted by linear theory.

1. Introduction

The goal of this paper is to investigate the influence of nonlinearity on plasma heating and on the absorption of magnetohydrodynamic (MHD) waves in the region of Alfvén resonance. This heating is known to be due to the fact that a disturbance of MHD type near the surface where its frequency is equal to the local Alfvén frequency becomes small in scale, and in such a form (i.e. a small-scale disturbance of Alfvén type) it transmits some of its energy via the collision mechanism to the thermal motion of particles. The physical meaning of the real processes that are of necessity accompanied by the emergence of an Alfvén resonance can be quite different. In the magnetosphere, these are FMS waves propagating across magnetic shells (Southwood 1974), cavity modes (Southwood and Kivelson 1986), and surface modes (Chen and Hasegawa 1974); on the Sun, they are the various disturbances in the corona and in solar arches (Ionson 1978); and in laboratory plasma heating devices, they are low-frequency waves used in plasma heating (Tataronis and Grossman 1973; Grossman and Tataronis 1973). The absorption of MHD waves in the magnetosphere implies the transformation of large-scale small-amplitude disturbances to localized (near particular magnetic surfaces) small-scale geomagnetic pulsations observed on the ground. Plasma heating in this region makes some contribution to energization of particles and to their precipitation into the ionosphere. Resonant absorption and heating is also recognized as the mechanism responsible for the heating of solar arches. The character of a small-scale resonance disturbance, which in its properties may be arbitrarily classified as a disturbance of Alfvén type, is determined by which effect(s) that bring us beyond the model of ideal magnetohydrodynamics are predominant in the resonance region. We will neglect the effects causing a disturbance of the Alfvén type to escape the resonance surface by assuming that the dominant small effects are the resistivity $\mu \neq 0$ and the viscosity $\nu \neq 0$. Such a case was rather thoroughly investigated within the linear approximation; specifically, it was found that the rate of resonant heating due to the presence of finite dissipation is independent of the value of the dissipative coefficients (Hollweg 1987; Davila 1987). We shall consider a very simple model of the medium that allows for the existence of the Alfvén resonance. This is a plasma with density that is inhomogeneous across a homogeneous magnetic field, i.e. $\rho_0 = \rho_0(x)$ and $\mathbf{B}_0 = (0, 0, B_0)$. It will be assumed that there exists a disturbance of the form $\mathbf{\Psi}(x)e^{-i\omega t + k_z z + k_y y}$. This can be, for example, a penetrating (from the transparent region) FMS wave or a disturbance caused by some source on some surface across the magnetic field. In the opaque region $(\omega^2/v_A^2 - k_z^2 - k_y^2 < 0)$ on some surface $x = x_0$, the condition $\omega^2/v_A^2 = k_z^2$ is satisfied. Our intent here is to obtain the matching conditions, and to calculate the rate of resonant absorption and heating, based on the technique for analytical investigation of the Alfvén resonance proposed by Dmitrienko (1997, 1999). It relies on the assumption that away from the resonance surface, the nonlinearity and dissipation can be neglected in the input equations. Near the resonance surface, the equations with dissipation and nonlinearity are simplified because of a possibility of restricting our consideration to the first term of an expansion in a series of functions describing the inhomogeneity of the medium; after that, the outer and inner solutions are matched. In linear theory, essentially the same approach has been used in many publications: away from the resonance surface, the influence of the small dissipation was considered unimportant, and the dissipation was taken into account only near the surface. A new element of this study, in addition to taking the nonlinearity into account, is the procedure of formalizing the problem suggested by Dmitrienko (1997, 1999) and further developed in this paper.

As will be shown later in this paper, nonlinearity reduces resonant heating compared with what is given by linear theory under the same conditions. It should be noted in this connection that a decrease in Ohmic heating with an increase of the driver amplitude in the nonlinear disturbance was revealed by simulating the heating of coronal arches through a numerical solution of nonlinearized MHD equations by Ofman and Davila (1995). However, the decrease in heating reported by Ofman and Davila (1995) manifested itself for a confined (in the z direction) disturbance; in this paper, we consider a disturbance propagating along this direction. Unfortunately, even this difference in the statement of the problem does not permit us to compare in a meaningful manner the results of Ofman and Davila (1995) and those obtained in this paper. Furthermore, Ofman and Davila totally neglect the gas kinetic pressure of plasma, whereas, within the framework of the problem considered in this paper, this pressure is taken into account for a correct determination of the zeroth harmonic.

2. Basic equations

We restrict our consideration to one-fluid MHD in the form

$$\partial_t \mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{B}) + \mu \nabla^2 \mathbf{B},$$
$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} = -\frac{T \boldsymbol{\nabla} n}{m_i n} + \frac{(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{B}}{4\pi m_i n} + \nu \nabla^2 \mathbf{v} \tag{2.1}$$
$$\partial_t n + \boldsymbol{\nabla} \cdot (n\mathbf{v}) = 0.$$

Here μ is the resistivity and ν is the ion viscosity. We remark from the outset that the presence of gas kinetic pressure in (2.1) lets us take it into account when establishing a total pressure balance in the situation where there is magnetic pressure produced by a disturbance – and only such a role will be taken into consideration here, i.e. we shall take into account the fact that pressure exists in the plasma only when calculating the zeroth harmonic of the disturbance. Linear effects associated with the presence of pressure will be neglected. This means that the linear limit of our nonlinear theory is the linearized system (2.1) with $T = 0$. Such a system describes the Alfvén resonance, in which only dissipative effects are important; it is they that determine, within the linear approximation, the spatial scale of the disturbance (Hollweg 1987; Davila 1987). In describing the disturbance, we shall be using the quantities

$$\mathbf{b} = \frac{\mathbf{B} - \mathbf{B}_0}{B_0}, \qquad \mathbf{u} = \frac{k}{\omega}\mathbf{v}, \qquad r = \frac{n_0(x)}{n_0(x_0)}.$$

We also denote $\theta = -\omega t + k_z z + k_y y$. Besides, it will be assumed that

$$n_0(x) = n_0(x_0)\left(1 + \frac{x - x_0}{l} + \dots\right),$$

and we denote $\xi = (x - x_0)/l$, where, obviously, $l^{-1} = (dr_0/dx)|_{x=x_0}$.

If, within the framework of linear theory, not only gas kinetic pressure but also dissipative effects are neglected, then for $b_z(\xi)$ we obtain a second-order differential equation (Southwood 1974):

$$\partial^2_{\xi\xi} b_z - \partial_\xi \ln\left(\frac{\omega^2}{k_z^2 v_A^2} - 1\right) \partial_\xi b_z + (k_z l)^2 \left(\frac{\omega^2}{k_z^2 v_A^2} - 1 - \frac{k_y^2}{k_z^2}\right) b_z = 0, \tag{2.2}$$

for which $\xi = 0$ is a regular singular point. When $\xi \to 0$,

$$b_z = A_\pm \left(\frac{k_z k_y l^2}{2}\xi^2 + O(\xi^4)\right) + B_\pm \left(\frac{k_z}{k_y} + \frac{k_z k_y l^2}{2}\xi^2 \ln|\xi| + O(\xi^4)\right). \tag{2.3}$$

The + and − signs refer to the regions $\xi > 0$ and $\xi < 0$ respectively. The coefficients A and B correspond to two linearly independent solutions, one of which is regular, while the other has a logarithmic singularity; (2.3) corresponds to such a choice of these solutions where the phase of the logarithm in the second solution is included in the coefficient A.

The influence of the dissipation and nonlinearity in the region $\xi/\Delta \to \infty$ (where Δ is the scale of the resonance layer) from (2.1) can be taken into account by perturbation theory. Assuming that the disturbance amplitude is characterized by a small parameter ϵ, one can obtain the expressions with estimates of the corrections for dissipation and nonlinearity for $b_x(\xi)$ and $b_y(\xi)$, the particular form of which will be used in our subsequent treatment:

$$b_x = \epsilon \left\{ ik_y l \left\{ A_\pm \left[1 + O(\xi^2)\right] + B_\pm \left[\ln|\xi| + O(\xi^2)\right] \right\} + O(\mu\xi^{-3}) + O(\nu\xi^{-3}) \right\} + O(\epsilon^3\xi^{-3}), \tag{2.4}$$

$$b_y = -\epsilon \left\{ A_\pm \left[\tfrac{1}{2}k_y^2 l^2 \xi + O(\xi^3)\right] + B_\pm \left[\xi^{-1} + \tfrac{1}{2}k_y^2 l^2 \xi \ln|\xi| + O(\xi^3)\right] + O(\mu\xi^{-4}) + O(\nu\xi^{-4}) \right\} + O(\epsilon^3\xi^{-4}). \tag{2.5}$$

By comparing, when $\xi \to 0$, the main contribution to the solution and, independently, each of the corrections, it is possible to calculate the dissipative and nonlinear scales as scales at which the corresponding corrections are compared with the main contribution. With such a determination, the resistive and viscous scales are given by $\Delta_\mu = (\mu/\omega l^2)^{1/3}$ and $\Delta_\nu = (\nu/\omega l^2)^{1/3}$ respectively, as they must (Hollweg 1987; Davila 1987), and the nonlinear scale turns out to be of order $\epsilon^{2/3}$ (Dmitrienko 1997). Note that the specific character of the nonlinearity in the case under consideration is such that corrections for nonlinearity cannot be calculated without taking the dissipation into account (Dmitrienko 1997). So the nonlinear terms of (2.4) and (2.5) have been calculated by using the linear equations, derived on the basis of the dissipative linear scale Δ_L. In linear theory, such equations were repeatedly obtained using an essentially allied method, even if formalized in another way (see e.g. Goossens et al. 1995). The main (of order $(\mu + \nu)^0$) terms of the asymptotic forms of their solutions when $(\xi/\Delta_L) \to \infty$ are sufficient for obtaining the nonlinear corrections and nonlinear scale.

From here on, for the resonance layer we shall be using the nonlinear spatial scale. The values of the disturbance components as functions of the new variable

$$\zeta = \epsilon^{-2/3}\xi \tag{2.6}$$

are expanded in series in terms of $\epsilon^{1/3}$:

$$\Psi_i = \begin{cases} \epsilon\Psi_i^{(1)}(\zeta,\theta,\tau) + \epsilon^{4/3}\Psi_i^{(4/3)}(\zeta,\theta,\tau) + \dots, & i = 1,4, \\ \epsilon^{1/3}\Psi_i^{(1/3)}(\zeta,\theta,\tau) + \epsilon^{2/3}\Psi_i^{(2/3)}(\zeta,\theta,\tau) + \dots, & i = 2,5, \\ \epsilon^{2/3}\Psi_i^{(2/3)}(\zeta,\theta,\tau) + \epsilon\Psi_i^{(1)}(\zeta,\theta,\tau) + \dots, & i = 3,6,7 \end{cases} \tag{2.7}$$

(with $\mathbf{\Psi} = (\mathbf{b}, \mathbf{u}, r)$). The boundary condition when $\zeta \to \infty$ for the main order of the y component of the magnetic field disturbance,

$$b_y^{(1/3)} = H_y e^{i\theta} + \text{c.c.}, \tag{2.8}$$

follows from (2.5):

$$H_y = -B_\pm\zeta^{-1} + O\left(\frac{\zeta^{-4}\mu}{\epsilon^2}\right) + O\left(\frac{\zeta^{-4}\nu}{\epsilon^2}\right) + O(\zeta^{-4}). \tag{2.9}$$

It is possible to obtain (with certain limitations to be specified later) the nonlinear equation containing H_y only. Since its derivation does not contain any essential differences from the derivation of the nonlinear equation in Dmitrienko (1997), we

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give it here in its final form:

$$i\frac{\Delta_L^3}{\epsilon^2}\partial^2_{\zeta\zeta}H_y - \left(\zeta - \frac{1}{\beta^*}|H_y|^2\right)H_y = \text{const}, \tag{2.10}$$

$$\Delta_L = \left(\frac{\mu+\nu}{\omega l^2}\right)^{1/3}, \qquad \beta^* = \frac{4\pi n_0(x_0)T}{B_0^2}.$$

For the boundary conditions (2.9) to be satisfied, it is necessary to put

$$\text{const} = B_+ = B_- = B, \tag{2.11}$$

which yields the matching of B_+ and B_-, and, incidentally, makes it possible to refine the meaning of the parameter ϵ introduced above. This characterizes, as is clear from, for example, (2.11) and (2.3), when $B \sim 1$, the disturbance amplitude of the z component of the magnetic field in the resonance layer, $B_z \sim \epsilon B B_0$. The matching of (2.11) coincides with what was obtained in linear theory on an earlier occasion (Southwood and Kivelson 1986). In view of (2.11), (2.10) takes the form:

$$i\frac{\Delta_L^3}{\epsilon^2}\,\partial^2_{\zeta\zeta}H_y - \left(\zeta - \frac{1}{\beta^*}|H_y|^2\right)H_y = B. \tag{2.12}$$

The nonlinear term in (2.12) is caused by the interaction of the fundamental harmonic with the zeroth harmonic of the density perturbation generated by it. No other components of the zeroth harmonic are involved in (2.12), by virtue of the fact that the condition $\mu/\nu \ll \beta^{*-1}$ is assumed to be satisfied. In this case (Dmitrienko 1997), one may assume that the contribution of the density is dominant among the contributions to the nonlinear term from different disturbance components at the zeroth harmonic. An important property of (2.12) is that the nonlinear term does not involve the contribution from the second harmonic. The contribution from the second harmonic to the nonlinear term is zero because the contributions from the different disturbance components compensate each other exactly (in the order required for (2.12)) (Dmitrienko 1997). The constant B can be taken as real and positive. On substituting

$$\zeta' = \beta^{*1/3}B^{-2/3}\zeta,$$

$$G = (\beta^* B)^{-1/3}H_y,$$

$$\lambda = \frac{\Delta_L^3}{\Delta_N^3}, \qquad \Delta_L = \left(\frac{\mu+\nu}{\omega l^2}\right)^{1/3}, \qquad \Delta_N = \beta^{*-1/3}(\epsilon B)^{2/3},$$

(2.12) is brought to the form

$$i\lambda\,\partial^2_{\zeta'\zeta'}G - (\zeta' - |G|^2)G = 1. \tag{2.13}$$

The parameter λ characterizes the relative roles of dissipation and nonlinearity; its value is determined by the ratio of the linear dissipative scale to the nonlinear spatial scale.

3. Matching through the resonance layer and heating rate

To obtain the matching of the coefficients A_+ and A_-, we now turn to (2.4). We shall use H_x similarly to H_y. The difference of the values of $b_x^{(1)}$ taken at the points symmetric about the resonance surface does not contain the term $\ln|\xi|$:

$$H_x(\zeta_0) - H_x(-\zeta_0) = ik_y l(A_+ - A_-) + O(\mu\epsilon^{-2}\zeta^{-3}) + O(\nu\epsilon^{-2}\zeta^{-3}) + O(\zeta^{-3}), \quad (3.1)$$

on the other hand, in accordance with the ordering of the values of the disturbance components from (2.7) and the scaling in (2.6), the relation

$$\partial_\zeta b_x^{(1)} + k_y l\, \partial_\theta b_y^{(1/3)} = 0, \quad (3.2)$$

holds in the resonance layer, so that

$$H_x(\zeta_0) - H_x(-\zeta_0) = -ik_y l \int_{-\zeta_0}^{\zeta_0} H_y \, d\zeta. \quad (3.3)$$

When $\zeta_0 \to \infty$, we obtain

$$\frac{A_+ - A_-}{B} = -⨍ G\, d\zeta'. \quad (3.4)$$

One can make sure that when $\lambda \to \infty$, corresponding to linear theory, the solution (2.13) gives a linear matching in the form

$$A_+ - A_- = -\pi i B. \quad (3.5)$$

This was obtained for the first time (in a different form, however) by Southwood and Kivelson (1986). In the linear case, the jump of the coefficient A does not depend on the value of the dissipative coefficients (Davila 1987). This is evident from the fact that the parameter λ can be deleted from the linearized forms of (2.13) and (3.4) on substituting $h = \Delta_L^{-1/3}\xi$ and $P = G\lambda^{1/3}$. In the nonlinear case, a dependence on λ is inevitable; the jump of A depends through λ on the disturbance amplitude and the value of the dissipative coefficients.

The formal problem of matching the disturbance through the resonance layer is closely associated with the problem of plasma heating in this layer. The Ohmic heating of the resonance layer is

$$h_\mu = \frac{\mu B_0^2}{2\pi l} \int \left|\frac{\partial b_y}{\partial \xi}\right|^2 d\xi.$$

We shall calculate only the zeroth harmonic of the heating rate. It is worthwhile introducing, for the main order of the heating rate $h_\mu^{(2)}$, the dimensionless heating rate

$$H_\mu^{(2)} = \frac{\Delta_\mu^3}{\Delta_N^3} \int_{-\infty}^{\infty} \left|\frac{\partial G}{\partial \zeta'}\right|^2 d\zeta', \qquad h_\mu^{(2)} = \frac{B_0^2 \omega l}{2\pi} B^2 H_\mu^{(2)}.$$

In a similar manner, for the heating rate and dimensionless heating rate of ions we shall use $h_\nu^{(2)}$ and $H_\nu^{(2)}$. Let a total heating rate $h^{(2)} = h_\mu^{(2)} + h_\nu^{(2)}$ be expressed in terms of a total dimensionless heating rate H:

$$H = \lambda \int_{-\infty}^{\infty} \left|\frac{\partial G}{\partial \zeta'}\right|^2 d\zeta'. \quad (3.6)$$

On multiplying (2.13) by G^* and the complex conjugate of (2.13) by G, and subtracting the latter equation obtained from the former, we get

$$i\lambda \int_{-\infty}^{\infty} \left(G^* \frac{\partial^2 G}{\partial \zeta'^2} + \text{c.c.}\right) d\zeta' = 2i\, \mathrm{Im}\left(\frac{A_+ - A_-}{B}\right).$$

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Thus, for the nonlinear resonant heating rate,

$$H = -\mathrm{Im}\left(\frac{A_+ - A_-}{B}\right). \tag{3.7}$$

For the linear theory, from here,

$$H = \pi. \tag{3.8}$$

For a nonlinear disturbance, the heating rate can be calculated numerically; results of such calculations are presented in Section 5.

4. The absorption coefficient

So far we have not used any assumptions about the character of the inhomogeneity profile of the medium; in the resonance layer, it was sufficient for us to have the first term of the expansion in a Taylor series of the function describing the density of the medium. For this reason, (2.11), (2.13), and (3.4) are applicable to Alfvén resonances of different origins, both for a confined mode such as the cavity mode, for the surface mode, and for the propagating MHD wave. Here we restrict ourselves to integrating (2.11), (2.13), and (3.4) into the problem of FMS wave propagation in an inhomogeneous plasma, and this inhomogeneity will be taken to be such that in some region in the coordinate x when $\xi > \xi_r$, this wave is a propagating one, there is one point of reflection ξ_r, and behind it when $\xi < \xi_r$ there is an opaque region. This region involves a resonance surface $\xi = 0$. It is known that this wave leaves some of its energy in the resonance layer, and the energy flux density in the reflected wave is therefore smaller when compared with the incident wave. In the propagation region, the incident $F_I(\xi)$ and reflected $F_R(\xi)$ waves can be chosen as two independent solutions (2.2); the coefficients C_I and C_R, with which they are involved in the full solution, represent incidence and reflection coefficients respectively. As the full solution continues into the opaque region, it takes near the resonance surface the form (2.3). The relation between the incidence and reflection coefficients and the coefficients A_+ and B can be written in matrix form as

$$\begin{pmatrix} C_I \\ C_R \end{pmatrix} = \hat{\mathbf{D}} \begin{pmatrix} A_+ \\ B \end{pmatrix}, \tag{4.1}$$

whose coefficients are determined by the inhomogeneity profile, and are calculated from (2.2). Equation (2.2) is also used to calculate a relation between the coefficients A_- and B of the form

$$A_- = \delta B, \tag{4.2}$$

following from the boundary condition $b_z \to 0$ when $\xi \to -\infty$. To be able to compare with previously reported results for linear theory (Southwood and Kivelson 1986; Forslund et al. 1975), we choose the linear profile of the plasma density. In this case, when $\xi \gg \xi_r$, the incident and reflected waves are represented as

$$F_I(\xi) = (k_z l)^{1/2}\,\xi^{1/4}\exp\left\{-i\tfrac{2}{3}[(k_z l)^{2/3}\,\xi - \lambda_{ex}^2]^{3/2} - \tfrac{1}{4}i\pi\right\} + O(\xi^{-3/4}),$$

$$F_R(\xi) = (k_z l)^{1/2}\,\xi^{1/4}\exp\left\{i\tfrac{2}{3}[(k_z l)^{2/3}\,\xi - \lambda_{ex}^2]^{3/2} + \tfrac{1}{4}i\pi\right\} + O(\xi^{-3/4}),$$

where

$$\lambda_{ex} = k_y l (k_z l)^{-2/3}. \tag{4.3}$$

The absorption coefficient is

$$\kappa = \frac{|C_I|^2 - |C_R|^2}{|C_I|^2}. \tag{4.4}$$

Having been defined by (4.4), it represents the ratio of the energy flux density difference in the incident and reflected waves to the energy flux in the incident wave. However, it is not a direct characteristic of energy storage in the resonance layer, because the fraction of the energy that is lost by the fundamental harmonic leaves the resonance layer together with a weak ($\sim \epsilon^2$) flow along the direction of the inhomogeneity. The density of the energy flux associated with this flow, when $|\zeta| \to \infty$, is

$$q_{x0}^{(2)} = \frac{1}{4\pi}\frac{\omega}{k}B_0^2\left(2u_x^{(1)}b_z^{(1)} + u_x^{(2)}\right)_0 + O(\zeta^{(-1)}). \tag{4.5}$$

Here $u_x^{(1)}$, $b_z^{(1)}$, and $u_x^{(2)}$ denote the respective disturbance components in 'inner orders' following from (2.7). It follows from (2.1) that

$$u_x^{(1)} = -b_x^{(1)}, \qquad u_{x0}^{(2)} = -(u_x^{(1)}b_z^{(1)})_0 + \text{const}. \tag{4.6}$$

The first term on the right-hand side of (4.5) is the energy flux density directly in the fundamental harmonic of the wave, and the second term is the density of wave energy associated with the formation of flow in the x direction. The jump of the first flow is directed toward the resonance surface, and the jump of the second flow is aimed away from it. On substituting (4.5) and (4.6) into the law of conservation of energy, $h_{x0}^{(2)} = -q_{x0}^{(2)}|_{-\infty}^{\infty}$, we arrive again at the relation (3.7). From the properties of the matrix $\widehat{\mathbf{D}}$ (Dmitrienko 1999), one can obtain the relation

$$|C_I|^2 - |C_R|^2 = -B\,\mathrm{Im}(A_+ - A_-). \tag{4.7}$$

In view of (3.7) and (4.4), this means that

$$B^2H = \kappa|C_I|^2; \tag{4.8}$$

in the linear case, $B^2\pi = \kappa|C_I|^2$.

5. Results of numerical calculations

A numerical integration of (3.4) for Gy, which is also obtained by a numerical solution of (2.13), makes it possible to calculate the matching condition at a given ratio of the dissipative to nonlinear spatial scales (λ); this quantity is independent of any other parameters. As a test, the numerical solution of (2.13) without the nonlinear term was obtained (this corresponds to the linear case). The straight dashed line in Fig. 1 plots the result; it means that the value of the jump in the imaginary part of the coefficient A of the regular part of the solution is independent of λ in linear theory. In nonlinear theory (the solid curve in Fig. 1) the relation between A_+ and A_- changes essentially with a change of λ and $\mathrm{Im}(A_+ - A_-) \to 0$ when $\lambda \to \infty$, which signifies disappearances of the jump in the imaginary part of the coefficient A. This means, in the language of the phase of the logarithm, that its jump at the passage through the resonance layer tends to 0.

The curves plotted in Fig. 1, as a consequence of (3.7), express also the dependence of a total dimensionless heating rate H on λ. The straight line corresponds to linear theory: heating does not depend on λ. The solid curve gives the dependence of

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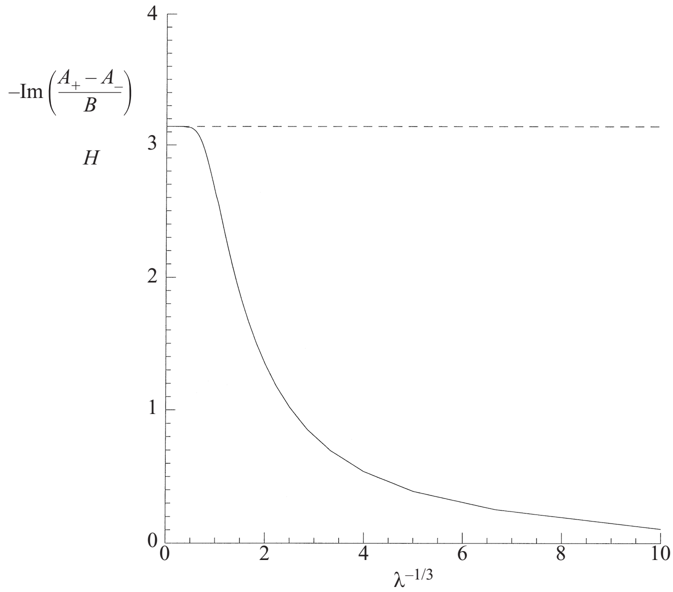

Figure 1. $\mathrm{Im}[(A_+ - A_-)/B]$ and H as functions of λ: Dashed line, linear theory; solid curve, nonlinear theory.

the total dimensionless heating on λ for nonlinear theory. It is evident that with decreasing λ, i.e. with decreasing ratio of the linear to nonlinear spatial scales, this rate decreases.

This decrease in the heating rate is similar to the decrease in heating rate that was found by Ofman and Davila (1995) in a numerical simulation of the heating of coronal arches, based on the nonlinear MHD equations, into which the source of a large-scale disturbance was introduced. With Dmitrienko (1997, 1999), in mind, it must be assumed that the difference of the cases considered by Ofman and Davila (1995) and in this paper means that there is no way of making any quantitative comparisons. On the one hand, Ofman and Davila neglected gas kinetic pressure and viscosity when establishing the zeroth harmonic; in such a case, according to Dmitrienko (1997), the non-stationary character of the disturbance, which is not considered in this paper, can play a role in establishing the zeroth harmonic. On the other hand, Dmitrienko (1997) and this paper investigate the disturbances propagating along the external magnetic field, while Ofman and Davila (1995) examine a standing disturbance, and there is as yet no possibility of clearly formulating to what differences in nonlinear effects this difference of the disturbances involved can lead. However, a qualitative explanation for this effect offered by Ofman and Davila (1995), i.e. a perturbation of the resonance condition because of the appearance of a nonlinear addition to the frequency, is applicable within the context of this paper as well. The nonlinear term in (2.13) appears because of the fact that in the resonance layer, equal roles are played by minor changes of unperturbed parameters (density) in the direction across this layer as a consequence of their inhomogeneity, and by changes of these parameters as a consequence of the presence of a disturbance in this layer. These latter may be treated as a change of the local Alfvén frequency under the influence of the nonlinearity.

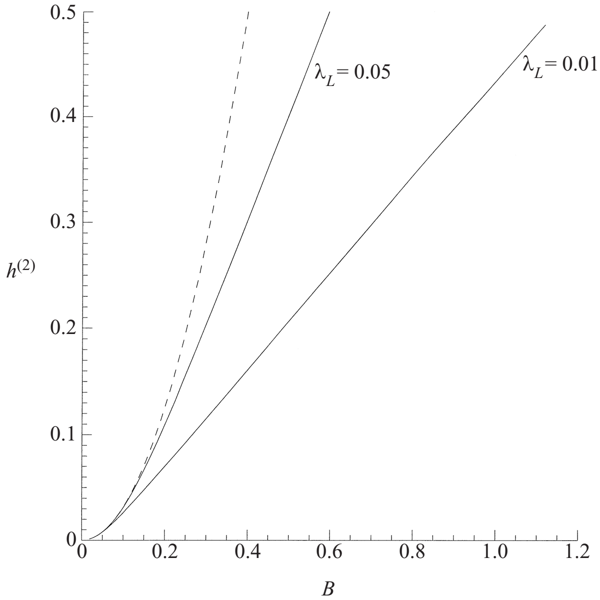

Figure 2. Dependence of the heating $h^{(2)}$ on B: Dashed curve, linear theory; solid curves, linear theory for $\lambda_L = \lambda|B|^2 = 0.05, 0.01$, respectively.

Although H depends only on one parameter, this parameter, in turn, depends on the other two parameters (the dissipative and nonlinear spatial scales), and a decrease in λ can be caused by, for example, an increase in disturbance amplitude with the invariable dissipation or by a decrease in the dissipative coefficients with the invariable amplitude. In either case, the ratio of the linear to nonlinear scales decreases, and the influence of the nonlinearity is enhanced. The first case is illustrated in Fig. 2. There it has been taken into account that the amplitude of the z component of the magnetic field disturbance in the resonance layer is $b_z \sim \epsilon B$, so it is possible (assuming ϵ to be unchanged) to change the amplitude by means of changing B. The heating $h^{(2)}$ contains a dependence on the parameter B apart from the one that was involved in H. Figure 2 presents the dependence of the heating $h^{(2)}$ on B, with the dissipative coefficients and β remaining unchanged; they can together be characterized by the parameter $\lambda_L = \lambda|B|^2$. It is seen that the quadratic growth of heating with increasing disturbance amplitude changes to a linear growth with subsequent increase; a change in the character of the dependence of $h^{(2)}$ on B occurs at $\lambda \sim 1$, i.e. at such an amplitude where the linear and nonlinear scales become equal. When neglecting the nonlinear term in (2.13), the dashed curve with quadratic growth was obtained.

It is clear that the decrease of resonant heating with decreasing ratio of the linear scale to the nonlinear one leads to a decrease in the absorption coefficient of the large-scale disturbance. In linear theory, the absorption coefficient depends on only one parameter, λ_{ex} (4.3). This parameter characterizes the large-scale disturbance, and determines, in particular, the connection of the amplitudes of the incidence FMS wave and the FMS disturbance in the resonance layer. In the nonlinear case, the absorption coefficient κ depends not only on λ_{ex} but on λ as well: $\kappa = \kappa(\lambda_{ex}, \lambda)$. We want to have a dependence of κ on λ_{ex} when the amplitude C_I is constant. It

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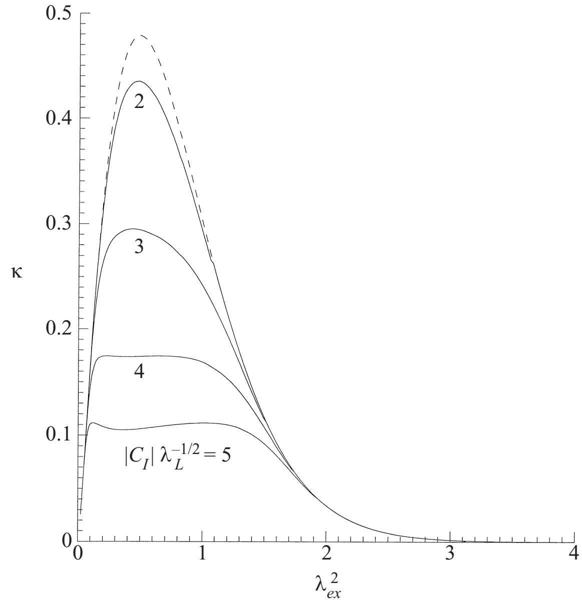

Figure 3. Dependence of the absorption coefficient on λ_{ex}: Dashed curve, linear theory; solid curves, nonlinear absorption at $|C_I|\lambda_L^{-1/2} = 2, 3, 4, 5$, respectively.

is possible to obtain this dependence on the basis of (2.13). From (4.1), (4.2), and (3.4) it follows that

$$C_I = \left[d_{11}\left(\delta - \fint G\, d\zeta'\right) + d_{12}\right] B,$$

so

$$\lambda = \frac{\lambda_L \left|d_{11}(\delta - \fint G\, d\zeta') + d_{12}\right|^2}{|C_I|^2}.$$

Numerical solution of this equation gives λ as $\lambda(\lambda_{ex}, \lambda_L/|C_I|^2)$. On substituting λ into (4.8), we get $\kappa = \kappa(\lambda_{ex}, \lambda_L/|C_I|^2)$. The last parameter depends in turn on the amplitude of the incident wave and the dissipation value. Figure 3 illustrates the dependence of the resonant absorption coefficient on λ_{ex}. The dashed curve corresponds to the linear case (such a curve was given in Southwood and Kivelson (1986), where it was taken from Forslund et al. (1975)), and the solid curves represent the nonlinear absorption at different values of $|C_I|\lambda_L^{-1/2}$. It is seen that with a decrease of this parameter, the curves become more flattened, with an increasingly less pronounced maximum.

6. Conclusions

This paper is a continuation of the study, initiated in Dmitrienko (1997, 1999), of the influence of nonlinear effects on the processes accompanying the propagation of an MHD mode in the region where its frequency coincides with the local Alfvén velocity. We have obtained the matching conditions for such a mode through the resonance layer in which the influence of both the dissipation (resistivity and viscosity) and nonlinearity is important. The MHD mode outside the resonance layer

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is described by a linear combination of two linearly independent solutions, one of which is regular while the other has a singularity. The coefficients of these solutions are generally different on different sides of the resonance surface. A complete description of the mode calls for a matching of these coefficients through the resonance layer. In this paper, it has been found that one matching condition is the condition of constancy of the coefficient of the singular solution. This property of a resonant disturbance arises in both the nonlinear and nonlinear cases; for the linear case, it is even known as a conservation law (in a different form, however) (Goossens and Ruderman 1995). According to its physical meaning, such a matching is a formalization of the concept (which was often taken as a postulate in earlier studies) of a minor influence of a resonant Alfvén disturbance on a large-scale disturbance generating it, the amplitude of which in the resonance region is just characterized by the coefficient of the singular solution. The other matching condition, as shown in this paper, is the jump of the coefficient of the regular solution. This jump is expressed in terms of an integral in the sense of the principal value of a function describing a resonant disturbance, and depends on the ratio of the dissipative to nonlinear spatial scales that is characterized by the parameter λ. In the linear case (when the linear dissipative scale is much larger than the nonlinear scale, i.e. $\lambda \gg 1$), it becomes a jump of the logarithm phase at the transition through the resonance layer, and is $-\pi i$; as λ decreases, its imaginary part decreases and tends to 0. The value of the imaginary part of the jump of the coefficient of the regular solution in the nonlinear case, as in the linear case, is connected with the loss of energy by the large-scale MHD disturbance in the resonance layer. In this paper, we have established a relation between the imaginary part of the jump of the coefficient of the regular solution and the rate of resonant heating of the plasma. It was found that if a dimensionless resonant heating rate that depends on λ only is introduced, then the ratio of the jump of the imaginary part of the regular solution to the coefficient of the singular solution is exactly equal to this dimensionless resonant heating rate. Such a simple relationship holds because the only linear small effect that has been taken into account in this paper is the dissipation, and the energy that is lost by a large-scale disturbance transforms to thermal energy of particles. In the presence of non-stationarity, but with dissipative effects being unimportant (Dmitrienko 1999), the jump in the imaginary part of the regular solution is associated with the accumulation of energy in the resonance layer in the form of Alfvén waves. In the last case, we are dealing with a non-stationary resonance layer, which can be linear or nonlinear depending on the ratio of the linear non-stationary to nonlinear scales. The case considered in this paper, however, may be characterized as a nonlinear dissipative resonance layer. In a linear dissipative layer (when $\lambda \gg 1$), the dimensionless rate of resonant heating does not depend on the value of the dissipative coefficients; in the nonlinear case, it decreases with decreasing λ.

The rate of actual physical resonant heating depends, in addition to λ, on the amplitude of the MHD mode in the resonance layer; it increases with increasing amplitude, but this growth is linear as against quadratic in the linear case. The decrease in the resonant heating rate due to the nonlinearity effect might, perhaps, be interpreted as resulting from a violation of the resonance condition of the Alfvén and large-scale MHD waves because of the nonlinear addition to the local Alfvén frequency.

Formulas have also been obtained that relate the heating rate to the absorption coefficient of the large-scale compressible mode and to the jump of the plasma flow velocity across the layer. It has been shown that these quantities are both dependent only on λ and on the parameter λ_{ex} that characterizes the degree of penetration of the compressible mode to the resonance surface; as λ decreases, both of them decrease and tend to zero.

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