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INTERPRETABILITY LOGICS AND GENERALISED VELTMAN SEMANTICS

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Abstract. We obtain modal completeness of the interpretability logics $\mathbf{IL}\mathsf{P}_0$ and $\mathbf{IL}\mathsf{R}$ w.r.t. generalised Veltman semantics. Our proofs are based on the notion of full labels [2]. We also give shorter proofs of completeness w.r.t. the generalised semantics for many classical interpretability logics. We obtain decidability and finite model property w.r.t. the generalised semantics for $\mathbf{IL}\mathsf{P}_0$ and $\mathbf{IL}\mathsf{R}$. Finally, we develop a construction that might be useful for proofs of completeness of extensions of $\mathbf{IL}\mathsf{W}$ w.r.t. the generalised semantics in the future, and demonstrate its usage with $\mathbf{IL}\mathsf{W}^* = \mathbf{IL}\mathsf{WM}_0$.

§1. Introduction.

1.1. Interpretability logics. It is well known that sufficiently strong formal theories T can reason about their own provability. The usual way to do this is through a certain Σ_1-predicate that formalises provability, usually denoted as Prov_T. For example, the following is provable in T:

$$\mathrm{Prov}_T\left(\overline{\left\ulcorner \neg \mathrm{Prov}_T\left(\overline{\ulcorner \bot \urcorner}\right)\right\urcorner}\right) \to \mathrm{Prov}_T\left(\overline{\ulcorner \bot \urcorner}\right),$$

that is, (the formalised version of) Gödel's second incompleteness theorem. It was also Gödel who first noticed that many interesting properties or Prov_T can be expressed in a simple modal language, where $\Box$ stands for Prov_T, and usages of $\overline{\ulcorner \cdot \urcorner}$ are left implicit. Gödel's second incompleteness theorem can be expressed more succinctly in this way:

$$\Box\neg\Box\bot \to \Box\bot.$$

Examples of other properties of Prov_T expressible in a modal language are $\Box(A \to B) \to (\Box A \to \Box B)$ and $\Box A \to \Box\Box A$ (where A and B are arbitrary sentences). The idea of treating provability predicate as a modality was also considered by Kripke and Montague. The correct choice of axioms, based on (the formalised version of) Löb's theorem, was seriously considered by several logicians independently: Boolos, de Jongh, Magari, Sambin, and Solovay.

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The provability logic **GL** (Gödel, Löb) is a modal propositional logic with the single unary modal operator $\Box$. The axioms of the system **GL** are all propositional tautologies (in the new language), and all instances of the schemas K: $\Box(A \to B) \to (\Box A \to \Box B)$, and L: $\Box(\Box A \to A) \to \Box A$. The inference rules of **GL** are modus ponens and necessitation $A/\Box A$. All interpretability logics we consider are extensions, both in terms of their language and their theoremhood, of **GL**.

Solovay [11] proved the arithmetical completeness theorem for **GL**. This theorem holds for all extensions of $\mathsf{I}\Delta_0 + \mathsf{EXP}$, where EXP is the sentence formalising the totality of exponentiation. This theorem shows that the language of provability logic **GL** is too weak to distinguish between most of the theories that are usually considered.[1] For example, whether a theory is finitely axiomatisable does not affect the theory's provability logic.

Other formal properties, beside provability, have been explored through modal or semimodal systems. For example, interpretability, Π_n-conservativity and interpolability.

We consider the usual modal treatment of interpretability: interpretability logics. Let us briefly describe what is usually meant by "interpretability" in the context of interpretability logics. Roughly, the theory S interprets the theory T if there is a natural way of translating the language of T into the language of S in such a way that the translations of the theorems of T are provable in S. In a sufficiently strong formal theory T in the language $\mathcal{L}_T$, one can construct a binary interpretability predicate Int_T. This predicate expresses that one finite extension of T interprets another finite extension of T.

Modal logics for interpretability were first studied by Hájek (1981) and Švejdar (1983). Visser introduced the modal logic **IL** (interpretability logic), a modal logic with a binary modal operator representing interpretability, in 1990 [13]. This operator is the only addition to the language; i.e., the language of interpretability logics is given by

$$A ::= \bot \mid p \mid A \to A \mid A \rhd A,$$

where p ranges over a countable set of propositional variables. Other Boolean connectives are defined as abbreviations, as usual. We treat $\rhd$ as having higher priority than $\to$, but lower than other logical connectives. Since $\Box B$ too can be defined (over **IL**) as an abbreviation (expanded to $\neg B \rhd \bot$), we do not formally include $\Box$ in the language. Similarly, we do not include $\Diamond$ in the language, where $\Diamond B$ stands for $\neg\Box\neg B$. If A is constructed in this way, we will say that A is a modal formula.

Any mapping $A \mapsto A^*$, with A a modal formula and $A^* \in \mathcal{L}_T$, such that:

- it commutes with logical connectives;
- if p is a propositional variable, p^* is a sentence;
- $(A \rhd B)^* = \mathsf{Int}_T\left(\overline{\ulcorner A^* \urcorner}, \overline{\ulcorner B^* \urcorner}\right)$, where $\overline{\ulcorner X \urcorner}$ is the numeral of the Gödel number of X;

is called *an arithmetical interpretation*.

[1] However, it is possible that the provability logics of theories below $\mathsf{I}\Delta_0 + \mathsf{EXP}$ differ from **GL**.

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The *interpretability logic of a theory* T, denoted by $\mathbf{IL}(\mathsf{T})$, is the set of all modal formulas A such that $T \vdash A^*$ for all arithmetical interpretations. While there are open questions regarding interpretability logics of certain theories, it is known that they all extend the basic system **IL**.

DEFINITION 1. The interpretability logic **IL** is axiomatised by the following axiom schemas.

- classical tautologies (in the new language);

(K) $\Box(A \to B) \to (\Box A \to \Box B)$;
(L) $\Box(\Box A \to A) \to \Box A$;
(J1) $\Box(A \to B) \to A \rhd B$;
(J2) $(A \rhd B) \wedge (B \rhd C) \to A \rhd C$;
(J3) $(A \rhd C) \wedge (B \rhd C) \to A \vee B \rhd C$;
(J4) $A \rhd B \to (\Diamond A \to \Diamond B)$;
(J5) $\Diamond A \rhd A$.

Rules of inference are modus ponens and necessitation $A/\Box A$.

In this context, the axiom (K) is a formalisation of the deduction theorem. Axiom (L) is a formalisation of Löb's theorem. Particularly interesting are the axioms (J4) and (J5). The axiom (J4) says that relative interpretability implies relative consistency. The axiom (J5) is the arithmetised model existence lemma: from a model of the consistency of A it is possible to unravel a model of A.

We say that a *modal* formula A is valid in a formal theory T if $T \vdash A^*$ for every arithmetical interpretation $*$. A modal theory S is sound w.r.t. T if all its theorems are valid in T. A modal theory S is complete w.r.t. T if it proves exactly those formulas that are valid in T. For the proof that the system **IL** is sound w.r.t. any reasonable formal theory, see [13].

The system **IL** is, unlike **GL**, arithmetically incomplete w.r.t. any reasonable theory. For example, **IL** does not prove all instances of $A \rhd B \to A \rhd B \wedge \Box\neg A$, which are all valid in every reasonable theory. To achieve completeness, we have to study extensions of the basic system **IL**. Extensions are built by adding new axiom schemas, the so-called principles of interpretability. Two principles and the corresponding extensions of **IL** are of particular interest because they are *the interpretability logic* of many interesting theories.

Montagna's principle M: $A \rhd B \to A \wedge \Box C \rhd B \wedge \Box C$ is valid in theories proving full induction. We denote by $\mathbf{IL}\mathsf{M}$ the system obtained by adding all instances of the principle M to the system **IL** as new axioms. Berarducci [1] and Shavrukov [10] independently proved that $\mathbf{IL}(\mathsf{T}) = \mathbf{IL}\mathsf{M}$, if T is an essentially reflexive theory. The persistence principle P: $A \rhd B \to \Box(A \rhd B)$ is valid in finitely axiomatisable theories. Visser [13] proved the arithmetical completeness of $\mathbf{IL}\mathsf{P}$ w.r.t. any finitely axiomatisable theory containing $\mathsf{I}\Delta_0 + \mathsf{SUPEXP}$, where SUPEXP asserts the totality of superexponentiation (tetration). Thus the interpretability logic $\mathbf{IL}\mathsf{M}$ of first-order Peano arithmetic differs from the interpretability logic $\mathbf{IL}\mathsf{P}$ of Gödel-Bernays set theory. It is still an open problem what is the interpretability logic of weaker theories like $\mathsf{I}\Delta_0 + \mathsf{EXP}, \mathsf{I}\Delta_0 + \Omega_1$ and PRA.

In particular, one can ask what is the set of principles valid in all reasonable theories.[2] This set is usually denoted as $\mathbf{IL}(\mathsf{All})$. Note that this does not mean that there has to be a theory T that attains $\mathbf{IL}(\mathsf{All})$ as its interpretability logic, i.e., $\mathbf{IL}(\mathsf{T}) = \mathbf{IL}(\mathsf{All})$. Clearly $\mathbf{IL} \subseteq \mathbf{IL}(\mathsf{All}) \subseteq \mathbf{IL}\mathsf{P} \cap \mathbf{IL}\mathsf{M}$. In fact, we know that both inclusions are proper. The ongoing search for $\mathbf{IL}(\mathsf{All})$ is the main motivation behind studying extensions of $\mathbf{IL}$ today. Studying modal properties of lower bounds of $\mathbf{IL}(\mathsf{All})$ turns out to be useful for finding new principles within $\mathbf{IL}(\mathsf{All})$. For example, the principle R was discovered while trying to prove modal completeness of $\mathbf{IL}\mathsf{P}_0\mathsf{W}$ [5] (we will define all these principles later). See the most recent development [6] for an overview of the progress that has been made in the search for $\mathbf{IL}(\mathsf{All})$.

Definitions and further details regarding interpretability and interpretability logics in general can be found in e.g., [14]. The remainder of this paper deals with modal semantics for interpretability logics.

1.2. Semantics. The most commonly used semantics for the interpretability logic $\mathbf{IL}$ and its extensions is the *Veltman semantics* (or *ordinary Veltman semantics*).

Definition 2 ([3], Definition 1.2). A Veltman frame $\mathfrak{F}$ is a structure $(W, R, \{S_w : w \in W\})$, where W is a nonempty set, R is a transitive and converse well-founded binary relation on W and for all $w \in W$ we have:

a) $S_w \subseteq R[w]^2$, where $R[w] = \{x \in W : w\,R\,x\}$;
b) S_w is reflexive on $R[w]$;
c) S_w is transitive;
d) if $w\,R\,u\,R\,v$ then $u\,S_w\,v$.

The standard logic of (the formalised) provability, the logic $\mathbf{GL}$, is complete w.r.t. the semantics based on the so-called $\mathbf{GL}$-frames (where $w \Vdash \Box A$ iff.: $w\,R\,x$ implies $x \Vdash A$). All interpretability logics that we study here conservatively extend the logic of provability. So, it should not be surprising that (W, R) in the preceding definition is precisely a $\mathbf{GL}$-frame. For reasons already explained earlier, we will usually work as if the symbol $\Box$ is not in the language.

A *Veltman model* is a quadruple $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$, where the first three components form a Veltman frame. The forcing relation $\Vdash$ is extended as usual in Boolean cases, and $w \Vdash A \rhd B$ holds if and only if for all u such that wRu and $u \Vdash A$ there exists v such that $u\,S_w\,v$ and $v \Vdash B$.

In what follows we will mainly use a different semantics, which we will refer to as the *generalised Veltman semantics*. R. Verbrugge [12] defined this specific generalisation of Veltman semantics. The main purpose of its introduction, and until recently the only usage, was to show that certain extensions of $\mathbf{IL}$ are independent, by Verbrugge [12], Vuković [16] and Goris and Joosten [5].

Definition 3. A generalised Veltman frame $\mathfrak{F}$ is a structure $(W, R, \{S_w : w \in W\})$, where W is a nonempty set, R is a transitive and converse well-founded binary relation on W and for all $w \in W$ we have:

a) $S_w \subseteq R[w] \times (\mathcal{P}(R[w]) \setminus \{\emptyset\})$;
b) S_w is quasi-reflexive: $w\,R\,u$ implies $u\,S_w\,\{u\}$;

[2] "Reasonable" usually means "an extension of S^1_2 or $\mathsf{I}\Delta_0 + \Omega_1$", or "as weak as possible under the condition that $\mathbf{IL}(\mathsf{All})$ remains elegantly axiomatisable".

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c) S_w is quasi-transitive: if $u\,S_w\,V$ and $v\,S_w\,Z_v$ for all $v \in V$, then $u\,S_w\,(\bigcup_{v\in V} Z_v)$;
d) if $w\,R\,u\,R\,v$, then $u\,S_w\,\{v\}$;
e) monotonicity: if $u\,S_w\,V$ and $V \subseteq Z \subseteq R[w]$, then $u\,S_w\,Z$.

A *generalised Veltman model* is a quadruple $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$, where the first three components form a generalised Veltman frame. Now $w \Vdash A \rhd B$ holds if and only if for all u such that $w\,R\,u$ and $u \Vdash A$ there exists V such that $u\,S_w\,V$ and $V \Vdash B$. By $V \Vdash B$ we mean that $v \Vdash B$ for all $v \in V$.

1.3. Principles, completeness, and decidability. Let us review some relevant results and approaches. When we need to refer to an extension of **IL** (an arbitrary extension if not stated otherwise), we will write **IL**X.

Let (X) (resp. $(\mathsf{X})_{\text{gen}}$) denote a formula of first-order or higher-order logic such that for all ordinary (resp. generalised) Veltman frames $\mathfrak{F}$ the following holds:

$$\mathfrak{F} \Vdash \mathsf{X} \text{ if and only if } \mathfrak{F} \models (\mathsf{X}) \text{ (resp. } \mathfrak{F} \models (\mathsf{X})_{\text{gen}}).$$

Formulas (X) and $(\mathsf{X})_{\text{gen}}$ are called characteristic properties (or frame conditions) of the given logic **IL**X. The class of all ordinary (resp. generalised) Veltman frames $\mathfrak{F}$ such that $\mathfrak{F} \models (\mathsf{X})$ (resp. $\mathfrak{F} \models (\mathsf{X})_{\text{gen}}$) is called the characteristic class of (resp. generalised) frames for **IL**X. If $\mathfrak{F} \models (\mathsf{X})_{\text{gen}}$ we also say that the frame $\mathfrak{F}$ possesses the property $(\mathsf{X})_{\text{gen}}$. We say that an ordinary (resp. generalised) Veltman model $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ is an **IL**X-model (resp. $\mathbf{IL}_{\text{gen}}$X-model), or that model $\mathfrak{M}$ possesses the property (X) (resp. $(\mathsf{X})_{\text{gen}}$), if the frame $(W, R, \{S_w : w \in W\})$ possesses the property (X) (resp. $(\mathsf{X})_{\text{gen}}$). A logic **IL**X will be said to be complete with respect to the ordinary (resp. generalised) semantics if for all modal formulas A we have that validity of A over all **IL**X-frames (resp. all $\mathbf{IL}_{\text{gen}}$X-frames) implies $\mathbf{IL}\mathsf{X} \vdash A$.

We say that **IL**X has the finite model property (FMP) w.r.t. ordinary (resp. generalised) semantics if for each formula A satisfiable in some **IL**X-model (resp. $\mathbf{IL}_{\text{gen}}$X-model), A is also satisfiable in some finite **IL**X-model (resp. $\mathbf{IL}_{\text{gen}}$X-model).

If we include results from the current paper, we have the following table. Here, *o* stands for ordinary Veltman semantics, and *g* for generalised Veltman semantics (as defined earlier).

	Principle	Compl. (o)	Compl. (g)	FMP (o)	FMP (g)
M	$A \rhd B \to A \wedge \Box C \rhd B \wedge \Box C$	+	+	+	+
$\mathsf{M_0}$	$A \rhd B \to \Diamond A \wedge \Box C \rhd B \wedge \Box C$	+	+	?	+
P	$A \rhd B \to \Box(A \rhd B)$	+	+	+	+
$\mathsf{P_0}$	$A \rhd \Diamond B \to \Box(A \rhd B)$	-	+	?	+
R	$A \rhd B \to \neg(A \rhd \neg C) \rhd B \wedge \Box C$	?	+	?	+
W	$A \rhd B \to A \rhd B \wedge \Box\neg A$	+	+	+	+
W*	$A \rhd B \to B \wedge \Box C \rhd B \wedge \Box C \wedge \Box\neg A$	+	+	?	+

De Jongh and Veltman proved the completeness of the logics **IL**, **IL**M and **IL**P w.r.t. their characteristic classes of ordinary (and finite) Veltman frames in [3]. As is usual for extension of the provability logic **GL**, all completeness proofs suffer

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from compactness-related issues. One way to go about this is to define a (large enough) adequate set of formulas and let worlds be maximal consistent subsets of such sets (used e.g., in [3]). With interpretability logics and the ordinary semantics, worlds have not been identified with (only) sets of formulas. It seems that with the ordinary semantics it is sometimes necessary to duplicate worlds in order to build models for certain consistent sets (see e.g., [3]). In [7], de Jongh and Veltman proved completeness of the logic **IL**W w.r.t. its characteristic class of ordinary (and finite) Veltman frames.

Goris and Joosten introduced a more robust approach to proving completeness of interpretability logics, the *construction method* [4], [5]. In this type of proofs, one builds models step by step, and the final model is retrieved as a union. While closer to the intuition and more informative than the standard proofs, these proofs are hard to produce and verify due to their size. (They might have been shorter if tools from [2] have been used from the start.) For the purpose for which they were invented (completeness of $\mathbf{IL}\mathsf{M}_0$ and $\mathbf{IL}\mathsf{W}^*$ w.r.t. the ordinary semantics) they are still the only known tools.

The completeness of interpretability logics w.r.t. the generalised semantics is an easy consequence of the completeness w.r.t. the ordinary semantics. In [9] and [8], the filtration technique was used to prove the finite model property of **IL** and its extensions **IL**M, $\mathbf{IL}\mathsf{M}_0$ and $\mathbf{IL}\mathsf{W}^*$ w.r.t. the generalised semantics. In those papers, generalised semantics was used because certain issues occur when trying to merge multiple worlds into one in ordinary Veltman models. Those explorations yielded some decidability results.

The aim of this paper is to show completeness (w.r.t. the generalised semantics) and decidability of some interpretability logics. We introduce a very direct type of proofs of completeness; similar to [3] in their general approach. We use *smart labels* from [2] for this purpose. An example that illustrates benefits of using the generalised semantics will be given in the section dedicated to $\mathbf{IL}\mathsf{M}_0$.

The main new results of this paper are completeness and finite model property (and thus decidability) of **IL**R and $\mathbf{IL}\mathsf{P}_0$. The principle R is important because it forms the basis of the, at the moment, best explicit candidate for **IL**(All). Results concerning the principle $\mathbf{IL}\mathsf{P}_0$ are interesting in a different way; they answer an old question: is there an unravelling technique that transforms generalised **IL**X-models to ordinary **IL**X-models, that preserves satisfaction of relevant characteristic properties? The answer is *no*: we find $\mathbf{IL}\mathsf{P}_0$ to be complete w.r.t. the generalised semantics, but it is known to be incomplete w.r.t. the ordinary semantics.

Other results include reproving some known facts with, in some cases, much shorter proofs (**IL**, **IL**P, **IL**M, $\mathbf{IL}\mathsf{M}_0$). Of particular interest is the logic **IL**W, which was known to be complete and decidable, but for which we nevertheless reprove completeness w.r.t. the generalised semantics using our approach. We will explain our motivation for doing so in the section dedicated to **IL**W and $\mathbf{IL}\mathsf{W}^*$.

§2. Completeness w.r.t. the generalised semantics. In what follows, "formula" will always mean "modal formula". If the ambient logic in some context is **IL**X, a maximal consistent set w.r.t. **IL**X will be called an **IL**X-MCS. Let us now introduce *smart labels* from [2].

Definition 4 ([2], a slightly modified Definition 3.1). Let w and u be some **IL**X-MCS's, and let S be an arbitrary set of formulas. We write $w \prec_S u$ if for any finite $S' \subseteq S$ and any formula A we have that $A \rhd \bigvee_{G \in S'} \neg G \in w$ implies $\neg A, \Box\neg A \in u$.

Note that the small differences between our Definition 4 and Definition 3.1 [2] do not affect the results of [2] that we use.[3]

Definition 5 ([2], p. 4). Let w be an **IL**X-MCS, and S an arbitrary set of formulas. Put:

$$w_S^{\Box} = \{\Box\neg A : \exists S' \subseteq S, S' \text{ finite}, A \rhd \bigvee_{G \in S'} \neg G \in w\};$$

$$w_S^{\boxdot} = \{\neg A, \Box\neg A : \exists S' \subseteq S, S' \text{ finite}, A \rhd \bigvee_{G \in S'} \neg G \in w\}.$$

Thus, $w \prec_S u$ if and only if $w_S^{\boxdot} \subseteq u$. If $S = \emptyset$ then $w_\emptyset^{\Box} = \{\Box\neg A : A \rhd \bot \in w\}$. Since w is maximal consistent, usages of this set usually amount to the same as the usages of the set $\{\Box A : \Box A \in w\}$.

We will usually write $w \prec u$ instead of $w \prec_\emptyset u$.

Lemma 6 ([2], Lemma 3.2). *Let w, u and v be some* **IL**X-*MCS's, and let S and T be some sets of formulas. Then we have*:

a) *if $S \subseteq T$ and $w \prec_T u$, then $w \prec_S u$;*
b) *if $w \prec_S u \prec v$, then $w \prec_S v$;*
c) *if $w \prec_S u$, then $S \subseteq u$.*

We will tacitly use the preceding lemma in most of our proofs.

The following two lemmas can be used to construct (or in our case, find) a MCS with the required properties.

Lemma 7 ([2], Lemma 3.4). *Let w be an* **IL**X-*MCS, and let $\neg(B \rhd C) \in w$. Then there is an* **IL**X-*MCS u such that $w \prec_{\{\neg C\}} u$ and $B, \Box\neg B \in u$.*

Lemma 8 ([2], Lemma 3.5). *Let w and u be some* **IL**X-*MCS's such that $B \rhd C \in w$, $w \prec_S u$ and $B \in u$. Then there is an* **IL**X-*MCS v such that $w \prec_S v$ and $C, \Box\neg C \in v$.*

Let B be a formula, and w a world in a generalised Veltman model. We write $[B]_w$ for $\{u : w R u \text{ and } u \Vdash B\}$.

In the remainder of the current paper, we will assume that $\mathcal{D}$ is always a finite set of formulas, closed under taking subformulas and single negations, and $\top \in \mathcal{D}$. The following definition is central to most of the results of this paper.

Definition 9. Let X be a subset of $\{\mathsf{M}, \mathsf{M_0}, \mathsf{P}, \mathsf{P_0}, \mathsf{R}\}$. We say that $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ is the **IL**X-structure for a set of formulas $\mathcal{D}$ if:

[3]The difference is a different strategy of ensuring converse well-foundedness for the relation R. Instead of asking for the existence of some $\Diamond F \in w \setminus u$ whenever $w R u$, as is usual in the context of provability (and interpretability) logics, we will go for a stronger condition (see Definition 9). Since we will later put $R := \prec$, this choice of ours is reflected already at this point.

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$$W = \{w : w \text{ is an } \mathbf{IL}\mathsf{X}\text{-MCS and for some } G \in \mathcal{D}, G \wedge \Box\neg G \in w\};$$
$$w\,R\,u \Leftrightarrow w \prec u;$$
$$u\,S_w\,V \Leftrightarrow w\,R\,u, V \subseteq R[w], (\forall S)(w \prec_S u \Rightarrow (\exists v \in V) w \prec_S v);$$
$$w \Vdash p \Leftrightarrow p \in w.$$

LEMMA 10. *If* $\mathbf{IL}\mathsf{X} \nvdash \neg A$ *then there is an* $\mathbf{IL}\mathsf{X}$*-MCS* w *such that* $A \wedge \Box\neg A \in w$.

PROOF. We are to show that $\{A \wedge \Box\neg A\}$ is an $\mathbf{IL}\mathsf{X}$-consistent set. Suppose $A, \Box\neg A \vdash \bot$. It follows that $\vdash \Box\neg A \to \neg A$. Applying generalisation (necessitation) gives $\vdash \Box(\Box\neg A \to \neg A)$. The Löb axiom implies $\vdash \Box\neg A$. Now, $\vdash \Box\neg A$ and $A, \Box\neg A \vdash \bot$ imply $A \vdash \bot$, i.e., $\vdash \neg A$, a contradiction. ⊣

LEMMA 11. *Let* X *be a subset of* $\{\mathsf{M}, \mathsf{M_0}, \mathsf{P}, \mathsf{P_0}, \mathsf{R}\}$. *The* $\mathbf{IL}\mathsf{X}$*-structure* $\mathfrak{M}$ *for a set of formulas* $\mathcal{D}$ *is a generalised Veltman model. Furthermore, the following holds*:

$$\mathfrak{M}, w \Vdash G \ \text{ if and only if } \ G \in w,$$

for all $G \in \mathcal{D}$ *and* $w \in W$.

PROOF. Let us verify that the $\mathbf{IL}\mathsf{X}$-structure $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ for $\mathcal{D}$ is a generalised Veltman model. Since $\mathbf{IL}\mathsf{X} \nvdash \bot$ and $\top \in \mathcal{D}$, Lemma 10 implies $W \neq \emptyset$.

Transitivity of R is immediate. To see converse well-foundedness, assume there are more than $|\mathcal{D}|$ worlds in an R-chain. Then there are x and y with $x\,R\,y$ and for some $G \in \mathcal{D}$, $G, \Box\neg G \in x, y$. However, $\Box\neg G \in x$ and $G \in y$ obviously contradict the assumption that $x\,R\,y$ ($x \prec y$).

Next, let us prove the properties of S_w for $w \in W$. Clearly $S_w \subseteq R[w] \times \mathcal{P}(R[w])$. If $x\,S_w\,V$, then $w \prec_\emptyset x$ implies there is at least one element v in V (with $w \prec_\emptyset v$). Quasi-reflexivity and monotonicity are obvious. Next, assume $w\,R\,x\,R\,u$ and $w \prec_S x$. Lemma 6 and $w \prec_S x \prec u$ imply $w \prec_S u$. Thus $xS_w\{u\}$. It remains to prove quasi-transitivity. Assume $x\,S_w\,V$ and $v\,S_w\,U_v$ for all $v \in V$. Put $U = \bigcup_v U_v$. We claim that $x\,S_w\,U$. We have $U \subseteq R[w]$. Assume $w \prec_S x$. This and $x\,S_w\,V$ imply there is $v \in V$ such that $w \prec_S v$. This and $v\,S_w\,U_v$ imply there is $u \in U_v$ (thus also $u \in U$) such that $w \prec_S u$.

Let us prove the truth lemma with respect to the formulas contained in $\mathcal{D}$. The claim is proved by induction on the complexity of $G \in \mathcal{D}$. We will only consider the case $G = B \rhd C$.

Assume $B \rhd C \in w$, $w\,R\,u$ and $u \Vdash B$. Induction hypothesis implies $B \in u$. We claim that $u\,S_w\,[C]_w$. Clearly $[C]_w \subseteq R[w]$. Assume $w \prec_S u$. Lemma 8 implies there is an $\mathbf{IL}\mathsf{X}$-MCS v with $w \prec_S v$ and $C, \Box\neg C \in v$ (thus also $w\,R\,v$ and $v \in W$). Induction hypothesis implies $\mathfrak{M}, v \Vdash C$.

To prove the converse, assume $B \rhd C \notin w$. Lemma 7 implies there is u with $w \prec_{\{\neg C\}} u$ and $B, \Box\neg B \in u$ (thus $u \in W$). It is immediate that $w\,R\,u$ and the induction hypothesis implies that $u \Vdash B$. Assume $u\,S_w\,V$. We are to show that $V \nVdash C$. Since $w \prec_{\{\neg C\}} u$ and $u\,S_w\,V$, there is $v \in V$ such that $w \prec_{\{\neg C\}} v$. Lemma 6 implies $\neg C \in v$. The induction hypothesis implies $v \nVdash C$; thus $V \nVdash C$. ⊣

THEOREM 12. *Let* $X \subseteq \{\mathsf{M}, \mathsf{M_0}, \mathsf{P}, \mathsf{P_0}, \mathsf{R}\}$. *Assume that for every set* $\mathcal{D}$ *the* $\mathbf{IL}\mathsf{X}$*-structure for* $\mathcal{D}$ *possesses the property* $(\mathsf{X})_{gen}$. *Then* $\mathbf{IL}\mathsf{X}$ *is complete w.r.t.* $\mathbf{IL}_{gen}\mathsf{X}$*-models.*

PROOF. Let A be a formula such that $\nvdash \neg A$. Lemma 10 implies there is an **IL**X-MCS w such that $A \wedge \Box\neg A \in w$. Let $\mathcal{D}$ have the usual properties, and contain A. Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be the **IL**X-structure for $\mathcal{D}$. Since $A \wedge \Box\neg A \in w$ and $A \in \mathcal{D}$, we have $w \in W$. Lemma 11 implies $\mathfrak{M}, w \nVdash \neg A$. ⊣

COROLLARY 13. *The logic* **IL** *is complete w.r.t. the generalised semantics.*

Note that the method presented in [17] now implies completeness of **IL** w.r.t. ordinary Veltman models. Unfortunately (but necessarily, as our result for **IL**P_0 shows), this method does not preserve characteristic properties in general.

In the following sections we prove (or reprove) the completeness of the following logics w.r.t. the generalised semantics: **IL**M, **IL**M_0, **IL**P, **IL**P_0, **IL**R, **IL**W, and **IL**W^*.

2.1. The logic ILM. Completeness of the logic **IL**M w.r.t. the generalised semantics is an easy consequence of the completeness of **IL**M w.r.t. the ordinary semantics, first proved by de Jongh and Veltman [3]. Another proof of the same result was given by Goris and Joosten, using the construction method [5].

Verbrugge determined the characteristic property $(\mathsf{M})_{\text{gen}}$ in [12]:

$$u\, S_w\, V \Rightarrow (\exists V' \subseteq V)(u\, S_w\, V' \;\&\; R[V'] \subseteq R[u]).$$

LEMMA 14 ([2], Lemma 3.7). *Let w and u be some* **IL**M-*MCS's, and let S be a set of formulas. If $w \prec_S u$ then $w \prec_{S \cup u^{\Box}_{\emptyset}} u$.*

THEOREM 15. *The logic* **IL**M *is complete w.r.t.* $\mathbf{IL}_{gen}\mathsf{M}$-*models.*

PROOF. Given Theorem 12, it suffices to show that for any set $\mathcal{D}$, the **IL**M-structure for $\mathcal{D}$ possesses the property $(\mathsf{M})_{\text{gen}}$. Let $(W, R, \{S_w : w \in W\}, \Vdash)$ be the **IL**M-structure for $\mathcal{D}$.

Let $u\, S_w\, V$ and take $V' = \{v \in V : w \prec_{u^{\Box}_{\emptyset}} v\}$. We claim $u\, S_w\, V'$ and $R[V'] \subseteq R[u]$. Suppose $w \prec_S u$. Lemma 14 implies $w \prec_{S \cup u^{\Box}_{\emptyset}} u$. Since $u\, S_w\, V$, we have that there is $v \in V$ with $w \prec_{S \cup u^{\Box}_{\emptyset}} v$. So, $v \in V'$. Thus, $u\, S_w\, V'$.

Now let $v \in V'$ and $z \in W$ be such that $v\, R\, z$. Since $v \in V'$, $w \prec_{u^{\Box}_{\emptyset}} v$. Then for all $\Box B \in u$ we have $\Box B \in v$. Since $v\, R\, z$, we have $B, \Box B \in z$. So, $u \prec z$ i.e., $u\, R\, z$. ⊣

2.2. The logic ILM$_0$. Modal completeness of **IL**M_0 w.r.t. ordinary Veltman semantics was proved in [4] by Goris and Joosten. Certain difficulties encountered in this proof were our main motivation for using generalised Veltman semantics. We will sketch one of these difficulties and show in what way the generalised semantics overcomes it.

Characteristic property $(\mathsf{M}_0)_{\text{gen}}$ (see [8]):

$$w\, R\, u\, R\, x\, S_w\, V \;\Rightarrow\; (\exists V' \subseteq V)(u\, S_w\, V' \;\&\; R[V'] \subseteq R[u])).$$

LEMMA 16 ([2], Lemma 3.9). *Let w, u, and x be* **IL**M_0-*MCS's, and S an arbitrary set of formulas. If $w \prec_S u \prec x$ then $w \prec_{S \cup u^{\Box}_{\emptyset}} x$.*

To motivate our proving of completeness (of **IL**M_0, but also in general) w.r.t. the generalised semantics, let us sketch a situation for which there are clear benefits in working with the generalised semantics. We do this only now because **IL**M_0 is

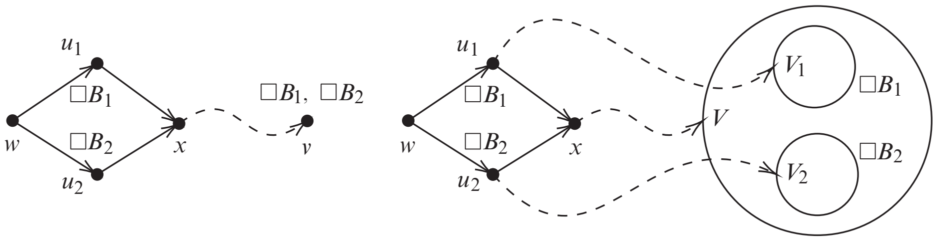

FIGURE 1. Left: extending an ordinary Veltman model. Right: extending a generalised Veltman model. Straight lines represent R-transitions, while curved lines represent S_w-transitions. Full lines represent the starting configuration, and dashed lines represent transitions that are to be added.

sufficiently complex to display (some of) these benefits. Suppose we are building models step by step (as in the *construction method* [4]), and worlds w, u_1, u_2, and x occur in the configuration displayed in Figure 1. Furthermore, suppose we need to produce an S_w-successor v of x.

With the ordinary semantics, we need to ensure that for our S_w-successor v, for each $\Box B_1 \in u_1$ and $\Box B_2 \in u_2$, we have $\Box B_1, \Box B_2 \in v$. It is not obvious that such construction is possible. In case of $\mathbf{IL}\mathsf{M}_0$, it was successfully solved in [4] by preserving the invariant that sets of boxed formulas in u_i are linearly ordered. This way, finite (quasi-)models can always be extended by only looking at the last u_i. With the generalised semantics, we need to produce a whole set of worlds V, but the requirements on each particular world are less demanding. For each u_i, there has to be a corresponding $V_i \subseteq V$ with $\Box B_i$ contained (true) in every world of V_i. Lemma 16 gives a recipe for producing such worlds.

THEOREM 17. *The logic* $\mathbf{IL}\mathsf{M}_0$ *is complete w.r.t.* $\mathbf{IL}_{gen}\mathsf{M}_0$*-models.*

PROOF. Given Theorem 12, it suffices to show that for any set $\mathcal{D}$, the $\mathbf{IL}\mathsf{M}_0$-structure for $\mathcal{D}$ possesses the property $(\mathsf{M}_0)_{\text{gen}}$. Let $(W, R, \{S_w : w \in W\}, \Vdash)$ be the $\mathbf{IL}\mathsf{M}_0$-structure for $\mathcal{D}$.

Assume $w\,R\,u\,R\,x\,S_w\,V$ and take $V' = \{v \in V : w \prec_{u^{\Box}_{\emptyset}} v\}$. We claim that $u\,S_w\,V'$ and $R[V'] \subseteq R[u]$. Obviously $V' \subseteq V \subseteq R[w]$. Assume $w \prec_S u$. Lemma 16 and $w \prec_S u \prec x$ imply $w \prec_{S \cup u^{\Box}_{\emptyset}} x$. Now $x\,S_w\,V$ and the definition of S_w imply there is $v \in V$ such that $w \prec_{S \cup u^{\Box}_{\emptyset}} v$. Lemma 6 implies $w \prec_{u^{\Box}_{\emptyset}} v$. So, $v \in V'$.

It remains to verify that $R[V'] \subseteq R[u]$. Let $v \in V'$ and $z \in W$ be worlds such that $v\,R\,z$. Since $w \prec_{u^{\Box}_{\emptyset}} v$, for all $\Box B \in u$ we have $\Box B \in v$, and since $v\,R\,z$, it follows that $\Box B, B \in z$. Thus, $u \prec z$ i.e., $u\,R\,z$. ⊣

2.3. The logic ILP. As in the case of the logic $\mathbf{IL}\mathsf{M}$, the completeness of $\mathbf{IL}\mathsf{P}$ w.r.t. the generalised semantics is an easy consequence of the completeness of $\mathbf{IL}\mathsf{P}$ w.r.t. the ordinary semantics, first proved by de Jongh and Veltman [3].

Verbrugge determined the characteristic property $(\mathsf{P})_{\text{gen}}$ in [12]:

$$wRw'RuS_wV \Rightarrow (\exists V' \subseteq V)\, uS_{w'}V'.$$

LEMMA 18 ([2] Lemma 3.8). *Let w, x, and u be some* **IL**P*-MCS's, and let S and T be arbitrary sets of formulas. If $w \prec_S x \prec_T u$ then $w \prec_{S\cup x_T^{\boxdot}} u$.*

THEOREM 19. *The logic* **IL**P *is complete w.r.t.* **IL**$_{gen}$P*-models.*

PROOF. Given Theorem 12, it suffices to show that for any set $\mathcal{D}$, the **IL**P-structure for $\mathcal{D}$ possesses the property $(\mathsf{P})_{\text{gen}}$. Let $(W, R, \{S_w : w \in W\}, \Vdash)$ be the **IL**P-structure for $\mathcal{D}$.

Let $wRw'RuS_wV$ and take $V' = V \cap R[w']$. We claim $uS_{w'}V'$. Let T be arbitrary such that $w' \prec_T u$. Lemma 18 and $w \prec_\emptyset w' \prec_T u$ imply $w \prec_{w'^{\boxdot}_T} u$. Now, uS_wV implies that there is a $v \in V$ with $w \prec_{w'^{\boxdot}_T} v$. Let $A \rhd \neg \bigwedge T' \in w'$ for some finite $T' \subseteq T$. Then $\neg A, \Box\neg A \in w'^{\boxdot}_T$. Lemma 6 and $w \prec_{w'^{\boxdot}_T} v$ imply $\neg A, \Box\neg A \in v$. Thus $w' \prec_T v$. Finally, $V' \subseteq R[w']$ holds by assumption, thus $uS_{w'}V'$. ⊣

2.4. The logic ILP$_0$. The interpretability principle $\mathsf{P}_0 = A \rhd \Diamond B \to \Box(A \rhd B)$ is introduced in J. Joosten's master thesis in 1998. In [5] it is shown that the interpretability logic **IL**P$_0$ is incomplete w.r.t. Veltman models. Since we will show that **IL**P$_0$ is complete w.r.t. the generalised semantics, this is the first example of an interpretability logic complete w.r.t. the generalised semantics, but incomplete w.r.t. the ordinary semantics.

Characteristic property $(\mathsf{P}_0)_{\text{gen}}$ was determined in [5]. A slightly reformulated version:

$$wRxRuS_wV \;\&\; (\forall v \in V)R[v] \cap Z \neq \emptyset \;\Rightarrow\; (\exists Z' \subseteq Z)uS_xZ'.$$

The following technical lemma is almost obvious.

LEMMA 20. *Let x be an* **IL**X*-MCS, A a formula, and T a finite set of formulas. Let B_G be an arbitrary formula, and T_G an arbitrary finite set of formulas, for every $G \in T$. Furthermore, assume:*

a) $A \rhd \bigvee_{G\in T} B_G \in x$;
b) $(\forall G \in T)\; B_G \rhd \bigvee_{H\in T_G} \neg H \in x$.

Then we have $A \rhd \bigvee_{H\in S'} \neg H \in x$, where $S' = \bigcup_{G\in T} T_G$.

PROOF. Let $G \in T$. Since $T_G \subseteq S'$, clearly $\vdash \bigvee_{H\in T_G} \neg H \rhd \bigvee_{H\in S'} \neg H$. The requirement b) and the axiom (J2) imply $B_G \rhd \bigvee_{H\in S'} \neg H \in x$. Now $|T| - 1$ applications of the axiom (J3) give $\bigvee_{G\in T} B_G \rhd \bigvee_{H\in S'} \neg H \in x$. Finally, apply the requirement a) and the axiom (J2). ⊣

Next we need a labelling lemma for **IL**P$_0$. This is where we use the technical lemma above.

LEMMA 21. *Let w, x, and u be some* **IL**P$_0$*-MCS's, and let S be a set of formulas. If $w \prec x \prec_S u$ then $w \prec_{x^{\boxdot}_S} u$.*

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Proof. Let A be an arbitrary formula. Let $T \subseteq x_S^{\Box}$ be a finite set such that $A \rhd \bigvee_{G\in T} \neg G \in w$. We will prove that $\neg A, \Box\neg A \in u$. If $G \in T$ $(\subseteq x_S^{\Box})$, then $G = \Box\neg B_G$, for some formula B_G. Thus $A \rhd \bigvee_{G\in T} \neg\Box\neg B_G \in w$, and by easy inferences and maximal consistency: $A \rhd \bigvee_{G\in T} \Diamond B_G \in w$, and $A \rhd \Diamond \bigvee_{G\in T} B_G \in w$. Applying P_0 gives $\Box(A \rhd \bigvee_{G\in T} B_G) \in w$. The assumption $w \prec x$ implies $A \rhd \bigvee_{G\in T} B_G \in x$. For each $G \in T$ $(\subseteq x_S^{\Box})$ there is a finite subset T_G of S such that $B_G \rhd \bigvee_{H\in T_G} \neg H \in x$. Let $S' = \bigcup_{G\in T} T_G$. Clearly S' is a finite subset of S. Lemma 20 implies $A \rhd \bigvee_{H\in S'} \neg H \in x$. Finally, $S' \subseteq S$ and the assumption $x \prec_S u$ imply $\neg A, \Box\neg A \in u$. ⊣

The following simple observation is useful both for **IL**P_0 and **IL**R.

Lemma 22. *Let w, x, v, and z be some* **IL**X-*MCS's, and let S be a set of formulas. If $w \prec_{x_S^{\Box}} v \prec z$ then $x \prec_S z$.*

Proof. Let S' be a finite subset of S with $A \rhd \bigvee_{G\in S'} \neg G \in x$. Then $\Box\neg A \in x_S^{\Box}$. Now $w \prec_{x_S^{\Box}} v$ and Lemma 6 imply $\Box\neg A \in v$. Since $v \prec z$, we have $\neg A, \Box\neg A \in z$. ⊣

Theorem 23. *The logic* **IL**P_0 *is complete w.r.t.* **IL**$_{gen}\mathsf{P}_0$-*frames.*

Proof. Given Theorem 12, it suffices to show that for any set $\mathcal{D}$, the **IL**P_0-structure for $\mathcal{D}$ possesses the property $(\mathsf{P}_0)_{\text{gen}}$. Let $(W, R, \{S_w : w \in W\}, \Vdash)$ be the **IL**P_0-structure for $\mathcal{D}$.

Assume $w\,R\,x\,R\,u\,S_w\,V$ and $R[v] \cap Z \neq \emptyset$ for each $v \in V$. We will prove that there is $Z' \subseteq Z$ such that uS_xZ'.

Let S be a set of formulas such that $w \prec x \prec_S u$. Lemma 21 implies $w \prec_{x_S^{\Box}} u$. Since $u\,S_w\,V$, there is $v \in V$ such that $w \prec_{x_S^{\Box}} v$. Since $R[v] \cap Z \neq \emptyset$, choose a world $z_S \in R[v] \cap Z$. Now $w \prec_{x_S^{\Box}} v \prec z_S$ and Lemma 22 imply $x \prec_S z_S$. Put $Z' = \{z_S : S$ is a set of formulas such that $x \prec_S u\}$. Clearly $Z' \subseteq Z$. So, $Z' \subseteq R[x]$, and since for each set S such that $x \prec_S u$ we have $x \prec_S z_S$, it follow that $u\,S_x\,Z'$. ⊣

In [17] a possibility was explored of transforming a generalised Veltman model to an ordinary Veltman model, such that these two models are bisimilar (in some aptly defined sense). A natural question is whether such transformation exists if we add the requirement that characteristic properties are preserved. The example of **IL**P_0 shows that there are **IL**$_{\text{gen}}\mathsf{P}_0$-models with no (bisimilar or otherwise) counterpart **IL**P_0-models.

2.5. The logic ILR. Completeness of **IL**R w.r.t. ordinary Veltman semantics is an open problem (see [2]), but completeness w.r.t. the generalised semantics is not yet resolved either. In this section we will prove that **IL**R is complete w.r.t. the generalised semantics.

Characteristic property $(\mathsf{R})_{\text{gen}}$ was determined in [5]. A slightly reformulated version:

$$w\,R\,x\,R\,u\,S_w\,V \Rightarrow (\forall C \in \mathcal{C}(x,u))(\exists U \subseteq V)(x\,S_w\,U \;\&\; R[U] \subseteq C),$$

where $\mathcal{C}(x,u) = \{C \subseteq R[x] : (\forall Z)(u\,S_x\,Z \Rightarrow Z \cap C \neq \emptyset)\}$ is the family of "choice sets".

Lemma 24 ([2], Lemma 3.10). *Let w, x and u be some* **IL**R-*MCS's, and let S and T be arbitrary sets of formulas. If $w \prec_S x \prec_T u$ then $w \prec_{S\cup x_T^{\Box}} u$.*

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THEOREM 25. *The logic* **IL**R *is complete w.r.t.* $\mathbf{IL}_{gen}$R-*models.*

PROOF. Given Theorem 12, it suffices to show that for any set $\mathcal{D}$, the **IL**R-structure for $\mathcal{D}$ possesses the property $(\mathsf{R})_{\text{gen}}$. Let $(W, R, \{S_w : w \in W\}, \Vdash)$ be the **IL**R-structure for $\mathcal{D}$.

Assume $w\,R\,x\,R\,u\,S_w\,V$ and $C \in \mathcal{C}(x,u)$. We are to show that $(\exists U \subseteq V)(x\,S_w\,U \;\&\; R[U] \subseteq C)$. We will first prove an auxiliary claim:

$$(\forall S)\big(w \prec_S x \;\Rightarrow\; (\exists v \in V)(w \prec_{S \cup x^{\Box}_{\emptyset}} v \;\&\; R[v] \subseteq C)\big).$$

So, let S be arbitrary such that $w \prec_S x$, and suppose (for a contradiction) that for every $v \in V$ with $w \prec_{S\cup x^{\Box}_{\emptyset}} v$, we have $R[v] \nsubseteq C$, that is, there is some $z_v \in R[v] \setminus C$. Let $Z = \{z_v : v \in V, w \prec_{S\cup x^{\Box}_{\emptyset}} v\}$. We claim that $u\,S_x\,Z$. Let T be arbitrary such that $x \prec_T u$, and we should prove that there exists $z \in Z$ such that $x \prec_T z$. From $w \prec_S x \prec_T u$ and Lemma 24 it follows that $w \prec_{S\cup x^{\Box}_{T}} u$. Since $u\,S_w\,V$, there is $v \in V$ with $w \prec_{S\cup x^{\Box}_{T}} v$. Now, $x^{\Box}_{\emptyset} \subseteq x^{\Box}_{T}$ and Lemma 6 imply $w \prec_{S\cup x^{\Box}_{\emptyset}} v$, so there is a world $z_v \in Z$ as defined earlier. Furthermore, $w \prec_{x^{\Box}_{T}} v \prec z_v$ and Lemma 22 imply $x \prec_T z_v$. To prove $u\,S_x\,Z$ it remains to verify that $Z \subseteq R[x]$. Let $z_v \in Z$ be arbitrary and apply Lemma 6 and Lemma 22 as before. Now, $u\,S_x\,Z$ and $C \in \mathcal{C}(x,u)$ imply $C \cap Z \neq \emptyset$, contradicting the definition of Z. This concludes the proof of the auxiliary claim.

Let $U = \{v \in V : w \prec_{x^{\Box}_{\emptyset}} v \text{ and } R[v] \subseteq C\}$. Auxiliary claim implies $U \neq \emptyset$. If $w \prec_S x$, auxiliary claim implies there is $v \in U$ such that $w \prec_{S\cup x^{\Box}_{\emptyset}} v$ and $R[v] \subseteq C$, so $v \in U$. Thus $x\,S_w\,U$. It is clear that $R[U] \subseteq C$. ⊣

§3. The logics ILW and ILW*. To prove that **IL**W is complete, one could try to find a sufficiently strong "labelling lemma" and utilise Definition 9. One candidate might be the following condition:

$$w \prec_S u \;\Rightarrow\; (\exists G \in \mathcal{D}) w \prec_{S\cup\{\Box\neg G\}} u \text{ and } G \in u,$$

where $\mathcal{D}$ is finite, closed under subformulas and such that each $w \in W$ contains A_w and $\Box\neg A_w$ for some $A_w \in \mathcal{D}$.

Since we weren't successful in finding a sufficiently strong labelling lemma for **IL**W, we will use a modified version of Definition 9 to work with **IL**W and its extensions. This way we won't require a labelling lemma, but we lose generality in the following sense. To prove the completeness of **IL**XW, for some X, it no longer suffices to simply show that the structure defined in Definition 9 has the required characteristic property (when each world is an **IL**X-MCS). Instead, the characteristic property of **IL**X has to be shown to hold on the modified structure. So, to improve compatibility with proofs based on Definition 9, we should prove the completeness of **IL**W with as similar definition to Definition 9 as possible. That is what we do in the remainder of this section. This approach turns out to be good enough for **IL**W* (**IL**WM$_0$). We didn't succeed in using it to prove the completeness of **IL**WR. However, to the best of our knowledge, **IL**WR might not be complete at all.

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In [5] the (complement of the) characteristic class for **IL**W is given by the condition Not-W such that for any generalised Veltman frame $\mathfrak{F}$ we have that

$$\mathfrak{F} \models \text{Not-W} \text{ if and only if } \mathfrak{F} \not\Vdash \mathsf{W}.$$

Another condition is $(\mathsf{W})_{\text{gen}}$ from [8]:

$$u\,S_w\,V \Rightarrow (\exists V' \subseteq V)(u\,S_w\,V' \;\&\; R[V'] \cap S_w^{-1}[V] = \emptyset).$$

We will use (this formulation of) $(\mathsf{W})_{\text{gen}}$ in what follows. In the proof of completeness of logic **IL**W we will use the following two lemmas. In what follows, **IL**WX denotes an arbitrary extension of **IL**W.

Lemma 26 ([2], Lemma 3.12). *Let w be an* **IL**WX-*MCS, and B and C formulas such that $\neg(B \rhd C) \in w$. Then there is an* **IL**WX-*MCS u such that $w \prec_{\{\Box\neg B, \neg C\}} u$ and $B \in u$.*

Lemma 27 ([2], Lemma 3.13). *Let w and u be some* **IL**WX-*MCS, B and C some formulas, and S a set of formulas such that $B \rhd C \in w, w \prec_S u$ and $B \in u$. Then there is an* **IL**WX-*MCS v such that $w \prec_{S \cup \{\Box\neg B\}} v$ and $C, \Box\neg C \in v$.*

Given a binary relation R, let $\dot{R}[x] = R[x] \cup \{x\}$.

Definition 28. Let X be W or W*. We say that $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ is the **IL**X-structure for a set of formulas $\mathcal{D}$ if:

$$W = \{w : w \text{ is an } \mathbf{IL}\mathsf{X} - \text{MCS and for some } G \in \mathcal{D},\ G \wedge \Box\neg G \in w\};$$

$$w\,R\,u \Leftrightarrow w \prec u;$$

$$u\,S_w\,V \Leftrightarrow w\,R\,u, V \subseteq R[w] \text{ and one of the following holds:}$$

$$(a)\quad V \cap \dot{R}[u] \neq \emptyset;$$

$$(b)\quad (\forall S)(w \prec_S u \Rightarrow (\exists v \in V)(\exists G \in \mathcal{D} \cap \cup\dot{R}[u])\ w \prec_{S \cup \{\Box\neg G\}} v);$$

$$w \Vdash p \Leftrightarrow p \in w.$$

Lemma 29. *Let* X *be* W *or* W*. **IL**X-*structure $\mathfrak{M}$ for $\mathcal{D}$ is a generalised Veltman model. Furthermore, the following holds*:

$$\mathfrak{M}, w \Vdash G \text{ if and only if } G \in w,$$

for each $G \in \mathcal{D}$ and $w \in W$.

Proof. Let us first verify that the **IL**X-structure $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ for $\mathcal{D}$ is a generalised Veltman model. All the properties, except for quasi-transitivity, have easy proofs (see the proof of Lemma 11).

Let us prove the quasi-transitivity. Assume $u\,S_w\,V$, and $v\,S_w\,U_v$ for all $v \in V$. Put $U = \bigcup_{v \in V} U_v$. We claim that $u\,S_w\,U$. Clearly $U \subseteq R[w]$. To prove $u\,S_w\,U$ we will distinguish the cases (a) and (b) from the definition of the relation S_w for $u\,S_w\,V$.

In the case (a), we have $v_0 \in V$ for some $v_0 \in \dot{R}[u]$. We will next distinguish two cases from the definition of $v_0\,S_w\,U_{v_0}$.

In the case (aa) we have $x \in U_{v_0}$ for some $x \in \dot{R}[v_0]$. Since $v_0 \in \dot{R}[u]$, we then have $x \in \dot{R}[u]$. Since $x \in U_{v_0} \subseteq U$, then $U \cap \dot{R}[u] \neq \emptyset$. So, we have $u\,S_w\,U$, as required.

In the case (ab) we have:

$$(\forall S)(w \prec_S v_0 \Rightarrow (\exists x \in U_{v_0})(\exists G \in \mathcal{D} \cap \bigcup \dot{R}[v_0])\ w \prec_{S\cup\{\Box\neg G\}} x).$$

To prove $u\, S_w\, U$ in this case, we will use the case (b) from the definition of the relation S_w. Assume $w \prec_S u$. Then we have $w \prec_S u \prec v_0$ or $w \prec_S u = v_0$. Either way, possibly using Lemma 6, we have $w \prec_S v_0$, and so there are $x \in U_{v_0}$ and $G \in \mathcal{D} \cap \bigcup \dot{R}[v_0]$ with $w \prec_{S\cup\{\Box\neg G\}} x$. Since $u\, R\, v_0$ or $u = v_0$, we have $\dot{R}[v_0] \subseteq \dot{R}[u]$. So, the claim follows.

In the case (b), we have:

$$(\forall S)(w \prec_S u \Rightarrow (\exists v \in V)(\exists G \in \mathcal{D} \cap \bigcup \dot{R}[u])\ w \prec_{S\cup\{\Box\neg G\}} v).$$

To prove $u\, S_w\, U$ we will use the case (b) from the definition of the relation S_w. Assume $w \prec_S u$. Then there are $v_0 \in V$ and $G \in \mathcal{D} \cap \bigcup \dot{R}[u]$ such that $w \prec_{S\cup\{\Box\neg G\}} v_0$. From $v_0 \in V$ it follows that $v_0\, S_w\, U_{v_0}$. We will next distinguish between the possible cases in the definition of $v_0\, S_w\, U_{v_0}$.

In the first case (ba) we have $U_{v_0} \cap \dot{R}[v_0] \neq \emptyset$, i.e., there is $x \in U_{v_0} \cap \dot{R}[v_0]$. Then $w \prec_{S\cup\{\Box\neg G\}} v_0 = x$ or $w \prec_{S\cup\{\Box\neg G\}} v_0 \prec x$. In both cases (possibly using Lemma 6) we have $w \prec_{S\cup\{\Box\neg G\}} x$.

In the case (bb):

$$(\forall S')(w \prec_{S'} v_0 \Rightarrow (\exists x \in U_{v_0})(\exists G' \in \mathcal{D} \cap \bigcup \dot{R}[v_0])\ w \prec_{S'\cup\{\Box\neg G'\}} x).$$

From $w \prec_{S\cup\{\Box\neg G\}} v_0$ it follows that there are some $x \in U_{v_0}$ and $G' \in \mathcal{D} \cap \bigcup \dot{R}[v_0]$ such that $w \prec_{S\cup\{\Box\neg G, \Box\neg G'\}} x$. Lemma 6 implies $w \prec_{S\cup\{\Box\neg G\}} x$, as required.

We claim that for each formula $G \in \mathcal{D}$ and each world $w \in W$ the following holds:

$$\mathfrak{M}, w \Vdash G \text{ if and only if } G \in w.$$

The claim is proved by induction on the complexity of G. The only nontrivial case is when $G = B \rhd C$.

Assume $B \rhd C \in w$, wRu and $u \Vdash B$. Induction hypothesis implies $B \in u$. We claim that $uS_w[C]_w$. Clearly $[C]_w \subseteq R[w]$. Assume $w \prec_S u$. Lemma 27 implies that there is an **IL**X-MCS v with $w \prec_{S\cup\{\Box\neg B\}} v$ and $C, \Box\neg C \in v$ (thus $v \in W$). Since $C \in v$, the induction hypothesis implies $v \Vdash C$. Since $w \prec v$, i.e., $w\, R\, v$, then $v \in [C]_w$. Now, $B \in \mathcal{D}$ and $B \in u$ imply $B \in \mathcal{D} \cap \bigcup \dot{R}[u]$. Thus $uS_w[C]_w$ holds, by the clause (b) from the definition.

To prove the converse, assume $B \rhd C \notin w$. Since w is an **IL**X-MCS, $\neg(B \rhd C) \in w$. Lemma 26 implies there is u with $w \prec_{\{\Box\neg B, \neg C\}} u$ and $B \in u$. Lemma 6 implies $\Box\neg B \in u$. So, $B \wedge \Box\neg B \in u$; thus $u \in W$. The induction hypothesis implies $u \Vdash B$. Let $V \subseteq R[w]$ be such that $u\, S_w\, V$. We will find a world $v \in V$ such that $w \prec_{\{\neg C\}} v$. We will distinguish the cases (a) and (b) from the definition of the relation S_w. Consider the case (a). Let v be an arbitrary node in $V \cap \dot{R}[u]$. If $v = u$, clearly $w \prec_{\{\Box\neg B, \neg C\}} v$. If $u\, R\, v$, then we have $w \prec_{\{\Box\neg B, \neg C\}} u \prec v$. Lemma 6 implies $w \prec_{\{\Box\neg B, \neg C\}} v$. Consider the case (b). From $w \prec_{\{\Box\neg B, \neg C\}} u$ and the definition of S_w it follows that there is $v \in V$ and a formula $D \in \mathcal{D}$ such that $w \prec_{\{\Box\neg B, \neg C, \Box\neg D\}} v$. In both cases we have $w \prec_{\{\neg C\}} v$; thus $C \notin v$. Induction hypothesis implies $v \nVdash C$; whence $V \nVdash C$, as required. ⊣

THEOREM 30. *The logic* **IL**W *is complete w.r.t.* $\mathbf{IL}_{gen}$W*-models.*

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PROOF. In the light of Lemma 29, it suffices to show that the **IL**W-structure $\mathfrak{M}$ for $\mathcal{D}$ possesses the property $(\mathsf{W})_{\text{gen}}$. Recall the characteristic property $(\mathsf{W})_{\text{gen}}$:

$$u\, S_w\, V \Rightarrow (\exists V' \subseteq V)(u\, S_w\, V' \;\&\; R[V'] \cap S_w^{-1}[V] = \emptyset).$$

Suppose for a contradiction that there are w, u, and V such that:

$$u\, S_w\, V \;\&\; (\forall V' \subseteq V)(u\, S_w\, V' \Rightarrow R[V'] \cap S_w^{-1}[V] \neq \emptyset). \tag{1}$$

Let $\mathcal{V}$ denote all such sets V (we keep w and u fixed).

Let $n = 2^{|\mathcal{D}|}$. Fix any enumeration $\mathcal{D}_0, \dots, \mathcal{D}_{n-1}$ of $\mathcal{P}(\mathcal{D})$ that satisfies $\mathcal{D}_0 = \emptyset$. We define a new relation S_w^i for all $0 \leq i < n$, $y \in W$ and $U \subseteq W$ as follows:

$$y\, S_w^i\, U \iff y\, S_w\, U,\; \mathcal{D}_i \subseteq \bigcup \dot{R}[y],\; U \subseteq \left[\bigvee_{G \in \mathcal{D}_i} \Box\neg G \right]_w .$$

Let $y \in W$ and $U \subseteq R[w]$ be arbitrary. Let us prove that $y S_w U$ implies the following:

$$(\exists U' \subseteq U)(\exists i < n)\; y\, S_w^i\, U'. \tag{2}$$

If $y\, S_w\, U$ holds by (a) from the definition of S_w, the set $U \cap \dot{R}[y]$ is nonempty. Pick arbitrary $z \in U \cap \dot{R}[y]$ and put $U' = \{z\}$. We have either $w\, R\, y\, R\, z$ or $y = z$. If $w\, R\, y\, R\, z$, we have $y\, S_w\, \{z\}$. Otherwise $y = z$. Now quasi-reflexivity implies $y\, S_w\, \{z\}$. Since $y \in W$, there is a formula $G \in \mathcal{D}$ such that $G \wedge \Box\neg G \in y$. Fix $i < n$ such that $\mathcal{D}_i = \{G\}$. Clearly $\mathcal{D}_i \subseteq \bigcup \dot{R}[y]$. Since $z \in U$ and $y\, S_w\, U$, clearly $U' \subseteq R[w]$. Since $y = z$ or $y\, R\, z$, we also have $\Box\neg G \in z$. Truth lemma implies $U' \Vdash \Box\neg G$; since if $z\, R\, t$, $G \notin t$, (truth lemma is applied here) $t \nVdash G$, so $z \Vdash \Box\neg G$. Thus $U' \subseteq [\Box\neg G]_w$, and $y\, S_w^i\, U'$.

If $y\, S_w\, U$ holds by (b) from the definition of S_w, take:

$$U' = \{z \in U : (\exists G \in \mathcal{D} \cap \bigcup \dot{R}[y])\; w \prec_{\{\Box\neg G\}} z\};$$

$$\mathcal{D}_i = \{G \in \mathcal{D} \cap \bigcup \dot{R}[y] : (\exists z \in U)\; w \prec_{\{\Box\neg G\}} z\}.$$

In other words, U' is the image of the mapping that is implicitly present in the definition of the relation S_w (clause (b)): for each S, pick a world v_S (to be included in U'), and a formula G_S (to be included in $\mathcal{D}_i$).

Let $m < n$ be maximal such that there are $U \in \mathcal{V}$ and $U' \subseteq U$ with the following properties:

(i) $(\forall x \in U)[(\exists y \in R[x])(\exists Z \subseteq U)(\exists i \leq m)\; y\, S_w^i\, Z \Rightarrow x \notin U']$;
(ii) $(\forall x \in W)(x\, S_w\, U \Rightarrow x\, S_w\, U')$.

Since $\mathcal{D}_0 = \emptyset$, we have $[\bigvee_{G \in \mathcal{D}_0} \Box\neg G]_w = [\bot]_w = \emptyset$. So there are no $Z \subseteq [\bigvee_{G \in \mathcal{D}_0} \Box\neg G]_w$ such that $y\, S_w\, Z$ for some $y \in W$. So, if we take $m = 0$ and $U' = U$ for any $U \in \mathcal{V}$, (i) and (ii) are trivially satisfied.

Since n is finite and conditions (i) and (ii) are satisfied for at least one value m, there must be a maximal $m < n$ with the required properties.

Let us first prove that $m < n - 1$. Assume the opposite, that is (since $m < n$), $m = n - 1$. Then there are $U \in \mathcal{V}$ and $U' \subseteq U$ such that the conditions (i) and (ii)

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are satisfied for $m = n-1$. Since $U \in \mathcal{V}$, we have $u\,S_w\,U$. The condition (ii) implies $u\,S_w\,U'$. Now $U \in \mathcal{V}$, $U' \subseteq U$ and $u\,S_w\,U'$ imply $R[U'] \cap S_w^{-1}[U] \neq \emptyset$. Thus there are $x \in U'$ and $y \in R[x]$ such that $y\,S_w\,U$. Now (ii) implies $y\,S_w\,U'$. The earlier remark (2) implies that there is $Z \subseteq U'$ and $i < n$ such that $y\,S_w^i\,Z$. Since $m = n-1$, it follows that $i \leq m$. The condition (i) implies $x \notin U'$, a contradiction. Thus $m < n-1$.

Let us now prove that m is, contrary to the assumption, not maximal, by showing that $m+1$ satisfies (i) and (ii). Let $U \in \mathcal{V}$ and $U' \subseteq U$ be some sets such that the conditions (i) and (ii) are satisfied for m. Denote:

$$Y = \{x \in U' : (\exists y \in R[x])(\exists Z \subseteq U')\ y\,S_w^{m+1}\,Z\}.$$

Let us prove that $m+1$ also satisfies (i) and (ii) with U' instead of U, and $U' \setminus Y$ instead of U'. We should first show that $U' \in \mathcal{V}$. So, suppose that $u\,S_w\,T \subseteq U'$. Now, $T \subseteq U' \subseteq U$ and $U \in \mathcal{V}$ imply that there are some $v \in T$ and $z \in R[v]$ such that $z\,S_w\,U$. The property (ii) for m (with sets U and U') implies $z\,S_w\,U'$. So, $R[T] \cap S_w^{-1}[U'] \neq \emptyset$, as required.

Now let us verify the property (i) for the newly defined sets (U' and $U' \setminus Y$). Let $x \in U', y \in R[x], Z \subseteq U', i \leq m+1$ be arbitrary such that $y\,S_w^i\,Z$. If $i \leq m$, the property (i) for m implies $x \notin U'$, so in particular, $x \notin U' \setminus Y$. If $i = m+1$, then $x \in Y$. Thus $x \notin U' \setminus Y$ and the condition (i) is satisfied.

It remains to prove (ii). Take arbitrary $x \in W$ such that $x\,S_w\,U'$. For every $y \in Y$, the definition of Y implies the existence of some $z_y \in R[y]$ and $U_y \subseteq U'$ such that $z_y\,S_w^{m+1}\,U_y$. From the definition of the relation S_w^{m+1} we have $\mathcal{D}_{m+1} \subseteq \bigcup \dot{R}[z_y]$. Now, $y\,R\,z_y$ and the truth lemma imply $y \Vdash \Diamond G$, for each $G \in \mathcal{D}_{m+1}$. From the definition of the relation S_w^{m+1} and $z_y\,S_w^{m+1}\,U_y$ we have $U_y \subseteq [\bigvee_{G \in \mathcal{D}_{m+1}} \Box\neg G]_w$. So, the following holds:

$$Y \Vdash \bigwedge_{G \in \mathcal{D}_{m+1}} \Diamond G \quad \text{and} \quad U_y \Vdash \bigvee_{G \in \mathcal{D}_{m+1}} \Box\neg G,$$

for all $y \in Y$. Thus, $U_y \cap Y = \emptyset$, for every $y \in Y$. For every $y \in U' \setminus Y$ put $U_y = \{y\}$. Again, $U_y \cap Y = \emptyset$. Note that $\bigcup_{y \in U'} U_y = U' \setminus Y$. Now $x\,S_w\,U'$ and quasi-transitivity imply $x\,S_w\,U' \setminus Y$.

The fact that (i) and (ii) hold for $m+1$ contradicts the maximality of m. ⊣

Goris and Joosten proved in [4] the completeness of **IL**W* (**IL**WM$_0$) w.r.t. ordinary Veltman semantics.

Theorem 31. *The logic* **IL**W* *is complete w.r.t.* **IL**$_{gen}$W*-*models.*

Proof. With Lemma 29, it suffices to prove that the **IL**W*-structure for $\mathcal{D}$ possesses the properties $(\mathsf{W})_{\text{gen}}$ and $(\mathsf{M}_0)_{\text{gen}}$, for each appropriate $\mathcal{D}$. So, let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be the **IL**W*-structure for $\mathcal{D}$. Theorem 30 shows that the model $\mathfrak{M}$ possesses the property $(\mathsf{W})_{\text{gen}}$. It remains to show that it possesses the property $(\mathsf{M}_0)_{\text{gen}}$.

Assume $w\,R\,u\,R\,x\,S_w\,V$. We claim that there is $V' \subseteq V$ such that $u\,S_w\,V'$ and $R[V'] \subseteq R[u]$.

First, consider the case when $x\,S_w\,V$ holds by the clause (a) from the definition of S_w. So there is $v \in V$ such that $x = v$ or $x\,R\,v$. In both cases, $w\,R\,u\,R\,v$, and so $u\,S_w\,\{v\}$. It is clear that $R[v] \subseteq R[x] \subseteq R[u]$. So it suffices to take $V' = \{v\}$.

Otherwise, $x S_w V$ holds by the clause (b). Take $V' = \{v \in V : w \prec_{u^\square_\emptyset} v\}$. Clearly, $V' \subseteq V \subseteq R[w]$. Assume $w \prec_S u$. Now $w \prec_S u \prec x$ and Lemma 16 imply $w \prec_{S \cup u^\square_\emptyset} x$. The definition of xS_wV (clause (b)) implies there is $G \in \mathcal{D} \cap \bigcup \dot{R}[x]$ (so $G \in \mathcal{D} \cap \bigcup \dot{R}[u]$) and $v \in V$ such that $w \prec_{S \cup u^\square_\emptyset \cup \{\square \neg G\}} v$, thus also $v \in V'$. In particular, $w \prec_{S \cup \{\square \neg G\}} v$. Since S was arbitrary, $u S_w V'$. It remains to verify that $R[V'] \subseteq R[u]$. Assume $V' \ni v R z$. Since $w \prec_{u^\square_\emptyset} v$, for all $\square B \in u$ we have $\square B \in v$, and since $v R z$, it follows that $\square B, B \in z$. Thus, $u \prec z$ i.e., $u R z$. ⊣

In [8] it is shown that **IL**W* possesses finite model property w.r.t. generalised Veltman models. To show decidability, (stronger) completeness w.r.t. ordinary Veltman models was used, but the Theorem 31 would suffice for this purpose.

§4. Finite model property and decidability. For **IL**, **IL**M, **IL**P, and **IL**W, the original completeness proofs were proofs of completeness w.r.t. appropriate finite models [3], [7]. For these logics, the FMP w.r.t. the ordinary semantics and decidability are immediate (and completeness and the FMP w.r.t. the generalised semantics are easily shown to follow from these results). These completeness proofs use *truncated* maximal consistent sets, that is, sets that are maximal consistent with respect to the so-called *adequate* set. The principal requirement is that this set is finite. Already with **IL**M, defining adequacy is not trivial (see [3]).

For more complex logics, not much is known about the FMP w.r.t. the ordinary semantics. The filtration method can be used with generalised models to obtain finite models. This approach was successfully used to prove the FMP of **IL**M_0 and **IL**W* w.r.t. the generalised semantics [8, 9]. A drawback of this approach is in that the FMP w.r.t. the ordinary semantics does not follow from the FMP w.r.t. the generalised semantics. Decidability can be obtained from the FMP w.r.t. either semantics (unless the logic in question is incomplete w.r.t. the ordinary semantics). At the moment it is not clear whether the choice of semantics would affect our ability to produce results regarding computational complexity of provability and consistency of **IL**X.

Let us overview basic notions and results of [9] and [8]. Let A be a formula. If A equals $\neg B$ for some B, then $\sim A$ is B, otherwise $\sim A$ is $\neg B$. We need to slightly extend the definition of adequate sets[4] that was used in [9]. The modified version will satisfy all the old properties.

Definition 32. Let $\mathcal{D}$ have the usual the properties: a finite set of formulas that is closed under taking subformulas and single negations, and $\top \in \mathcal{D}$. We say that a set of formulas $\Gamma_\mathcal{D}$ is an adequate set w.r.t. $\mathcal{D}$ if it satisfies the following conditions:

1. $\Gamma_\mathcal{D}$ is closed under taking subformulas;
2. if $A \in \Gamma_\mathcal{D}$ then $\sim A \in \Gamma_\mathcal{D}$;
3. $\bot \rhd \bot \in \Gamma_\mathcal{D}$;
4. $A \rhd B \in \Gamma_\mathcal{D}$ if A is an antecedent or succedent of some $\rhd$-formula in $\Gamma_\mathcal{D}$, and so is B;

[4]Note that this is a different notion of *adequacy* than the one used for completeness proofs in [3], [7], and [4].

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5. if $A \in \mathcal{D}$ then $\Box\neg A \in \Gamma_{\mathcal{D}}$;
6. $\Gamma_{\mathcal{D}}$ should be the smallest set satisfying the preceding properties.

Since $\mathcal{D}$ is finite, $\Gamma_{\mathcal{D}}$ is finite too. Next we require the concept of bisimulations between generalised models.

Definition 33 ([15]). A bisimulation between generalised Veltman models $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ and $\mathfrak{M}' = (W', R', \{S'_{w'} : w' \in W'\}, \Vdash)$ is a nonempty relation $Z \subseteq W \times W'$ such that:

(at) if $w\,Z\,w'$, then $w \Vdash p$ if and only if $w' \Vdash p$, for all propositional variables p;
(forth) if $w\,Z\,w'$ and $w\,R\,u$, then there is $u' \in W'$ such that $w'\,R'\,u'$, $u\,Z\,u'$ and for all $V' \subseteq W'$ such that $u'\,S'_{w'}\,V'$ there is $V \subseteq W$ such that $u\,S_w\,V$ and for all $v \in V$ there is $v' \in V'$ with vZv';
(back) if $w\,Z\,w'$ and $w'\,R'\,u'$, then there is $u \in W$ such that $w\,R\,u$, $u\,Z\,u'$ and for all $V \subseteq W$ such that $u\,S_w\,V$ there is $V' \subseteq W'$ such that $u'\,S'_{w'}\,V'$ and for all $v' \in V'$ there is $v \in V$ with $v\,Z\,v'$.

Given a generalised Veltman model $\mathfrak{M}$, the union of all bisimulations on $\mathfrak{M}$, denoted by $\sim_{\mathfrak{M}}$, is the largest bisimulation on $\mathfrak{M}$, and $\sim_{\mathfrak{M}}$ is an equivalence relation [15].

An $\sim_{\mathfrak{M}}$-equivalence class of $w \in W$ will be denoted by $[w]$. For any set of worlds V, put $\widetilde{V} = \{[w] : w \in V\}$.

A filtration of $\mathfrak{M}$ through $\Gamma_{\mathcal{D}}$, $\sim_{\mathfrak{M}}$ is any generalised Veltman model $\widetilde{\mathfrak{M}} = (\widetilde{W}, \widetilde{R}, \{\widetilde{S}_{[w]} : w \in W\}, \Vdash)$ such that for all $w \in W$ and $A \in \Gamma_{\mathcal{D}}$ we have $w \Vdash A$ if and only if $[w] \Vdash A$ (we denote both forcing relations as $\Vdash$, as there is no risk of confusion).

The following lemma combines key results of [9] (Lemma 2.3, Theorems 2.4 and Theorem 3.2).

Lemma 34. *Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be a generalised Veltman model, and $\sim_{\mathfrak{M}}$ the largest bisimulation on $\mathfrak{M}$. Define:*

(1) *$[w]\,\widetilde{R}\,[u]$ if and only if for some $w' \in [w]$ and $u' \in [u]$, $w'\,R\,u'$ and there is $\Box A \in \Gamma_{\mathcal{D}}$ such that $w' \nVdash \Box A$ and $u' \Vdash \Box A$;*
(2) *$[u]\,\widetilde{S}_{[w]}\,\widetilde{V}$ if and only if $[w]\,\widetilde{R}\,[u]$, $\widetilde{V} \subseteq \widetilde{R}[[w]]$, and for all $w' \in [w]$ and $u' \in [u]$ such that $w'\,R\,u'$ we have $u'\,S_{w'}\,V'$ for some V' such that $\widetilde{V'} \subseteq \widetilde{V}$;*
(3) *for all propositional variables $p \in \Gamma_{\mathcal{D}}$ put $[w] \Vdash p$ if and only if $w \Vdash p$, and interpret propositional variables $q \notin \Gamma_{\mathcal{D}}$ arbitrarily (e.g., put $[w] \nVdash q$ for all $[w] \in \widetilde{W}$).*

Then $\widetilde{\mathfrak{M}} = (\widetilde{W}, \widetilde{R}, \{\widetilde{S}_{[w]} : w \in W\}, \Vdash)$ is a filtration of $\mathfrak{M}$ through $\Gamma_{\mathcal{D}}, \sim_{\mathfrak{M}}$. The model $\widetilde{\widetilde{\mathfrak{M}}}$ is finite.

Lemma 34 implies that **IL** has the FMP w.r.t. the generalised semantics. To prove that a specific extension has the FMP, it remains to show that filtration preserves its characteristic property.

Since we are going to use **IL**X-structures as the starting models $\mathfrak{M}$, we can make use of their properties. In particular, we do not have to make sure that there is a formula $\Box A$ such that $x \nVdash \Box A$ and $y \Vdash \Box A$ when we want to show that $x\,R\,y$ implies $[x]\,\widetilde{R}\,[y]$.

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LEMMA 35. *Let* $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ *be a generalised Veltman model such that for some set* $\mathcal{D}$ (*with the usual properties*) *we have that for all* $w \in W$ *there is some* $B \in \mathcal{D}$ *such that* $w \Vdash B \wedge \Box\neg B$. *Then*:

1. *for all* $[w] \in \widetilde{W}$ *there is some* $B \in \mathcal{D}$ *such that* $[w] \Vdash B \wedge \Box\neg B$;
2. *for all* $x, y \in W$, *if* $x\,R\,y$ *then* $[x]\,\widetilde{R}\,[y]$.

PROOF. To see that the first claim holds, note that if $B \in \mathcal{D}$ then $\Box\neg B \in \Gamma_{\mathcal{D}}$, and filtration preserves the truth value of formulas within $\Gamma_{\mathcal{D}}$.

For the second claim, we should find a formula $\Box A \in \Gamma_{\mathcal{D}}$ such that $x \nVdash \Box A$ and $y \Vdash \Box A$. Since $y \in W$, there is a formula $B \in \mathcal{D}$ such that $y \Vdash B \wedge \Box\neg B$. Since $x\,R\,y$, we have $x \nVdash \Box\neg B$. Since $B \in \mathcal{D}$, we have $\Box\neg B \in \Gamma_{\mathcal{D}}$. Thus we can take $A = \neg B$. ⊣

Note that all previously defined **IL**X-structures (Definitions 9 and 28) satisfy requirements of the preceding lemma. Given $w \in W$, there is a formula $B \in \mathcal{D}$ such that $B \wedge \Box\neg B \in w$. Depending on X, we can use Lemma 11 or Lemma 29 to conclude $w \Vdash B \wedge \Box\neg B$ ($\Box\neg B$ might not be in $\mathcal{D}$, but $\sim B$ is; since $R[w] \Vdash \sim B$, we must have $w \Vdash \Box\neg B$).

LEMMA 36. *Let* $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ *be an* **IL**P_0*-model such that for some set* $\mathcal{D}$ (*with the usual properties*) *we have that for all* $w \in W$ *there is some* $B \in \mathcal{D}$ *such that* $w \Vdash B \wedge \Box\neg B$. *Let* $\sim_{\mathfrak{M}}$ *be the largest bisimulation on* $\mathfrak{M}$. *Then the filtration* $\widetilde{\mathfrak{M}}$ *as defined in Lemma 34 possesses the property* $(\mathsf{P}_0)_{gen}$.

PROOF. Assume $[w]\,\widetilde{R}\,[x]\,\widetilde{R}\,[u]\,\widetilde{S}_{[w]}\,V$ and $\widetilde{R}[[v]] \cap Z \neq \emptyset$ for each $[v] \in V$. We claim that there exists $Z' \subseteq Z$ such that $[u]\,\widetilde{S}_{[x]}\,Z'$.

Since $[w]\,\widetilde{R}\,[x]$, there are $w_0 \in [w]$ and $x_0 \in [x]$ such that $w_0\,R\,x_0$. Let $x' \in [x]$ and $u' \in [u]$ be any worlds such that $x'\,R\,u'$. The condition (back) implies that there is a world $u_{x',u'}$ such that $x_0\,R\,u_{x',u'}$ and $u_{x',u'} \sim_{\mathfrak{M}} u'$. Now, $[u]\,\widetilde{S}_{[w]}\,V$, $u_{x',u'} \in [u]$ and $w_0\,R\,u_{x',u'}$ imply there is a set $V_{x',u'}$ such that $u_{x',u'}\,S_{w_0}\,V_{x',u'}$ and $\widetilde{V_{x',u'}} \subseteq V$. Since $\widetilde{R}[[v]] \cap Z \neq \emptyset$ for each $[v] \in V$, we have $\widetilde{R}[[v]] \cap Z \neq \emptyset$ for each $v \in V_{x',u'}$. For each $v \in V_{x',u'}$, choose a world z_v such that $[z_v] \in \widetilde{R}[[v]] \cap Z$. Now $[v]\,\widetilde{R}\,[z_v]$ implies that there are some $v' \in [v]$ and $z'_v \in [z_v]$ such that $v'\,R\,z'_v$. Applying (back), we can find a world z''_v such that $v\,R\,z''_v$ and $z'_v \sim_{\mathfrak{M}} z''_v$. Put $Z_{x',u'} = \{z''_v : v \in V_{x',u'}\}$. Note that we have $R[v] \cap Z_{x',u'} \neq \emptyset$ for each $v \in V_{x',u'}$.

Applying $(\mathsf{P}_0)_{\text{gen}}$ gives $u_{x',u'}\,S_{x_0}\,Z'_{x',u'}$ for some $Z'_{x',u'} \subseteq Z_{x',u'}$. Clearly $\widetilde{Z'_{x',u'}} \subseteq \widetilde{Z_{x',u'}} \subseteq Z$. Continuing our first application of (back), there is a set $Z''_{x',u'}$ such that $u'\,S_{x'}\,Z''_{x',u'}$, and for each $z'' \in Z''_{x',u'}$ there is $z' \in Z'_{x',u'}$ such that $z' \sim_{\mathfrak{M}} z''$. This implies $\widetilde{Z''_{x',u'}} \subseteq \widetilde{Z'_{x',u'}}$. Let $T = \bigcup_{x' \in [x], u' \in [u], x'Ru'} Z''_{x',u'}$ and $Z' = \widetilde{T}$. It is easy to see that $Z' \subseteq Z$. Lemma 35 implies $Z' \subseteq \widetilde{R}[[x]]$. We have $u'\,S_{x'}\,Z''_{x',u'}$ with $\widetilde{Z''_{x',u'}} \subseteq Z'$ for all $x' \in [x]$ and $u' \in [u]$ with $x'\,R\,u'$. Thus, $[u]\,\widetilde{S}_{[x]}\,Z'$. ⊣

COROLLARY 37. **IL**P_0 *is decidable.*

PROOF. Since **IL**P_0 is complete, it remains to show that it has the finite model property. Let $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ be the **IL**P_0-structure for an appropriate $\mathcal{D}$, and apply Lemma 36. As the resulting model $\widetilde{\mathfrak{M}}$ itself also satisfies the conditions of Lemma 36 (see Lemma 35), we can apply Lemma 36 once more, and by Lemma 34 obtain a finite model. ⊣

Let us prove the same for **IL**R.

LEMMA 38. *Let* $\mathfrak{M} = (W, R, \{S_w : w \in W\}, \Vdash)$ *be an* **IL**R*-model such that for some set* $\mathcal{D}$ *(with the usual properties) we have that for all* $w \in W$ *there is some* $B \in \mathcal{D}$ *such that* $w \Vdash B \wedge \Box\neg B$. *Let* $\sim_{\mathfrak{M}}$ *be the largest bisimulation on* $\mathfrak{M}$. *Then the filtration* $\widetilde{\mathfrak{M}}$ *as defined in Lemma 34 possesses the property* $(\mathsf{R})_{gen}$.

PROOF. Assume $[w]\,\widetilde{R}\,[x]\,\widetilde{R}\,[u]\,\widetilde{S}_{[w]}\,V$, and let $C \in \mathcal{C}([x],[u])$ be an arbitrary choice set. We are to prove that there is a set U such that $\widetilde{U} \subseteq V, [x]\,\widetilde{S}_{[w]}\,\widetilde{U}$ and $\widetilde{R}[\widetilde{U}] \subseteq C$.

Put $C_{x'} = \{c \in R[x'] : [c] \in C\}$ for all $x' \in [x]$.

Let us first prove that for some $x_0 \in [x], u_0 \in [u]$ with $x_0\,R\,u_0$ we have $C_{x_0} \in \mathcal{C}(x_0, u_0)$. Suppose not. Then for all $x' \in [x], u' \in [u]$ with $x'\,R\,u'$, there is a set $Z_{x',u'}$ such that $u'\,S_{x'}\,Z_{x',u'}$ with $Z_{x',u'} \cap C_{x'} = \emptyset$. Put $Z = \bigcup_{x' \in [x], u' \in [u], x'\,Ru'} Z_{x',u'}$. Lemma 35 implies $\widetilde{Z} \subseteq \widetilde{R}[[x]]$. Thus $[u]\,\widetilde{S}_{[x]}\,\widetilde{Z}$. Since $C \in \mathcal{C}([x],[u])$, there is $z \in Z$ such that $[z] \in C \cap \widetilde{Z}$. Thus $z \in Z_{x',u'}$ for some $x' \in [x], u' \in [u]$ and $x'\,R\,u'$. The definition of $C_{x'}$ implies $z \in C_{x'}$. Thus, $Z_{x',u'} \cap C_{x'} \neq \emptyset$, a contradiction.

Now we claim that for all $y \in [x]$ there is $u_y \sim u_0$ with $y\,R\,u_y$ and $C_y \in \mathcal{C}(y, u_y)$. Since $y \sim_{\mathfrak{M}} x_0$ and $x_0\,R\,u_0$, the (back) condition implies that there is a world u_y such that $u_y \sim_{\mathfrak{M}} u_0$ and $y\,R\,u_y$ (and other properties that we will return to later). We will show that $C_y \in \mathcal{C}(y, u_y)$. Let Z' be such that $u_y\,S_y\,Z'$, and we are to prove that $C_y \cap Z' \neq \emptyset$. The earlier instance of (back) condition for u_y further implies that there is a set Z with $u_0\,S_{x_0}\,Z$, and for all $z \in Z$ there is $z' \in Z'$ with $z \sim_{\mathfrak{M}} z'$. Let $z \in Z \cap C_{x_0}$ be an arbitrary element (which exists because, as we proved, C_{x_0} is a choice set). Then there is $z' \in Z'$ such that $z' \sim_{\mathfrak{M}} z$. Since $[z] \in C$, i.e., $[z'] \in C$, we have $z' \in C_y$. In particular, $Z' \cap C_y \neq \emptyset$. Thus $C_y \in \mathcal{C}(y, u_y)$.

Let us prove that there is a set U such that $\widetilde{U} \subseteq V$, $[x]\,\widetilde{S}_{[w]}\,\widetilde{U}$ and $\widetilde{R}[\widetilde{U}] \subseteq C$. Let $w' \in [w]$ and $y \in [x]$ be such that $w'\,R\,y$. Since $[u]\,\widetilde{S}_{[w]}\,V$, there is a set $V_{w',y}$ such that $u_y\,S_{w'}\,V_{w',y}$ and $\widetilde{V_{w',y}} \subseteq V$. Applying $(\mathsf{R})_{\text{gen}}$ with C_y, there is $U_{w',y} \subseteq V_{w',y}$ such that $y\,S_{w'}\,U_{w',y}$ and $R[U_{w',y}] \subseteq C_y$. Let $U = \bigcup_{w' \in [w], y \in [x], w'Ry} U_{w',y}$. Clearly $\widetilde{U} \subseteq V$. Lemma 35 implies $\widetilde{U} \subseteq \widetilde{R}[[w]]$. The definition of $\widetilde{S}_{[w]}$ implies $[x]\,\widetilde{S}_{[w]}\,\widetilde{U}$.

It remains to verify that $\widetilde{R}[\widetilde{U}] \subseteq C$. Let $t \in U$ and $z \in W$ be such that $[t]\,\widetilde{R}\,[z]$. Then we have $t \in U_{w',y}$ for some $w' \in [w]$ and $y \in [x]$. Since $[t]\,\widetilde{R}\,[z]$, there are $t' \in [t]$ and $z' \in [z]$ with $t'\,R\,z'$. The condition (forth) implies that there is z'' such that $t\,R\,z''$ and $z' \sim_{\mathfrak{M}} z''$. Since $R[U_{w',y}] \subseteq C_y$ and $z'' \in R[U_{w',y}]$, we have $z'' \in C_y$. The definition of C_y implies $[z''] \in C$, or equivalently, $[z] \in C$. ⊣

COROLLARY 39. **IL**R *is decidable.*

§5. Conclusions and future work. Let us briefly recapitulate our results. We introduced a new type of completeness proofs for interpretability logics. Our proofs are based on the notion of full labels [2], which encapsulate more information regarding the relation between two given maximal consistent sets than the classical notion of criticality. Combined with the robustness of the generalised semantics, this approach allows for shorter and more natural proofs in some cases (most notably **IL**M_0). We prove completeness of **IL**R, where R is a recently discovered principle valid in all reasonable theories. We also prove completeness of $\mathbf{IL}\mathsf{P}_0$, a logic known to be incomplete w.r.t. the ordinary semantics. For logics that we find to be complete, we also prove the finite model property and decidability.

Future work concerns the related questions of completeness and finite model property and decidability. The largest subset of $\mathbf{IL}(\mathsf{All})$ for which we have a completeness result w.r.t. the ordinary semantics is $\mathbf{IL}\mathsf{W}^*$ [4]. In this line of research it is natural to ask:

1. Is $\mathbf{IL}\mathsf{R}$ complete w.r.t. the ordinary semantics [5]?

The most promising method for obtaining this result, out of currently available methods, is the *construction method*. This method was previously used to prove the completeness of $\mathbf{IL}\mathsf{M}_0$ and $\mathbf{IL}\mathsf{W}^*$ [4].

If we look at the corresponding situation in the generalised semantics, there are two different and mutually incomparable subsets of $\mathbf{IL}(\mathsf{All})$ that we now know to be complete: $\mathbf{IL}\mathsf{W}^*$ [4] and $\mathbf{IL}\mathsf{R}$ (current paper). We do not know whether the combination of these, the logic $\mathbf{IL}\mathsf{R}^*$ ($= \mathbf{IL}\mathsf{WR} = \mathbf{IL}\mathsf{WRM}_0$), is complete or incomplete. At the moment, we do not even know whether a complete superset of $\mathbf{IL}\mathsf{WR}$ that is also a subset of $\mathbf{IL}(\mathsf{All})$ even exists. (Although, it would be surprising if one did not exist.) Thus one natural step is to tackle the following question:

2. Is $\mathbf{IL}\mathsf{WR}$ or one of its sufficiently weak extensions complete?

There are preliminary indications that $\mathbf{IL}\mathsf{WR}$ may be incomplete. We expect to address this question in future work. Another open problem concerns the two recently discovered series of principles [6]. The logic $\mathbf{IL}\mathsf{WR}$ enriched with these principles is the best explicit candidate for $\mathbf{IL}(\mathsf{All})$ (however, it is an unlikely candidate, see [6]). The first step in this direction would be to determine the following:

3. What are the two series' frame conditions w.r.t. the generalised semantics?

The next step towards the completeness proof would be to determine the labelling lemmas corresponding to these principles.

4. What are the two series' labelling lemmas?

The criteria of what constitutes an *appropriate* labelling lemma in this context is simply the lemma's usefulness in proving completeness.

5. Are logics of form $\mathbf{IL}\mathsf{X}$ complete, where $\mathsf{X} \subseteq \{\mathsf{W}, \mathsf{R}_1, \mathsf{R}_2, \dots, \mathsf{R}^1, \mathsf{R}^2, \dots\}$, $\mathsf{X} \neq \{W\}, \{R\}$?

There is some intrinsic interest in exploring semantics for interpretability logics (e.g., to ease reasoning, or even provide decision procedures). There is a more palpable interest in this too. Occasionally new principles of interpretability are found not by arithmetical considerations, but rather by (i) determining which principles are required in order to establish completeness, or by (ii) modifying the frame conditions for known principles, and extracting formulas defining the modified conditions. For example, both the principle R [5] and the two series [6] were found through semantic means.[5]

[5]There are some indications that trying to establish completeness for $\mathbf{IL}\mathsf{WR}$ may result in new principles too.

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Let us turn to the finite model property and decidability. In all known cases of (decidable) interpretability logics, the simplest way to show decidability is via the generalised semantics [9], [8]. Decidability does not seem to be a problematic issue in the context of interpretability logics; currently there is no known complete logic that is not known to be decidable too. Furthermore, taking into account the results of this paper, we know e.g., that **IL**WR has the finite model property, and so if it is complete, it has to be decidable. Thus the next natural question regarding the finite model property concerns the two series. The most straightforward way of obtaining the finite model property is via filtrations, which presupposes that we have already answered the question 5.

6. Are the principles $\mathsf{R}_1, \mathsf{R}_2, \dots, \mathsf{R}^1, \mathsf{R}^2, \dots$ preserved under filtration? (See Lemma 34.)

At the moment, and as long as we do not provide a better approximation of **IL**(All), this may also be the only open question regarding the finite model property.[6]

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REFERENCES

[1] A. Berarducci, *The interpretability logic of Peano arithmetic*, this Journal, vol. 55 (1990), no. 3, pp. 1059–1089.

[2] M. Bílková, E. Goris, and J.J. Joosten. *Smart labels*, ***Liber Amicorum for Dick de Jongh*** (L. Afanasiev and M. Marx, editors), Institute for Logic, Language and Computation, Amsterdam, 2004.

[3] D.H.J. de Jongh and F.J.M.M. Veltman. *Provability logics for relative interpretability*, ***Mathematical Logic, Proceedings of the Heyting 1988 Summer School in Varna, Bulgaria*** (P. P. Petkov, editor), Plenum Press, Boston, 1990, pp. 31–42.

[4] E. Goris and J. J. Joosten, *Modal matters for interpretability logics*. ***Logic Journal of the IGPL***, vol. 16 (2008), no. 4, pp. 371–412.

[5] ———, *A new principle in the interpretability logic of all reasonable arithmetical theories*. ***Logic Journal of the IGPL***, vol. 19 (2011), no. 1, pp. 1–17.

[6] ———, *Two new series of principles in the interpretability logic of all reasonable arithmetical theories*, this Journal, vol. 85 (2020), pp. 1–26.

[7] D.H.J. de Jongh and F.J.M.M. Veltman, *Modal completeness of ILW*, ***Essays Dedicated to Johan van Benthem on the Occasion of His 50th Birthday*** (J. Gerbrandy, M. Marx, M. de Rijke, and Y. Venema, editors), Amsterdam University Press, Amsterdam, 1999.

[8] L. Mikec, T. Perkov, and M. Vuković, *Decidability of interpretability logics ILM_0 and ILW**. ***Logic Journal of the IGPL***, vol. 25 (2017), no. 5, pp. 758–772.

[9] T. Perkov and M. Vuković, *Filtrations of generalized Veltman models*. ***Mathematical Logic Quarterly***, vol. 62 (2016), no. 4–5, pp. 412–419.

[10] V. Shavrukov, ***The Logic of Relative Interpretability Over Peano Arithmetic***, Steklov Mathematical Institute, Moscow, 1988.

[11] R. M. Solovay, *Provability interpretations of modal logic*. ***Israel Journal of Mathematics***, vol. 25 (1976), no. 3, pp. 287–304.

[6] If, in a somewhat unexpected turn of events, a logic axiomatised by a combination of these principles does have the finite model property, but this is not demonstrable by Lemma 34, there would be another important question: can we add W to this logic while retaining the finite model property?

[12] L.C. Verbrugge, Verzamelingen-Veltman frames en modellen (set Veltman frames and models). Unpublished manuscript, Amsterdam, 1992.

[13] A. Visser, *Interpretability logic*, ***Mathematical Logic, Proceedings of the Heyting 1988 Summer School in Varna, Bulgaria*** (P. P. Petkov, editor), Plenum Press, Boston, 1990, pp. 175–209.

[14] A. Visser, *An overview of interpretability logic*, ***Advances in Modal Logic, vol. 1*** (M. Kracht, M. de Rijke, H.Wansing, and M. Zakharyaschev, editors), CSLI Publications, Stanford, CA, 1998, pp. 307–359.

[15] D. Vrgoč and M. Vuković, *Bisimulations and bisimulation quotients of generalized Veltman models*. ***Logic Journal of the IGPL***, vol. 18 (2010), no. 6, pp. 870–880.

[16] M. Vuković, *The principles of interpretability*. ***Notre Dame Journal of Formal Logic***, vol. 40 (1999), no. 2, pp. 227–235.

[17] ———, *Bisimulations between generalized Veltman models and Veltman models*. ***Mathematical Logic Quarterly***, vol. 54 (2008), no. 4, pp. 368–373.

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