Bull. Aust. Math. Soc. **110** (2024), 554–563
doi:10.1017/S0004972724000297

MULTIPLE LOOSE MAPS BETWEEN GRAPHS

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(Received 22 January 2024; accepted 20 March 2024; first published online 22 April 2024)

Abstract

Given maps $f_1, \ldots, f_n : X \to Y$ between (finite and connected) graphs, with $n \geq 3$ (the case $n = 2$ is well known), we say that they are *loose* if they can be deformed by homotopy to coincidence free maps, and *totally loose* if they can be deformed by homotopy to maps which are two by two coincidence free. We prove that: (i) if Y is not homeomorphic to the circle, then any maps are totally loose; (ii) otherwise, any maps are loose and they are totally loose if and only if they are homotopic.

2020 *Mathematics subject classification*: primary 55M20.

Keywords and phrases: coincidence, graphs, multiple maps, loose maps.

1. Introduction and main theorem

Let $f_1, \ldots, f_n : X \to Y$ be $n \geq 2$ (continuous) maps between topological spaces. We consider two kinds of coincidences for these maps:

- the *multiple coincidence set*, $\mathrm{Coin}(f_1, \ldots, f_n) = \{x \in X : f_1(x) = \cdots = f_n(x)\}$;
- the *partial coincidence set*, $\mathrm{C}_2(f_1, \ldots, f_n) = \bigcup_{i \neq j} \mathrm{Coin}(f_i, f_j)$.

Let $f = (f_1, \ldots, f_n) : X \to Y^n$ be the map defined by $f(x) = (f_1(x), \ldots, f_n(x))$, where Y^n denotes the Cartesian product of n copies of Y. Then:

- $\mathrm{Coin}(f_1, \ldots, f_n) = \varnothing$ if and only if $f(X) \subset Y^n \setminus \Delta$, where Δ is the diagonal in Y^n, that is, $\Delta = \{(y, \ldots, y) \in Y^n : y \in Y\}$;
- $\mathrm{C}_2(f_1, \ldots, f_n) = \varnothing$ if and only if $f(X) \subset \mathrm{Conf}_n(Y)$, where $\mathrm{Conf}_n(Y)$ is the n-points configuration space of Y, that is,

$$\mathrm{Conf}_n(Y) = \{(y_1, \ldots, y_n) \in Y^n : y_i \neq y_j \text{ if } i \neq j\}.$$

We study, in a specific setting, the conditions under which the maps $f_1, \ldots, f_n$ can be deformed by homotopy to avoid coincidences of one kind or another. To make the language easier, we say that the maps $f_1, \ldots, f_n$ are:

- *loose* if there exist $f_1' \simeq f_1, \ldots, f_n' \simeq f_n$ such that $\mathrm{Coin}(f_1', \ldots, f_n') = \varnothing$;
- *totally loose* if there exist $f_1' \simeq f_1, \ldots, f_n' \simeq f_n$ such that $\mathrm{C}_2(f_1', \ldots, f_n') = \varnothing$.

The goal is to provide conditions under which given maps $f_1, \ldots, f_n : X \to Y$ between graphs are loose or totally loose. By a *graph*, we mean a finite and connected one-dimensional CW-complex.

For $n \geq 3$ maps $f_1, \ldots, f_n$ to be loose, it suffices that two of them are loose. In fact, if (without loss of generality) f_1, f_2 are loose, then there exist maps $f_1' \simeq f_1$ and $f_2' \simeq f_2$ such that $\mathrm{Coin}(f_1', f_2') = \varnothing$, which forces $\mathrm{Coin}(f_1', f_2', f_3, \ldots, f_n) = \varnothing$. However, for $f_1, \ldots, f_n$ to be totally loose, it is necessary that f_i, f_j be loose whenever $i \neq j$, with the 'loosening' being realised by a same set of maps $f_1', \ldots, f_n'$.

Of course, *totally loose* implies *loose*. The converse is not true; for instance, consider the maps $f_1, f_2, f_3 : S^1 \to S^1$ defined by f_1 = identity and $f_2 = f_3$ = constant. The triple f_1, f_2, f_3 is loose, since the pair (f_2, f_3) is loose (f_2 and f_3 can be deformed to different constant maps). However, f_1, f_2, f_3 are not totally loose; in fact, since $\deg(f_1) = 1$, and f_2 and f_3 are constant, it follows from [3, Proposition 8.4, page 23] that the pairs (f_1, f_2) and (f_1, f_3) have one essential coincidence each.

For $n = 2$, we have $\mathrm{Coin}(f_1, f_2) = \mathrm{C}_2(f_1, f_2)$ and *loose* = *totally loose*. In this case, [1, Theorem 3.1] and [4, Theorem 2.4] provide the following solution.

THEOREM 1.1. *Let $f_1, f_2 : X \to Y$ be maps between graphs. If Y is not homeomorphic to the circle, then f_1, f_2 are loose. Otherwise, f_1, f_2 are loose if and only if $f_1 \simeq f_2$.*

The following theorem answers the problem for $n \geq 3$.

THEOREM 1.2 (Main Theorem). *Let $f_1, \ldots, f_n : X \to Y$ be maps between graphs, $n \geq 3$.*

(1) *If Y is not homeomorphic to the circle, then $f_1, \ldots, f_n$ are totally loose.*
(2) *If Y is homeomorphic to the circle, then $f_1, \ldots, f_n$ are loose. Furthermore, $f_1, \ldots, f_n$ are totally loose if and only if they are all homotopic.*

In each of Sections 2–4, we prove a part of Theorem 1.2.

2. When the range is not a circle

Let $f_1, \ldots, f_n : X \to Y$ be maps between graphs and suppose that Y is not homeomorphic to the circle S^1. We will prove that $f_1, \ldots, f_n$ are totally loose.

If Y is contractible, the result is trivial. Otherwise, we can change the cellular decomposition of Y, if necessary, so that no edge is a loop and every edge has a vertex of degree at least three. Figure 1 illustrates such a change of cellular decomposition.

Let $\alpha_1, \ldots, \alpha_p$ be the edges of X. Each α_i can be identified with a (one-to-one, except possibly at the final points) parametrisation $\hat{\alpha}_i : [0,1] \to X$. So a map $f : X \to Y$ may be seen as a family $f^1, \ldots, f^p : [0,1] \to Y$ of maps defined as follows: if $x \in \alpha_k$, then $x = \hat{\alpha}_k(t_x)$ for some $t_x \in [0,1]$ and $f^k(x) = f(\hat{\alpha}_k(t_x))$.

Following [4], we can suppose (up to homotopy) that $C_2(f_1, \ldots, f_n)$ is finite and does not contain vertices of X and that $f(C_2(f_1, \ldots, f_n))$ does not contain vertices of Y.

Suppose $x_0 \in C_2(f_1, \ldots, f_n)$. Then there exists a unique edge α of X such that $x_0 \in \alpha$ (in fact, x_0 is in the interior of α) and a unique $t_0 \in (0,1)$ such that $x_0 = \hat{\alpha}(t_0)$.

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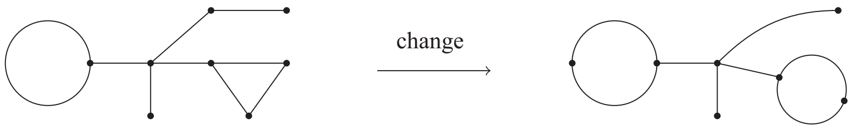

FIGURE 1. A change of cellular decomposition.

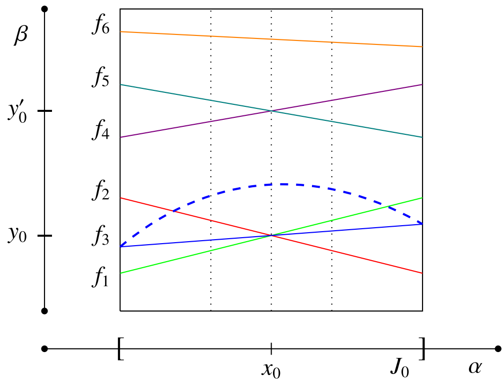

FIGURE 2. The graph of the maps $f_i|_{J_0}$.

Furthermore, there exists $\varepsilon > 0$ such that $I_0 = [t_0 - \varepsilon, t_0 + \varepsilon] \subset (0, 1)$ and for $J_0 = \hat{\alpha}(I_0)$, one has $J_0 \cap C_2(f_1, \ldots, f_n) = \{x_0\}$.

Without loss of generality, we can suppose that $x_0 \in \mathrm{Coin}(f_1, f_2)$. Let β be the unique edge of Y containing the point $y_0 = f_1(x_0) = f_2(x_0)$ in its interior. Without loss of generality, we can suppose that, for a certain $n_0 \in \{2, \ldots, n\}$, we have $f_i(x_0) \in \beta$ for $1 \leq i \leq n_0$ and $f_j(x_0) \notin \beta$ for $j > n_0$. We can decrease ε, if necessary, so that

$$f_i(J_0) \subset \mathrm{int}(\beta) \text{ for } 1 \leq i \leq n_0 \quad \text{and} \quad f_j(J_0) \subset Y \setminus \beta \text{ for } j > n_0.$$

Then there exists an open set $V \subset Y$ such that $\beta \subset V$ and $f_j(J_0) \subset Y \setminus \overline{V}$ for $j > n_0$.

We will prove that $f_1, \ldots, f_{n_0}$ can be deformed just in the interior of J_0 to maps $f_1' \simeq f_1, \ldots, f_{n_0}' \simeq f_{n_0}$ such that $J_0 \cap C_2(f_1', \ldots, f_{n_0}') = \emptyset$ and $f_i'(J_0) \subset V$, which forces

$$J_0 \cap C_2(f_1', \ldots, f_{n_0}', f_{n_0+1}, \ldots, f_n) = \emptyset.$$

Each restricted map $f_i|_{J_0}$ can be seen as a map from the closed arc $J_0 \subset \mathrm{im}(\alpha)$ into a closed arc $K_0 \subset \mathrm{int}(\beta)$. Hence, by composing this map with parametrisations of α and β, it can be seen as a map between closed intervals, and hence we can draw its graph as a subset of a rectangle. Figure 2 shows a scenario in which $n_0 = 6$ and we have $f_1(x_0) = f_2(x_0) = f_3(x_0) = y_0 \neq y_0' = f_4(x_0) = f_5(x_0)$ and f_6 coincides with none of the other maps in J_0.

We remark that the map f_3, whose graph corresponds to the blue solid line, can be deformed to a map f_3', whose graph corresponds to the blue dashed line. This generates

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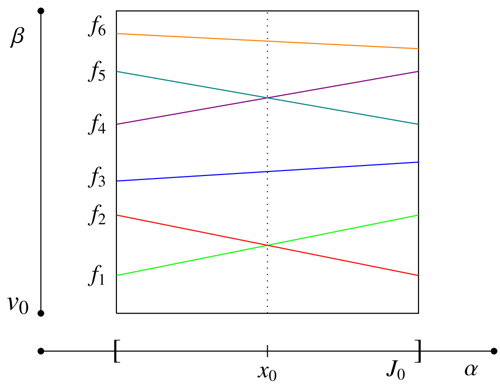

FIGURE 3. Maps with just a simple coincidence.

two new partial coincidences. However, we can decrease the interval I_0, and so the arc J_0, in such a way that x_0 is the unique partial coincidence in J_0.

This kind of deformation can be done in a general situation. Therefore, up to homotopy, we can assume that all the partial coincidences are *simple* coincidences, that is, coincidences for pairs of maps, but not for triples.

For instance, in Figure 2, after deforming f_3 to f_3', x_0 is a coincidence for the pairs (f_1, f_2) and (f_4, f_5), but there are no three maps that coincide at x_0. Of course, the map f_4 could be deformed so that the coincidence for (f_4, f_5) goes from x_0 to a nearby point x_1, and thus x_0 is a coincidence just for (f_1, f_2). However, this procedure is not relevant. In fact, we will show that we can annihilate a simple coincidence, even if it is a coincidence for more than one pair of maps. Therefore, a general situation will look like the one shown in Figure 3.

Thus, we have reduced the problem to the annihilation of isolated simple coincidences. We solve this problem using the idea of Staecker in [4].

One of the vertices of β, say v_0, is also a vertex of two more edges of Y, say γ and σ. For each pair of maps f_i, f_j for which x_0 is a coincidence, we will deform these maps just on the interior of J_0 as follows: we push f_i to γ without leaving the neighbourhood V and then we pull back, and we push f_j to σ without leaving V and then we pull back. Figure 4 shows how this can be done for each pair f_i, f_j to annihilate the coincidence. In Figure 4, solid lines represent images in β, dashed lines in γ and dotted lines in σ. Thus, the maps f_1, f_3 and f_4 are 'deflected' through γ, and the maps f_2 and f_5 are 'deflected' through σ. The map f_6 does not need to be deformed.

We remark that the map f_3 can be either deformed through γ or deformed through σ, independently of the other maps, since it does not coincide with any other. However, for the pair f_1, f_2, if f_1 is deformed through γ, then f_2 must be deformed through σ and *vice versa*. The same happens for the pair f_4, f_5. We emphasise that the choice of which edge will be used to deform f_1 does not interfere with the choice of which edge will be used to deform f_4. This means that the simple coincidences can be annihilated

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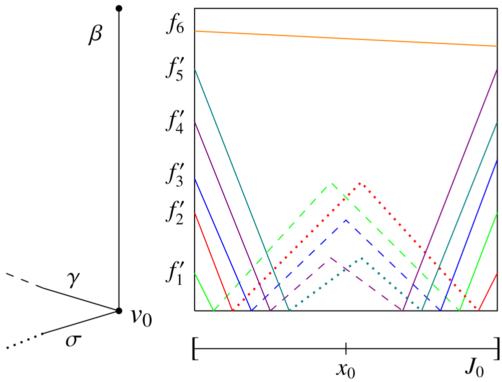

FIGURE 4. Deformations to annihilate coincidences.

independently of each other. Therefore, it is easy to see that the procedure works regardless of the number of pairs of maps with coincidences, that is, the procedure works for any n_0.

Finally, since the point x_0 is arbitrary, the same technique can be used for each isolated partial coincidence $x \in C_2(f_1, \ldots, f_n)$. Therefore, all the partial coincidences can be annihilated by way of local deformations of the maps (without producing new coincidences).

3. More than two maps into the circle are loose

In this section, we prove the first statement of item (2) of Theorem 1.2.

We consider the circle S^1 with its minimal cellular decomposition, namely, $S^1 = s^0 \cup s^1$. We take a 0-cell x_0 in X and the 0-cell s^0 in S^1 to be base points for X and S^1, respectively. Up to homotopy, each map $f : X \to S^1$ given *a priori* may be supposed to be cellular and so based (that is, $f(x_0) = s^0$). Thus, f induces a homomorphism

$$f_\# : \pi_1(X, x_0) \to \pi_1(S^1, s^0).$$

To prove that any maps $f_1, \ldots, f_n : X \to S^1$, with $n \geq 3$, are loose, it is sufficient to prove that f_1, f_2, f_3 are loose. Moreover, as we have seen, we may suppose that these maps take x_0 to s^0. We consider the map

$$f : X \to M = S^1 \times S^1 \times S^1 \quad \text{given by } f(x) = (f_1(x), f_2(x), f_3(x)).$$

We take $s_0 = (s^0, s^0, s^0) \in M$ to be the base point of M. Then f is a based map and so it induces the homomorphism $f_\# : \pi_1(X, x_0) \to \pi_1(M, s_0)$, which is given, up to the natural isomorphism $\pi_1(M, s_0) \approx \pi_1(S^1, s^0) \times \pi_1(S^1, s^0) \times \pi_1(S^1, s^0)$ by $f_\# = (f_{1_\#}, f_{2_\#}, f_{3_\#})$.

The space $M = S^1 \times S^1 \times S^1$ corresponds to the quotient space obtained from the cube $Q = [0, 1]^3$ via the identifications $(x, y, 0) \sim (x, y, 1)$, $(x, 0, z) \sim (x, 1, z)$ and $(0, y, z) \sim (1, y, z)$, that is, the identifications of the opposite faces.

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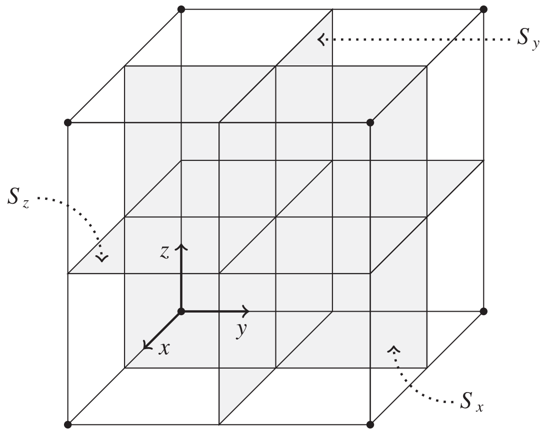

FIGURE 5. The cube Q with three distinguished squares.

Figure 5 illustrates the cube Q and three inside squares, namely: S_x (with equation $x = 1/2$, and so orthogonal to the x-axis), S_y (with equation $y = 1/2$, and so orthogonal to the y-axis) and S_z (with equation $z = 1/2$, and so orthogonal to the z-axis). After the identifications of the opposite faces of Q, each of these squares becomes a two-dimensional torus embedded into the three-dimensional torus M, labelled T_x, T_y and T_z, respectively.

To obtain the space $M \setminus \Delta$, we delete the eight vertices of Q (since all of them correspond to the point s_0 in M, which belongs to the diagonal of Q) and also the diagonal of Q. Only then do we identify the opposite faces. After deleting the vertices of Q and before identifying the opposite faces, we can 'break walls' from each vertex inside the corresponding octant, so providing a strong deformation retraction of $Q \setminus \{\text{vertices}\}$ onto $S_x \cup S_y \cup S_z$. After identifying the opposite faces, this provides a strong deformation retraction of $M \setminus \{s_0\}$ onto $T_x \cup T_y \cup T_z$. Since the diagonal of Q meets $S_x \cup S_y \cup S_z$ just in the central point $p_0 = (1/2, 1/2, 1/2)$, it follows that the construction gives a strong deformation retraction of $M \setminus \Delta$ onto $(T_x \cup T_y \cup T_y) \setminus \{p_0\}$. In its turn, $(T_x \cup T_y \cup T_y) \setminus \{p_0\}$ has a strong deformation retraction (by way of a 'radial retraction') to the graph L illustrated in Figure 6 on the right. The dashed edge c does not belong to L. Moreover, the two vertices of each edge must be identified with each other.

Following [2, Section 3], we attach an arc $c \subset M$ (which meets Δ just in s_0 and meets L just in a point) to L so obtaining a graph $G = L \cup c \subset M$ and a strong deformation retraction of $\{s_0\} \cup (M \setminus \Delta)$ onto G. The graph G is homotopy equivalent to the bouquet of seven circles (see Figure 7). It follows that $\pi_1(G, s_0)$ is the free group $F_7 = F(z_1, \ldots, z_7)$ of rank 7, in which the generators correspond to the loops indicated in the second column of Table 1. Let $\ell : (G, s_0) \hookrightarrow (M, s_0)$ be the natural inclusion. The third column of Table 1 indicates the images $\ell_\#(z_i) \in \pi_1(M, s_0) \approx \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. It is obvious that $\ell_\#$ is surjective.

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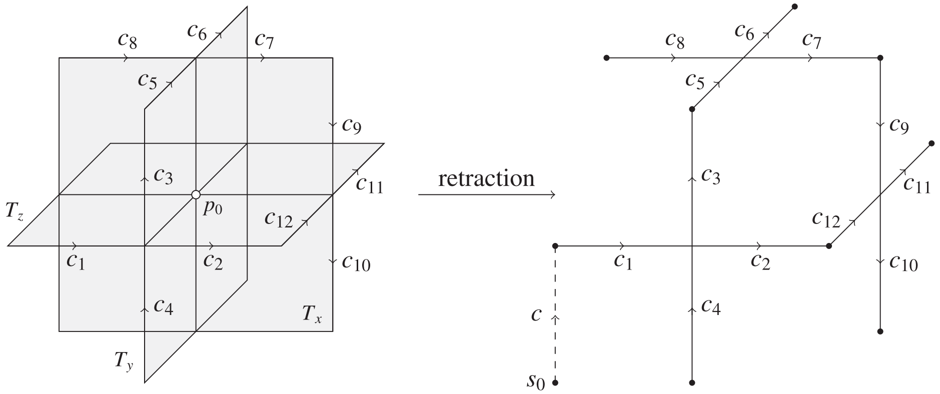

FIGURE 6. The retractions $M \setminus \Delta \to (T_x \cup T_y \cup T_y) \setminus \{p_0\} \to L$.

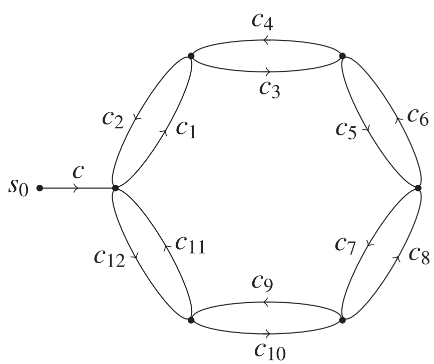

FIGURE 7. The graph $G = L \cup c \subset M$.

TABLE 1. The homomorphism $\ell_\#$.

Generator	Loop	$\ell_\#(\cdot)$
z_1	$cc_1c_2\bar{c}$	$(0, 1, 0)$
z_2	$cc_1c_3c_4\bar{c}_1\bar{c}$	$(0, 0, 1)$
z_3	$cc_1c_3c_5c_6\bar{c}_3\bar{c}_1\bar{c}$	$(1, 0, 0)$
z_4	$cc_1c_3c_5c_7c_8\bar{c}_5\bar{c}_3\bar{c}_1\bar{c}$	$(0, 1, 0)$
z_5	$cc_1c_3c_5c_7c_9c_{10}\bar{c}_7\bar{c}_5\bar{c}_3\bar{c}_1\bar{c}$	$(0, 0, -1)$
z_6	$cc_1c_3c_5c_7c_9c_{11}c_{12}\bar{c}_9\bar{c}_7\bar{c}_5\bar{c}_3\bar{c}_1\bar{c}$	$(1, 0, 0)$
z_7	$cc_1c_3c_5c_7c_9c_{11}\bar{c}$	$(1, 1, 0)$

To fill the third column of the table, we consider the correspondence between the generators $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ of $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \approx \pi_1(M, s_0)$ and the based loops that traverse the 3-torus M in the x-direction, in the y-direction and in the z-direction, respectively. As indicated in Figure 5, we consider the standard oriented basis for the

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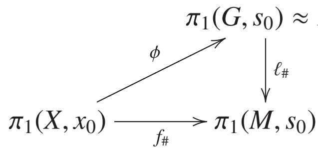
three-dimensional space. Thus, c_5c_6 and $c_{12}c_{11}$ are in the x-direction, c_1c_2 and c_8c_7 are in the y-direction, and c_4c_3 and $\bar{c}_{10}\bar{c}_9$ are in the z-direction.

Since the group $\pi_1(X, x_0)$ is free and $\ell_\#$ is surjective, there exists a homomorphism $\phi : \pi_1(X, x_0) \to F_7$ such that $\ell_\# \circ \phi = f_\#$. In fact, for each free generator σ of $\pi_1(X, x_0)$, we choose a word $w(\sigma) \in F_7$ such that $\ell_\#(w(\sigma)) = f_\#(\sigma)$. Then we extend the function $\sigma \mapsto w(\sigma)$ to a homomorphism $\phi : \pi_1(X, x_0) \to F_7$ satisfying $\ell_\# \circ \phi = f_\#$. This gives the commutative diagram:



Since X and G are graphs, ϕ may be realised as the homomorphism induced on fundamental groups by a cellular map $\varphi : (X, x_0) \to (G, s_0)$, that is, $\phi = \varphi_\#$.

For each $i = 1, 2, 3$, we consider the composite $f_i'' = p_i \circ \ell \circ \varphi : X \to G \to M \to S^1$, where $p_i : M \to S^1$ is the projection onto the ith coordinate. Then

$$f''_{i_\#} = p_{i_\#} \circ \ell_\# \circ \varphi_\# = p_{i_\#} \circ f_\# = f_{i_\#}.$$

It follows that $f_i'' \simeq f_i$ (since the homotopy classes of maps from a graph into the circle are uniquely defined by the homomorphisms induced on the fundamental groups).

Now we consider the composite

$$f' = \iota \circ \kappa \circ r \circ \varphi : X \to G \to L \hookrightarrow M \setminus \Delta \hookrightarrow M,$$

where ι and κ are the natural inclusions and $r : G \to L$ is the natural strong deformation retraction (namely, the one obtained by retracting the arc c through itself). It is obvious that $\iota \circ \kappa \circ r \simeq \ell$ and hence $f' \simeq f'' = (f_1'', f_2'', f_3'')$.

Finally, for each $i = 1, 2, 3$, we define $f_i' = p_i \circ f' : X \to M \to S^1$. Then $f_i' \simeq f_i$. Therefore, we have defined maps $f_1', f_2', f_3' : X \to S^1$ such that $f_1' \simeq f_1$, $f_2' \simeq f_2$, $f_3' \simeq f_3$ and $\mathrm{Coin}(f_1', f_2', f_3') = \emptyset$, since $f' = (f_1', f_2', f_3') : X \to M$ lifts to $\kappa \circ r \circ \varphi$ through ι.

4. Totally loose maps into the circle are homotopic

In this section, we prove the second statement of item (2) of Theorem 1.2.

Let $f_1, \ldots, f_n : X \to S^1$ be maps from a graph into the circle, which we may suppose to be cellular and so based. As in Section 3, we consider the based spaces (X, x_0) and (S^1, s^0).

Let $\{a_1, b_1\}, \ldots, \{a_k, b_k\}$ be all the $k := n!\,/2(n-2)!$ subsets of $\{1, \ldots, n\}$ of cardinality two. For each $i = 1, \ldots, k$, let $p_i = (a_i, b_i)$ be a fixed ordering for the elements a_i, b_i. We define the map $f : X \to (S^1 \times S^1)^k$ by setting

$$f(x) = (f_{a_1}(x), f_{b_1}(x), f_{a_2}(x), f_{b_2}(x), \ldots, f_{a_k}(x), f_{b_k}(x)).$$

The space $S^1 \times S^1 \setminus \Delta$ strong deformation retracts to a subspace $L \subset S^1 \times S^1$ homeomorphic to S^1. This subspace may be extended, by attaching an arc c, to a graph

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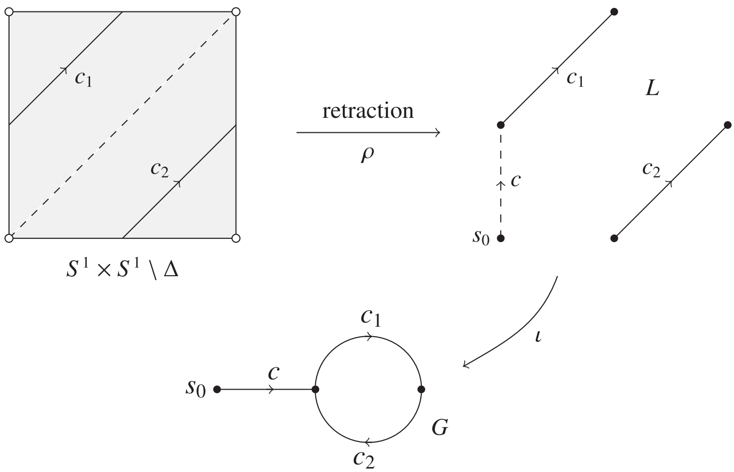

FIGURE 8. The maps $S^1 \times S^1 \setminus \Delta \to L \hookrightarrow G$.

$G = L \cup c \subset S^1 \times S^1$ containing the point $s_0 = (s^0, s^0)$ and so that there exists a strong deformation retraction $G \to L$ (see Figure 8).

The natural inclusion $\ell : G \to S^1 \times S^1$ induces the homomorphism

$$\ell_\# : \mathbb{Z} \approx \pi_1(G, s_0) \to \pi_1(S^1 \times S^1, s_0) \approx \mathbb{Z} \times \mathbb{Z} \quad \text{given by } 1 \mapsto (1,1),$$

since the loop $cc_1c_2\bar{c}$ makes a longitudinal turn and a latitudinal turn in the torus $S^1 \times S^1$ (see Figure 8 again). This homomorphism was described in [1, Section 3].

It follows that the inclusion $\ell^k : G^k \hookrightarrow (S^1 \times S^1)^k$ induces the homomorphism

$$\ell^k_\# : \mathbb{Z}^k \approx \pi_1(G^k, s_0^k) \to \pi_1((S^1 \times S^1)^k, s_0^k) \approx (\mathbb{Z} \times \mathbb{Z})^k$$

given by $e_1 \mapsto (1,1,0,0,\dots,0,0), \dots, e_k \mapsto (0,0,\dots,0,0,1,1)$, where $\{e_1,\dots,e_k\}$ is the canonical base of the free abelian group $\mathbb{Z}^k$.

Suppose $f_1,\dots,f_n$ are totally loose. Then there exist maps $f_1'' \simeq f_1, \dots, f_n'' \simeq f_n$ such that $\mathrm{im}(f'') \subset (S^1 \times S^1 \setminus \Delta)^k$, where

$$f'' = (f''_{a_1}, f''_{b_1}, f''_{a_2}, f''_{b_2}, \dots, f''_{a_k}, f''_{b_k}) : X \to (S^1 \times S^1)^k.$$

For each index i, we define the map $f_i' : (X, x_0) \to (G, s_0)$ to be a cellular approximation of the composite $\iota \circ \rho \circ f_i'' : X \to G$, where $\rho : S^1 \times S^1 \setminus \Delta \to L$ is a strong deformation retraction and $\iota : L \hookrightarrow G$ is the natural inclusion. Then $\ell \circ f_i' \simeq f_i$, which implies that the induced homomorphism $f_{i_\#} = \ell_\# \circ f'_{i_\#} : \pi_1(X, x_0) \to \pi_1(S^1 \times S^1, s_0)$.

Consider the map

$$f' = (f'_{a_1}, f'_{b_1}, f'_{a_2}, f'_{b_2}, \dots, f'_{a_k}, f'_{b_k}) : X \to G^k.$$

We have $f_\# = \ell^k_\# \circ f'_\#$, which forces

$$\mathrm{im}(f_\#) \subset \mathrm{im}(\ell^k_\#) = \langle (1,1,0,0,\dots,0,0), \dots, (0,0,\dots,0,0,1,1) \rangle.$$

Thus, $f_{a_{i\#}} = f_{b_{i\#}}$ for each $i = 1, \ldots, k$ and, since we are dealing with maps between graphs, it follows that $f_{a_i} \simeq f_{b_i}$. Therefore, the maps $f_1, \ldots, f_n$ are all homotopic.

Conversely, we will prove that n copies of a map $f : X \to S^1$ are totally loose, which is equivalent to saying that any homotopic maps $f_1, \ldots, f_n : X \to S^1$ are totally loose. We consider n points $0 = t_1 < t_2 < \cdots < t_n$ of the interval $[0, 1)$. For each $j = 1 \ldots, n$, we define the map $f'_j : X \to S^1$ by setting $f'_j(x) = f(x)e^{2\pi i t_j}$. In other words, f_j is the composite of f with the clockwise rotation of angle $2\pi t_j$. Of course, each $f'_j \simeq f$ and, moreover, $\mathrm{C}_2(f'_1, \ldots, f'_n) = \emptyset$.

Acknowledgment

The author thanks the anonymous referee for valuable comments and suggestions.

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