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Dirichlet and Neumann boundary conditions for the p-Laplace operator: what is in between?

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Let $p \in (1, \infty)$ and let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary. We characterize on $L^2(\Omega)$ all order-preserving semigroups that are generated by convex, lower semicontinuous, local functionals and are sandwiched between the semigroups generated by the p-Laplace operator with Dirichlet and Neumann boundary conditions. We show that every such semigroup is generated by the p-Laplace operator with Robin-type boundary conditions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. It is well known that for every $p \in (1, \infty)$ the diffusion equation governed by the p-Laplace operator

$$\left.\begin{aligned} u_t - \Delta_p u = 0 &\quad \text{in } (0,\infty) \times \Omega, \\ u(0,\cdot) = u_0 &\quad \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

with $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, is well posed in $L^2(\Omega)$ if it is complemented by Dirichlet boundary conditions

$$u = 0 \quad \text{on } (0,\infty) \times \partial\Omega \quad \text{(perfectly conducting boundary)}$$

or by Neumann boundary conditions

$$|\nabla u|^{p-2} \frac{\partial u}{\partial \nu} = 0 \quad \text{on } (0,\infty) \times \partial\Omega \quad \text{(perfectly isolating boundary).}$$

The two problems can be rewritten as abstract gradient systems of the form

$$\dot{u} + \partial\varphi(u) \ni 0 \quad \text{on } \mathbb{R}_+, \qquad u(0) = u_0,$$

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with $\varphi\colon L^2(\Omega) \to [0,+\infty]$ being a convex and lower semicontinuous functional and $\partial\varphi$ being its subgradient; the well-posedness of such gradient systems follows from a classical result by Minty. In the case of Dirichlet boundary conditions, this functional is

$$\varphi_{\mathrm{D}}(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x & \text{if } u \in W_0^{1,p}(\Omega) \cap L^2(\Omega), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.2}$$

and in the case of Neumann boundary conditions it is

$$\varphi_{\mathrm{N}}(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x & \text{if } u \in W^{1,p}(\Omega) \cap L^2(\Omega), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.3}$$

The associated gradient systems give rise to strongly continuous semigroups of nonlinear (linear if $p = 2$) contractions on $L^2(\Omega)$, denoted by S_{D} and S_{N}, respectively. It is well known that both semigroups are order preserving, and that S_{D} is dominated by the semigroup S_{N} in the sense that

$$|S_{\mathrm{D}}(t)u| \leqslant S_{\mathrm{N}}(t)|u| \quad \text{for every } t \geqslant 0 \text{ and every } u \in L^2(\Omega).$$

In other words, the diffusion governed by the p-Laplace operator with Neumann boundary conditions (perfectly isolating boundary; the total energy $\int_\Omega u$ is conserved) dominates the diffusion governed by the p-Laplace operator with Dirichlet boundary conditions (perfectly conducting boundary; the energy dissipates through the boundary).

We also consider the functional φ on $L^2(\Omega)$ defined by

$$\varphi(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu & \text{if } u \in D(\varphi), \\ +\infty & \text{otherwise,} \end{cases} \tag{1.4}$$

with effective domain

$$D(\varphi) = \left\{ u \in W^{1,p}(\Omega) \cap L^2(\Omega) \colon \int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu < \infty \right\},$$

where μ is a regular Borel measure on $\partial\Omega$, and $B\colon \partial\Omega \times \mathbb{R} \to [0,+\infty]$ is a Borel function which is measurable in the first variable and lower semicontinuous and *bi-monotone* (that is, non-increasing on $]-\infty, 0]$ and non-decreasing on $[0,+\infty[$) in the second variable. This functional is equal to φ_{D} or to φ_{N} if $\mu = \sigma$ is the surface measure on $\partial\Omega$ and if

$$B(x,s) = \begin{cases} +\infty & \text{if } s \neq 0, \\ 0 & \text{if } s = 0, \end{cases}$$

in the case of Dirichlet boundary conditions, and

$$B(x,s) = 0$$

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in the case of Neumann boundary conditions. In any case, if φ is convex and lower semicontinuous, then $\partial\varphi$ is a realization of the p-Laplace operator with generalized Robin-type boundary conditions formally given by

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\,\mathrm{d}\sigma + \beta(x,u)\,\mathrm{d}\mu \ni 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

If $\mu = \sigma$ or, more generally, if μ is absolutely continuous with respect to σ (hence, $\mathrm{d}\mu = \alpha(x)\,\mathrm{d}\sigma$), then (1.5) reduces to the classical Robin boundary conditions. For more details on this formulation of the boundary conditions, we refer the reader to [1, 2, 6–8, 17, 18] and the references therein, and to § 3.1. The subgradient $\partial\varphi$ generates a nonlinear semigroup S of contractions on $L^2(\Omega)$. Under appropriate further conditions on μ and B (see theorem 2.1 for the precise conditions), the semigroup S is sandwiched between the semigroups S_{D} and S_{N} in the sense that

$$|S_{\mathrm{D}}(t)u| \leqslant S(t)|u| \quad \text{and} \quad |S(t)u| \leqslant S_{\mathrm{N}}(t)|u| \quad \text{for every } t \geqslant 0 \text{ and } u \in L^2(\Omega). \tag{1.6}$$

In this sense, the diffusion governed by the p-Laplace operator with Robin-type boundary conditions is intermediate between the diffusions governed by the p-Laplace operator with Dirichlet and Neumann boundary conditions.

The aim of this paper is to prove the converse of (1.6). More precisely, we show that if S is a semigroup on $L^2(\Omega)$ generated by the subgradient of a convex and lower semicontinuous functional φ on $L^2(\Omega)$, if S is sandwiched between S_{D} and S_{N} in the sense of (1.6) and if φ satisfies a natural locality condition (so that its subgradient is a local operator), then φ is necessarily of the form (1.4) for some μ and some B (theorem 2.1). This is a possible answer to the question 'Dirichlet and Neumann boundary conditions: what is in between?', which was asked in the title of [2]. Arendt and Warma [2] gave an answer to this question in the linear case, that is, in the case of the diffusion governed by the Laplace operator and when all semigroups are C_0-semigroups of linear, self-adjoint operators. Our paper (and the title of our paper) is clearly motivated by the question considered in [2], and by the question of whether or not a similar characterization of sandwiched semigroups holds in the context of a nonlinear diffusion equation.

We outline the plan of the paper. In § 2 we give some preliminaries and state the main result of this paper (theorem 2.1). Concerning the proof of the main result, we follow in some sense the idea of proof of the corresponding result in the linear case [2, theorem 4.1]. Our proof thus depends heavily on a characterization of domination and order preservation of semigroups in terms of properties of the generating functionals and on a Riesz-type representation theorem for convex, lower semicontinuous functionals. These two results are also stated in § 2. Characterizations of domination and order preservation of nonlinear semigroups generated by subdifferentials go back to Brézis and Pazy [9], but the formulation which is appropriate for our purposes (theorem 2.2) is taken from [3]. There also exist several Riesz-type representation theorems for nonlinear functionals (see, for example, [14–16,22]), but no appropriate result is stated for lower semicontinuous functionals on the Sobolev space $W^{1,p}(\Omega)$. Therefore, we state and prove such a result (theorem 2.3), which seems, to the best of our knowledge, to be new and may have its own independent interest. Sections 4 and 5 are devoted to the proofs of theorems 2.1 and 2.3, while in § 3 we include a discussion to clarify the conditions from the main theorem.

2. Main result

Let H be a real Hilbert space with inner product $(\cdot,\cdot)_H$, and let $\varphi\colon H \to (-\infty,+\infty]$ be a convex and lower semicontinuous (l.s.c.) functional with *effective domain*

$$D(\varphi) := \{u \in H\colon \varphi(u) < \infty\}.$$

By a classical result of Minty [24] (see also [9, 19, 25, 28]), every convex, l.s.c. functional φ on H *generates* a strongly continuous semigroup $S = (S(t))_{t\geqslant 0}$ of (in general nonlinear) contractions on $\overline{D(\varphi)}$. This means that there exists a unique family $S = (S(t))$ of contractions on $\overline{D(\varphi)}$ such that for every $u_0 \in \overline{D(\varphi)}$ the trajectory $u := S(\cdot)u_0$ is the unique strong solution of the following abstract gradient system:

$$\begin{aligned} u &\in C(\mathbb{R}_+; H) \cap W^{1,\infty}_{\mathrm{loc}}((0,\infty); H),\\ \dot u + \partial\varphi(u) &\ni 0 \quad \text{almost everywhere on } \mathbb{R}_+,\\ u(0) &= u_0. \end{aligned}$$

Here, the *subgradient* $\partial\varphi$ at a point $u \in D(\varphi)$ is defined by

$$\partial\varphi(u) := \{f \in H\colon \varphi(u+w) - \varphi(u) \geqslant (f,w)_H \text{ for every } w \in H\}.$$

In fact, throughout the following, $H = L^2(\Omega)$ for some bounded domain $\Omega \subseteq \mathbb{R}^N$ with Lipschitz continuous boundary, and all functionals on $L^2(\Omega)$ have dense effective domain, unless otherwise stated. The space $L^2(\Omega)$ is a real Hilbert lattice for the natural ordering. It thus makes sense to consider the following properties of a semigroup or a pair of semigroups. We say that a semigroup $S = (S(t))_{t\geqslant 0}$ on $L^2(\Omega)$ is *order preserving*, if

$$S(t)u \leqslant S(t)v \text{ for all } t \geqslant 0 \quad \text{whenever } u, v \in L^2(\Omega) \text{ and } u \leqslant v. \tag{2.1}$$

Moreover, if $S_1 = (S_1(t))_{t\geqslant 0}$ and $S_2 = (S_2(t))_{t\geqslant 0}$ are two semigroups on $L^2(\Omega)$, then we say that S_1 *is dominated by* S_2 and we write $S_1 \preccurlyeq S_2$ if

$$|S_1(t)u| \leqslant S_2(t)|u| \quad \text{for all } u \in L^2(\Omega) \text{ and } t \geqslant 0.$$

We say that a functional $\varphi\colon L^2(\Omega) \to (-\infty,+\infty]$ is *local*[1] if, for every $u, v \in L^2(\Omega)$,

$$|u| \wedge |v| = 0 \quad \Longrightarrow \quad \varphi(u+v) = \varphi(u) + \varphi(v). \tag{2.2}$$

Here, $u \wedge v$ denotes the (pointwise) infimum of the functions u and v. Note that every local functional necessarily vanishes in 0. By abuse of language, we call a semigroup *local* if it is generated by a local functional.

The functionals φ_{D} and $\varphi_{\mathrm{N}}\colon L^2(\Omega) \to [0,+\infty]$ defined in (1.2) and (1.3) are convex, l.s.c. and local. Their effective domains $D(\varphi_{\mathrm{D}}) = W^{1,p}_0(\Omega) \cap L^2(\Omega)$ and $D(\varphi_{\mathrm{N}}) = W^{1,p}(\Omega) \cap L^2(\Omega)$ are both dense in $L^2(\Omega)$. Hence, the semigroups S_{D} and S_{N} generated by the two functionals φ_{D} and φ_{N} are both defined on the whole space $L^2(\Omega)$. The subgradients $\partial\varphi_{\mathrm{D}}$ and $\partial\varphi_{\mathrm{N}}$ are realizations of the p-Laplace operator with Dirichlet and Neumann boundary conditions, respectively. It is well known that the semigroups S_{D} and S_{N} are both order preserving and that S_{D} is dominated by S_{N} (see [26] for $p = 2$ and [3] for general p).

[1] In the literature, one can also find the term *additive*.

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The *p-capacity* of a set $A \subseteq \mathbb{R}^N$ is given by

$$\operatorname{Cap}_p(A) := \inf\{\|u\|^p_{W^{1,p}(\mathbb{R}^N)} : u \in W^{1,p}(\mathbb{R}^N) \text{ and there exists } O \subseteq \mathbb{R}^N \text{ open, such that } A \subseteq O \text{ and } u \geqslant 1 \text{ a.e. on } O\}. \tag{2.3}$$

A set $A \subseteq \mathbb{R}^N$ is called *p-polar* if $\operatorname{Cap}_p(A) = 0$. A statement $P(x)$ is said to hold *p-quasi-everywhere* on $B \subseteq \mathbb{R}^N$ if there exists a p-polar set $A \subseteq \mathbb{R}^N$ such that the statement $P(x)$ holds for every $x \in B \setminus A$. A function $u\colon B \to \mathbb{R}$ ($B \subseteq \mathbb{R}^N$) is said to be *p-quasi-continuous* if, for every $\varepsilon > 0$, there exists an open set $O \subseteq \mathbb{R}^N$ such that $\operatorname{Cap}_p(O) < \varepsilon$ and u restricted to $B \setminus O$ is continuous. It is well known that every $u \in W^{1,p}(\Omega)$ admits a p-quasi-continuous representative $\tilde{u}\colon \bar{\Omega} \to \mathbb{R}$. This p-quasi-continuous representative is unique up to a p-polar set, that is, every two p-quasi-continuous representatives coincide p-quasi-everywhere on $\bar{\Omega}$. Throughout the following, we identify each function $u \in W^{1,p}(\Omega)$ with a p-quasi-continuous representative. A subset G of $\mathbb{R}^N$ is said to be *p-quasi-open* if for every $\varepsilon > 0$, there exists an open set $O \subseteq \mathbb{R}^N$ such that $\operatorname{Cap}_p(O) < \varepsilon$ and $G \cup O$ is open.

Despite the fact that the p-capacity is not a Borel measure (the p-capacity is not σ-additive), we say that the measure μ is *absolutely continuous with respect to the p-capacity* if for every p-polar Borel set $A \subseteq \bar{\Omega}$, one has $\mu(A) = 0$.

Finally, a function $B\colon \mathbb{R} \to]-\infty, +\infty]$ is called *bi-monotone* if it is non-increasing on $]-\infty, 0]$ and non-decreasing on $[0, +\infty[$.

The following is the main theorem of this paper.

Theorem 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$, and let $p \in (1, \infty)$. Let S be the semigroup generated by a convex, l.s.c. functional $\varphi\colon L^2(\Omega) \to [0, +\infty]$, and let S_{D} and S_{N} be the semigroups generated by the Dirichlet p-Laplace operator and the Neumann p-Laplace operator, respectively. Then the following assertions are equivalent.*

(i) *The semigroup S is local, order preserving and $S_{\mathrm{D}} \preccurlyeq S \preccurlyeq S_{\mathrm{N}}$.*

(ii) *There exist a finite, regular Borel measure μ on $\partial\Omega$, which is absolutely continuous with respect to the p-capacity, and a Borel function $B\colon \partial\Omega \times \mathbb{R} \to [0, +\infty]$ satisfying*

$$\left.\begin{array}{ll} B(\cdot, s) \text{ is measurable} & \text{for every } s \in \mathbb{R}, \\ B(x, 0) = 0 & \text{for } \mu\text{-a.e. } x \in \partial\Omega, \\ B(x, \cdot) \text{ is lower semicontinuous} & \text{for } \mu\text{-a.e. } x \in \partial\Omega, \\ B(x, \cdot) \text{ is bi-monotone} & \text{for } \mu\text{-a.e. } x \in \partial\Omega, \end{array}\right\} \tag{H}$$

such that

$$\varphi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p \,\mathrm{d}x + \int_{\partial\Omega} B(x, u)\,\mathrm{d}\mu \quad \text{for all } u \in D(\varphi).$$

The proof of theorem 2.1 is carried out in § 5, while § 3 is reserved for a discussion of theorem 2.1. The proof of the implication (ii) ⇒ (i) is based on the following theorem and is relatively straightforward. The following theorem, due to Barthélemy [3], is an application, also using clever convexity arguments, of the characterization that a semigroup generated by a convex, l.s.c. functional $\varphi\colon L^2(\Omega) \to [0,+\infty]$ leaves a closed, convex set invariant. This characterization goes back to Brézis and Pazy [10] (see also [9, § IV.4], [13] and the references therein). We state the theorem for densely defined functionals only.

Theorem 2.2 (Barthélemy). *Let $\varphi, \varphi_1, \varphi_2\colon L^2(\Omega) \to (-\infty,+\infty]$ be three convex and l.s.c. functionals with dense effective domains. Let S, S_1 and S_2 be the semigroups generated by φ, φ_1 and φ_2, respectively.*

(a) *If the functional φ is non-negative, then the semigroup S is order preserving if and only if for all $u, v \in L^2(\Omega)$ one has*

$$\varphi(u \wedge v) + \varphi(u \vee v) \leqslant \varphi(u) + \varphi(v) \tag{2.4}$$

(see théorème 2.1 of [3]).

(b) *If the semigroup S_2 is order preserving, then the semigroup S_1 is dominated by the semigroup S_2, that is, $S_1 \preccurlyeq S_2$, if and only if for every $u, v \in L^2(\Omega)$, $v \geqslant 0$,*

$$\varphi_1((|u| \wedge v)\operatorname{sgn}(u)) + \varphi_2(|u| \vee v) \leqslant \varphi_1(u) + \varphi_2(v) \tag{2.5}$$

(see théorème 3.3 of [3]).

The difficult part in theorem 2.1 is the implication (i) $\Rightarrow$ (ii). Its proof uses the above characterization of order preservation and domination, but, in addition, it uses the following Riesz-type representation theorem which may have interest independent of the application given in this paper. Similar representation theorems for various classes of functionals on various functions spaces are included in [11, 12, 14–16, 20–22, 27, 30] and the references therein.

Let $p \in (1,\infty)$. We denote by $W^{1,p}(\Omega)^+$ the positive cone in $W^{1,p}(\Omega)$. Given a functional $\psi\colon W^{1,p}(\Omega)^+ \to [0,+\infty]$, we call $D(\psi) = \{u \in W^{1,p}(\Omega)^+ \colon \psi(u) < +\infty\}$ its *effective domain*. The *effective support* of the functional ψ is the set

$$\operatorname{supp}[\psi] := \bar{\Omega} \setminus \{x \in \bar{\Omega}\colon \text{there exists a neighbourhood } U \text{ of } x \text{ such that for every } u \in D(\psi) \text{ with } \operatorname{supp}[u] \subseteq U \text{ one has } \psi(u) = 0\}.$$

We say that the functional ψ is *monotone* if, for every $u, v \in W^{1,p}(\Omega)^+$,

$$u \leqslant v \quad \Longrightarrow \quad \psi(u) \leqslant \psi(v). \tag{2.6}$$

Analogously to functionals defined on $L^2(\Omega)$, we say that ψ is *local* if, for every $u, v \in W^{1,p}(\Omega)^+$,

$$u \wedge v = 0 \quad \Longrightarrow \quad \psi(u + v) = \psi(u) + \psi(v). \tag{2.7}$$

Theorem 2.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary. For every functional $\psi\colon W^{1,p}(\Omega)^+ \to [0,+\infty]$, the following assertions are equivalent.*

(i) *The functional ψ is lower semicontinuous, monotone, local and, for every $u, v \in D(\psi)$, one has $u \vee v, u \wedge v \in D(\psi)$ and*

$$\psi(u \vee u) + \psi(u \wedge v) \leqslant \psi(u) + \psi(v). \tag{2.8}$$

(ii) *There exist a finite, regular Borel measure* μ *with* $\operatorname{supp}[\mu] \subseteq \operatorname{supp}[\psi]$*, which is absolutely continuous with respect to the* p*-capacity, and a Borel function* $B\colon \bar{\Omega} \times \mathbb{R}_+ \to [0, +\infty]$ *satisfying*

$$\left.\begin{array}{ll} B(\cdot, s) \text{ is measurable} & \text{for every } s \in \mathbb{R}, \\ B(x, 0) = 0 & \text{for } \mu\text{-a.e. } x \in \bar{\Omega}, \\ B(x, \cdot) \text{ is lower semicontinuous} & \text{for } \mu\text{-a.e. } x \in \bar{\Omega}, \\ B(x, \cdot) \text{ is monotone} & \text{for } \mu\text{-a.e. } x \in \bar{\Omega}, \end{array}\right\} \tag{H$^+$}$$

such that

$$\psi(u) = \int_{\bar{\Omega}} B(x, u)\,\mathrm{d}\mu \quad \text{for all } u \in D(\psi).$$

Section 4 is devoted to the proof of this theorem.

Remark 2.4.

(a) Note that the representing measure μ and function B are not unique. For example, given a representing measure μ and a representing function B, and given any Borel measurable weight $w\colon \bar{\Omega} \to \mathbb{R}_+$ which is bounded from above and from below (away from zero), the weighted measure $w\,\mathrm{d}\mu$ and the function B/w also represent ψ.

(b) On the other hand, the proof of theorem 2.3 shows that for any pair ψ_1, ψ_2 of lower semicontinuous, monotone, local functionals satisfying the inequality (2.8) one can find a *common* representing measure μ with $\operatorname{supp}[\mu] \subseteq \operatorname{supp}[\psi_1] \cup \operatorname{supp}[\psi_2]$ and two representing functions B_1 and B_2 satisfying condition (H$^+$) such that

$$\psi_i(u) = \int_{\bar{\Omega}} B_i(x, u)\,\mathrm{d}\mu \quad \text{for every } u \in D(\psi_i),\ i = 1, 2.$$

It suffices to take, for example, $\mu = \mu_1 + \mu_2$, where μ_1 and μ_2 are two representing measures for ψ_1 and ψ_2, respectively, the existence of which is guaranteed by theorem 2.3.

(c) Theorem 2.3 remains true if the space $W^{1,p}(\Omega)$ is replaced by the *a priori* smaller space $W^{1,p}(\Omega) \cap L^2(\Omega)$. This is trivially true for $p \geqslant 2$, since then the two spaces actually coincide. If $p < 2$, essentially the same proof works. At first glance, it seems necessary to replace everywhere the p-capacity by the following $(p, 2)$-capacity, which is for subsets $A \subseteq \mathbb{R}^N$ defined by

$$\begin{aligned} \operatorname{Cap}_{(p,2)}(A) = \inf\{\|u\|^p_{W^{1,p}(\mathbb{R}^N)} + \|u\|^2_{L^2(\mathbb{R}^N)} \colon u \in W^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N) \\ \text{and there exists } O \subseteq \mathbb{R}^N \text{ open, such that } A \subseteq O \\ \text{and } u \geqslant 1 \text{ a.e. on } O\}. \end{aligned}$$

However, it is actually not necessary to do this since for $p \leqslant 2$ the p-capacity and the $(p, 2)$-capacity are equivalent in the sense that, for every subset $A \subseteq \mathbb{R}^N$,

$$\operatorname{Cap}_p(A) \leqslant \operatorname{Cap}_{(p,2)}(A) \leqslant 2\operatorname{Cap}_p(A).$$

Here, the first inequality is obvious from the respective definitions of the two capacities, while for the second inequality one has to notice that the definition of the $(p,2)$-capacity does not change if one takes the infimum over functions satisfying $0 \leqslant u \leqslant 1$ everywhere (and $u = 1$ a.e. on O) and that $\|u\|^2_{L^2(\mathbb{R}^N)} \leqslant \|u\|^p_{L^p(\mathbb{R}^N)}$ whenever $0 \leqslant u \leqslant 1$ and $p \leqslant 2$.

3. Discussion of the conditions in theorem 2.1

3.1. Interpretation of the generalized Robin-type boundary conditions

Let B and μ be as in theorem 2.1(ii). Assume, for simplicity, that $B(x,\cdot)$ is convex for μ-a.e. $x \in \partial\Omega$. Denote by $\beta(x,\cdot) = \partial B(x,\cdot)$ the *subgradient* of the functional $B(x,\cdot)$, that is, for $s \in D(B(x,\cdot))$,

$$\beta(x,s) = \{\tau \in \mathbb{R} \colon B(x,s+\xi) - B(x,s) \geqslant \tau\xi \text{ for every } \xi \in \mathbb{R}\}.$$

Let $f \in L^2(\Omega)$. We say that a function $u \in W^{1,p}(\Omega) \cap L^2(\Omega)$ is a *weak solution* of the elliptic problem

$$\left.\begin{aligned} -\Delta_p u &= f \quad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\,\mathrm{d}\sigma + \beta(x,u)\,\mathrm{d}\mu &\ni 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{3.1}$$

if $-\Delta_p u = f$ in the sense of distributions, if

$$\int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu < +\infty$$

and if, for every $w \in W^{1,p}(\Omega) \cap L^2(\Omega)$,

$$\int_{\partial\Omega} (B(x,u+w) - B(x,u))\,\mathrm{d}\mu \geqslant \int_\Omega fw\,\mathrm{d}x - \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w\,\mathrm{d}x. \tag{3.2}$$

The relation between this inequality and the boundary condition in (3.1) becomes clear if one replaces f by $-\Delta_p u$, recalls Green's formula,

$$\int_\Omega \Delta_p u w\,\mathrm{d}x + \int_\Omega |\nabla u|^{p-2}\nabla u \nabla w\,\mathrm{d}x = \int_{\partial\Omega} |\nabla u|^{p-2}\frac{\partial u}{\partial\nu} w\,\mathrm{d}\sigma, \quad w \in W^{1,p}(\Omega)$$

(which holds for sufficiently smooth functions u) and uses the definition of the subgradient.

Let φ be as in theorem 2.1(ii), that is,

$$\varphi(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu & \text{if } u \in D(\varphi), \\ +\infty & \text{otherwise,} \end{cases}$$

where the effective domain is given by

$$D(\varphi) = \left\{u \in W^{1,p}(\Omega) \cap L^2(\Omega); \int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu < \infty\right\}.$$

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Proposition 3.1. *Let $f \in L^2(\Omega)$. Then $u \in W^{1,p}(\Omega) \cap L^2(\Omega)$ is a weak solution of (3.1) if and only if $u \in D(\varphi)$ and $f \in \partial\varphi(u)$.*

Proof. Assume that $u \in D(\varphi)$ and $f \in \partial\varphi(u)$. Then, by the definition of $D(\varphi)$, $u \in W^{1,p}(\Omega) \cap L^2(\Omega)$,

$$\int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu < +\infty$$

and, as a consequence of the definition of the subgradient $\partial\varphi$, for every $w \in W^{1,p}(\Omega) \cap L^2(\Omega)$,

$$\frac{1}{p}\int_\Omega (|\nabla(u+w)|^p - |\nabla u|^p)\,\mathrm{d}x + \int_{\partial\Omega}(B(x,u+w) - B(x,u))\,\mathrm{d}\mu \geqslant \int_\Omega fw\,\mathrm{d}x.$$

In particular, when we replace w by tw ($t > 0$), divide the inequality by t and let t tend to 0, we obtain that, for every $w \in W^{1,p}(\Omega) \cap L^2(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w\,\mathrm{d}x + \liminf_{t\to 0+}\int_{\partial\Omega}\frac{B(x,u+tw) - B(x,u)}{t}\,\mathrm{d}\mu \geqslant \int_\Omega fw\,\mathrm{d}x. \quad (3.3)$$

In particular, for every test function $w \in \mathcal{D}(\Omega)$,

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla w\,\mathrm{d}x \geqslant \int_\Omega fw\,\mathrm{d}x.$$

Since this inequality is true for w and $-w$ ($w \in \mathcal{D}(\Omega)$), one actually has equality, and therefore $-\Delta_p u = f$ in the sense of distributions. Finally, by convexity of $B(x,\cdot)$, the inequality (3.3) holds for every $w \in W^{1,p}(\Omega) \cap L^2(\Omega)$ if and only if the inequality (3.2) holds for every $w \in W^{1,p}(\Omega) \cap L^2(\Omega)$. Hence, u is a weak solution of (3.1).

Conversely, let $u \in W^{1,p}(\Omega) \cap L^2(\Omega)$ be a weak solution of (3.1). Then, by definition of the weak solution and by definition of $D(\varphi)$, $u \in D(\varphi)$. Moreover, the inequality (3.2) holds for every $w \in W^{1,p}(\Omega) \cap L^2(\Omega)$, that is

$$\varphi(u+w) - \varphi(u) \geqslant \int_\Omega fw\,\mathrm{d}x \quad \text{for every } w \in W^{1,p}(\Omega) \cap L^2(\Omega).$$

Clearly, this inequality also holds trivially for $w \in L^2(\Omega) \setminus W^{1,p}(\Omega)$. Hence, $f \in \partial\varphi(u)$. □

3.2. Regularity of Ω

Theorems 2.1 and 2.3 remain true if Ω is a bounded domain with the $W^{1,p}$-*extension property* (for more details on the extension property we refer the reader to [23, 29]). This means that there exists a bounded, linear extension operator

$$W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^N).$$

We note that the $W^{1,p}$-extension property depends on p. If Ω has Lipschitz continuous boundary $\partial\Omega$, as we assumed in theorems 2.1 and 2.3, then Ω has the $W^{1,p}$-extension property for every $p \in (1,\infty)$ [29, ch. VI, theorem 5, p. 181]. The Lipschitz continuity of the boundary (or the $W^{1,p}$-extension property) is important in several places in which we deal with the p-capacity. Note that the p-capacity is defined by means of $W^{1,p}(\mathbb{R}^N)$ functions.

If Ω does not have a Lipschitz continuous boundary but is just an arbitrary bounded, open set, one may replace the p-capacity by the *relative p-capacity* and replace the Sobolev space $W^{1,p}(\Omega)$ by the space

$$\tilde{W}^{1,p}(\Omega) = \overline{W^{1,p}(\Omega) \cap C(\bar{\Omega})}^{W^{1,p}(\Omega)}.$$

The relative p-capacity $\mathrm{Cap}_{p,\Omega}$ is for subsets $A \subseteq \bar{\Omega}$ defined by

$$\mathrm{Cap}_{p,\Omega}(A) := \inf\{\|u\|^p_{W^{1,p}(\Omega)} : u \in \tilde{W}^{1,p}(\Omega) \text{ and there exists } O \subseteq \mathbb{R}^N \text{ open, such that } A \subseteq O \text{ and } u \geqslant 1 \text{ a.e. on } O \cap \bar{\Omega}\};$$

see [1] for the case $p = 2$ and [5, 7] for general $p \in [1, \infty[$. Up to these changes, and up to replacing the functional φ_{N} (and the associated semigroup S_{N}) by the functional

$$\tilde{\varphi}_{\mathrm{N}}(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p \,\mathrm{d}x & \text{if } u \in \tilde{W}^{1,p}(\Omega) \cap L^2(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

(and the associated semigroup $\tilde{S}_{\mathrm{N}}$), the main results in this paper (theorems 2.1 and 2.3) hold with essentially the same proofs.

3.3. The special case of quadratic forms ($p = 2$ and B quadratic)

A particular situation occurs when all the functionals in theorem 2.1 are assumed to be quadratic. A functional $\varphi\colon L^2(\Omega) \to (-\infty, +\infty]$ is *quadratic* if there exists a symmetric, bilinear form $(a, D(a))$ such that

$$\varphi(u) = \begin{cases} \frac{1}{2}a(u,u) & \text{if } u \in D(a), \\ +\infty & \text{otherwise.} \end{cases}$$

Note that $D(\varphi) = D(a)$ is a linear space in this case. A quadratic functional φ is convex and l.s.c. if and only if the associated form $(a, D(a))$ is positive and closed. In this case, $A := \partial\varphi$ is a linear, self-adjoint, non-negative operator and the associated semigroup S is a C_0-semigroup of linear, self-adjoint contractions. The functionals φ_{D} and φ_{N} are quadratic if and only if $p = 2$, and then $\partial\varphi_{\mathrm{D}}$ and $\partial\varphi_{\mathrm{N}}$ are the realizations of the Laplace operator with Dirichlet and Neumann boundary conditions, respectively. In the case when all functionals are quadratic, our theorem 2.1 should be compared with [2, theorem 4.1], at least in the situation when Ω is a bounded domain with Lipschitz continuous boundary $\partial\Omega$ (in [2], Ω is an arbitrary open set). Theorem 4.1 in [2] characterizes all the symmetric local semigroups that are sandwiched between the semigroups S_{D} and $\tilde{S}_{\mathrm{N}}$. However, there, the generating bilinear form $(a, D(a))$ and the measure μ are both assumed to satisfy an additional regularity condition, namely that $D(a) \cap C(\bar{\Omega})$ is dense in $(D(a), \|\cdot\|_a)$ and that μ is *admissible* [2, definition 2.3]. Our theorem 2.1 shows that these regularity conditions may be dropped.

In the general situation of theorem 2.1, we are not actually able to show that $D(\varphi) \cap C(\bar{\Omega})$ is dense in $D(\varphi)$ (in the $W^{1,p}(\Omega)$ topology, for example). We are not even sure whether $D(\varphi)$ contains non-trivial continuous functions at all. This problem is also the reason why we cannot use a Riesz-type representation theorem for functionals defined on $C(\bar{\Omega})^+$, such as the representation theorems in [12, 30]. Instead, we are forced to use a Riesz-type representation theorem on $W^{1,p}(\Omega)^+$ (that is, theorem 2.3).

If in the situation of theorem 2.1 one assumes in addition that the functional φ is *continuous* on $W^{1,p}(\Omega)$, then in theorem 2.3 one obtains that $B(x, \cdot)$ is continuous. In this situation, it is possible to show that the functional φ is also regular in the sense that for every $u \in D(\varphi)$ there exists a sequence $(u_n) \subseteq D(\varphi) \cap C(\bar{\Omega})$ converging to u in $W^{1,p}(\Omega)$ and satisfying $\lim_{n\to\infty} \varphi(u_n) = \varphi(u)$.

Finally, we note that if $p = 2$ in theorem 2.1, i.e. the case of the Laplace operator, even if S_{D} and S_{N} are linear semigroups, theorem 2.1 shows that there are nonlinear sandwiched semigroups generated by the Laplace operator with nonlinear boundary conditions of the form

$$\frac{\partial u}{\partial \nu}\,\mathrm{d}\sigma + \beta(x, u)\,\mathrm{d}\mu \ni 0.$$

3.4. Non-negativity of the functional φ

While the generation theorem by Minty applies for general convex, l.s.c. functionals $\varphi\colon L^2(\Omega) \to]-\infty, +\infty]$, the functional in theorem 2.1 is assumed to be convex, l.s.c. *and* non-negative. However, assuming condition (i) in theorem 2.1, φ is automatically non-negative. In fact, the domination $S \preccurlyeq S_{\mathrm{N}}$ implies that 0 is an equilibrium point of the semigroup S (that is, $S(t)0 = 0$ for every $t \geqslant 0$). This is equivalent to $0 \in \partial\varphi(0)$. Since φ is convex, this in turn is equivalent to the fact that φ attains its minimum in 0. Now, the further assumption that φ is local implies $\varphi(0) = 0$. Hence, φ is necessarily non-negative.

3.5. Locality of φ

The assumption that the functional φ is local is also necessary. In fact, there are semigroups S generated by convex, l.s.c. and non-local functionals φ such that $S_{\mathrm{D}} \preccurlyeq S \preccurlyeq S_{\mathrm{N}}$. Clearly, such functionals cannot be of the integral form as in theorem 2.1(ii). Examples of such non-local functionals exist even in the quadratic case, that is, when all semigroups are linear [2, example 4.5].

3.6. Relation between the lower semicontinuity of φ and properties of B and μ

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Let μ be a finite, regular Borel measure on $\partial\Omega$ (no further assumption on μ) and let $B\colon \partial\Omega \times \mathbb{R} \to [0, +\infty]$ be a Borel function satisfying hypothesis (H). Fix $p \in (1, \infty)$, and consider the functional $\varphi\colon L^2(\Omega) \to [0, +\infty]$ given by

$$\varphi(u) = \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p\,\mathrm{d}x + \int_{\partial\Omega} B(x, u)\,\mathrm{d}\mu & \text{if } u \in D(\varphi), \\ +\infty & \text{otherwise,} \end{cases} \tag{3.4}$$

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where the effective domain

$$D(\varphi) := \left\{ u \in W^{1,p}(\Omega) \cap L^2(\Omega) \colon \int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu < \infty \right\}.$$

THEOREM 3.2. *If the functional φ defined above is lower semicontinuous on $L^2(\Omega)$, then for every p-polar set $K \subseteq \partial\Omega$ and every $u \in D(\varphi)$ one has*

$$\int_K B(x,u)\,\mathrm{d}\mu = 0. \tag{3.5}$$

Proof. Let $K \subseteq \partial\Omega$ be a p-polar set, and let $u \in D(\varphi)$. We first assume that K is compact and that u is non-negative and essentially bounded. Since K is p-polar, there exists a sequence $(v_n) \subseteq W^{1,p}(\mathbb{R}^N)$ such that

$$0 \leqslant v_n \leqslant \|u\|_{L^\infty(\Omega)} \quad \text{everywhere on } \bar{\Omega}, \qquad v_n = \|u\|_{L^\infty(\Omega)} \quad \text{on } K$$

and

$$\lim_{n\to\infty} \|v_n\|_{W^{1,p}(\mathbb{R}^N)} = 0.$$

Now let $u_n := v_n \wedge u$. Then

$$0 \leqslant u - u_n \leqslant u \quad \text{everywhere on } \bar{\Omega}, \qquad u - u_n = 0 \quad \text{on } K$$

and

$$\lim_{n\to\infty} \|u_n\|_{W^{1,p}(\Omega)} = 0.$$

By the bounded convergence theorem, we also have

$$\lim_{n\to\infty} \|u - u_n\|_{L^2(\Omega)} = 0.$$

Since φ is lower semicontinuous on $L^2(\Omega)$, we obtain

$$\frac{1}{p}\int_\Omega |\nabla u|^p + \int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu \leqslant \liminf_{n\to\infty} \left(\frac{1}{p}\int_\Omega |\nabla (u-u_n)|^p + \int_{\partial\Omega} B(x,u-u_n)\,\mathrm{d}\mu \right).$$

The inequality $u - u_n \leqslant u$ (everywhere on $\bar{\Omega}$), the equality $u - u_n = 0$ on K, the assumption $B(x,0) = 0$ and the bi-monotonicity of B imply that, for every n,

$$\int_{\partial\Omega} B(x,u-u_n)\,\mathrm{d}\mu \leqslant \int_{\partial\Omega\setminus K} B(x,u)\,\mathrm{d}\mu.$$

The two preceding inequalities and the convergence $\|u_n\|_{W^{1,p}(\Omega)} \to 0$ together imply

$$\int_K B(x,u)\,\mathrm{d}\mu \leqslant 0.$$

Since $B \geqslant 0$, we thus obtain (3.5).

The equality (3.5) for arbitrary non-negative $u \in D(\varphi)$ (but compact K) follows by an approximation with the sequence $(u \wedge n)$, using also the lower semi-continuity of φ and the monotonicity of B. If K is not compact (but p-polar) and if $u \in D(\varphi)$ is non-negative, then we obtain (3.5) from the inner regularity of the measure $B(x,u)\,\mathrm{d}\mu$.

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Similarly, using the bi-monotonicity of B, one shows that (3.5) holds for every p-polar set $K \subseteq \partial\Omega$ and every nonpositive $u \in D(\varphi)$. Finally, if K is an arbitrary p-polar set and $u \in D(\varphi)$ is arbitrary, too, then the previous steps imply

$$\begin{aligned}\int_K B(x,u)\,\mathrm{d}\mu &= \int_K B(x,u^+ - u^-)\,\mathrm{d}\mu \\ &= \int_K B(x,u^+)\,\mathrm{d}\mu + \int_K B(x,-u^-)\,\mathrm{d}\mu \\ &= 0,\end{aligned}$$

where we have also used the fact that if $u \in D(\varphi)$, then $u^+ \in D(\varphi)$. □

With a slight abuse of language, theorem 3.2 says that if the functional φ given by (3.4) is lower semicontinuous, then the weighted measure $B\,\mathrm{d}\mu$ is necessarily absolutely continuous with respect to the p-capacity (in the sense made precise in theorem 3.2). We point out that the stronger property that the *unweighted* measure μ is absolutely continuous with respect to the p-capacity cannot be expected in the general situation of theorem 3.2 (take, for example, $B = 0$ and μ a measure which is not absolutely continuous with respect to the p-capacity). At the same time, we point out that our main theorem (theorem 2.1) does state the existence of a representing measure μ that is absolutely continuous with respect to the p-capacity.

In the literature (see, for example, [6–8, 18] for the nonlinear case and [1, 2, 17] for the linear case) parabolic and elliptic equations associated with the functional φ defined in (3.4) have been investigated. It has been assumed in the literature cited here that $B(x,\cdot)$ is convex for μ-a.e. $x \in \partial\Omega$, and that the measure μ is absolutely continuous with respect to the p-capacity. Theorem 3.2 shows that this is a natural assumption (for obtaining not only well-posedness of the associated evolution problem, for example, but also existence and regularity of weak solutions to associated elliptic problems).

The following result is a partial converse of theorem 3.2.

THEOREM 3.3. *Assume that $B\,\mathrm{d}\mu$ is absolutely continuous with respect to the p-capacity in the sense that, for every p-polar set $K \subseteq \partial\Omega$ and every $u \in D(\varphi)$, one has*

$$\int_K B(x,u)\,\mathrm{d}\mu = 0. \tag{3.6}$$

If the functional φ given by (3.4) is convex, φ is lower semicontinuous on $L^2(\Omega)$.

Proof. We have to show that for every $c \in \mathbb{R}$ the set $\{\varphi \leqslant c\}$ is closed in $L^2(\Omega)$. So, fix $c \in \mathbb{R}$. Let $\mathcal{A}$ be a closed (bounded) ball in $L^2(\Omega)$ and let $\mathcal{C} := \{\varphi \leqslant c\} \cap \mathcal{A}$. Let $(u_n) \subseteq \mathcal{C}$ and $u \in W^{1,p}(\Omega) \cap L^2(\Omega)$ be such that

$$\lim_{n\to\infty} \|u_n - u\|_{W^{1,p}(\Omega)\cap L^2(\Omega)} = 0.$$

The convergence in $L^2(\Omega)$ implies that $u \in \mathcal{A}$. The convergence in $W^{1,p}(\Omega)$ implies, after passing to a subsequence, that $u_n \to u$ p-quasi-everywhere, i.e. $u_n \to u$ everywhere except possibly on a p-polar set $K \subseteq \partial\Omega$. Hence, by assumption on B and μ,

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}\int_\Omega |\nabla u|^p + \int_{\partial\Omega} B(x,u)\,\mathrm{d}\mu \\
&= \frac{1}{p}\int_\Omega |\nabla u|^p + \int_{\partial\Omega\setminus K} B(x,u)\,\mathrm{d}\mu && \text{(by (3.6))} \\
&\leqslant \frac{1}{p}\int_\Omega |\nabla u|^p + \int_{\partial\Omega\setminus K} \liminf_{n\to\infty} B(x,u_n)\,\mathrm{d}\mu && \text{(B is l.s.c.)} \\
&\leqslant \liminf_{n\to\infty}\left(\frac{1}{p}\int_\Omega |\nabla u_n|^p + \int_{\partial\Omega\setminus K} B(x,u_n)\,\mathrm{d}\mu\right) && \text{(by Fatou's lemma)} \\
&= \liminf_{n\to\infty}\left(\frac{1}{p}\int_\Omega |\nabla u_n|^p + \int_{\partial\Omega} B(x,u_n)\,\mathrm{d}\mu\right) && \left(\text{as } \int_K B(x,u_n)\,\mathrm{d}\mu = 0\right) \\
&= \liminf_{n\to\infty} \varphi(u_n).
\end{aligned}$$

This shows $u \in \{\varphi \leqslant c\}$. In particular, we have shown that $\mathcal{C}$ is closed in $W^{1,p}(\Omega)\cap L^2(\Omega)$. By convexity of φ, the set $\mathcal{C}$ is, in addition, convex. Hence, by Mazur's theorem, $\mathcal{C}$ is weakly closed in $W^{1,p}(\Omega)\cap L^2(\Omega)$. Next, since B is non-negative, the norm $u \mapsto \|\nabla u\|_{L^p(\Omega)} + \|u\|_{L^2(\Omega)}$ is bounded on $\mathcal{C}$. This norm is equivalent to the canonical norm on $W^{1,p}(\Omega)\cap L^2(\Omega)$: for $p \leqslant 2$, this is always true, while for $p > 2$ we may use that Ω has a Lipschitz boundary and apply the theorem in [23, § 1.1]. Since the embedding $W^{1,p}(\Omega)\cap L^2(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, we obtain that $\mathcal{C}$ is bounded and weakly closed, and hence closed, in $L^2(\Omega)$. Since $\mathcal{A}$ was an arbitrary closed ball in $L^2(\Omega)$, this shows that $\{\varphi \leqslant c\}$ is closed in $L^2(\Omega)$. □

4. Proof of theorem 2.3

In this section, we prove theorem 2.3. We start by proving the implication in theorem 2.3, which is relatively straightforward.

Proof of theorem 2.3, (ii) ⇒ (i). Assume that assertion (ii) holds. The monotonicity of the function $B(x,\cdot)$ (for μ-almost every $x \in \bar{\Omega}$, assumption (H^+)) and the monotonicity of the integral imply that the functional ψ is monotone.

We show that ψ is lower semicontinuous. Let $(u_n) \subseteq W^{1,p}(\Omega)^+$ be a sequence which converges to $u \in W^{1,p}(\Omega)^+$. By considering a subsequence, if necessary, we may assume that (u_n) converges to u p-quasi-everywhere, that is, there exists a p-polar set $A \subseteq \bar{\Omega}$ such that (u_n) converges to u everywhere on $\bar{\Omega}\setminus A$ (where possibly A is a larger p-polar set). Since, for μ-almost every $x \in \bar{\Omega}$, the function $B(x,\cdot)$ is lower semicontinuous, we obtain $B(x,u(x)) \leqslant \liminf_{n\to\infty} B(x,u_n(x))$ for every $x \in \bar{\Omega}\setminus A$. Using Fatou's lemma and the fact that, by assumption, the measure μ is absolutely continuous with respect to Cap_p (this implies that (3.5) holds for every p-polar set $K \subseteq \bar{\Omega}$ and every $u \in D(\psi)$), we therefore obtain that

$$\begin{aligned}
\psi(u) &= \int_{\bar{\Omega}} B(x,u)\,\mathrm{d}\mu = \int_{\bar{\Omega}\setminus A} B(x,u)\,\mathrm{d}\mu \\
&\leqslant \int_{\bar{\Omega}\setminus A} \liminf_{n\to\infty} B(x,u_n(x))\,\mathrm{d}\mu
\end{aligned}$$

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$$\leqslant \liminf_{n\to\infty} \int_{\bar{\Omega}\setminus A} B(x, u_n(x))\, \mathrm{d}\mu$$
$$= \liminf_{n\to\infty} \int_{\bar{\Omega}} B(x, u_n(x))\, \mathrm{d}\mu = \liminf_{n\to\infty} \psi(u_n).$$

Hence, ψ is lower semicontinuous.

Let $u, v \in D(\psi) \subseteq W^{1,p}(\Omega)^+$. Clearly, $u \vee v, u \wedge v \in W^{1,p}(\Omega)^+$. From the equality

$$\begin{aligned}\psi(u\vee v)+\psi(u\wedge v) &= \int_{\bar{\Omega}} B(x, u\vee v)\, \mathrm{d}\mu + \int_{\bar{\Omega}} B(x, u\wedge v)\, \mathrm{d}\mu \\ &= \int_{\{u\leqslant v\}} B(x, v)\, \mathrm{d}\mu + \int_{\{u\leqslant v\}} B(x, u)\, \mathrm{d}\mu \\ &\quad + \int_{\{u>v\}} B(x, u)\, \mathrm{d}\mu + \int_{\{u>v\}} B(x, v)\, \mathrm{d}\mu \\ &= \int_{\bar{\Omega}} B(x, u)\, \mathrm{d}\mu + \int_{\bar{\Omega}} B(x, v)\, \mathrm{d}\mu \\ &= \psi(u) + \psi(v)\end{aligned}$$

we obtain that $u \vee v, u \wedge v \in D(\psi)$ and that (2.8) holds (even with equality).

Finally, let $u, v \in W^{1,p}(\Omega)^+$ be such that $u \wedge v = 0$. Then $u = v = 0$ on $\operatorname{supp}[u] \cap \operatorname{supp}[v]$. Since $B(x, 0) = 0$, we obtain that

$$\begin{aligned}\psi(u+v) &= \int_{\bar{\Omega}} B(x, u+v)\, \mathrm{d}\mu \\ &= \int_{\operatorname{supp}[u]} B(x, u)\, \mathrm{d}\mu + \int_{\operatorname{supp}[v]} B(x, v)\, \mathrm{d}\mu - \int_{\operatorname{supp}[u]\cap\operatorname{supp}[v]} B(x, u+v)\, \mathrm{d}\mu \\ &= \int_{\operatorname{supp}[u]} B(x, u)\, \mathrm{d}\mu + \int_{\operatorname{supp}[v]} B(x, v)\, \mathrm{d}\mu \\ &= \int_{\bar{\Omega}} B(x, u)\, \mathrm{d}\mu + \int_{\bar{\Omega}} B(x, v)\, \mathrm{d}\mu \\ &= \psi(u) + \psi(v).\end{aligned}$$

Hence, ψ is local. □

To prove the converse implication (i) ⇒ (ii), we proceed stepwise, in the form of several lemmas.

Throughout the following, we denote by $\mathcal{B}$ the Borel σ-algebra of $\bar{\Omega}$. The set of all compact subsets of $\bar{\Omega}$ is denoted by $\mathcal{K}$. We assume also that the functional ψ satisfies theorem 2.3(i).

For $\delta > 0$ and every subset $K \subseteq \bar{\Omega}$ we define

$$K^{\delta} := \{x \in \bar{\Omega} \colon d(x, K) \leqslant \delta\}.$$

With this definition, for every compact subset $K \in \mathcal{K}$ we define

$$\mathcal{R}(K) := \{\varrho \in W^{1,\infty}(\mathbb{R}^N) \colon \text{there exists } \delta > 0 \text{ such that } \varrho \geqslant 1 \text{ on } K^{\delta}\}.$$

Note that every function $\varrho \in W^{1,\infty}(\mathbb{R}^N)$ admits a Lipschitz continuous representative; the condition $\varrho \geqslant 1$ is to be understood as a pointwise inequality everywhere for this unique representative.

Definition 4.1. For each $u \in D(\psi)$ we define a non-negative set function $\mu_u(\cdot)$ on $\mathcal{K}$ by setting

$$\mu_u(K) = \inf_{\varrho \in \mathcal{R}(K)} \psi(u\varrho), \quad K \in \mathcal{K}. \tag{4.1}$$

We remark that for every $\varrho \in \mathcal{R}(K)$ one has $\varrho \wedge 1 \in \mathcal{R}(K)$. Therefore, in the definition of $\mu_u(K)$ it suffices to take the infimum over all functions $\varrho \in \mathcal{R}(K)$ satisfying $0 \leqslant \varrho \leqslant 1$ everywhere and $\varrho = 1$ on some K^δ. In particular, by the monotonicity of ψ, $\mu_u(K)$ is finite for every compact $K \subseteq \bar{\Omega}$.

Lemma 4.2 (finite additivity). *Let $u \in D(\psi)$, and let $K_1, K_2 \subseteq \bar{\Omega}$ be two compact sets such that $K_1 \cap K_2 = \emptyset$. Then*

$$\mu_u(K_1 \cup K_2) = \mu_u(K_1) + \mu_u(K_2).$$

Proof. Let $K_1, K_2 \subseteq \bar{\Omega}$ be two compact sets such that $K_1 \cap K_2 = \emptyset$. Then $d(K_1, K_2) > 0$. We can therefore find two functions $\varrho_i \in \mathcal{R}(K_i)$, $i = 1, 2$, such that $0 \leqslant \varrho_i \leqslant 1$ and $\varrho_1 \wedge \varrho_2 = 0$; such a pair of functions ϱ_1, ϱ_2 can easily be constructed by taking appropriate convolutions of characteristic functions and test functions.

Let $\varrho \in \mathcal{R}(K_1 \cup K_2)$. The monotonicity and locality of ψ implies

$$\begin{aligned}
\psi(u\varrho) &\geqslant \psi(u\varrho(\varrho_1 + \varrho_2)) \\
&= \psi(u\varrho\varrho_1) + \psi(u\varrho\varrho_2) \\
&\geqslant \mu_u(K_1) + \mu_u(K_2).
\end{aligned}$$

Since $\varrho \in \mathcal{R}(K_1 \cup K_2)$ was arbitrary, this implies $\mu_u(K_1 \cup K_2) \geqslant \mu_u(K_1) + \mu_u(K_2)$.

Now let $\varrho_i' \in \mathcal{R}(K_i)$, $i = 1, 2$. Then, again, by monotonicity and locality,

$$\begin{aligned}
\psi(u\varrho_1') + \psi(u\varrho_2') &\geqslant \psi(u\varrho_1'\varrho_1) + \psi(u\varrho_2'\varrho_2) \\
&= \psi(u(\varrho_1'\varrho_1 + \varrho_2'\varrho_2)) \\
&\geqslant \mu_u(K_1 \cup K_2).
\end{aligned}$$

Since $\varrho_i' \in \mathcal{R}(K_i)$, $i = 1, 2$, were arbitrary, this implies

$$\mu_u(K_1) + \mu_u(K_2) \geqslant \mu_u(K_1 \cup K_2).$$

□

Lemma 4.3 (monotonicity). *Let $u \in D(\psi)$ and let $K_1, K_2 \subseteq \bar{\Omega}$ be two compact sets such that $K_1 \subseteq K_2$. Then*

$$\mu_u(K_1) \leqslant \mu_u(K_2).$$

Proof. This follows immediately from the definition of μ_u and the inclusion

$$\mathcal{R}(K_1) \supseteq \mathcal{R}(K_2).$$

□

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Lemma 4.4. *Let $u \in D(\psi)$ and let $K_1, K_2 \subseteq \bar{\Omega}$ be two compact sets. Then*

$$\mu_u(K_1 \cup K_2) + \mu_u(K_1 \cap K_2) \leqslant \mu_u(K_1) + \mu_u(K_2).$$

Proof. Let $\varrho_i \in \mathcal{R}(K_i)$ $(i = 1, 2)$. Then, by assumption (2.8),

$$\begin{aligned}\psi(u\varrho_1) + \psi(u\varrho_2) &\geqslant \psi(u(\varrho_1 \vee \varrho_2)) + \psi(u(\varrho_1 \wedge \varrho_2)) \\ &\geqslant \mu_u(K_1 \cup K_2) + \mu_u(K_1 \cap K_2).\end{aligned}$$

Since $\varrho_i \in \mathcal{R}(K_i)$, $i = 1, 2$, were arbitrary, this implies

$$\mu_u(K_1) + \mu_u(K_2) \geqslant \mu_u(K_1 \cup K_2) + \mu_u(K_1 \cap K_2).$$

□

Lemma 4.5 (outer regularity). *Let $u \in D(\psi)$, let (K_m) be a decreasing sequence of compact subsets of $\bar{\Omega}$, and let $K := \bigcap_m K_m$. Then*

$$\lim_{m\to\infty} \mu_u(K_m) = \mu_u(K). \tag{4.2}$$

Proof. First, the monotonicity of the set function μ_u (lemma 4.3) implies that $\lim_{m\to\infty} \mu_u(K_m)$ exists and $\lim_{m\to\infty} \mu_u(K_m) \geqslant \mu_u(K)$. In order to prove the converse inequality, observe that for every $\delta, \delta' > 0$ with $\delta > \delta'$ there exists $m_0 \in \mathbb{N}$ such that for every $m \geqslant m_0$ one has $K_m^{\delta'} \subseteq K^{\delta}$ (here we use that (K_m) is decreasing and $K = \bigcap_m K_m$). In particular, for every $\varrho \in \mathcal{R}(K)$ there exists $m_0 \in \mathbb{N}$ such that for every $m \geqslant m_0$ one has $\varrho \in \mathcal{R}(K_m)$. As a consequence

$$\psi(u\varrho) \geqslant \mu_u(K_m) \quad \text{for every } m \geqslant m_0$$

or

$$\psi(u\varrho) \geqslant \lim_{m\to\infty} \mu_u(K_m).$$

Since this inequality holds for every $\varrho \in \mathcal{R}(K)$, this proves

$$\mu_u(K) \geqslant \lim_{m\to\infty} \mu_u(K_m).$$

□

Lemma 4.6. *For every $u \in D(\psi)$ the set function μ_u can be extended uniquely to a finite, regular Borel measure on $\bar{\Omega}$ (again denoted by μ_u in the following). Moreover, $\mu_u(\bar{\Omega}) = \psi(u)$ and $\operatorname{supp}[\mu_u] \subseteq \operatorname{supp}[\psi]$.*

Proof. By lemmas 4.2–4.5, μ_u is a regular content on $\mathcal{K}$. The fact that μ_u extends to a regular Borel measure (which we again denote by μ_u) follows from standard measure theory, including the theory of measures on topological spaces (see, for example, [4, Kapitel I, § 3; Kapitel IV]). The set $\bar{\Omega}$ is compact and every function in $\mathcal{R}(\bar{\Omega})$ is greater than or equal to 1 on $\bar{\Omega}$. From here and the definition of μ_u it follows easily that $\mu_u(\bar{\Omega}) = \psi(u) < +\infty$ for every $u \in D(\psi)$. The inclusion $\operatorname{supp}[\mu_u] \subseteq \operatorname{supp}[\psi]$ is a straightforward consequence of the definition of μ_u and the definition of the effective support of ψ. □

Lemma 4.7 (monotonicity of μ_u). *Let $u, v \in D(\psi)$ be such that $u \leqslant v$. Then $\mu_u \leqslant \mu_v$.*

Proof. The monotonicity of ψ and the definition of the measures μ_u and μ_v imply $\mu_u(K) \leqslant \mu_v(K)$ for every compact subset $K \subseteq \bar{\Omega}$. The claim then follows from the inner regularity of μ_u and μ_v. □

Lemma 4.8. *Let $(u_n) \subseteq D(\psi)$ and $u \in D(\psi)$ be such that $u_n \leqslant u$ and $\lim_{n\to\infty} u_n = u$ in $W^{1,p}(\Omega)$. Then*

$$\lim_{n\to\infty} \mu_{u_n}(G) = \mu_u(G) \quad \textit{for every } G \in \mathcal{B}. \tag{4.3}$$

Proof. By lemma 4.7, the domination $u_n \leqslant u$ implies $\mu_u(G) - \mu_{u_n}(G) \geqslant 0$ for every $G \in \mathcal{B}$. Hence, for every $G \in \mathcal{B}$,

$$\begin{aligned} 0 &\leqslant \limsup_{n\to\infty}(\mu_u(G) - \mu_{u_n}(G)) \\ &\leqslant \limsup_{n\to\infty}(\mu_u(G) - \mu_{u_n}(G) + \mu_u(G^c) - \mu_{u_n}(G^c)) \\ &= \limsup_{n\to\infty}(\mu_u(\bar{\Omega}) - \mu_{u_n}(\bar{\Omega})) \\ &= \limsup_{n\to\infty}(\psi(u) - \psi(u_n)) \\ &\leqslant 0, \end{aligned}$$

where in the last inequality we have used the lower semi-continuity of ψ. The preceding chain of inequalities implies the claim. □

Lemma 4.9. *For every $u \in D(\psi)$ the measure μ_u is absolutely continuous with respect to the p-capacity.*

Proof. Let $K \subseteq \bar{\Omega}$ be a p-polar set, that is, $\mathrm{Cap}_p(K) = 0$. We have to show that $\mu_u(K) = 0$. By inner regularity of μ_u, we may assume K to be compact. Moreover, by replacing u by $u \wedge n \in D(\psi)$ (with $n \in \mathbb{N}$ large enough), and by using lemma 4.8, we see that we may in addition assume that u is essentially bounded. We assume both in the following.

Since $\mathrm{Cap}_p(K) = 0$, there exists a sequence (O_n) of open sets in $\mathbb{R}^N$ and a sequence $(w_n) \subseteq W^{1,p}(\mathbb{R}^N) \cap C_c(\mathbb{R}^N)$ such that

$$0 \leqslant w_n \leqslant 1, \quad w_n = 1 \text{ on } O_n \supseteq K \quad \text{and} \quad \lim_{n\to\infty} \|w_n\|_{W^{1,p}(\mathbb{R}^N)} = 0.$$

We claim that we can choose (O_n) and (w_n) such that

$$K \subseteq O_{n+1} \subseteq \bar{O}_{n+1} \subseteq O_n, \quad K = \bigcap_n O_n \quad \text{and} \quad \mathrm{supp}[w_{n+1}] \subseteq O_n.$$

First, observe that by replacing O_n by the smaller set $O_1 \cap \cdots \cap O_n$, we may assume that the sequence (O_n) is decreasing. Clearly, $K \subseteq \bigcap_n O_n$. If there exists $x \in (\bigcap_n O_n) \setminus K$, then we may replace O_n by the smaller open set $O_n \setminus \overline{B(x,r)}$, where $r > 0$ is sufficiently small so that one still has $K \subseteq O_n \setminus \overline{B(x,r)}$ (recall that K is compact and that x has positive distance to K). In this way we can eliminate every $x \in (\bigcap_n O_n) \setminus K$ and finally obtain $K = \bigcap_n O_n$. By using the facts that

K is compact and that the sets O_n are open, it is straightforward to construct a subsequence of (O_n) (which we still denote by O_n) such that $K \subseteq O_{n+1} \subseteq \bar{O}_{n+1} \subseteq O_n$. Now let $(z_k) \subseteq W^{1,\infty}(\mathbb{R}^N)$ be a sequence of functions satisfying $0 \leqslant z_k \leqslant 1$, $z_k = 1$ on O_{k+1} and $\operatorname{supp}[z_k] \subseteq O_k$. For every k we can find $m_k \geqslant k+1$ such that, for every $m \geqslant m_k$,

$$\|z_k w_m\|_{W^{1,p}(\mathbb{R}^N)} \leqslant \frac{1}{k}.$$

Note that, for every $m \geqslant m_k$, $0 \leqslant z_k w_m \leqslant 1$, $z_k w_m = 1$ on O_m and $\operatorname{supp}[z_k w_m] \subseteq O_k$. Now, by replacing w_n by $z_{k_n} w_{m_n}$ for some appropriate sequences (k_n), (m_n) (so that $k_{n+1} \geqslant m_n$), we obtain the desired claim.

By passing to a further subsequence, if necessary, we may finally assume that (w_n) and (∇w_n) converge pointwise almost everywhere to 0. Then it is straightforward to check that

$$\lim_{n\to\infty} \|u w_n\|_{W^{1,p}(\Omega)} = 0.$$

Moreover, for every $n \in \mathbb{N}$,

$$\begin{aligned} \psi(u) &\geqslant \psi(u - uw_n + uv_{n+1}) && \text{(by monotonicity of } \psi) \\ &= \psi(u - uw_n) + \psi(uv_{n+1}) && \text{(by locality of } \psi) \\ &\geqslant \psi(u - uw_n) + \mu_u(K) && \text{(by definition of } \mu_u). \end{aligned} \tag{4.4}$$

Since $\lim_{n\to\infty}(u - uw_n) = u$ and $u - uw_n \leqslant u$, the monotonicity and the lower semi-continuity of ψ imply

$$\lim_{n\to\infty} \psi(u - uw_n) = \psi(u).$$

Hence, by passing to the limit in the inequality (4.4), we obtain $\psi(u) \geqslant \psi(u) + \mu_u(K)$, that is, $\mu_u(K) = 0$. Since K was an arbitrary p-polar set, this shows that μ_u is absolutely continuous with respect to the p-capacity. □

LEMMA 4.10. *For every $u \in D(\psi)$ one has $\mu_u(\{u = 0\}) = 0$ (here $\{u = 0\}$ denotes the null-set of a p-quasi-continuous representative of u; it is unique up to a p-polar set).*

Proof. Let $u \in D(\psi)$. By replacing u by $u \wedge n \in D(\psi)$ (for $n \in \mathbb{N}$ sufficiently large) and by using lemma 4.8, we see that we may assume that u is essentially bounded. This will be done in the following.

We show that we can find a sequence $(u_n) \subseteq D(\psi)$ such that $u_n \leqslant u$,

$$\lim_{n\to\infty} u_n = u \quad \text{in } W^{1,p}(\Omega)$$

and $u_n = 0$ in a neighbourhood of $\{u = 0\}$. In fact, for every $n \in \mathbb{N}$ the set $U_n := \{u < 1/n\}$ is p-quasi-open, that is, there exists a sequence (O_j) of open sets such that $U_n \cup O_j$ is open and $\lim_{j\to\infty} \operatorname{Cap}_p(O_j) = 0$. Associated with (O_j) there exists a sequence $(w_j) \subseteq W^{1,p}(\mathbb{R}^N)$ such that $0 \leqslant w_j \leqslant 1$, $w_j = 1$ on O_j and

$$\lim_{j\to\infty} \|w_j\|_{W^{1,p}(\mathbb{R}^N)} = 0.$$

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Note that $\lim_{j\to\infty}\|(u-1/n)^+w_j\|_{W^{1,p}(\mathbb{R}^N)}=0$ for every $n\in\mathbb{N}$. Now, for $n\in\mathbb{N}$, one may take $u_n=(u-1/n)^+(1-w_{j_n})$ with $j_n\in\mathbb{N}$ large enough and one obtains the desired sequence.

Let $K\subseteq\{u=0\}$ be a compact set. Since $u_n=0$ in a neighbourhood of $\{u=0\}$, it follows that $K\subseteq\bar{\Omega}\backslash\operatorname{supp}[u_n]$. By definition of μ_{u_n}, since $\bar{\Omega}\backslash\operatorname{supp}[u_n]$ is relatively open (i.e. open with respect to the relative topology), and since $\psi(0)=0$, one has $\mu_{u_n}(K)=0$. By lemma 4.8, $\mu_u(K)=\lim_{n\to\infty}\mu_{u_n}(K)=0$. The claim then follows from the inner regularity of μ_u. □

LEMMA 4.11. *Let $u,v\in D(\psi)$. Then $\mu_u(G)\leqslant\mu_v(G)$ for every Borel set*

$$G\subseteq\{u\leqslant v\}.$$

Proof. The proof is given in three steps.

STEP 1. We first prove the inequality $\mu_u(G)\leqslant\mu_v(G)$ for every Borel set $G\subseteq\{u<v\}$. By p-quasi-continuity of u and v, the set $U:=\{u<v\}$ is quasi-open. For every $n\in\mathbb{N}$, let O_n be an open set with $\operatorname{Cap}_p(O_n)<1/n$, and $U_n:=U\cup O_n$ is open. Associated with (O_n) there exists a sequence $(w_n)\subseteq W^{1,p}(\mathbb{R}^N)$ such that $0\leqslant w_n\leqslant 1$, $w_n=1$ p-quasi-everywhere on O_n and

$$\lim_{n\to\infty}\|w_n\|_{W^{1,p}(\mathbb{R}^N)}=0.$$

Let $u_n:=(u\wedge n)(1-w_n)$ and $v_n:=(v\wedge n)(1-w_n)$. Then $u_n\leqslant v_n$ on the open set U_n with $u_n=v_n=0$ p-quasi-everywhere on O_n. Moreover, $u_n\leqslant u$ and $v_n\leqslant v$. Since U_n is open, for every compact set $K\subseteq U\subseteq U_n$ there exists $\delta>0$ such that $K^\delta\subseteq U_n$. It follows from the monotonicity of ψ and the definition of the measures μ_{u_n} and μ_{v_n} that $\mu_{u_n}(K)\leqslant\mu_{v_n}(K)$ for every compact subset $K\subseteq U$. Passing to the limit and using lemma 4.8, we obtain that $\mu_u(K)\leqslant\mu_v(K)$ for every compact set $K\subseteq\{u<v\}$. The inequality $\mu_u(G)\leqslant\mu_v(G)$ for Borel sets $G\subseteq\{u<v\}$ then follows from the inner regularity of μ_u and μ_v.

STEP 2. Now let G be a Borel set in $\{u\leqslant v \text{ and } 0<v\}$. Let $(\lambda_n)\subseteq\mathbb{R}_+$ be a sequence such that $\lambda_n<1$ and $\lim_{n\to\infty}\lambda_n=1$. Then, for every n one has

$$\{u\leqslant v \text{ and } 0<v\}\subseteq\{\lambda_n u<v\}.$$

Hence, by step 1, for every n,

$$\mu_{\lambda_n u}(G)\leqslant\mu_v(G).$$

Clearly, $\lambda_n u\leqslant u$ and $\lim_{n\to\infty}\lambda_n u=u$ in $W^{1,p}(\Omega)$. Hence, by lemma 4.8,

$$\mu_u(G)=\lim_{n\to\infty}\mu_{\lambda_n u}(G)\leqslant\mu_v(G).$$

STEP 3. Finally, let G be an arbitrary Borel set in $\{u\leqslant v\}$. Then, by step 2 and lemma 4.10,

$$\begin{aligned}\mu_u(G)&=\mu_u(G\cap\{0<v\})+\mu_u(G\cap\{0=v\})\\&\leqslant\mu_v(G\cap\{0<v\})+\mu_v(G\cap\{0=v\})\\&=\mu_v(G),\end{aligned}$$

which is the claim. □

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In the remainder of this section, fix a sequence $(w_n) \subseteq D(\psi)$ such that $\{w_n : n\}$ is dense in $D(\psi)$ (such a sequence exists since the space $W^{1,p}(\Omega)$ is separable). Then we define a measure μ on $\mathcal{B}$ by

$$\mu(G) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\mu_{w_n}(G)}{1 + \mu_{w_n}(\bar{\Omega})}, \quad G \in \mathcal{B}. \tag{4.5}$$

It is clear that μ is a finite, regular Borel measure on $\bar{\Omega}$.

LEMMA 4.12. *The measure μ is absolutely continuous with respect to the p-capacity.*

Proof. This follows directly from lemma 4.9 and the definition of the measure μ. □

LEMMA 4.13. *Every measure μ_u, $u \in D(\psi)$, is absolutely continuous with respect to μ.*

Proof. Let $G \in \mathcal{B}$ be such that $\mu(G) = 0$. It follows from the definition of μ and the positivity of μ_{w_n} that $\mu_{w_n}(G) = 0$ for every $n \in \mathbb{N}$.

Now let $u \in D(\psi)$. There exists a subsequence (w_{n_k}) such that $\lim_{k\to\infty} w_{n_k} = u$ in $W^{1,p}(\Omega)$. Define $u_k := w_{n_k} \wedge u$. Then $u_k \leqslant w_{n_k}$ and lemma 4.7 implies that $\mu_{u_k}(G) = 0$. Moreover, $u_k \leqslant u$ and $\lim_{k\to\infty} u_k = u$ in $W^{1,p}(\Omega)$. From this and lemma 4.8 we obtain

$$\mu_u(G) = \lim_{k\to\infty} \mu_{u_k}(G) = 0.$$

Hence, μ_u is absolutely continuous with respect to μ. □

By the preceding lemma and by the Radon–Nikodým theorem, for every $u \in D(\psi)$ there exists a non-negative Borel measurable function $B_u = B_u(x)$ such that

$$\mu_u(G) = \int_G B_u(x)\,\mathrm{d}\mu(x), \quad G \in \mathcal{B}.$$

LEMMA 4.14. *Let u_n, $u, v \in D(\psi)$. Then*

(a) *if $u_n \leqslant u$ and $\lim_{n\to\infty} u_n = u$ in $W^{1,p}(\Omega)$, then $\lim_{n\to\infty} B_{u_n} = B_u$ μ-almost everywhere,*

(b) *$B_u = 0$ μ-almost everywhere on $\{u = 0\}$,*

(c) *$B_u \leqslant B_v$ μ-almost everywhere on $\{u \leqslant v\}$.*

Proof. This lemma is an immediate consequence of lemmas 4.8, 4.10 and 4.11. □

Recall that we identify each function $w_n \in D(\varphi)$ with a p-quasi-continuous representative; note that we may assume that this representative is non-negative everywhere. For every $x \in \bar{\Omega}$ we define the set $W(x) := \{w_n(x) : n \in \mathbb{N}\}$. Let $I(x)$ be the closed convex hull of $W(x)$, that is, the smallest, closed interval which contains $W(x)$. Then, for every $x \in \bar{\Omega}$ and every $s \in \mathbb{R}_+$ we define

$$B(x,s) = \begin{cases} \sup_n B_{w_n}(x) 1_{\{w_n < s\}}(x) & \text{if } s \in I(x), \\ +\infty & \text{if } s \notin I(x). \end{cases}$$

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Lemma 4.15. *The function $B\colon \bar{\Omega}\times\mathbb{R}_+\to[0,+\infty]$ defined above satisfies hypothesis (H$^+$). Moreover, for every $u\in D(\psi)$ one has $B(\cdot,u(\cdot)) = B_u(\cdot)$ μ-almost everywhere on $\bar{\Omega}$.*

Proof. For every $s\in\mathbb{R}_+$ the set

$$\{x\in\bar{\Omega}\colon s\in I(x)\} = \Big\{x\in\bar{\Omega}\colon s\leqslant \sup_n w_n(x)\Big\}$$

is a Borel set. From this and from the definition of B one obtains that for every $s\in\mathbb{R}_+$ the function $B(\cdot,s)$ is measurable.

It follows readily from the definition of B that $B(x,0)=0$ for every $x\in\bar{\Omega}$ (since the sets $\{w_n<0\}$ are empty and therefore $1_{\{w_n<0\}}=0$ for every n). Moreover, since the sets $\{w_n<s\}$ are increasing with $s\in\mathbb{R}_+$, the function $B(x,\cdot)$ is monotone for every $x\in\bar{\Omega}$. Finally, for every $x\in\bar{\Omega}$, every $s\in I(x)\setminus\{0\}$ and every $\varepsilon>0$ there exists, by definition of the supremum, n such that $w_n(x)<s$ and $B(x,s)-\varepsilon\leqslant B_{w_n}(x)\leqslant B(x,s)$. This implies $B(x,s)-\varepsilon\leqslant B(x,s')\leqslant B(x,s)$ for every $w_n(x)<s'\leqslant s$. As a consequence, $B(x,\cdot)$ is lower semicontinuous for every $x\in\bar{\Omega}$. Thus, B satisfies hypothesis (H$^+$).

Next, we show the second part of the statement. Let $u\in D(\psi)$. By lemma 4.14(c), there exists a set A_u of μ-measure zero such that for every n and every $x\in\{w_n\leqslant u\}\setminus A_u$ one has $B_{w_n}(x)\leqslant B_u(x)$. As a consequence, $B(\cdot,u(\cdot))\leqslant B_u(\cdot)$ μ-almost everywhere on $\bar{\Omega}$. In order to show the converse inequality, we first note that, by lemma 4.14(b), $B_u(\cdot)=0=B(x,0)$ μ-almost everywhere on $\{u=0\}$. Hence, it remains to show that the inequality $B(\cdot,u(\cdot))\geqslant B_u(\cdot)$ holds almost everywhere on $\{u>0\}$.

Let $(\lambda_m)\subseteq[0,1[$ be a strictly increasing sequence such that $\lim_{m\to\infty}\lambda_m=1$. Recall that the sequence (w_n) is dense in the effective domain $D(\psi)$ (with respect to the $W^{1,p}(\Omega)$ topology) and that every sequence converging in $W^{1,p}(\Omega)$ admits a subsequence that converges p-quasi-everywhere. Hence, for every m there exists a subsequence $(w_{\alpha_m(n)})_n$ such that $\lim_{n\to\infty} w_{\alpha_m(n)}=\lambda_m u$ p-quasi-everywhere. In particular,

$$u(x) > \lim_{n\to\infty} w_{\alpha_{m+1}(n)} = \lambda_{m+1}u > \lambda_m u \quad p\text{-quasi-everywhere on } \{u>0\}.$$

Since μ is absolutely continuous with respect to the p-capacity, the above convergence holds μ-almost everywhere. As a consequence, by lemma 4.14(c), for every m,

$$\begin{aligned} B(x,u(x)) &= \sup_{w_n(x)<u(x)} B_{w_n}(x)\\ &\geqslant \limsup_{n\to\infty} B_{w_{\alpha_{m+1}(n)}}(x)\\ &\geqslant B_{\lambda_m u}(x) \quad \mu\text{-almost everywhere on } \{u>0\}.\end{aligned}$$

Since $B_{\lambda_m u}\to B_u$ μ-almost everywhere on $\bar{\Omega}$ by lemma 4.14(a), we thus obtain the remaining inequality $B_u(\cdot)\leqslant B(\cdot,u(\cdot))$ μ-almost everywhere on $\{u>0\}$. □

Proof of theorem 2.3, (i) ⇒ (ii). Let $(w_n)\subseteq D(\psi)$ be dense in $D(\psi)$, let μ be the Borel measure and let $B\colon\bar{\Omega}\times\mathbb{R}\to[0,+\infty]$ be the function defined above. It follows from lemma 4.6, that $\operatorname{supp}[\mu]\subseteq\operatorname{supp}[\psi]$. By lemma 4.12, μ is absolutely continuous with respect to the p-capacity. By lemma 4.15, the function B satisfies hypothesis (H$^+$). Now, by lemma 4.15, for every $u \in D(\psi)$ we have $B(\cdot, u(\cdot)) = B_u(\cdot)$ μ-almost everywhere on $\bar{\Omega}$. By the definition of B_u this means

$$\psi(u) = \int_{\bar{\Omega}} B_u(x)\,\mathrm{d}\mu(x) = \int_{\bar{\Omega}} B(x, u(x))\,\mathrm{d}\mu(x) \quad \text{for every } u \in D(\psi).$$

□

The proof of Theorem 2.3 is now complete.

5. Proof of theorem 2.1

In this section we give the proof of theorem 2.1. We call a functional $\psi\colon W^{1,p}(\Omega) \to]-\infty, \infty]$ *bi-monotone* if, for every $u, v \in D(\psi)$,

$$\left.\begin{aligned} 0 \leqslant u \leqslant v \quad &\Longrightarrow \quad \psi(u) \leqslant \psi(v), \\ u \leqslant v \leqslant 0 \quad &\Longrightarrow \quad \psi(u) \geqslant \psi(v). \end{aligned}\right\} \tag{5.1}$$

Proof of theorem 2.1, (ii) ⇒ (i). Let φ, μ and B be as in (ii).

Since $B(x, 0) = 0$ for μ-a.e. $x \in \partial\Omega$, then it is clear that the functional φ is local and hence, S is a local semigroup.

We show that S is order preserving. Since the functional φ is non-negative, by theorem 2.2(b), it suffices to show that φ satisfies (2.4) for every $u, v \in L^2(\Omega)$. Since the inequality (2.4) trivially holds if u or v does not belong to $D(\varphi)$, we may assume that $u, v \in D(\varphi)$. Then, by Stampacchia's lemma,

$$\begin{aligned} \int_\Omega |\nabla(u \vee v)|^p\,\mathrm{d}x &+ \int_\Omega |\nabla(u \wedge v)|^p\,\mathrm{d}x \\ &= \int_{\{u<v\}} (|\nabla v|^p + |\nabla u|^p)\,\mathrm{d}x + \int_{\{u>v\}} (|\nabla u|^p + |\nabla v|^p)\,\mathrm{d}x \\ &\quad + \int_{\{u=v\}} (|\nabla v|^p + |\nabla u|^p)\,\mathrm{d}x \\ &= \int_\Omega (|\nabla u|^p + |\nabla v|^p)\,\mathrm{d}x \end{aligned} \tag{5.2}$$

and

$$\begin{aligned} \int_{\partial\Omega} B(x, u \vee v)\,\mathrm{d}\mu &+ \int_{\partial\Omega} B(x, u \wedge v)\,\mathrm{d}\mu \\ &= \int_{\{u<v\}} (B(x, v) + B(x, u))\,\mathrm{d}\mu + \int_{\{u>v\}} (B(x, u) + B(x, v))\,\mathrm{d}\mu \\ &\quad + \int_{\{u=v\}} (B(x, u) + B(x, v))\,\mathrm{d}\mu \\ &= \int_{\partial\Omega} (B(x, u) + B(x, v))\,\mathrm{d}\mu. \end{aligned} \tag{5.3}$$

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Combining (5.2) and (5.3) we get that

$$\varphi(u \wedge v) + \varphi(u \vee v) = \varphi(u) + \varphi(v).$$

Hence, by theorem 2.2(b), S is order preserving.

We show that $S_{\mathrm{D}} \preccurlyeq S$. Since the semigroup S is order preserving, it suffices to show that φ_{D} and φ satisfy (2.5). Let $u, v \in L^2(\Omega)$, $v \geqslant 0$. If $u \notin W_0^{1,p}(\Omega)$ or $v \notin D(\varphi)$, then the inequality (2.5) is trivially satisfied. So assume that $u \in W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and $v \in D(\varphi)$. Note that $u = 0$ p-quasi-everywhere on $\partial\Omega$. Proceeding in a manner similar to that above, we thus obtain that

$$\begin{aligned}
\varphi_{\mathrm{D}}((|u| \wedge v)\operatorname{sgn}(u)) + \varphi(|u| \vee v)& \\
&= \frac{1}{p}\int_{\{|u|<v\}} (|\nabla u|^p + |\nabla v|^p)\,\mathrm{d}x + \int_{\partial\Omega\cap\{|u|<v\}} B(x, v)\,\mathrm{d}\mu \\
&\quad + \frac{1}{p}\int_{\{|u|\geqslant v\}} (|\nabla(v\operatorname{sgn}(u))|^p + |\nabla(|u|)|^p)\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega\cap\{|u|\geqslant v\}} B(x, |u|)\,\mathrm{d}\mu \\
&\leqslant \varphi_{\mathrm{D}}(u) + \varphi(v).
\end{aligned}$$

Hence, $S_{\mathrm{D}} \preccurlyeq S$.

Finally, we show that $S \preccurlyeq S_{\mathrm{N}}$. Since S_{N} is order preserving, it suffices to show that φ and φ_{N} satisfy (2.5). Indeed, let $u \in D(\varphi)$, $v \in W^{1,p}(\Omega) \cap L^2(\Omega)$, $v \geqslant 0$. Then it is clear that $(|u| \wedge v)\operatorname{sgn}(u), |u| \vee v \in W^{1,p}(\Omega) \cap L^2(\Omega)$. Since

$$\begin{aligned}
\int_{\partial\Omega} B(x, (|u| \wedge v)\operatorname{sgn}(u))\,\mathrm{d}\mu &= \int_{\{|u|\leqslant v\}} B(x, u)\,\mathrm{d}\mu + \int_{\{|u|>v\}} B(x, v\operatorname{sgn}(u))\,\mathrm{d}\mu \\
&\leqslant \int_{\{|u|\leqslant v\}} B(x, u)\,\mathrm{d}\mu + \int_{\{|u|>v\}} B(x, u)\,\mathrm{d}\mu \\
&= \int_{\partial\Omega} B(x, u)\,\mathrm{d}\mu < \infty
\end{aligned}$$

(where we have used the fact that B is bi-monotone), we have that $(|u| \wedge v)\operatorname{sgn}(u) \in D(\varphi)$. Now, calculating, we get that

$$\begin{aligned}
\varphi((|u| \wedge v)\operatorname{sgn}(u)) + \varphi_{\mathrm{N}}(|u| \vee v)& \\
&= \frac{1}{p}\int_{\{|u|<v\}} (|\nabla u|^p + |\nabla v|^p)\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega\cap\{|u|<v\}} B(x, u)\,\mathrm{d}\mu \\
&\quad + \frac{1}{p}\int_{\{|u|\geqslant v\}} (|\nabla(v\operatorname{sgn}(u))|^p + |\nabla(|u|)|^p)\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega\cap\{|u|\geqslant v\}} B(x, v\operatorname{sgn}(u))\,\mathrm{d}\mu
\end{aligned}$$

$$\begin{aligned}&\leqslant \varphi_{\mathrm{N}}(v)\frac{1}{p}\int_{\Omega}|\nabla u|^p\,\mathrm{d}x+\int_{\partial\Omega\cap\{|u|<v\}}B(x,u)\,\mathrm{d}\mu\\&\quad+\int_{\partial\Omega\cap\{|u|\geqslant v\}}B(x,v\,\mathrm{sgn}(u))\,\mathrm{d}\mu\\&\leqslant \varphi(u)+\varphi_{\mathrm{N}}(v),\end{aligned}$$

where we have again used the fact that B is bi-monotone. Hence, S is dominated by S_{N}.

The implication (ii)⇒(i) of theorem 2.1 is completely proved. □

Proof of theorem 2.1, (i) ⇒ (ii). Let S be the semigroup on $L^2(\Omega)$ generated by a convex and lower semicontinuous functional $\varphi\colon L^2(\Omega)\to[0,+\infty]$. Assume that S is local, order preserving and $S_{\mathrm{D}}\preccurlyeq S\preccurlyeq S_{\mathrm{N}}$.

Define the functional $\psi\colon W^{1,p}(\Omega)\cap L^2(\Omega)\to[0,+\infty]$ by

$$\psi(u)=\begin{cases}\varphi(u)-\varphi_{\mathrm{N}}(u) & \text{if } u\in D(\varphi),\\ +\infty & \text{otherwise.}\end{cases}\tag{5.4}$$

STEP 1. We claim ψ is lower semicontinuous, local, bi-monotone and, for every $u,v\in W^{1,p}(\Omega)\cap L^2(\Omega)$, one has

$$\psi(u\vee u)+\psi(u\wedge v)\leqslant\psi(u)+\psi(v).\tag{5.5}$$

First, since φ is lower semicontinuous on $W^{1,p}(\Omega)\cap L^2(\Omega)$ and since φ_{N} is continuous on that same space, it follows that ψ is lower semicontinuous on $W^{1,p}(\Omega)\cap L^2(\Omega)$.

Second, since φ_{N} is local and since, by assumption, φ is local, it follows that ψ is local, too.

Third, it follows from the domination $S\preccurlyeq S_{\mathrm{N}}$ and theorem 2.2(b) that

$$D(\varphi)\subseteq D(\varphi_{\mathrm{N}})=W^{1,p}(\Omega)\cap L^2(\Omega)$$

and

$$(\varphi-\varphi_{\mathrm{N}})(u)\leqslant(\varphi-\varphi_N)(v)\quad\text{for all } u,v\in D(\varphi)\text{ with } 0\leqslant u\leqslant v.\tag{5.6}$$

The domination $S\preccurlyeq S_{\mathrm{N}}$ and theorem 2.2(b) imply in addition that

$$(\varphi-\varphi_{\mathrm{N}})(v)\leqslant(\varphi-\varphi_N)(u)\quad\text{for all } u,v\in D(\varphi)\text{ with } u\leqslant v\leqslant 0.\tag{5.7}$$

Hence, the functional ψ is bi-monotone.

Finally, let $u,v\in W^{1,p}(\Omega)\cap L^2(\Omega)$. By Stampacchia's lemma,

$$\varphi_{\mathrm{N}}(u\vee v)+\varphi_{\mathrm{N}}(u\wedge v)=\varphi_{\mathrm{N}}(u)+\varphi_{\mathrm{N}}(v).$$

Since S is order preserving, we also have

$$\varphi(u\vee v)+\varphi(u\wedge v)\leqslant\varphi(u)+\varphi(v).$$

The last two relations yield (5.5).

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Step 2. Let $\psi_1, \psi_2 \colon (W^{1,p}(\Omega) \cap L^2(\Omega))^+ \to [0, +\infty]$ be defined by $\psi_1(u) := \psi(u)$ and $\psi_2(u) := \psi(-u)$ $(u \in (W^{1,p}(\Omega) \cap L^2(\Omega))^+)$. By step 1, the functionals ψ_1 and ψ_2 both satisfy the hypotheses in theorem 2.3(i) (with the space $W^{1,p}(\Omega)$ replaced by the space $W^{1,p}(\Omega) \cap L^2(\Omega)$). It follows from theorem 2.3 together with parts (b) and (c) of remark 2.4 that there exist a finite, regular Borel measure μ on $\bar{\Omega}$ that is absolutely continuous with respect to the p-capacity, and two functions $B_1, B_2 \colon \bar{\Omega} \times \mathbb{R}_+ \to [0, +\infty]$ satisfying hypothesis (H^+) such that

$$\psi_i(u) = \int_{\bar{\Omega}} B_i(x, u(x))\,\mathrm{d}\mu \quad \text{for every } u \in D(\psi_i),\ i = 1, 2. \tag{5.8}$$

For every $x \in \bar{\Omega}$ and every $s \in \mathbb{R}$ we set

$$B(x, s) := \begin{cases} B_1(x, s) & \text{if } s \geqslant 0, \\ B_2(x, -s) & \text{if } s < 0. \end{cases} \tag{5.9}$$

It is readily seen that B satisfies hypothesis (H). Since ψ is local, we obtain, for every $u \in D(\psi) = D(\varphi)$,

$$\psi(u) = \psi(u^+ - u^-) = \psi(u^+) + \psi(-u^-) = \psi_1(u^+) + \psi_2(u^-),$$

and hence, by (5.8) and (5.9),

$$\begin{aligned} \psi(u) &= \int_{\bar{\Omega}} B_1(x, u^+(x))\,\mathrm{d}\mu + \int_{\bar{\Omega}} B_2(x, u^-(x))\,\mathrm{d}\mu \\ &= \int_{\bar{\Omega}} B(x, u)\,\mathrm{d}\mu \quad \text{for every } u \in D(\varphi). \end{aligned}$$

We have just shown that

$$\varphi(u) = \varphi_{\mathrm{N}}(u) + \psi(u) = \frac{1}{p}\int_{\Omega} |\nabla u|^p\,\mathrm{d}x + \int_{\bar{\Omega}} B(x, u)\,\mathrm{d}\mu \quad \text{for every } u \in D(\varphi).$$

Step 3. It follows from the domination $S_{\mathrm{D}} \preccurlyeq S$ and theorem 2.2(b) that

$$W_0^{1,p}(\Omega) \cap L^2(\Omega) = D(\varphi_{\mathrm{D}}) \subseteq D(\varphi)$$

and

$$(\varphi_{\mathrm{D}} - \varphi)(u) \leqslant (\varphi_{\mathrm{D}} - \varphi)(v) \quad \text{for all } u, v \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \text{ with } 0 \leqslant u \leqslant v.$$

The domination $S_{\mathrm{D}} \preccurlyeq S$ and theorem 2.2(b) imply in addition that

$$(\varphi_{\mathrm{D}} - \varphi)(v) \leqslant (\varphi_{\mathrm{D}} - \varphi)(u) \quad \text{for all } u, v \in W_0^{1,p}(\Omega) \cap L^2(\Omega) \text{ with } u \leqslant v \leqslant 0.$$

These two inequalities, together with the inequalities (5.6) and (5.7), imply that the functionals φ_{D}, φ and φ_{N} coincide on $D(\varphi_{\mathrm{D}}) = W_0^{1,p}(\Omega) \cap L^2(\Omega)$, that is,

$$\begin{aligned} \varphi_{\mathrm{D}}(u) &= \varphi(u) = \varphi_{\mathrm{N}}(u) \\ &= \frac{1}{p}\int_{\Omega} |\nabla u|^p\,\mathrm{d}x \quad \text{for every } u \in D(\varphi_{\mathrm{D}}) = W_0^{1,p}(\Omega) \cap L^2(\Omega). \end{aligned}$$

It follows that the effective support of the functional ψ is a subset of $\partial\Omega$. In particular, by theorem 2.3, $\operatorname{supp}[\mu] \subseteq \partial\Omega$. Hence,

$$\varphi(u) = \frac{1}{p}\int_{\Omega} |\nabla u|^p \, \mathrm{d}x + \int_{\partial\Omega} B(x,u)\, \mathrm{d}\mu \quad \text{for every } u \in D(\varphi). \tag{5.10}$$

□

This completes the proof of Theorem 2.1.

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