Proceedings of the Edinburgh Mathematical Society (2017) **60**, 947–954
DOI:10.1017/S0013091516000511

RINGS OF POLYNOMIALS WITH ARTINIAN COEFFICIENTS

F. E. A. JOHNSON

Department of Mathematics, University College London, Gower Street, London WC1E 6BT, UK (feaj@math.ucl.ac.uk)

(Received 28 July 2015)

Abstract We study the extent to which the *weak Euclidean* and *stably free cancellation* properties hold for rings of Laurent polynomials $A[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$ with coefficients in an Artinian ring A.

Keywords: stably free cancellation; weakly Euclidean ring; Laurent polynomials

2010 *Mathematics subject classification:* Primary 20C07; 16D40
Secondary 19A13

Introduction

Recall that a module S over a ring Λ is said to be *stably free* when $S \oplus \Lambda^a \cong \Lambda^b$ for some positive integers a, b. We say that Λ has *stably free cancellation* (SFC) when any stably free Λ-module is free. Elementary duality considerations show that this property is left–right symmetric. We show that Artinian rings have the SFC property. More generally, we study the extent to which the SFC property holds for the rings

$$L_n(A) = A[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$$

of Laurent polynomials in n variables $t_1 \dots, t_n$ with coefficients in an Artinian ring A. Here we do not assume that A is commutative but we do require that the variables t_i commute both among themselves and with the coefficients in A. When A is Artinian the Jacobson radical $\mathrm{rad}(A)$ is nilpotent [**9**, p. 81] and the quotient $A/\mathrm{rad}(A)$ is isomorphic to a product of matrix rings:

$$A/\mathrm{rad}(A) \cong M_{d_1}(D_1) \times \cdots \times M_{d_m}(D_m), \qquad (*)$$

where $D_1, \dots, D_m$ are division rings and $d_1, \dots, d_m$ are positive integers. The Artinian ring A is said to satisfy the *Eichler condition* (see [**11**, pp. 174–175]) when, in the decomposition $(*)$, D_i is commutative whenever $d_i = 1$. We strengthen this condition as follows: say that A is *strongly Eichler* when in $(*)$ each division algebra D_i is commutative. We then have the following.

Theorem I. *If the Artinian ring A is strongly Eichler, then $L_n(A)$ has the SFC property for all $n \geqslant 1$.*

There is a corresponding property that has strong stability implications for automorphisms of free modules. A ring Λ is *weakly Euclidean** (see [**6**, Chapter 1]) when, for all $d \geqslant 2$, any $X \in \mathrm{GL}_d(\Lambda)$ can be written as a product

$$X = E_1 \cdot \dots \cdot E_n \cdot \Delta_d(\lambda),$$

where each E_i is an elementary transvection and $\Delta_d(\lambda)$ is an elementary diagonal matrix with $\lambda \in \Lambda^*$. Here Λ^* denotes the group of invertible elements in the ring Λ. We say that the Artinian ring A is *very strongly Eichler* when in the decomposition $(*)$ each D_i is commutative and, in addition, each $d_i \geqslant 2$.

Theorem II. *If the Artinian ring A is very strongly Eichler, then $L_n(A)$ is weakly Euclidean for all $n \geqslant 1$.*

Both Theorem I and Theorem II would seem to be best possible. In relation to Theorem I, a result of Ojanguran and Sridharan [**8**] shows that, for $n \geqslant 2$, $L_n(D)$ fails to have the SFC property whenever the division ring D is non-commutative. Regarding Theorem II, when $n \geqslant 2$ the so-called Cohn matrix (see [**2**, p. 26])

$$\begin{pmatrix} 1 + t_1 t_2 & -t_2^2 \\ t_1^2 & 1 - t_1 t_2 \end{pmatrix} \in \mathrm{GL}_2(L_2(\boldsymbol{F}))$$

fails to decompose as a product of elementary matrices over any field $\boldsymbol{F}$. A direct proof of this result may be found on p. 54 of Lam's book [**7**]. When $n = 1$ we nevertheless obtain the following useful result.

Theorem III. *If the ring A is Artinian, then $L_1(A)$ is weakly Euclidean; furthermore, if A also satisfies the Eichler condition, then $L_1(A)$ has the SFC property.*

Finite rings are Artinian and strongly Eichler; thus we have the following.

Theorem IV. *If the ring A is finite, then*

(i) *$L_n(A)$ has the SFC property for all $n \geqslant 1$; and, moreover,*

(ii) *$L_1(A)$ is weakly Euclidean.*

The results proved here all continue to hold if the rings $L_n(A)$ are replaced by the standard polynomial rings $P_m(A) = A[s_1, \dots, s_m]$ or even by rings of mixed type $A[s_1, \dots, s_m, t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$. However, as rings of the form $L_n(A)$ occur naturally as group rings $\boldsymbol{F}[\Phi \times C_\infty^n]$ when Φ is finite, the construction $L_n(A)$ seems more relevant to applications in non-simply connected homotopy theory (see [**6**, Chapter 11]).

* The terminology arises from the classical theorem of Smith [**10**], which we may state as saying that an integral domain with a Euclidean algorithm is weakly Euclidean.

https://doi.org/10.1017/S0013091516000511 Published online by Cambridge University Press

1. The weak Euclidean property for $L_1(A)$

Given a ring Λ and an integer $d \geqslant 2$, there is a canonical Λ-basis $\{\varepsilon^{(d)}(r,s)\}_{1 \leqslant r,s \leqslant d}$ for the ring of $d \times d$ matrices $M_d(\Lambda)$ given by

$$\varepsilon^{(d)}(r,s)_{tu} = \delta_{rt}\delta_{su};$$

that is, $\varepsilon^{(d)}(r,s)$ is the $d \times d$ matrix with 1 in the (r,s)th position and 0 elsewhere. By an *elementary matrix of type I* in $M_d(\Lambda)$ we mean one of the form

$$E(r,s;\lambda) = I_d + \lambda\varepsilon^{(d)}(r,s) \quad (r \neq s,\ \lambda \in \Lambda).$$

By an *elementary matrix of type II* in $M_d(\Lambda)$ we mean one of the form

$$\Delta_d(\lambda) = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \quad (\lambda \in \Lambda^*).$$

Formally, we have $\Delta_d(\lambda) = I_d + (\lambda - 1)\varepsilon^{(d)}(1,1)$, where $\lambda \in \Lambda^*$. We say that Λ is *weakly Euclidean* when for $d \geqslant 2$ each invertible matrix $X \in \mathrm{GL}_d(\Lambda)$ can be written in the form

$$X = E \cdot \Delta_d(\lambda),$$

where E is a product of elementary matrices of type I over Λ and $\lambda \in \Lambda^*$. A ring homomorphism $\varphi\colon A \to B$ has the *lifting property for units* when the induced map $\phi_*\colon A^* \to B^*$ is surjective. We say φ has the *strong lifting property for units** when, in addition, the following holds for $\alpha \in A$:

$$\alpha \in A^* \iff \varphi(\alpha) \in B^*.$$

It is straightforward to see the following.

Proposition 1.1. *Let $\varphi\colon A \to B$ be a surjective ring homomorphism; if* $\mathrm{Ker}(\varphi)$ *is nilpotent, then φ has the strong lifting property for units.*

Elsewhere [**6**, Proposition 2.43, p. 21] we have shown the following.

Proposition 1.2. *Let $\varphi\colon A \to B$ be a surjective ring homomorphism where B is weakly Euclidean; if φ has the strong lifting property for units, then A is also weakly Euclidean.*

* The referee pointed out that the strong lifting property for φ may be restated as saying that φ has the lifting property and is a local morphism in the sense of Camps and Dicks [**1**].

Thus we have the following.

Proposition 1.3. *Let $\varphi\colon A \to B$ be a surjective ring homomorphism with nilpotent kernel; if B is weakly Euclidean, then A is also weakly Euclidean.*

Proposition 1.4. *Let $D_1, \dots, D_m$ be (possibly non-commutative) division rings; then $M_{d_1}(D_1[t, t^{-1}]) \times \cdots \times M_{d_m}(D_m[t, t^{-1}])$ is weakly Euclidean for any positive integers $d_1, \dots, d_m$.*

Proof. If D_i is a division ring, then $D_i[t, t^{-1}]$ is a (possibly non-commutative) integral domain that admits a Euclidean algorithm (see [**3**]). It is now straightforward to see that matrix rings $M_{d_i}(D_i[t, t^{-1}])$ are also weakly Euclidean (see [**6**, p. 22]). The required conclusion now follows as the class of weakly Euclidean rings is closed under finite direct products. □

Theorem 1.5. *Let A be an Artinian ring; then $A[t, t^{-1}]$ is weakly Euclidean.*

Proof. The radical $\mathrm{rad}(A)$ of the Artinian ring A is nilpotent (see [**9**, p. 81]). Consequently, $\mathrm{rad}(A)[t, t^{-1}]$ is a nilpotent ideal in $A[t, t^{-1}]$. Moreover,

$$A/\mathrm{rad}(A) \cong M_{d_1}(D_1) \times \cdots \times M_{d_m}(D_m)$$

for some division rings $D_1, \dots, D_m$ so that

$$A[t, t^{-1}]/\mathrm{rad}(A)[t, t^{-1}] \cong M_{d_1}(D_1[t, t^{-1}]) \times \cdots \times M_{d_m}(D_m[t, t^{-1}]).$$

The desired conclusion now follows from Remark 1.3 and Proposition 1.4. □

2. Suslin's theorem and proof of Theorem II

We shall use the following theorem of Suslin [**7**, **12**].

Theorem 2.1. *Let $\boldsymbol{F}$ be a field and let $k \geqslant 3$; then any $X \in \mathrm{GL}_k(L_n(\boldsymbol{F}))$ can be written in the form*

$$X = E_1 \cdots E_m \cdot \Delta_k(\lambda),$$

where $\lambda \in L_n(\boldsymbol{F})^*$ and each $E_i \in \mathrm{GL}_k(L_n(\boldsymbol{F}))$ is an elementary matrix of type I.

We note that the unit group $L_n(\boldsymbol{F})^*$ consists simply of elements of the form $\alpha \cdot t_i^{e_i}$, where $\alpha \in \boldsymbol{F}^*$ and e_i is an integer [**6**, Appendix C].

Fixing a ring Λ and an integer $q \geqslant 2$, we study elementary matrices over the rings $\Omega = M_d(M_q(\Lambda))$. Write

$$\mathcal{E}(i,j)_{kl} = \delta_{ik}\delta_{jl} I_q,$$

where I_q is the identity matrix in $M_q(\Lambda)$; then $\{\mathcal{E}(i,j)\}_{1 \leqslant i,j \leqslant d}$ is a basis for $M_d(M_q(\Lambda))$ over $M_q(\Lambda)$. When $M_q(\Lambda)$ is considered as the base ring, we write $\bullet$ for a matrix product over $M_q(\Lambda)$. Elementary matrices of type I in $\mathrm{GL}_d(M_q(\Lambda))$ then take the form

$$\bar{E}(i,j;Z) = \tilde{I} + Z \bullet \mathcal{E}(i,j),$$

https://doi.org/10.1017/S0013091516000511 Published online by Cambridge University Press

where $\tilde{I}$ denotes the identity matrix in $M_d(M_q(\Lambda))$ and $Z \in M_q(\Lambda)$. Likewise, elementary matrices of type II in $\mathrm{GL}_d(M_q(\Lambda))$ take the form

$$\Delta_d(Z) = \begin{pmatrix} Z & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix},$$

where $Z \in \mathrm{GL}_q(\Lambda) = M_q(\Lambda)^*$. In the special case where $Z \in \mathrm{GL}_q(\Lambda)$ is itself an elementary matrix of type II over Λ,

$$Z = \Delta_q(\lambda) = \begin{pmatrix} \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \ddots & & \\ & & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

with $\lambda \in \Lambda^*$, we write $\bar{\Delta}_{d,q}(\lambda) = \Delta_d(\Delta_q(\lambda)) \in \mathrm{GL}_d(M_q(\Lambda))$.

When $d \geqslant 2$ there is a mapping, 'block decomposition', $\nu\colon M_{dq}(\Lambda) \to M_d(M_q(\Lambda))$ defined as follows: if $X = (x_{rs})_{1 \leqslant r,s \leqslant dq} \in M_{dq}(\Lambda)$ and $1 \leqslant i, j \leqslant d$, then

$$\nu(X) = (X(i,j))_{1 \leqslant i,j \leqslant d},$$

where $X(i,j) \in M_q(\Lambda)$ is given by $X(i,j)_{kl} = x_{q(i-1)+k,q(j-1)+l}$; moreover, we have the following.

Proposition 2.2. *For any ring Λ, $\nu\colon M_{dq}(\Lambda) \to M_d(M_q(\Lambda))$ is a ring isomorphism.*

To record the relationship between the various elementary matrices under block decomposition we first observe that there are unique functions

$$\upsilon\colon \{1, \dots, dq\} \to \{1, \dots, d\}, \qquad \rho\colon \{1, \dots, dq\} \to \{1, \dots, q\}$$

defined by the requirement $t + q = q\upsilon(t) + \rho(t)$ for $1 \leqslant t \leqslant dq$. It is straightforward to verify that

$$\nu(\varepsilon^{(dq)}(r,s)) = \varepsilon^{(q)}(\rho(r), \rho(s)) \bullet \mathcal{E}(\upsilon(r), \upsilon(s)). \tag{2.1}$$

The inverse relation is perhaps clearer: namely,

$$\nu^{-1}(\varepsilon^{(q)}(a,b) \bullet \mathcal{E}(i,j)) = \varepsilon^{(dq)}(q(i-1)+a, q(j-1)+b). \tag{2.2}$$

From (2.1) we note that

$$\nu(E(r,s;\lambda)) = \bar{E}(\upsilon(r), \upsilon(s); \lambda\varepsilon(\rho(r), \rho(s))) \quad (\lambda \in \Lambda). \tag{2.3}$$

Likewise, we have

$$\nu(\Delta_{dq}(\lambda)) = \bar{\Delta}_{d,q}(\lambda) \quad (\lambda \in \Lambda^*). \tag{2.4}$$

We first consider the rings $L_n(\boldsymbol{F}) = \boldsymbol{F}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]$, where $\boldsymbol{F}$ is a field.

https://doi.org/10.1017/S0013091516000511 Published online by Cambridge University Press

Theorem 2.3. *Let $d, q \geqslant 1$ be integers such that $dq \geqslant 3$. If $X \in \mathrm{GL}_d(M_q(L_n(\boldsymbol{F})))$, then X can be expressed as a product*

$$X = \bar{E}_1 \bullet \cdots \bullet \bar{E}_m \bullet \bar{\Delta}_{d,q}(\delta),$$

where $\bar{E}_1, \ldots, \bar{E}_m \in \mathrm{GL}_d(M_q(L_n(\boldsymbol{F})))$ are elementary of type I and $\delta \in L_n(\boldsymbol{F})^*$.

Proof. Put $\Lambda = L_n(\boldsymbol{F})$. If $X \in \mathrm{GL}_d(M_q(\Lambda))$, put $\hat{X} = \nu^{-1}(X) \in \mathrm{GL}_{dq}(\Lambda)$. By Suslin's theorem, $\hat{X}$ can be expressed as a product

$$\hat{X} = E_1 \cdots E_m \cdot \Delta(\lambda),$$

where $\lambda \in L_n(\boldsymbol{F})^*$ and each $E_i \in \mathrm{GL}_{dq}(L_n(\boldsymbol{F}))$ is an elementary matrix of type I. Thus

$$\nu(\hat{X}) = \nu(E_1) \bullet \cdots \bullet \nu(E_m) \bullet \nu(\Delta(\lambda))$$

so that, writing $\bar{E}_i = \nu(E_i)$ we have $X = \bar{E}_1 \bullet \cdots \bullet \bar{E}_m \bullet \bar{\Delta}_{d,q}(\delta)$. □

Corollary 2.4. *If $\boldsymbol{F}$ is a field, then $M_q(L_n(\boldsymbol{F}))$ is weakly Euclidean for each $q \geqslant 2$.*

The weak Euclidean property is preserved under finite direct products. Moreover, the construction L_n commutes with both direct products and with the functor $\Lambda \mapsto M_q(\Lambda)$; hence we have the following.

Corollary 2.5. *$L_n[M_{q_1}(\boldsymbol{F}_1) \times \cdots \times M_{q_m}(\boldsymbol{F}_m)]$ is weakly Euclidean whenever $\boldsymbol{F}_1, \ldots, \boldsymbol{F}_m$ are fields and $q_1, \ldots, q_m$ are integers greater than or equal to 2.*

Theorem 2.6. *If the Artinian ring A is very strongly Eichler, then $L_n(A)$ is weakly Euclidean for $n \geqslant 2$.*

Proof. Write $A/\mathrm{rad}(A) \cong M_{q_1}(\boldsymbol{F}_1) \times \cdots \times M_{q_m}(\boldsymbol{F}_m)$ for some fields $\boldsymbol{F}_1, \ldots, \boldsymbol{F}_m$ and integers $q_1, \ldots, q_m \geqslant 2$. Then $L_n(\mathrm{rad}(A))$ is a nilpotent ideal in $L_n(A)$ and

$$L_n(A)/L_n(\mathrm{rad}(A)) \cong L_n[M_{q_1}(\boldsymbol{F}_1) \times \cdots \times M_{q_m}(\boldsymbol{F}_m)].$$

The desired conclusion now follows from Remark 1.3 and Corollary 2.5. □

Theorem II is now the conjunction of Theorem 1.5 and Theorem 2.6.

3. Proof of Theorems I, III and IV

The following is a straightforward deduction from Nakayama's lemma (see [**6**, pp. 170–171]).

Proposition 3.1. *Let $\varphi \colon \Lambda \to \Omega$ be a surjective ring homomorphism such that* $\mathrm{Ker}(\varphi)$ *is nilpotent; if Ω satisfies SFC, then so too does Λ.*

Suppose that A is an Artinian ring such that

$$A/\mathrm{rad}(A) \cong M_{d_1}(D_1) \times \cdots \times M_{d_m}(D_m),$$

where $D_1, \ldots, D_m$ are division rings. We shall apply Proposition 3.1 in the case when $\Lambda = L_n(A)$, $\Omega = L_n(A)/L_n(\mathrm{rad}(A))$ and φ is the natural mapping. Then

$$\Omega \cong M_{d_1}(L_n(D_1)) \times \cdots \times M_{d_m}(L_n(D_m)).$$

We showed in [**5**] that Ω has the SFC property provided each D_i is commutative; that is, provided A is strongly Eichler. Thus from Proposition 3.1 we obtain the following.

Proposition 3.2. *If the ring A is Artinian and strongly Eichler, then $L_n(A)$ has the SFC property.*

As we observed in the introduction, Ojanguran and Sridharan proved in [**8**] that $L_n(D)$ fails the SFC property whenever $n \geqslant 2$ and the division ring D is non-commutative. However, in the case $n = 1$ one may show that $L_1(\mathcal{D}) = \mathcal{D}[t, t^{-1}]$ has SFC regardless of whether the division ring D is commutative or not. Indeed, in that case, $\mathcal{D}[t, t^{-1}]$ is projective free (see [**4**] or [**5**, Proposition 2.9]). The SFC property is now preserved under finite direct products and passage to matrix rings [**6**, pp. 171–173]. Thus $M_{d_1}(L_1(D_1)) \times \cdots \times M_{d_m}(L_1(D_m))$ has the SFC property. From Proposition 3.1 we get the following.

Proposition 3.3. *If the ring A is Artinian, then $L_1(A)$ has the SFC property.*

The conjunction of Proposition 3.2 and Proposition 3.3 is Theorem I of the introduction.

Any finite ring A is trivially Artinian so that $A/\mathrm{rad}(A) \cong M_{d_1}(D_1) \times \cdots \times M_{d_m}(D_m)$, where $D_1, \ldots, D_m$ are finite division rings. However, a celebrated theorem of Wedderburn (see [**13**, p. 1]) now shows that each D_i is commutative; that is, we have the following.

Corollary 3.4. *Any finite ring is Artinian and strongly Eichler.*

Thus from Theorem 1.5, Proposition 3.2 and Remark 3.4 we have the following.

Corollary 3.5. *Let A be a finite ring; then*

(i) *$L_n(A)$ has the SFC property for all $n \geqslant 1$;*

(ii) *$L_1(A)$ is weakly Euclidean.*

We may regard the coefficient ring A as a degenerate case $A = L_0(A)$. Thus, suppose that A is Artinian and write $A/\mathrm{rad}(A) \cong M_{d_1}(D_1) \times \cdots \times M_{d_m}(D_m)$, where $D_1, \ldots, D_m$ are division rings. Then each $M_{d_1}(D_1)$ is weakly Euclidean and has the SFC property. As both these properties are closed under finite direct products, $A/\mathrm{rad}(A)$ is weakly Euclidean and has the SFC property. However, $\mathrm{rad}(A)$ is nilpotent so that, from Remark 1.3 and Proposition 3.1, we conclude the following, which should be well known but is difficult to locate explicitly in the literature.

Corollary 3.6. *Any Artinian ring is weakly Euclidean and has the SFC property.*

Acknowledgements. We thank the referee, whose careful observations have helped to clarify a number of statements.

References

1. R. Camps and W. Dicks, On semilocal rings, *Israel J. Math.* **81** (1993), 203–211.
2. P. M. Cohn, On the structure of the GL_2 of a ring, *Publ. Math. IHES* **30** (1966), 5–53.
3. P. M. Cohn, *Free rings and their relations* (Academic Press, 1985).
4. W. Dicks and E. D. Sontag, Sylvester domains, *J. Pure Appl. Alg.* **13** (1978), 243–275.
5. F. E. A. Johnson, Stably free modules over rings of Laurent polynomials, *Arch. Math.* **97** (2011), 307–317.
6. F. E. A. Johnson, *Syzygies and homotopy theory* (Springer, 2011).
7. T. Y. Lam, *Serre's problem on projective modules* (Springer, 2006).
8. M. Ojanguran and R. Sridharan, Cancellation of Azumaya algebras, *J. Alg.* **18** (1971), 501–505.
9. I. Reiner, *Maximal orders* (Academic Press, 1975).
10. H. J. S. Smith, On systems of linear indeterminate equations and congruences, *Phil. Trans. R. Soc. Lond.* **151** (1861), 293–326.
11. R. G. Swan, Strong approximation and locally free modules, in *Ring theory and algebra III* (ed. B. McDonald), pp. 153–223 (Marcel Dekker, New York, 1980).
12. A. A. Suslin, On the structure of the special linear group over polynomial rings, *Math. USSR Isvestiya* **11** (1977), 221–238.
13. A. Weil, *Basic number theory* (Springer, 1973).