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On stability and asymptotic behaviours for a degenerate Landau–Lifshitz equation

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In this paper we study the problem concerning stability and asymptotic behaviours of solutions for a degenerate Landau–Lifshitz equation in micromagnetics involving only the non-local magnetostatic energy. Due to the lack of derivative estimates, we do not have the compactness needed for strong convergence and the natural convergence is weak* convergence. By formulating the problem in a new framework of differential inclusions, we show that the Cauchy problems for such an equation are not stable under the weak* convergence of initial data. For the asymptotic behaviours of weak solutions, we establish an estimate on the weak* ω-limit sets that is valid for all initial data satisfying the saturation condition.

1. Introduction

The Landau–Lifshitz theory of micromagnetics is a well-known theory for the equilibrium states or evolution of the magnetization of ferromagnetic materials under the formulation of a total energy that contains several competing energy contributions; we refer the reader to [3, 16, 17] and references therein for comprehensive expositions and further developments of such a theory.

In this paper, we study a special dynamic *Landau–Lifshitz equation* that models the evolution of the magnetization of a ferromagnetic material under only the *magnetostatic energy* contribution. Specifically, we study the stability and asymptotic behaviours of solutions to the Cauchy problem

$$\left.\begin{aligned}\partial_t \boldsymbol{M} = \gamma \boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}} + \frac{\alpha\gamma}{|\boldsymbol{M}|}\boldsymbol{M} \times (\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}}),\\ \boldsymbol{M}(x,0) = \boldsymbol{M}_0(x), \quad x \in \Omega,\ t > 0,\end{aligned}\right\} \tag{1.1}$$

where $\boldsymbol{M} = \boldsymbol{M}(x,t) \in \mathbb{R}^3$ is the magnetization field vector of a ferromagnetic material occupying a bounded open set $\Omega \subset \mathbb{R}^3$, $\boldsymbol{a} \times \boldsymbol{b}$ stands for the cross product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ in $\mathbb{R}^3$, $\gamma < 0$ is the electron gyromagnetic ratio, $\alpha > 0$ is the Landau–Lifshitz phenomenological damping parameter and $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}$ is the non-local magnetostatic field induced by $\boldsymbol{M}$ on the whole of $\mathbb{R}^3$ through the Maxwell equation

$$\operatorname{curl} \boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = \boldsymbol{0}, \quad \operatorname{div}(\boldsymbol{\mathcal{H}}_{\boldsymbol{M}} + \boldsymbol{M}\chi_\Omega) = 0 \quad \text{in } \mathbb{R}^3. \tag{1.2}$$

Equation (1.1) can be written as an equivalent *Landau–Lifshitz–Gilbert equation*

$$\partial_t \boldsymbol{M} = \gamma(1+\alpha^2)\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}} + \frac{\alpha}{|\boldsymbol{M}|}\boldsymbol{M} \times \partial_t \boldsymbol{M} \tag{1.3}$$

with a new electron gyromagnetic ratio and a rate-dependent damping.

The part of the energy corresponding to (1.1) is the non-local magnetostatic energy defined by

$$\mathcal{I}(\boldsymbol{M}) = \frac{1}{2}\int_{\mathbb{R}^3} |\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}|^2 \,\mathrm{d}x = -\frac{1}{2}\int_{\Omega} \boldsymbol{M} \cdot \boldsymbol{\mathcal{H}}_{\boldsymbol{M}} \,\mathrm{d}x, \tag{1.4}$$

where the second equation follows from the Maxwell equation (1.2).

Most existing studies on Landau–Lifshitz equations have focused on problems with the so-called *exchange energy* [2, 4, 5, 16, 20]; such a problem is considered regular in the sense that derivative estimates and hence certain compactness results are available to the problem so that standard methods (for example, the Galerkin method) can be applied.

The reasons that we consider the no-exchange energy models are as follows: (1) in static energy minimization, the no-exchange energy model gives a good approximation for large ferromagnetic bodies, as studied in [8,11,19]; (2) in studying Landau–Lifshitz–Maxwell systems of electro-magnetics, the no-exchange energy models of Landau–Lifshitz equations turn out to be the quasi-stationary limit when the electric permittivity tends to zero, as justified in [9, 12, 13]; (3) for certain boundary-value problems, the no-exchange energy model is the singular limit of the Landau–Lifshitz equations as the exchange constant tends to zero, as studied in [5].

The Landau–Lifshitz equation (1.1) becomes degenerate in the sense that no suitable compactness is available due to the lack of derivative estimates that would play an important role in the study of stability and asymptotics of solutions [4, 13, 14]. Our study of the special model (1.1) should be considered as a first step in understanding the effects of degeneracy on the stability and asymptotics for general no-exchange energy models.

Existence and certain stability for degenerate no-exchange energy models of micromagnetics have been established by different methods in [5,9,12,13]. For example, following the method of [15], a strong stability has been proved in [9], which asserts that, given any $T, R > 0$, if initial data $\boldsymbol{M}_0^1$, $\boldsymbol{M}_0^2$ satisfy $\|\boldsymbol{M}_0^k\|_{L^\infty} \leqslant R$ $(k = 1, 2)$ and are sufficiently close in $L^2(\Omega)$, then the solutions $\boldsymbol{M}^1(x,t)$, $\boldsymbol{M}^2(x,t)$ to (1.1) will satisfy, for some $0 < \rho < 1$, $C > 0$ depending on T, R,

$$\sup_{0 \leqslant t \leqslant T} \|\boldsymbol{M}^2(\cdot,t) - \boldsymbol{M}^1(\cdot,t)\|_{L^2(\Omega)} \leqslant C\|\boldsymbol{M}_0^2 - \boldsymbol{M}_0^1\|^{\rho}_{L^2(\Omega)}.$$

Consequently, the solutions $\boldsymbol{M}^j(x,t)$ of (1.1) with initial data satisfying $\boldsymbol{M}_0^j \to \boldsymbol{M}_0$ in $L^2(\Omega;\mathbb{R}^3)$ and $\|\boldsymbol{M}_0^j\|_{L^\infty(\Omega)} \leqslant R$ must converge to the solution $\boldsymbol{M}(x,t)$ of (1.1) with initial datum $\boldsymbol{M}_0$.

However, for bounded initial data, due to the lack of derivative bounds, the natural convergence of solutions is weak* convergence. In this paper, we show that such a stability result fails under the *weak* convergence* of initial data.

THEOREM 1.1. *There exist initial data* $\boldsymbol{M}_0^j \overset{*}{\rightharpoonup} \bar{\boldsymbol{M}}$ *in* $L^\infty(\Omega;\mathbb{R}^3)$ *such that the solutions* $\boldsymbol{M}^j(x,t)$ *to* (1.1) *with initial data* $\boldsymbol{M}_0^j$ *weakly* converge to a limit* $\boldsymbol{M}(x,t)$ *that is not the solution of* (1.1) *with initial datum* $\bar{\boldsymbol{M}}$*.*

This result is obtained as a byproduct of studying the *stationary solutions* for Cauchy problem (1.1):

$$\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = \boldsymbol{0} \quad \text{almost everywhere (a.e.) } \Omega. \tag{1.5}$$

We say that $\bar{\boldsymbol{M}}$ is a *non-trivial* weak* limit of stationary solutions if $\bar{\boldsymbol{M}}$ is a (sequential) weak* limit of stationary solutions and satisfies $\bar{\boldsymbol{M}} \times \boldsymbol{\mathcal{H}}_{\bar{\boldsymbol{M}}} \neq \boldsymbol{0}$ on Ω. Any non-trivial weak* limit will provide an example for theorem 1.1. For example, suppose that there exists a sequence $\{\boldsymbol{M}_0^j\}$ satisfying (1.5) such that $\boldsymbol{M}_0^j \overset{*}{\rightharpoonup} \bar{\boldsymbol{M}}$ but $\bar{\boldsymbol{M}} \times \boldsymbol{\mathcal{H}}_{\bar{\boldsymbol{M}}} \neq \boldsymbol{0}$ on Ω. Then the function $\boldsymbol{M}^j(x,t) \equiv \boldsymbol{M}_0^j(x)$ solves the Cauchy problem (1.1) with initial datum $\boldsymbol{M}_0^j$ and satisfies $\boldsymbol{M}^j(x,t) \overset{*}{\rightharpoonup} \boldsymbol{M}(x,t)$ with $\boldsymbol{M}(x,t) \equiv \bar{\boldsymbol{M}}(x)$. Clearly, $\boldsymbol{M}(x,t)$ is not the solution to (1.1) with initial datum $\bar{\boldsymbol{M}}$.

The existence of non-trivial weak* limits is proved in § 3 by a special construction using ellipsoid domains (see theorem 3.3). It should be pointed out that special stationary solutions $\boldsymbol{M}$ of (1.5) defined by the linear equation $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = \boldsymbol{0}$ (equivalently $\operatorname{div}(\boldsymbol{M}\chi_\Omega) = 0$) or $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = -\boldsymbol{M}\chi_\Omega$ (equivalently $\operatorname{curl}(\boldsymbol{M}\chi_\Omega) = \boldsymbol{0}$) would never produce a non-trivial weak* limit because any weak* limit of such solutions must also satisfy the same linear equation; the existence of non-trivial weak* limits must be a consequence of the nonlinear nature of condition (1.5).

Another purpose of this paper is to study the asymptotic behaviours as $t \to \infty$ of solutions $\boldsymbol{M}(x,t)$ to (1.1) for initial data $\boldsymbol{M}_0 \in L^\infty(\Omega;\mathbb{R}^3)$. Due to the absence of exchange energy or electric field, only the L^∞-bound is available and the methods for asymptotics used in [4, 13, 14] do not apply in our case. Hence, the weak* limit points of the orbits along $t \to \infty$ are the natural object of study. We introduce the (sequential) *strong* and *weak** ω*-limit sets* of solution $M(x,t)$ as follows:

$$\omega(\boldsymbol{M}_0) = \{\bar{\boldsymbol{M}} \colon \exists t_j \uparrow \infty,\ \|\boldsymbol{M}(\cdot,t_j) - \bar{\boldsymbol{M}}\|_{L^2(\Omega)} \to 0\}, \tag{1.6}$$

$$\omega^*(\boldsymbol{M}_0) = \{\tilde{\boldsymbol{M}} \colon \exists t_j \uparrow \infty,\ \boldsymbol{M}(\cdot,t_j) \overset{*}{\rightharpoonup} \tilde{\boldsymbol{M}} \text{ in } L^\infty(\Omega;\mathbb{R}^3)\}. \tag{1.7}$$

Although not required for studying the Cauchy problem (1.1), the magnetization field $\boldsymbol{M}(x,t)$ is usually assumed to be *saturated*; that is, the length $|\boldsymbol{M}(x,t)|$ is constant over $\Omega \times \mathbb{R}^+$. Such a saturation assumption is part of the Landau–Lifshitz theory of ferromagnetism when the temperature is below a certain critical value [3, 16, 17]. One of the important features of the Landau–Lifshitz equations is that the saturation condition is preserved by the solution flows. In the case of our special Cauchy problem (1.1), this is valid for weak solutions: if $|\boldsymbol{M}_0(x)| = C$, a constant, on Ω, then the weak solution $\boldsymbol{M}(x,t)$ of (1.1) satisfies $|\boldsymbol{M}(x,t)| = C$ on Ω for all $t > 0$; see [9].

By rescaling, we focus on the solutions of (1.1) satisfying the saturation condition $|\boldsymbol{M}(x,t)| = 1$ on $\Omega \times \mathbb{R}^+$. The saturated stationary solutions are then characterized by the set

$$\mathcal{E}(\Omega) = \{\boldsymbol{M} \in L^\infty(\Omega;\mathbb{R}^3) \colon |\boldsymbol{M}| = 1,\ \boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = \boldsymbol{0} \text{ a.e. } \Omega\}. \tag{1.8}$$

This set $\mathcal{E}(\Omega)$ has been studied in [21]. In particular, the weak* closure of the set $\mathcal{E}(\Omega)$ has been shown to be contained in the set

$$\mathcal{F}(\Omega) = \{\boldsymbol{M} \in L^\infty(\Omega;\mathbb{R}^3)\colon |\boldsymbol{M}|^2 + 2|\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}}| \leqslant 1 \text{ a.e. } \Omega\}. \tag{1.9}$$

It is not known whether or not the weak* closure of $\mathcal{E}(\Omega)$ is exactly equal to the set $\mathcal{F}(\Omega)$.

As for the ω-limit sets $\omega(\boldsymbol{M}_0)$ and $\omega^*(\boldsymbol{M}_0)$ defined above, we have the following partial result.

THEOREM 1.2. *Suppose that* $|\boldsymbol{M}_0(x)| = 1$ *on* Ω. *Then*

$$\emptyset \neq \omega^*(\boldsymbol{M}_0) \subseteq \mathcal{F}(\Omega), \qquad \omega^*(\boldsymbol{M}_0) \cap \mathcal{E}(\Omega) = \omega(\boldsymbol{M}_0). \tag{1.10}$$

The proof of this theorem is given in § 4. The main idea relies on the global-in-time energy estimate for the solution $\boldsymbol{M} = \boldsymbol{M}(x,t)$,

$$\int_0^\infty \|\boldsymbol{M}_t\|^2_{L^2(\Omega)}\,\mathrm{d}t = C\int_0^\infty \|\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}}\|^2_{L^2(\Omega)}\,\mathrm{d}t < \infty,$$

that is, $\boldsymbol{M}_t \in L^2((0,\infty);L^2(\Omega;\mathbb{R}^3))$. However, this estimate is not sufficient to guarantee the strong convergence of $\boldsymbol{M}(\cdot,t)$ as $t \to \infty$; it would suffice for such a strong convergence if one could obtain $\boldsymbol{M}_t \in L^1((0,\infty);L^2(\Omega;\mathbb{R}^3))$ (see [14]).

Finally, we point out in passing that, *if* there exists an ω^*-limit point $\tilde{\boldsymbol{M}}$ in $\omega^*(\boldsymbol{M}_0)$ that does not satisfy (1.5), then one could obtain another counter-example for theorem 1.1 as follows. Suppose that $\boldsymbol{M}(x,t_j) \overset{*}{\rightharpoonup} \tilde{\boldsymbol{M}}(x)$ as $t_j \uparrow \infty$ and define $\boldsymbol{M}^j(x,t) = \boldsymbol{M}(x,t+t_j)$. Then $\boldsymbol{M}^j(x,t)$ solves (1.1) with initial datum $\boldsymbol{M}^j_0(x) = \boldsymbol{M}(x,t_j) \overset{*}{\rightharpoonup} \tilde{\boldsymbol{M}}(x)$. From the proof of theorem 1.2 (see (4.1)), $\boldsymbol{M}^j(x,t) \overset{*}{\rightharpoonup} \boldsymbol{M}(x,t) \equiv \tilde{\boldsymbol{M}}(x)$, which is not the solution of (1.1) with initial datum $\tilde{\boldsymbol{M}}$.

2. Helmholtz decompositions and differential inclusions

In this section we discuss some auxiliary results and set up our study in a new framework of the calculus of variations. Some of the results are only provided in a way that is sufficient for the purpose of this paper.

In what follows, let Ω be a bounded open set in $\mathbb{R}^3$ and let $\Omega^{\mathrm{c}} = \mathbb{R}^3 \setminus \bar{\Omega}$. Denote by $\mathbb{M}^{m\times n}$ the space of $m \times n$ real matrixes. Let $\Gamma(x) = 1/4\pi|x|$ be the *fundamental solution* for the Laplace equation on $\mathbb{R}^3$. Note that, for all $x \in \mathbb{R}^3$, $\Gamma(x-y) \in L^q(\Omega,\mathrm{d}y)$ for all $1 \leqslant q < 3$.

Given any function $M \in L^p(\Omega;\mathbb{R}^m)$ with $p > 3/2$, the *Newton potential* $F^\Omega_M = F_M$ of M on Ω is well defined as a regular integral by

$$F_M(x) = \int_\Omega M(y)\Gamma(x-y)\,\mathrm{d}y = \frac{1}{4\pi}\int_\Omega \frac{M(y)}{|x-y|}\,\mathrm{d}y \quad (x \in \mathbb{R}^3). \tag{2.1}$$

By partial differential equations theory [10], $F_M(x)$ satisfies $-\Delta F_M = M\chi_\Omega$ in the sense of distributions on $\mathbb{R}^3$; hence, a standard regularity estimate shows that $F_M \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^m)$. Moreover, since Ω is bounded, one has $\partial_{x_i}F_M \in W^{1,p}(\mathbb{R}^3;\mathbb{R}^m)$ for all $i = 1,2,3$.

We have the following regularity result, which uses some ideas from the proof of [5, lemma 4.1].

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LEMMA 2.1. *Let $D \subseteq \Omega \subset\subset B$, where D and B are bounded domains with smooth boundaries. Assume that $m \in W^{1,p}(D)$ for some $p > 1$ and $M = m\chi_D$. Then $F_M = F_M^{\Omega} \in W^{3,p}(D) \cap W^{3,p}(B \setminus \bar{D})$.*

Proof. Obviously, $F_M = F_M^{\Omega} = F_m^D$. Since $m \in W^{1,p}(D)$ and $p > 1$, by the Sobolev embedding theorem, it follows that $m \in L^{p^*}(D)$ for some $p^* > 3/2$, and hence $F_M \in W^{2,p^*}_{\text{loc}}(\mathbb{R}^3)$. The equation $-\Delta F_M = m\chi_D$ holds a.e. on $\mathbb{R}^3$. Given $k \in \{1, 2, 3\}$, let $\psi = \partial_{x_k} F_M$. We claim that

$$\psi \in W^{2,p}(D) \cap W^{2,p}(B \setminus \bar{D}), \tag{2.2}$$

which proves the lemma. Note that $\psi \in W^{1,p^*}_{\text{loc}}(\mathbb{R}^3)$ is a weak solution to the equation $-\Delta\psi = \partial_{x_k}(m\chi_D)$ on $\mathbb{R}^3$; that is,

$$\int_{\mathbb{R}^3} \nabla\psi \cdot \nabla\zeta = -\int_D m\zeta_{x_k}$$

for all test functions $\zeta \in C_0^\infty(\mathbb{R}^3)$. Since $m \in W^{1,p}(D)$, using integration by parts, this weak form of the equation can be written as

$$\int_{\mathbb{R}^3} \nabla\psi \cdot \nabla\zeta \,\mathrm{d}x = \int_D m_{x_k}\zeta \,\mathrm{d}x - \int_{\partial D} m\nu_k\zeta \,\mathrm{d}S, \tag{2.3}$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ denotes the outer unit normal on ∂D. This implies that ψ is harmonic on D^c; hence, $\psi|_{\partial B}$ is smooth. Also note that $m\nu_k \in W^{1-1/p,p}(\partial D)$. By the Sobolev trace theorem [1], for any bounded smooth domain G, the trace map $T\colon W^{2,p}(G) \to W^{2-1/p,p}(\partial G) \times W^{1-1/p,p}(\partial G)$ defined by $Tu = (u|_{\partial G}, (\partial u/\partial\nu)|_{\partial G})$ is onto. Therefore, one can find a function $\phi_2 \in W^{2,p}(B \setminus \bar{D})$ such that

$$\phi_2|_{\partial B} = \psi|_{\partial B}, \qquad \frac{\partial\phi_2}{\partial\nu}\Big|_{\partial D} = m\nu_k. \tag{2.4}$$

Once ϕ_2 is chosen, by the trace theorem again, one finds a function $\phi_1 \in W^{2,p}(D)$ such that

$$\phi_1|_{\partial D} = \phi_2|_{\partial D}, \qquad \frac{\partial\phi_1}{\partial\nu}\Big|_{\partial D} = 0. \tag{2.5}$$

Define the function ϕ on B by $\phi = \psi - \phi_1\chi_D - \phi_2\chi_{B\setminus\bar{D}}$. Then $\phi \in W_0^{1,p}(B)$ with $\nabla\phi = \nabla\psi - (\nabla\phi_1)\chi_D - (\nabla\phi_2)\chi_{B\setminus\bar{D}}$. Let $h = (m_{x_k} + \Delta\phi_1)\chi_D + (\Delta\phi_2)\chi_{B\setminus\bar{D}}$. Then, $h \in L^p(B)$. We now verify that the equation $-\Delta\phi = h$ is satisfied in the sense of distributions on B. Given any $\zeta \in C_0^\infty(B)$, we have

$$\begin{aligned}\int_B \nabla\phi \cdot \nabla\zeta &= \int_B \nabla\psi \cdot \nabla\zeta - \int_D \nabla\phi_1 \cdot \nabla\zeta - \int_{B\setminus\bar{D}} \nabla\phi_2 \cdot \nabla\zeta \\ &= I_1 + I_2 + I_3.\end{aligned}$$

By (2.3) and (2.4), the term I_1 is given by

$$I_1 = \int_D m_{x_k}\zeta \,\mathrm{d}x - \int_{\partial D} \frac{\partial\phi_2}{\partial\nu}\zeta \,\mathrm{d}S.$$

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By (2.5) and Green's identity, one can write terms I_2 and I_3 as

$$I_2 = \int_D \Delta\phi_1 \zeta \,\mathrm{d}x, \qquad I_3 = \int_{B\setminus\bar{D}} \Delta\phi_2 \zeta \,\mathrm{d}x + \int_{\partial D} \frac{\partial \phi_2}{\partial \nu} \zeta \,\mathrm{d}S.$$

Therefore,

$$I_1 + I_2 + I_3 = \int_D (m_{x_k} + \Delta\phi_1)\zeta \,\mathrm{d}x + \int_{B\setminus\bar{D}} \Delta\phi_2 \zeta \,\mathrm{d}x = \int_B h\zeta \,\mathrm{d}x.$$

Hence, $\int_B \nabla\phi \cdot \nabla\zeta \,\mathrm{d}x = \int_B h\zeta \,\mathrm{d}x$ holds for all $\zeta \in C_0^\infty(B)$. This shows that $\phi \in W_0^{1,p}(B)$ is the weak solution to the Dirichlet problem $-\Delta\phi = h$ in B, $\phi|_{\partial B} = 0$. Hence, by the standard elliptic estimate [10], $\phi \in W^{2,p}(B)$. Therefore, we obtain that $\psi|_D = \phi + \phi_1 \in W^{2,p}(D)$ and $\psi|_{B\setminus\bar{D}} = \phi + \phi_2 \in W^{2,p}(B \setminus \bar{D})$, as claimed in (2.2). This completes the proof. □

Now assume that $\boldsymbol{M} \in L^\infty(\Omega; \mathbb{R}^3)$. In this case, $\nabla \boldsymbol{F}_{\boldsymbol{M}} \in W^{1,p}(\mathbb{R}^3; \mathbb{M}^{3\times 3})$ for all $p > 3/2$. Moreover, one has the identity

$$\Delta \boldsymbol{F}_{\boldsymbol{M}} = \nabla(\operatorname{div} \boldsymbol{F}_{\boldsymbol{M}}) - \operatorname{curl}(\operatorname{curl} \boldsymbol{F}_{\boldsymbol{M}}) = -\boldsymbol{M}\chi_\Omega \quad \text{a.e. } \mathbb{R}^3.$$

Define $U_{\boldsymbol{M}} = (u_{\boldsymbol{M}}, \boldsymbol{v}_{\boldsymbol{M}})\colon \mathbb{R}^3 \to \mathbb{R}^4$, where

$$u_{\boldsymbol{M}} = -\operatorname{div} \boldsymbol{F}_{\boldsymbol{M}}, \qquad \boldsymbol{v}_{\boldsymbol{M}} = \operatorname{curl} \boldsymbol{F}_{\boldsymbol{M}}. \tag{2.6}$$

Then $U_{\boldsymbol{M}} \in W^{1,p}(\mathbb{R}^3; \mathbb{R}^4)$ for all $p > 3/2$ and verifies the *Helmholtz decompositions* for $\boldsymbol{M}\chi_\Omega$:

$$\boldsymbol{M}\chi_\Omega = \nabla u_{\boldsymbol{M}} + \operatorname{curl} \boldsymbol{v}_{\boldsymbol{M}} \quad \text{a.e. } \mathbb{R}^3. \tag{2.7}$$

Therefore, the non-local field $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}$ defined in (1.2) is given by

$$\boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = -\nabla u_{\boldsymbol{M}} \in L^p(\mathbb{R}^3; \mathbb{R}^3) \quad \forall p > 3/2.$$

In what follows, we identify the gradient matrix $\nabla U_{\boldsymbol{M}} = \binom{\nabla u_{\boldsymbol{M}}}{\nabla \boldsymbol{v}_{\boldsymbol{M}}} \in \mathbb{M}^{4\times 3}$.

Let $\delta\colon \mathbb{M}^{3\times 3} \to \mathbb{R}^3$ be the linear map such that

$$\boldsymbol{\delta}(\nabla \boldsymbol{v}(x)) = \operatorname{curl} \boldsymbol{v}(x) \quad \text{for all } \boldsymbol{v} \in C^\infty(\mathbb{R}^3; \mathbb{R}^3).$$

For $\xi \in \mathbb{M}^{4\times 3}$, write $\xi = \binom{\boldsymbol{a}}{B}$ with $\boldsymbol{a} \in \mathbb{R}^3$, $B \in \mathbb{M}^{3\times 3}$. The function $U_{\boldsymbol{M}}$ defined above always satisfies a *linear* partial differential inclusion on the exterior domain Ω^{c}:

$$\nabla U_{\boldsymbol{M}}(x) \in \mathcal{L} = \left\{ \begin{pmatrix} \boldsymbol{a} \\ B \end{pmatrix} : \boldsymbol{a} + \boldsymbol{\delta}(B) = 0 \right\}, \quad x \in \Omega^{\mathrm{c}}. \tag{2.8}$$

Related to the sets $\mathcal{E}(\Omega)$ and $\mathcal{F}(\Omega)$ defined above, we introduce the following subsets of $\mathbb{M}^{4\times 3}$:

$$\mathcal{K} = \left\{ \begin{pmatrix} \boldsymbol{a} \\ B \end{pmatrix} : \boldsymbol{a} \times \boldsymbol{\delta}(B) = 0,\ |\boldsymbol{a} + \boldsymbol{\delta}(B)| = 1 \right\}, \tag{2.9}$$

$$\mathcal{S} = \left\{ \begin{pmatrix} \boldsymbol{a} \\ B \end{pmatrix} : |\boldsymbol{a} + \boldsymbol{\delta}(B)|^2 + 2|\boldsymbol{a} \times \boldsymbol{\delta}(B)| \leqslant 1 \right\}. \tag{2.10}$$

One easily verifies the following characterization of sets $\mathcal{E}(\Omega)$ and $\mathcal{F}(\Omega)$.

Lemma 2.2. *Let $\boldsymbol{M} \in L^\infty(\Omega;\mathbb{R}^3)$. Then, $\boldsymbol{M} \in \mathcal{E}(\Omega)$ (or $\mathcal{F}(\Omega)$, respectively) if and only if $\nabla U_{\boldsymbol{M}}(x) \in \mathcal{K}$ (or $\mathcal{S}$, respectively) a.e. on Ω.*

Conversely, we can construct functions in $\mathcal{E}(\Omega)$ using solutions to certain partial differential inclusions.

Lemma 2.3. *Let $U = (u, \boldsymbol{v}) \in W^{1,2}(\mathbb{R}^3;\mathbb{R}^4)$ satisfy the non-homogeneous partial differential inclusion*

$$\nabla U(x) \in \mathcal{K}(x) \equiv \chi_\Omega(x)\mathcal{K} + \chi_{\Omega^c}(x)\mathcal{L} \quad \textit{a.e. on } \mathbb{R}^3. \tag{2.11}$$

Then the function $\boldsymbol{M} = \nabla u + \operatorname{curl} \boldsymbol{v}$ defined on Ω belongs to $\mathcal{E}(\Omega)$.

Proof. Since U satisfies inclusion (2.11), it easily follows that $|\boldsymbol{M}(x)| = 1$ on Ω and $\boldsymbol{M}\chi_\Omega = \nabla u + \operatorname{curl} \boldsymbol{v}$ a.e. on $\mathbb{R}^3$. Hence, $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = -\nabla u$ on $\mathbb{R}^3$, so, again by (2.11), $\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = \nabla u \times \operatorname{curl} \boldsymbol{v} = \boldsymbol{0}$ on Ω. This proves $\boldsymbol{M} \in \mathcal{E}(\Omega)$. □

We remark that for the function $\boldsymbol{M}$ defined in the lemma it always holds that $u_{\boldsymbol{M}} = u$, but $\boldsymbol{v}_{\boldsymbol{M}}$ may not be $\boldsymbol{v}$ because the inclusion (2.11) and function $\boldsymbol{M}$ remain unchanged if we replace $\boldsymbol{v}$ by $\boldsymbol{v} + \nabla\psi$ for any $\psi \in W_0^{2,2}(\mathbb{R}^3)$.

The set $\mathcal{S}$ defined above can be written as $\mathcal{S} = \{\xi \in \mathbb{M}^{4\times 3} \colon f(\xi) \leqslant 1\}$, where f is defined by

$$f(\xi) = |\boldsymbol{a} + \boldsymbol{\delta}(B)|^2 + 2|\boldsymbol{a} \times \boldsymbol{\delta}(B)|. \tag{2.12}$$

It was proved in [21] that the function f is *quasi-convex* on $\mathbb{M}^{4\times 3}$ in the sense of Morrey [18] (see [6] for a systematic study). In fact, this can also be proved by writing $f(\xi) = f_1(\xi) + f_2(\xi)$, where

$$f_1(\xi) = |\boldsymbol{a}|^2 + |\boldsymbol{\delta}(B)|^2 + 2|\boldsymbol{a} \times \boldsymbol{\delta}(B)|, \qquad f_2(\xi) = 2\boldsymbol{a} \cdot \boldsymbol{\delta}(B),$$

and noting that f_1 is convex and f_2 is null-Lagrangian; see [6].

The following result is a consequence of the quasi-convexity of the function $f(\xi)$ (see also [21, theorem 1.1]).

Lemma 2.4. *Let $\boldsymbol{M}_j \overset{*}{\rightharpoonup} \boldsymbol{M}$ in $L^\infty(\Omega;\mathbb{R}^3)$ as $j \to \infty$. If*

$$\lim_{j\to\infty} \int_\Omega (|1 - |\boldsymbol{M}_j|^2| + 2|\boldsymbol{M}_j \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}_j}|)\,\mathrm{d}x = 0, \tag{2.13}$$

then $\boldsymbol{M} \in \mathcal{F}(\Omega)$. In particular, the weak closure of $\mathcal{E}(\Omega)$ is contained in $\mathcal{F}(\Omega)$.*

Proof. We present a slightly simpler proof than the one given in [21]. Let $g(\xi) = (f(\xi) - 1)^+$ and $\Phi(\xi) = |1 - |\boldsymbol{a} + \boldsymbol{\delta}(B)|^2| + 2|\boldsymbol{a} \times \boldsymbol{\delta}(B)|$ on $\mathbb{M}^{4\times 3}$. Then g is quasi-convex in the sense of Morrey. Note that $\Phi(\xi) \geqslant f(\xi) - 1$, and hence $\Phi(\xi) \geqslant g(\xi)$. Moreover, for all $\boldsymbol{M} \in L^\infty(\Omega;\mathbb{R}^3)$, $|1 - |\boldsymbol{M}|^2| + 2|\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}}| = \Phi(\nabla U_{\boldsymbol{M}}(x))$. Hence, condition (2.13) implies that

$$\lim_{j\to\infty} \int_\Omega g(\nabla U_{\boldsymbol{M}_j}(x))\,\mathrm{d}x \leqslant \lim_{j\to\infty} \int_\Omega \Phi(\nabla U_{\boldsymbol{M}_j}(x))\,\mathrm{d}x = 0. \tag{2.14}$$

From $\boldsymbol{M}_j \overset{*}{\rightharpoonup} \boldsymbol{M}$ in $L^\infty(\Omega;\mathbb{R}^3)$, it follows that $U_{\boldsymbol{M}_j} \rightharpoonup U_{\boldsymbol{M}}$ in $W^{1,p}(\mathbb{R}^3;\mathbb{R}^4)$ for all $p > 3/2$. Since g is quasi-convex and $0 \leqslant g(\xi) \leqslant C(|\xi|^2 + 1)$, the functional

$G(U) = \int_\Omega g(\nabla U(x))\,\mathrm{d}x$ is weakly lower semi-continuous on $W^{1,2}(\Omega;\mathbb{R}^4)$ (see, for example, [6]). Therefore, (2.14) implies that $g(\nabla U_{\boldsymbol{M}}(x)) = 0$, and hence $\nabla U_{\boldsymbol{M}}(x) \in \mathcal{S}$ a.e. on Ω. By lemma 2.2, $\boldsymbol{M} \in \mathcal{F}(\Omega)$. □

To study the interior inclusion $\nabla U(x) \in \mathcal{K}$ on Ω, given any $Q > 0$, consider the open bounded set

$$\mathcal{A}_Q = \{\xi \in \mathbb{M}^{4\times 3} \colon |\xi| < Q,\ f(\xi) < 1\}. \tag{2.15}$$

The computations of [21, theorem 3.1] showed that for each $\xi \in \mathcal{A}_Q$ there exist two matrixes $\xi^{\pm} \in \mathcal{K}$ with $\operatorname{rank}(\xi^+ - \xi^-) \leqslant 1$ and a number $t \in (0,1)$ such that

$$\xi = \lambda\xi^- + (1-\lambda)\xi^+, \quad |\xi^{\pm}| \leqslant C(Q+1),$$

where C is a constant independent of Q and ξ. With this constant C, we define a compact subset of $\mathcal{K}$ by

$$\mathcal{K}_Q = \{\xi \in \mathcal{K} \colon |\xi| \leqslant C(Q+1)\}. \tag{2.16}$$

We have the following existence result; see [21, theorem 4.1] for the proof and further references.

Lemma 2.5. *For each $\xi \in \mathcal{A}_Q$, $\varepsilon > 0$ and bounded open set Ω, there exists $W \in \xi x + W_0^{1,\infty}(\Omega;\mathbb{R}^4)$ such that $\nabla W(x) \in \mathcal{K}_Q$ a.e. Ω, $\|W - \xi x\|_{L^\infty(\Omega)} < \varepsilon$.*

3. Existence of non-trivial weak* limits of $\mathcal{E}(\Omega)$: proof of theorem 1.1

In this section, we aim to show that the set $\mathcal{E}(\Omega)$ always has non-trivial weak* limits. Once this is proved, the proof of theorem 1.1 will follow, as described in the introduction.

We proceed with a general result concerning certain weak* limits of $\mathcal{E}(\Omega)$. The following result generalizes the similar result of [21, theorem 6.3] for the cases of constants $\boldsymbol{m}$ and ellipsoidal domains $D = \Omega$.

Theorem 3.1. *Let $D \subseteq \Omega$ be a domain with smooth boundary and also let $\boldsymbol{m} \in W^{1,p}(D;\mathbb{R}^3)$ for some $p > 3$. If $\boldsymbol{M} = \boldsymbol{m}\chi_D \in \mathcal{F}(\Omega)$, then $\boldsymbol{M}$ is in the weak* closure of $\mathcal{E}(\Omega)$.*

Proof. First of all, assume that we have proved that

$$\forall 0 < \sigma < 1,\ \boldsymbol{M}_\sigma = \sigma\boldsymbol{M} \text{ is in the weak* closure of } \mathcal{E}(\Omega). \tag{3.1}$$

Since the weak* topology of the unit ball of $L^\infty(\Omega;\mathbb{R}^3)$ is metrizable, there exists a metric ρ on the unit ball of $L^\infty(\Omega;\mathbb{R}^3)$ such that a sequence $\{N_j\}$ with $\|N_j\|_{L^\infty} \leqslant 1$ converges weakly* to N in $L^\infty(\Omega;\mathbb{R}^3)$ if and only if $\rho(N_j, N) \to 0$. Hence, by (3.1), for each $0 < \sigma < 1$ there exists a sequence $\{\boldsymbol{N}_j^\sigma\}$ in $\mathcal{E}(\Omega)$ such that $\rho(\boldsymbol{N}_j^\sigma, \boldsymbol{M}_\sigma) \to 0$ as $j \to \infty$. Clearly, $\rho(\boldsymbol{M}_\sigma, \boldsymbol{M}) \to 0$ as $\sigma \to 1^-$. Therefore, by a diagonalization method, there exists a sequence $\{\boldsymbol{M}^k\}$ in $\mathcal{E}(\Omega)$ such that $\rho(\boldsymbol{M}^k, \boldsymbol{M}) \to 0$ as $k \to \infty$; this shows that $\boldsymbol{M}$ is in the weak* closure of $\mathcal{E}(\Omega)$.

We now prove (3.1). Note that $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}_\sigma} = \sigma\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}$, and hence

$$|\boldsymbol{M}_\sigma|^2 + 2|\boldsymbol{M}_\sigma \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}_\sigma}| = \sigma^2(|\boldsymbol{M}|^2 + 2|\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}}|) \leqslant \sigma^2 < 1.$$

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Therefore, replacing $\boldsymbol{M}$ by $\sigma\boldsymbol{M}$, we may assume that the given function $\boldsymbol{M} = \boldsymbol{m}\chi_D$ satisfies, for some $0 < \varepsilon < 1$,

$$|\boldsymbol{M}|^2 + 2|\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}}| \leqslant 1 - \varepsilon \quad \text{a.e. } \Omega. \tag{3.2}$$

Since $\boldsymbol{M} = \boldsymbol{m}\chi_D$ and $\boldsymbol{m} \in W^{1,p}(D;\mathbb{R}^3)$ with $p > 3$, by lemma 2.1,

$$U_{\boldsymbol{M}} \in W^{2,p}(D;\mathbb{R}^4) \cap W^{2,p}(B \setminus \bar{D};\mathbb{R}^4),$$

where B is any bounded smooth domain containing $\bar{\Omega}$. Therefore, by the Sobolev embedding, $U_{\boldsymbol{M}} \in C^{1,\alpha}(\bar{D}_\pm;\mathbb{R}^4)$, where $D_+ = D$, $D_- = \Omega \setminus \bar{D}$ and $\alpha = 1 - 3/p \in (0,1)$. This implies that $U_{\boldsymbol{M}}$ is a *piecewise* $C^{1,\alpha}$-function on $\bar{\Omega}$. Hence, $\nabla U_{\boldsymbol{M}} \in L^\infty(\Omega)$. Assume that $\|\nabla U_{\boldsymbol{M}}\|_{L^\infty(\Omega)} < Q$ for some $Q > 0$. Consider the open set

$$\mathcal{A} = \{\xi \in \mathbb{M}^{4\times 3} \colon |\xi| < Q,\ f(\xi) < 1 - \varepsilon/2\},$$

where f is defined by (2.12). By (3.2), it follows that $f(\nabla U_{\boldsymbol{M}}(x)) \leqslant 1-\varepsilon < 1-\varepsilon/2$ a.e. on Ω. Hence,

$$\nabla U_{\boldsymbol{M}}(x) \in \mathcal{A} \quad \text{on } \bar{D}_+ \cup \bar{D}_-. \tag{3.3}$$

Note that the open set $\mathcal{A}$ is contained in the open set $\mathcal{A}_Q$ defined by (2.15). Since $U_{\boldsymbol{M}} \in C^1(\bar{D}_\pm;\mathbb{R}^4)$, from condition (3.3) and using an *approximation* result for C^1 functions (see [7, corollary 10.15]), it follows that there exist sequences of functions $\{A_k^\pm\}$ in $W^{1,\infty}(D_\pm;\mathbb{R}^4)$, with $A_k^\pm$ being *piecewise affine* on $D_\pm$, satisfying the following conditions: for all $k = 1, 2, \dots,$

$$\left.\begin{aligned} A_k^\pm &= U_{\boldsymbol{M}} \quad \text{on } \partial D_\pm, \\ \nabla A_k^\pm(x) &\in \mathcal{A} \quad \text{a.e. } x \in D_\pm, \\ \|A_k^\pm - U_{\boldsymbol{M}}\|_{W^{1,\infty}(D_\pm;\mathbb{R}^4)} &< 1/k. \end{aligned}\right\} \tag{3.4}$$

Write $A_k^\pm = \sum_{j=1}^\infty (\xi_{k,j}^\pm x + b_{k,j}^\pm)\chi_{\Omega_{k,j}^\pm}$, where $\{\Omega_{k,j}^\pm\}$ are disjoint open sets covering $D_\pm$ up to a null set, $\xi_{k,j}^\pm \in \mathcal{A} \subset \mathcal{A}_Q$, and $b_{k,j}^\pm \in \mathbb{R}^4$ are constants. Since $\xi_{k,j}^\pm \in \boldsymbol{A}_Q$, by lemma 2.5, there exists $W_{k,j}^\pm \in (\xi_{k,j}^\pm x + b_{k,j}^\pm) + W_0^{1,\infty}(\Omega_{k,j}^\pm;\mathbb{R}^4)$ with the property

$$\nabla W_{k,j}^\pm(x) \in \mathcal{K}_Q \quad \text{a.e. } \Omega_{k,j}^\pm, \quad \|W_{k,j}^\pm - (\xi_{k,j}^\pm x + b_{k,j}^\pm)\|_{L^\infty(\Omega_{k,j}^\pm)} < 1/k. \tag{3.5}$$

Finally, define the function $U_k \in W^{1,2}(\mathbb{R}^3;\mathbb{R}^4)$ by

$$U_k = \sum_{j=1}^\infty W_{k,j}^+ \chi_{\Omega_{k,j}^+} + \sum_{j=1}^\infty W_{k,j}^- \chi_{\Omega_{k,j}^-} + U_{\boldsymbol{M}}\chi_{\Omega^c}.$$

Let $U_k = (u_k, \boldsymbol{v}_k)$ and define $\boldsymbol{M}_k(x) = \nabla u_k(x) + \operatorname{curl} \boldsymbol{v}_k(x)$ a.e. $x \in \Omega$. By (3.5), $\nabla U_k(x) \in \mathcal{K}_Q \subset \mathcal{K}$ a.e. on Ω and $\nabla U_k(x) = \nabla U_{\boldsymbol{M}}(x) \in \mathcal{L}$ a.e. on Ω^c. Hence, by lemma 2.3, $\boldsymbol{M}_k \in \mathcal{E}(\Omega)$. Moreover, by (3.4) and (3.5), one has $U_k \overset{*}{\rightharpoonup} U_{\boldsymbol{M}}$ in $W^{1,\infty}(\Omega;\mathbb{R}^4)$. Hence, $\boldsymbol{M}_k \overset{*}{\rightharpoonup} \boldsymbol{M}$ in $L^\infty(\Omega;\mathbb{R}^3)$. This proves that $\boldsymbol{M}$ is in the weak* closure of $\mathcal{E}(\Omega)$. □

Corollary 3.2. *Let $D \subseteq \Omega$ be a domain with smooth boundary and also let $\boldsymbol{m} \in W^{1,p}(D;\mathbb{R}^3)$ for some $p > 3$. Then $\sigma\boldsymbol{m}\chi_D$ is in the weak* closure of $\mathcal{E}(\Omega)$ for all sufficiently small constants σ.*

Proof. Let $\boldsymbol{M} = \boldsymbol{m}\chi_D$. From the proof of the theorem, it follows that $\nabla U_{\boldsymbol{M}} \in L^\infty(\Omega)$, and hence $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}} \in L^\infty(\Omega)$. As in the proof of the theorem, for $\boldsymbol{M}_\sigma = \sigma\boldsymbol{M} = \sigma\boldsymbol{m}\chi_D$,

$$|\boldsymbol{M}_\sigma|^2 + 2|\boldsymbol{M}_\sigma \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}_\sigma}| = \sigma^2(|\boldsymbol{M}|^2 + 2|\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}}|) \leqslant \sigma^2 L,$$

where $L > 0$ is a constant depending on the $L^\infty(\Omega)$-norms of $\boldsymbol{m}$ and $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}$. Therefore, $\boldsymbol{M}_\sigma = \sigma\boldsymbol{m}\chi_D \in \mathcal{F}(\Omega)$ as long as $|\sigma| \leqslant L^{-1/2}$. Hence, by the previous theorem, $\boldsymbol{M}_\sigma = \sigma\boldsymbol{m}\chi_D$ is in the weak* closure of $\mathcal{E}(\Omega)$ for all $|\sigma| \leqslant L^{-1/2}$. □

The following result establishes the existence of non-trivial weak* limits of $\mathcal{E}(\Omega)$ and thus provides a proof of theorem 1.1.

THEOREM 3.3. *Let D be any non-spherical ellipsoid in Ω. Then there is a constant vector $\boldsymbol{m} \in \mathbb{R}^3$ such that $\boldsymbol{M} = \boldsymbol{m}\chi_D$ is in the weak* closure of $\mathcal{E}(\Omega)$ but $\boldsymbol{M}\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}} \neq \mathbf{0}$ on D.*

Proof. Let $\boldsymbol{m} \in \mathbb{R}^3$ and $\boldsymbol{M} = \boldsymbol{m}\chi_D$. Then the Newton potential $\boldsymbol{F}^\Omega_{\boldsymbol{M}} = \boldsymbol{F}_{\boldsymbol{M}}$ of $\boldsymbol{M}$ on Ω defined by (2.1) is the same as the Newton potential $F^D_{\boldsymbol{m}}$ of $\boldsymbol{m}$ on D. Hence, the magnetostatic field $\boldsymbol{\mathcal{H}}^\Omega_{\boldsymbol{M}} = \boldsymbol{\mathcal{H}}_{\boldsymbol{M}}$ defined by (1.2) is the same as the corresponding field $\boldsymbol{\mathcal{H}}^D_{\boldsymbol{m}}$ defined by (1.2) with $\boldsymbol{M}$, Ω replaced by $\boldsymbol{m}$, D. From the well-known results in potential theory for ellipsoid domains (see, for example, [19, 21] and the references therein),

$$\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}(x) = \boldsymbol{\mathcal{H}}^D_{\boldsymbol{m}}(x) = -\Lambda\boldsymbol{m} \quad \text{on } D,$$

where Λ is a symmetric positive-definite matrix, called the *demagnetization matrix* for the ellipsoid D. Moreover, $\Lambda = \lambda I$, with I the identity matrix in $\mathbb{M}^{3\times3}$, if and only if D is a ball. Therefore, since D is not a ball, Λ has a non-eigenvector $\bar{\boldsymbol{m}} \in \mathbb{R}^3\setminus\{0\}$. Hence, $\bar{\boldsymbol{m}} \times \Lambda\bar{\boldsymbol{m}} \neq \mathbf{0}$. By corollary 3.2, the function $\boldsymbol{M} = \sigma\bar{\boldsymbol{m}}\chi_D$ will be a weak* limit of $\mathcal{E}(\Omega)$ if $\sigma \neq 0$ is a sufficiently small constant. Such a weak* limit also satisfies $\boldsymbol{M} \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}} = -\sigma^2(\bar{\boldsymbol{m}} \times \Lambda\bar{\boldsymbol{m}}) \neq \mathbf{0}$ on D. This completes the proof. □

4. Asymptotic behaviours for (1.1): proof of theorem 1.2

In what follows, we assume that $\boldsymbol{M}_0 \in L^\infty(\Omega;\mathbb{R}^3)$ satisfies $|\boldsymbol{M}_0(x)| = 1$ on Ω. Let $\boldsymbol{M}(x,t)$ be the solution to (1.1) with initial datum $\boldsymbol{M}_0$. The ω-limit sets $\omega(\boldsymbol{M}_0)$ and $\omega^*(\boldsymbol{M}_0)$ are defined as in the introduction.

We start with the following useful lemma.

LEMMA 4.1. *For all sequences $t_n, s_n \to \infty$ such that $0 < s_n < t_n$ and $\{t_n - s_n\}$ is bounded, it follows that*

$$\lim_{n\to\infty} \|\boldsymbol{M}(\cdot,t_n) - \boldsymbol{M}(\cdot,s_n)\|_{L^2(\Omega)} = 0. \tag{4.1}$$

Proof. By (1.1), it follows that $|\boldsymbol{M}_t| = |\gamma|(1+\alpha^2)^{1/2}|\boldsymbol{M}\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}|$. So Hölder's inequality yields that, for all $0 < s < t$,

$$\|\boldsymbol{M}(\cdot,t) - \boldsymbol{M}(\cdot,s)\|^2_{L^2(\Omega)} \leqslant (t-s)\int_s^t \int_\Omega \gamma^2(1+\alpha^2)|\boldsymbol{M}\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}|^2 \,\mathrm{d}x\,\mathrm{d}\tau. \tag{4.2}$$

Furthermore, $(\mathrm{d}/\mathrm{d}t)(|\boldsymbol{M}|^2) = 2\boldsymbol{M}\cdot\boldsymbol{M}_t = 0$, and hence $|\boldsymbol{M}|^2$ is time-independent. Since $|\boldsymbol{M}_0(x)| = 1$ on Ω, it follows that $|\boldsymbol{M}(x,t)| = 1$ on $x \in \Omega$ for all $t \geqslant 0$.

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Let $\varphi(t) = \mathcal{I}(\boldsymbol{M}(\cdot,t))$, where $\mathcal{I}$ is the magnetostatic energy defined by (1.4). Then $\varphi(t) = -\frac{1}{2}\int_\Omega \boldsymbol{M}(x,t)\cdot\boldsymbol{\mathcal{H}}_{\boldsymbol{M}(x,t)}\,\mathrm{d}x$, and hence

$$\varphi'(t) = -\frac{1}{2}\int_\Omega (\boldsymbol{M}_t\cdot\boldsymbol{\mathcal{H}}_{\boldsymbol{M}} + \boldsymbol{M}\cdot\boldsymbol{\mathcal{H}}_{\boldsymbol{M}_t})\,\mathrm{d}x = -\int_\Omega \boldsymbol{M}_t\cdot\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}\,\mathrm{d}x, \tag{4.3}$$

where we have used the fact that $(\boldsymbol{\mathcal{H}}_{\boldsymbol{M}})_t = \boldsymbol{\mathcal{H}}_{\boldsymbol{M}_t}$ and

$$\int_\Omega \boldsymbol{P}\cdot\boldsymbol{\mathcal{H}}_{\boldsymbol{N}}\,\mathrm{d}x = \int_\Omega \boldsymbol{N}\cdot\boldsymbol{\mathcal{H}}_{\boldsymbol{P}}\,\mathrm{d}x \quad \forall \boldsymbol{P},\boldsymbol{N}\in L^2(\Omega;\mathbb{R}^3),$$

which follows readily from the definition (1.2) of $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}$. Therefore, by (1.1) and (4.3), using the identity $\boldsymbol{M}\times(\boldsymbol{M}\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}) = (\boldsymbol{M}\cdot\boldsymbol{\mathcal{H}}_{\boldsymbol{M}})\boldsymbol{M} - |\boldsymbol{M}|^2\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}$, it follows that

$$\varphi'(t) = \int_\Omega \frac{\alpha\gamma}{|\boldsymbol{M}|}|\boldsymbol{M}\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}|^2\,\mathrm{d}x = \int_\Omega \alpha\gamma|\boldsymbol{M}\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}|^2\,\mathrm{d}x.$$

From this, we have that φ is continuous in $t>0$ and

$$\varphi(t) = \varphi(0) + \alpha\gamma\int_0^t\int_\Omega |\boldsymbol{M}\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}|^2\,\mathrm{d}x\,\mathrm{d}s \quad \forall t>0. \tag{4.4}$$

Since the constant $\alpha\gamma<0$, φ is bounded and non-increasing on $(0,\infty)$. Hence, $\lim_{t\to\infty}\varphi(t)$ exists and is finite. Let $\beta = |\gamma|(1+\alpha^2)/\alpha > 0$. From (4.2) and (4.4), one obtains that, for all $0<s<t$,

$$\|\boldsymbol{M}(\cdot,t) - \boldsymbol{M}(\cdot,s)\|^2_{L^2(\Omega)} \leqslant \beta(t-s)|\varphi(t)-\varphi(s)|, \tag{4.5}$$

from which (4.1) follows. Furthermore, by (4.5), $\boldsymbol{M}(\cdot,t)$ is continuous on $t>0$ in $L^2(\Omega)$ and so is $\boldsymbol{M}\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}}$. Hence, by (4.4) we also have that φ is differentiable on $t>0$. □

Finally, we are able to complete the proof of theorem 1.2.

Proof of theorem 1.2. Since $|\boldsymbol{M}(x,t)| = 1$ on $x\in\Omega$ for all $t>0$, it easily follows that $\omega^*(\boldsymbol{M}_0)\neq\emptyset$.

We now prove $\omega^*(\boldsymbol{M}_0)\subseteq\mathcal{F}(\Omega)$. Let $\tilde{\boldsymbol{M}}\in\omega^*(\boldsymbol{M}_0)$ and assume that $\boldsymbol{M}(\cdot,t_j)\overset{*}{\rightharpoonup}\tilde{\boldsymbol{M}}$ in $L^\infty(\Omega;\mathbb{R}^3)$, where $t_j\to\infty$. Let $\varphi(t) = \mathcal{I}(\boldsymbol{M}(\cdot,t))$ be the function defined above. Since φ is differentiable on (t_j-1,t_j), let $s_j\in(t_j-1,t_j)$ be such that $\varphi'(s_j) = \varphi(t_j)-\varphi(t_j-1)\to 0$. Note that

$$\varphi'(s_j) = \int_\Omega \alpha\gamma|\boldsymbol{M}_j\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}_j}|^2\,\mathrm{d}x \quad \text{with } \boldsymbol{M}_j = \boldsymbol{M}(\cdot,s_j).$$

Since $\alpha\gamma<0$, it follows that $\boldsymbol{M}_j\times\boldsymbol{\mathcal{H}}_{\boldsymbol{M}_j}\to 0$ in $L^2(\Omega)$, so (2.13) is satisfied. By (4.1), $\boldsymbol{M}_j\overset{*}{\rightharpoonup}\tilde{\boldsymbol{M}}$ in $L^\infty(\Omega;\mathbb{R}^3)$, so, by lemma 2.4, $\tilde{\boldsymbol{M}}\in\mathcal{F}(\Omega)$, which proves $\omega^*(\boldsymbol{M}_0)\subseteq\mathcal{F}(\Omega)$.

If, in addition, $|\tilde{\boldsymbol{M}}(x)| = 1$ on Ω, then $\|\boldsymbol{M}(\cdot,t_j)\|_{L^2(\Omega)} = \|\tilde{\boldsymbol{M}}\|_{L^2(\Omega)} = |\Omega|^{1/2}$, and hence $\boldsymbol{M}(\cdot,t_j)\to\tilde{\boldsymbol{M}}$ in $L^2(\Omega;\mathbb{R}^3)$, so $\tilde{\boldsymbol{M}}\in\omega(\boldsymbol{M}_0)$, which establishes that $\omega^*(\boldsymbol{M}_0)\cap\mathcal{E}(\Omega)\subseteq\omega(\boldsymbol{M}_0)$.

To prove the reverse inclusion $\omega(\boldsymbol{M}_0)\subseteq\omega^*(\boldsymbol{M}_0)\cap\mathcal{E}(\Omega)$, since, clearly, $\omega(\boldsymbol{M}_0)\subseteq\omega^*(\boldsymbol{M}_0)$, it suffices to show $\omega(\boldsymbol{M}_0)\subseteq\mathcal{E}(\Omega)$. This can be proved in a similar way to above. For example, let $\bar{\boldsymbol{M}} \in \omega(\boldsymbol{M}_0)$ and $\boldsymbol{M}(\cdot, t_j) \to \bar{\boldsymbol{M}}$ in $L^2(\Omega)$, where $t_j \to \infty$. As above, let $s_j \in (t_j - 1, t_j)$ be such that $\varphi'(s_j) = \varphi(t_j) - \varphi(t_j - 1) \to 0$ with $\boldsymbol{M}_j \times \boldsymbol{\mathcal{H}}_{\boldsymbol{M}_j} \to 0$ in $L^2(\Omega)$, where $\boldsymbol{M}_j = \boldsymbol{M}(\cdot, s_j)$. By (4.1), $\boldsymbol{M}_j \to \bar{\boldsymbol{M}}$ in $L^2(\Omega)$, and hence $\boldsymbol{\mathcal{H}}_{\boldsymbol{M}_j} \to \boldsymbol{\mathcal{H}}_{\bar{\boldsymbol{M}}}$ in $L^2(\mathbb{R}^3; \mathbb{R}^3)$. Therefore, $\bar{\boldsymbol{M}} \times \boldsymbol{\mathcal{H}}_{\bar{\boldsymbol{M}}} = 0$ on Ω. Also, from the strong convergence, $|\bar{\boldsymbol{M}}(x)| = 1$ on Ω. Hence, $\bar{\boldsymbol{M}} \in \mathcal{E}(\Omega)$, establishing $\omega(\boldsymbol{M}_0) \subseteq \mathcal{E}(\Omega)$.

This completes the proof of theorem 1.2. □

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