AN ACTUARIAL BALANCE SHEET MODEL FOR DEFINED BENEFIT PAY-AS-YOU-GO PENSION SYSTEMS WITH DISABILITY AND RETIREMENT CONTINGENCIES

BY

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ABSTRACT

In this paper, we develop a theoretical basis for drawing up a "Swedish" type actuarial balance sheet for a defined benefit pay-as-you-go (DB PAYG) scheme with retirement and disability benefits. Our model enables us to obtain the system's expected average turnover duration, measure the scheme's solvency and explore the phenomenon identified as "pension reclassification", a widespread practice that masks the system's real status unless further pension information becomes available. The model is clearly linked to actuarial practice in social security and gives partial support to the practical adaptation of Swedish methodology carried out by OSFI (2012) in applying the concept of the contribution asset to the Canadian Pension Plan (CPP) balance sheet, which includes disability and survivor benefits.

KEYWORDS

Actuarial balance, disability, retirement, solvency, Sweden, transparency.

1. INTRODUCTION

Regularly compiling an official actuarial balance (AB) is standard practice in public Social Security Administrations (SSAs) in countries such as the USA (BOT (2013)), Japan (AAD (2009)), Sweden (Pensionsmyndigheten (2013)), Canada (OSFI (2010)), the UK (GAD (2010)) and Finland (Elo *et al.* (2010)).

According to Ménard *et al.* (2012) and Vidal-Meliá *et al.* (2010), the AB is becoming an instrument essential to the efficient running of PAYG pension systems. There are convincing reasons why Social Security Programs (SSPs) should have one:

✓ It tends to minimize the traditional difference between the planning horizons of whichever authority is in charge of the system and the system itself.

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- ✓ It should "force" politicians to be much more careful about what they say about the system, thereby reducing populism in pensions.
- ✓ Its findings are used to prompt decision-makers to take action to correct any financial imbalances in the schemes.
- ✓ Stakeholders will have a good idea of how far promises or commitments made to them regarding their pensions are being kept.
- ✓ Public interest in how the system is developing is strengthened, making it easier to introduce automatic balance mechanisms (ABMs).[1]
- ✓ It enables the impact of proposed reforms of the public pension system to be assessed with greater reliability.

When it comes to compiling the AB for PAYG systems, there are basically two options to choose from: what are known as the Swedish and US models, although the actuarial valuation report (AVR) on the Canadian Pension Plan (CPP) and the Japanese actuarial balance (JAB) also present relevant features.

The two models have very different characteristics and strengths. The so-called US model uses explicit projections to highlight future challenges to the financial side deriving basically from ageing, the expected increase in longevity and fluctuations in economic activity.

In the Swedish balance sheet (SBS) the main accounting entries are developed using the principles of double-entry bookkeeping and can briefly be described as showing the actuarial (im)balance in pension systems in understandable language in the shape of assets and liabilities, without needing to use explicit projections.[2] However, it can only be applied to the retirement contingency.

The AB for the Old-Age, Survivors and Disability Insurance (OASDI) program[3] has been compiled in the US since 1941. As Goss (2010) explains, it measures the difference in present value — discounted by the projected yield on trust fund assets — between income from contributions and spending on pensions over the next 75 years as a whole, expressed as a percentage of the present value of the contribution bases for that time horizon, taking into account that the level of financial reserves (trust fund) at the end of the time horizon reaches a magnitude of one year's expenditure. The US report, BOT (2013), also presents another summary measure called "open group unfunded liabilities", which indicates the size of any shortfall in present-value dollars.

In Canada, OSFI (2010), similar methodology to that applied in the US has been used to draw up an AVR on the Canadian Pension Plan (CPP)[4] every three years since 1966. This involves projecting revenue and expenditure over a period of 75 years with the aim of accurately assessing the future effects of historical and projected trends in demography and economic factors. The CPP is considered unsustainable if the projected steady-state contribution rate (SSCR) for the next 75 years needs to be greater than that established by law (currently 9.9 percent). The SSCR is the key financial measure for evaluating the CPP, specifically its adequacy and stability over time. It is defined as the lowest rate sufficient to ensure both the stabilization of the ratio of assets to the following year's expenditures over time and the long-term financial sustainability of the Plan without recourse to further rate increases.

The most relevant feature of the Japanese actuarial balance (JAB) is that it includes explicit measures for making the system sustainable in the sense that projected benefit payments for the time period covered by the actuarial balance, 95 years, cannot exceed the total revenue from contributions and subsidies for that period plus the accumulated funds existing at the beginning of the valuation period.

The AB sheet for the NDC[5] pension system has been compiled in Sweden[6] since 2001. The legal definitions and specific formulas applied in the Swedish system can be found in Pensionsmyndigheten (2013). The SBS does not include a disability contingency. In the Swedish system, disability pensions are paid from a sickness and accident insurance fund, and contributions on behalf of disabled contributors are paid to the state pension scheme by the central government. The Swedish pension system does not therefore require separate treatment for the disability contingency.

The SBS can be described as a financial statement listing the pension system's obligations to contributors and pensioners at a particular date, with the amounts of the various assets (financial and through contributions) which back up these commitments. For Settergren (2009), Swedish reporting on financial status bears great resemblance to the standard income statement and balance sheet of an insurance company. As we will see later, this balance sheet structure is perfectly valid for DB PAYG, especially if the contribution rates for different contingencies are clearly separated.

This paper will deal exclusively with a Swedish-type AB sheet model, looking especially at the two concepts that make the balance sheet possible: the system's expected average turnover duration (TD) and the contribution asset (CA). These concepts initially appear in connection with NDCs, the general outline of which can be found in Settergren (2001) and (2003), while in Settergren and Mikula (2005) both concepts are modelled in continuous time, giving theoretical support.

The search for valid expressions to apply to DB PAYG systems began with the paper by Boado-Penas *et al.* (2008), continuing with that by Vidal-Meliá *et al.* (2009), which in addition links to the concept of the ABM. The paper by Vidal-Meliá and Boado-Penas (2013) obtains the analytical properties of the CA and confirms its soundness as a measure of the assets of a PAYG scheme. However, all the papers cited limit themselves to the retirement contingency, which may be appropriate for defined contribution (DC) pension systems in which the contributory contingencies are clearly separated, but in DB PAYG systems there tends to be no clear separation between contingencies as far as contribution rates are concerned, and disability pensioners are often reclassified as retirement pensioners once they reach a certain age. Also, spending on disability pensions is considerable.[7]

Finally, with the aim of making a comparison with the official AVR of the CPP, OSFI (2012) and Billig and Ménard (2013) make a practical adaptation of the methodology used to compile the Swedish balance sheet. They draw up a modified ABS for the CPP that includes retirement pensions, disability and survivor benefits. However, the theoretical basis for making the adaptation is not developed in their papers. Indeed, the authors warn us that the exercise of compiling the balance sheet should be viewed simply as an illustration.

The aim of this paper is to develop a theoretical basis for applying a Swedish type ABS model to both the retirement and disability contingencies in a DB PAYG system. The possibility of compiling this type of ABS from the integrated perspective of both retirement and disability contingencies, which are closely linked and account for a very high proportion of pension spending in DB systems, has not previously been explored.

After this introduction, in Section 2 we present an ABS model for DB PAYG pension systems with disability and retirement contingencies and develop the main entries for both liabilities and assets. In Section 3, we compile the ABS using various reasonable assumptions for a numerical example representative of the system. The results for the system's assets and liabilities per contingency are also shown and special attention is paid to the phenomenon identified as pension reclassification. In Section 4, we discuss some practical issues that can be considered when choosing a value for G (the growth of the wage bill) for an already-functioning DB PAYG system. In Section 5, we list our main conclusions, and the paper ends with two appendices where we briefly describe the retirement and permanent disability contingencies in the pension systems of Canada, Sweden and the USA and develop the process for obtaining the analytical expressions for the system's turnover duration (TD), contribution asset (CA) and liabilities from the actuarial point of view.

2. The Actuarial Balance Sheet (ABS) Model for DB PAYG Pension Systems with Disability and Retirement Contingencies

In this section, we present an ABS for DB PAYG pension systems with disability and retirement contingencies. To do this, we develop the analytical expression of the contribution asset (CA), the main methodological innovation that enables the ABS of the PAYG system to be compiled.

According to Boado-Penas *et al.* (2008), the presence of the CA in the ABS is a counterargument against those who discredit pure and partial pay-as-you-go finance by claiming that it is always bankrupt or insolvent. This claim is based on accepting the system's liabilities but ignoring the assets implicit in contributions. Billig and Ménard (2013) point out that the CA recognizes that a PAYG system does not have any legal requirement to hold assets to fully guarantee its liabilities. Since such a system relies on contributions as a major source of its financing, this implies that the flow of future contributions represents an asset for the system. Also, the IAA (2012a) promotes the concept of choosing ABS methodology to actuaries who are "*performing, reviewing, advising on, or opining on actuarial valuations of SSPs*".

An alternative measure of PAYG scheme assets can also be found in the literature, what is termed the "quasi asset" by Jackson (2004) and the "hidden asset" by Valdés-Prieto (2005). These authors suggest that it is valid for drawing up the AB sheet of a DB PAYG scheme, but, as demonstrated by Vidal-Meliá and Boado-Penas (2013), the hidden asset supplies a solvency indicator which is not always consistent with the system's financial health.

The CA is based on the system's expected average turnover duration (TD). Lee (1994) began the formal development of the TD and described a framework for organizing, summarizing and interpreting data on transfer systems and the life cycle. Other pioneering papers which arrive at similar frameworks are Arthur and McNicoll (1978) and Willis (1988).

2.1. Description of the system and main assumptions

Our model is developed for a case in which the participants' lives last $(w-1-x_e)$ periods, $(w-1)$ is the highest age to which it is possible to survive and x_e is the age of entry into the system. In this case, when the system reaches the mature state, A generations of contributors, $(w-1-(x_e+A))$ generations of retirement pensioners and $(w-2-x_e)$ generations of disability pensioners coexist at each moment in time.

We adopt the hypothesis that at the earliest age at which one can contribute, x_e years, there are no disability pensioners. However, people become disabled throughout the period and start to receive a pension one year later, i.e. at age x_e+1 years.

We also assume that both the average (insured) wage and the population increase or decrease at an annual real rate of g and γ respectively, which means it must be assumed that real GDP and the wage bill also increase or decrease at rate $G=(1+g)\cdot(1+\gamma)-1$ and that pensions in payment increase or decrease at an annual rate of λ.

The pension system's parameters are considered to be in a mature state. As we progress, we will see that the "mature" condition implies that the dependency ratio (*dr*) stabilizes and the average pension-average contribution base quotient is constant for both contingencies due to the fact that they evolve at the rate of variation in wages. Hence, the total contribution rate $(\theta^D+\theta^R)$ that ensures equality between contribution revenue and pension expenditure is considered constant over time.

The contributor collective is open, i.e. the system has a guaranteed perpetual flow of new entrants, which means it has to be assumed that the system is ongoing and new entrants are taken into account to calculate the system's assets and liabilities. Both the age giving entitlement to retirement pension, "x_e+A", and the formula used for calculating retirement pension are constant, leading to a fixed replacement rate of size β.

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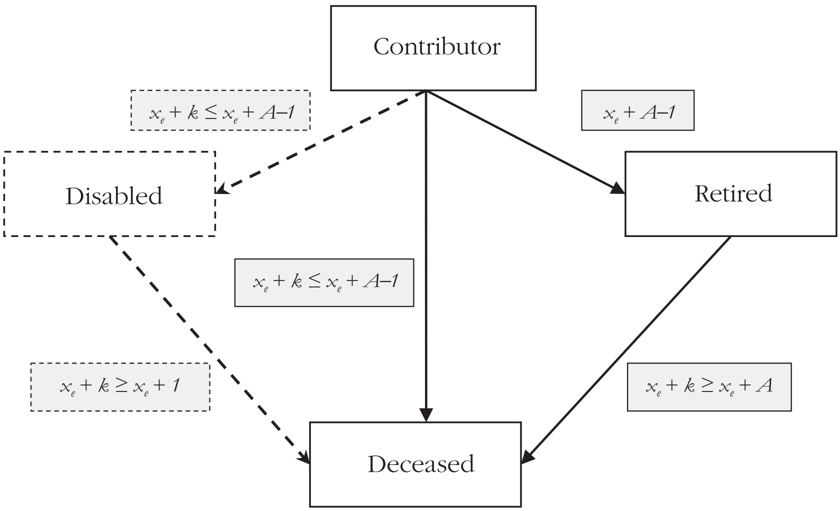
As regards disability pension, it is supposed that initially the ages that give entitlement to benefit are to be found in age interval $[x_e + 1, x_e + A]$ and that for each age within that interval the calculation formula is a percentage (or adjustment factor) of the wage base. The age interval is later widened to $[x_e + 1, w - 1]$.

Diagram 1 shows the relationships (transitions) between the various collectives (states) that will be separated in the model. The difference between this model and the one found in Vidal-Meliá and Boado-Penas (2013) is that a new state — disability — is introduced, along with the new relationships shown by dotted lines in the diagram.

Diagram 1

Contribution, disability, retirement and death



The demographic-financial structure at any moment "t" from the system's inception is given by:

1. -Age:

$$\begin{cases} \overbrace{x_e, x_e + 1, x_e + 2, \ldots\ldots\ldots\ldots, x_e + A - 1}^{\text{Contributors' ages}}, \underbrace{x_e + A, x_e + A + 1, \ldots\ldots\ldots w - 1}_{\text{Pensioners' ages (retirement)}} \\ \underbrace{x_e + 1, x_e + 2, \ldots\ldots\ldots\ldots, x_e + A - 1, x_e + A, x_e + A + 1, \ldots\ldots\ldots w - 1.}_{\text{Pensioners' ages (disability)}} \end{cases} \quad (1)$$

We adopt the assumption that the contributor cannot contribute and receive pension in the same year. However, if an individual becomes disabled

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at contribution age $x_e + k \in [x_e, x_e + A - 1]$, the corresponding disability pension will be payable at age $x_e + k + 1 \in [x_e + 1, x_e + A]$.

2. - Number of contributors by age at time t:

$$\left\{ N_{(x_e,t)}, N_{(x_e+1,t)}, \ldots\ldots\ldots\ldots\ldots\ldots, N_{(x_e+A-1,t)} \right\} = \left\{ N_{(x_e,0)} \cdot (1+\gamma)^t, N_{(x_e+1,0)} \cdot (1+\gamma)^t, \ldots\ldots\ldots\ldots\ldots\ldots, N_{(x_e+A-1,0)} \cdot (1+\gamma)^t \right\} \quad (2)$$

where $N_{(x_e+k,t)} = N_{(x_e,t)} \cdot {}_k p_{x_e}$ with ${}_k p_{x_e}$ being the stable-in-time ratio between the number of individuals of age x_e and $x_e + k$ years. Stable ratios or probabilities include the decrements due to death and disability associated with each age, with the possibility of a return to active life not being considered (practical disability model). It is a different matter when it comes to considering decrements or new entries due to migratory movements, these being included in parameter γ.

It is important to bear in mind that for age $x_e + k \in [x_e + A; w - 1]$, $N_{(x_e+k,t)}$ stands for the number of retirement pensioners of age $x_e + k$ in year t.

3. - Average wage (average contribution base) by age at time t:

$$\left\{ y_{(x_e,t)}, y_{(x_e+1,t)}, \ldots\ldots\ldots\ldots\ldots\ldots, y_{(x_e+A-1,t)} \right\} = \left\{ y_{(x_e,0)} \cdot (1+g)^t, y_{(x_e+1,0)} \cdot (1+g)^t, \ldots\ldots\ldots\ldots\ldots\ldots, y_{(x_e+A-1,0)} \cdot (1+g)^t \right\}. \quad (3)$$

The demographic framework above implies that the age-wage structure only undergoes proportional changes. The slope of the age-wage structure is constant.

The annual retirement pension is $P^r_{(x_e+A,1)} = \beta \cdot Y_{C,0}$, which is a set percentage, β, of the average contribution bases taking into account all the years (A) contributed, with $Y_{C,0} = \frac{\sum_{h=0}^{A-1} y_{(x_e+h,-A+h)}}{A}$. It will also be assumed that contributions and benefits are payable in advance.

4. - Number of disabled in age interval $[x_e + 1, x_e + A]$ at $t = 1$

$$I_{(x_e+k,1)} = N_{(x_e+k-1,0)} \cdot d_{x_e+k-1} = N_{(x_e,0)} \cdot {}_{k-1} p_{x_e} \cdot d_{x_e+k-1}, \quad (4)$$

where:

d_{x_e+k-1} is the probability that an individual of age $x_e + k - 1$ will suffer permanent disability without being able to return to active life and $I_{(x_e+k,1)}$ is the number of people who become disability pensioners in year 1 of age $x_e + k$, becoming disabled as far as the system is concerned because their disability really began in the previous period [0, 1).

For $t \geq 2$ and age interval $[x_e + 1, x_e + A]$ we need to consider two types of disabled people: those aged $x_e + k$ years who became disabled in the current year, $I^N_{(x_e+k,t)}$, and those whose disability began earlier or survivors aged $x_e + k$

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years who continue from previous years, $I^{s}_{(x_e+k,t)}$. The structure for the number of people who became disabled during the year in question is always given by:

$$\begin{aligned} I^{N}_{(x_e+k,t)} &= N_{(x_e+k-1,t-1)} \cdot d_{x_e+k-1} \\ &= N_{(x_e+k-1,0)} \cdot (1+\gamma)^{t-1} \cdot d_{x_e+k-1} = N_{(x_e,0)} \cdot (1+\gamma)^{t-1} \cdot {}_{k-1}p_{x_e} \cdot d_{x_e+k-1} \\ &= I_{(x_e+k,1)} \cdot (1+\gamma)^{t-1}. \end{aligned} \quad (5)$$

After $x_e + A + 1$ years, all the disabled in the system are by definition considered survivor disabled because, once the state of activity disappears, nobody can become disabled for the purposes of the system. Therefore, and always for $t \geq 2$, as far as the continuing disabled are concerned a distinction has to be made between two age intervals, $[x_e + 2,\ x_e + A]$[8] and from $x_e + A + 1$ years onwards.

The structure of the survivor disabled in $[x_e + 2,\ x_e + A]$, whose evolution will depend on the survival probabilities of a disabled person, $p^{d}_{x_e+k-1}$, which may be different from that for the active population, p_{x_e+k-1}, incorporates all those who became disabled in successive earlier periods and have survived.

In general, when all the disabled people who began in $t = 1$ have disappeared, this means that $t = w - x_e$, and therefore from here on in all this disability band we get $k < t$,

$$\begin{aligned} I^{S}_{(x_e+k,t)} &= I_{(x_e+k-1,t-1)} \cdot p^{d}_{x_e+k-1} = \left(I^{N}_{(x_e+k-1,t-1)} + I^{S}_{(x_e+k-1,t-1)}\right) \cdot p^{d}_{x_e+k-1} \\ &= \ldots = \sum_{s=1}^{k-1} I_{(x_e+s,1)} \cdot (1+\gamma)^{t-1-k+s} \cdot {}_{k-s}p^{d}_{x_e+s} = \sum_{s=1}^{k-1} I^{N}_{(x_e+s,t-k+s)} \cdot {}_{k-s}p^{d}_{x_e+s}. \end{aligned} \quad (6)$$

The total number of disabled for each age in t can be calculated by:

$$\begin{aligned} I_{(x_e+k,t)} &= I^{S}_{(x_e+k,t)} + I^{N}_{(x_e+k,t)} = \sum_{s=1}^{k} I_{(x_e+s,1)} \cdot (1+\gamma)^{t-1-k+s} \cdot {}_{k-s}p^{d}_{x_e+s} \\ &= \sum_{s=1}^{k} I^{N}_{(x_e+s,t-k+s)} \cdot {}_{k-s}p^{d}_{x_e+s}. \end{aligned} \quad (7)$$

From $x_e + A + 1$ years onwards no more new disabled people are taken into account, and so for age interval $[x_e + A + 1,\ w - 1]$, i.e. $k \in 1,\ w - 1 - (x_e + A)$,

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we get:

$$I_{(x_e+A+k,t)} = I_{(x_e+A,t-k)} \cdot {}_k p^d_{x_e+A}$$
$$= \left(\sum_{s=1}^{A} I_{(x_e+s,1)} \cdot (1+\gamma)^{t-k-1-A+s} \cdot {}_{A-s} p^d_{x_e+s} \right) \cdot {}_k p^d_{x_e+A}$$
$$I_{(x_e+A+k,t)} = \left(I^S_{(x_e+A,t-k)} + I^N_{(x_e+A,t-k)} \right) \cdot {}_k p^d_{x_e+A}$$
$$= \left(\sum_{s=1}^{A} I^N_{(x_e+s,t-k+s-A)} \cdot {}_{A-s} p^d_{x_e+s} \right) \cdot {}_k p^d_{x_e+A}. \tag{8}$$

If $\forall k \in [1, A]$, the initial annual disability pension (in $t = 1$) is $P^d_{(x_e+k,1)}$. Then, the pension amounts for the newly disabled in $t \geq 2$ and $\forall k \in [1, A]$ are calculated according to the following formula:

$$P^d_{(x_e+k,t)} = P^d_{(x_e+k,1)} \cdot (1+g)^{t-1} = b^d_k \cdot \bar{y}_{(x_e+k-1,0)} \cdot (1+g)^{t-1} = b^d_k \cdot \bar{y}_{(x_e+k-1,t-1)} \tag{9}$$

because $P^d_{(x_e+k,1)}$ is considered to be a variable percentage, b^d_k, of the contribution base of all the wages that contributions had been paid on, k years, at the age of becoming disabled, $\bar{y}_{(x_e+k-1,t-1)} = \frac{\sum_{h=0}^{k-1} y_{(x_e+h,-k+h+t)}}{k}$.

The amounts of the disability pensions for survivors from previous periods, $P^S_{(x_e+k,k-s,t)}$, also in $t \geq 2$, $\forall k \in [2, A]$ and $s \in \{Max\{1, k-t+1\}, \ldots, k-1\}$, where $x_e + k$ is the actual age of the disability pensioner and $x_e + s$ is the age at which the disability first began, would be obtained in accordance with this formula:

$$P^S_{(x_e+k,k-s,t)} = P^d_{(x_e+s,1)} \cdot (1+g)^{t-1-k+s} \cdot (1+\lambda)^{k-s}$$
$$= b^d_s \cdot \bar{y}_{(x_e+s,1)} \cdot (1+g)^{t-1-k+s} \cdot (1+\lambda)^{k-s}. \tag{10}$$

It can be seen that for each period t and for each age k there is a vector 1x(k-s) of old pension amounts, i.e. of as many components as the difference between the age used for calculating the benefit, k, and the age at which it first came into payment, s.

The disability pensions for ages $[x_e+A+1, w-1-x_e-A)$ are all for survivors as no newly disabled are considered, but by following them back to age $x_e + A$ they may come from newly disabled at that age or from survivor disabled from previous ages (a vector of 1×2), in such a way that:

$$P^d_{(x_e+A+k,t)} = \left(P^d_{(x_e+A,t-k)}, P^S_{(x_e+A,t-k)} \right) \cdot (1+\lambda)^k. \tag{11}$$

However, following (10), because $P^S_{(x_e+A,t-k)}$ is going to depend on the age at which the disability originally began, we get $s \in \{Max\{1, A+1-t+k\},$

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$\ldots, A-1\}$. And once we consider $P^d_{(x_e+A,t-k)}$, the final formula for $s \in \{Max\{1, A+1-t+k\}, \ldots, A\}$ will be:

$$P^d_{(x_e+A+k,\,A+k-s,t)} = \overbrace{P^d_{(x_e+A,1)}}^{b^d_A \cdot \bar{y}_{(x_e+A-1,0)}} \cdot (1+g)^{t-1-A-k+s} \cdot (1+\lambda)^{A+k-s}. \qquad (12)$$

Like we said for equation (11), $P^d_{(x_e+A+k,t)}$ is also a row vector, in this case of 1x(A-1-s) with $s \in \{Max\{1, A+1-t+k\}, A\}$.

2.2. The actuarial balance sheet

Taking into account the rules of the pension system and the demographic and economic framework described, the process for obtaining the system's liabilities from the actuarial point of view — the analytical expressions for the system's turnover duration (TD) and contribution asset (CA) — can be separated into five steps for the purposes of clarity, as shown in Appendix 2.

The idea that the financial position of a PAYG pension system would be presented in terms of assets and liabilities, as shown in Table 1, does not come naturally, and indeed it may take some getting used to.

The ABS is compiled using a type of closed group methodology (CGM) that has been modified to make it equivalent to open group methodology (OGM). CGM is widely applied in fully funded systems and assumes that no new entrants to the pension system are permitted. OGM is based on the assumption that the scheme is ongoing, and therefore future new entrants are included in the valuations. As stated above, in keeping with the PAYG nature of the system, we also adopt the assumption that the system has a guaranteed perpetual flow of new entrants.

The liabilities under OGM consist of the present values not only of benefits in payment and benefits expected to become payable for current participants, but also of benefits expected to become payable for new entrants. Hence the assets include the present values of expected future contributions made by or on behalf of current participants and also new entrants.

Our model can be considered "open group" at any particular year t because it takes new entrants into account and assumes that there will be contributions

TABLE 1

THE ABS OF A BALANCED PAYG SYSTEM AT YEAR t.

ASSETS	LIABILITIES
Contribution asset for disability $= CA_t^D$	Liability to pensioners for disability $= {}^D V_t^r$
	Liability to contributors for disability $= {}^D V_t^c$
Contribution asset for retirement $= CA_t^R$	Liability to pensioners for retirement $= {}^R V_t^r$
	Liability to contributors for retirement $= {}^R V_t^c$
Total assets $= CA_t^S$	Total liabilities $= V_t^S$

to meet the liabilities, but valuation formulas consider only pensioners and contributors at the valuation date. Our concept of "open group" is used from a dynamic perspective since the model enables us to draw up the ABS at any date t after the system reaches a mature state.

2.3. Entries on the liabilities side

The entries on the liabilities side are: liabilities to pensioners for retirement, ${}^{R}V_t^r$, and disability, ${}^{D}V_t^r$, and liabilities to contributors for retirement, ${}^{R}V_t^c$, and disability, ${}^{D}V_t^c$.

If we take into account formula (22) in Appendix 2, the liability to pensioners for disability can be expressed as:

$$
\begin{aligned}
{}^{D}V_t^r = {} & \overbrace{\sum_{k=1}^{A}\left(\sum_{s=1}^{k} P^d_{(x_e+s,t)}\cdot I_{(x_e+s,t)}\cdot\left[\frac{1+\lambda}{1+G}\right]^{k-s}\cdot{}_{k-s}p^d_{x_e+s}\right)\cdot{}^{d}\ddot{a}^{\lambda}_{x_e+k}}^{\text{present value of disability pensions (working ages)}} \\
& + \overbrace{\sum_{k=1}^{w-x_e-A-1}\left(\sum_{s=1}^{A} P^d_{(x_e+s,t)}\cdot I_{(x_e+s,t)}\cdot\left[\frac{1+\lambda}{1+G}\right]^{A+k-s}\cdot{}_{A+k-s}p^d_{x_e+s}\right)\cdot{}^{d}\ddot{a}^{\lambda}_{x_e+A+k}}^{\text{present value of disability pensions (retirement ages)}}
\end{aligned}
\tag{13}
$$

with ${}^{d}\ddot{a}^{\lambda}_{x_e+k}$ and ${}^{d}\ddot{a}^{\lambda}_{x_e+A+k}$ being the present value of a lifetime annuity for the disabled of 1 monetary unit per year payable in advance and growing at real rate λ, valued at age "x_e+k" years and age "x_e+A+k" years respectively, with a technical interest rate equal to $d=G$ (see Appendix 2).

For the retirement contingency, the liability to pensioners is equal to:

$$
{}^{R}V_t^r = P^r_{(x_e+A,t)}\cdot\sum_{k=0}^{w-xe-A-1} N_{(x_e+A+k,t)}\cdot\ddot{a}^{\lambda}_{x_e+A+k}\cdot\left[\frac{1+\lambda}{1+G}\right]^{k} \tag{14}
$$

with $\ddot{a}^{\lambda}_{x_e+A+k}$ being the present value of a lifetime annuity for the retiree of 1 monetary unit per year payable in advance and growing at real rate λ, valued at age "x_e+A+k" years, with a technical interest rate equal to $d=G$.

The liability to contributors for disability, whose payments have not yet begun but to whom a commitment has been made by virtue of the contributions already paid, is calculated using the prospective method:

$$
{}^{D}V_t^c = \overbrace{\sum_{k=1}^{A}\sum_{h=k}^{A} P^d_{(x_e+h,t)} \cdot I^N_{(x_e+h,t)} \cdot {}^{d}\ddot{a}^{\lambda}_{x_e+h} \cdot \left[\frac{(1+G)}{(1+d)}\right]^h}^{\text{present value of future disability pensions}}
- \underbrace{\theta^D \cdot \left(\sum_{k=0}^{A-1}\sum_{h=0}^{k} N_{(x_e+k,t)} \cdot y_{(x_e+k,t)} \cdot \left[\frac{(1+G)}{(1+d)}\right]^h\right)}_{\text{present value of future contributions}}. \quad (15)
$$

For the retirement contingency, according to Vidal-Meliá and Boado-Penas (2013), the liability to current contributors is equal to:

$$
{}^{R}V_t^c = \overbrace{P^r_{(x_e+A,t)} \cdot N_{(xe+A,t)} \cdot \ddot{a}^{\lambda}_{xe+A} \cdot \sum_{h=1}^{A}\left[\frac{(1+G)}{(1+d)}\right]^h}^{\text{present value of future retirement benefits}}
- \underbrace{\theta^R \cdot \left(\sum_{k=0}^{A-1}\sum_{h=0}^{k} N_{(xe+k,t)} \cdot y_{(xe+k,t)} \cdot \left[\frac{(1+G)}{(1+d)}\right]^h\right)}_{\text{present value of future contributions}}. \quad (16)
$$

The way these liabilities are defined appears to follow CGM, but in fact a modified version is applied to make it equivalent to OGM because the plan is considered to be ongoing and in a mature state.

2.4. Entries on the assets side and the solvency ratio

The system's contribution asset, CA_t^S, can be understood as the maximum level of liabilities that can be financed by the contribution rate determined for the system without extraordinary contributions from the sponsor. This is because in a balanced PAYG system the difference between the liabilities and current assets (zero financial assets) is simply the present value of future contributions, i.e. the system's contribution asset. Also, as the CA is derived from linking the assets and liabilities of the pension system in a mature state and in cash-flow equilibrium, the population data in the cross-section are identical to those applicable to longitudinal projections, so cross-sectional data are just as valid as longitudinal data.

Analytically, the system's contribution asset can be expressed as:

$$
\begin{aligned}
CA_t^S &= TD_t^S \cdot C_t^S = \left(A_r^S - A_c^S\right) \cdot C_t^S = \left(pt_r^S + pt_c^S\right) \cdot C_t^S \\
&= \overbrace{TD_t^R \cdot C_t^R}^{CA_t^R} + \overbrace{TD_t^D \cdot C_t^D}^{CA_t^D} = \overbrace{\left(pt_r^R + pt_c^R\right)}^{TD_t^R} \cdot C_t^R + \overbrace{\left(pt_r^D + pt_c^D\right)}^{TD_t^D} \cdot C_t^D \\
&= \left(A_r^R - A_c^R\right) \cdot C_t^R + \left(A_r^D - A_c^D\right) \cdot C_t^D. \qquad (17)
\end{aligned}
$$

The value of the system's contribution asset is the product of the system's turnover duration, TD_t^S, and the value of the contributions, C_t^S, made in that period for the retirement and disability contingencies.

The system's turnover duration can be calculated either as a weighted average of the TDs for both contingencies, the weighting being the spending on pensions by contingency as part of total spending, or as the difference between the weighted average of the average ages of disability $(A_r^D - A_c^D)$ and retirement $(A_r^R - A_c^R)$, the weightings here being spending on pensions per contingency as a part of total spending and the average age of the contributors.

The system's TD is also the sum of the weighted pay-in, pt_c^S, and pay-out, pt_r^S, durations of one monetary unit in the system for the year's contributions and is based on population data obtained from a cross-section, not from an explicit projection.

The TD for the system is interpreted as the number of years expected to elapse before the committed liabilities with contributors and pensioners for retirement and disability are completely renewed at the current contribution level.

As Lee (2006) points out, the TD synthesizes into a single number a great deal of information about the system's rules, the age distribution of the population, the age patterns of labour supply and earnings, survival and, in our model, disability rates too. For Goss (2010), it is often desirable to express the outcome of a complex process in a single number.

The value of CA_t^D is the product of turnover duration TD_t^D — formulas (31), (42) and (43) in Appendix 2 — and the value of the contributions made in that period for the disability contingency, C_t^D.

For the retirement contingency, the value of CA_t^R is the product of turnover duration TD_t^R — formulas (32), (33) and (44) in Appendix 2 — and the value of the contributions made in that period, C_t^R.

The solvency index (ratio), $SI_t^S = CA_t^S / V_t^S$, is equal to one in the case of a balanced pension system. At the date of the balance sheet, therefore, participants have a realistic expectation of receiving the benefits they expect — without the system's sponsor having to make periodic contributions — as long as the system's rules and the economic and demographic conditions prevailing at the time of valuation remain constant. Solvency is clearly never completely assured in the long term as neither the assets nor the liabilities are known in their entirety.

As Lee (2006) indicated for the case of the retirement contingency, when using this framework for actual, non-steady state situations, "*we have to imagine* *stopping time at two intervals and using a comparative static comparison between them*". This is the approach developed in practice. In the Swedish case, for example, Pensionsmyndigheten (2013), the ABS is compiled every year according to verifiable events and transactions, but it tends to provide a true and fair view because successive changes are included as they are registered in consecutive balance sheets. Consequently, as Auerbach and Lee (2011 and 2009) point out, the solvency indicator remains reasonably reliable.

Last but not least, the model makes it possible to obtain an actuarial income statement by contingency, thereby enriching the information on the sources from which future financial imbalances in the system may originate and making it easier to set the contribution rates that should be applied for each contingency. The results mainly depend on annual financial variations (treasury surpluses or deficits, return on financial assets and costs of liabilities), on the evolution of economic factors (contributors, contribution bases, the structure of the economic activity that has an impact on disability rates), on demographic factors (longevity of the various collectives) and on the rules of the pension system.

3. Numerical example

The numerical example presented in this section has been calculated using the closed formulas developed in Section 2 and Appendix 2. Our starting point is the numerical example developed by Vidal-Meliá and Boado-Penas (2013). They work with contributors and pensioners by age and contributions (wages) and a "mature" pension structure 36 years after the system's inception, assuming that g grows at an annual accumulative rate of 1%, the population grows at an annual accumulative rate of 2%, and the pension payable to pensioners at age 65 is 80% of the previous 40 years' contributions and constant in real terms ($\lambda = 0\%$).

With these conditions, see Table 2, the contribution rate for balance is 16.51% and the TD is 27.59 years (weighted average age of pensioners 73.32 years, weighted average age of contributors 45.72 years) distributed over 9.32 years for the pay-out and 18.28 years for the pay-in. The contributor-pensioner ratio is 4.5 and the financial ratio 0.7427. Hence according to formula (26) in Appendix 2, the product of these two ratios is the system's contribution rate.

Let us extend this initial system from the start by adding a disability contingency in which a contributor who becomes disabled receives a pension with a variable replacement rate that depends on age and contributions made. Hence a contributor who becomes disabled at age 64, the last age at which it is possible to contribute, would receive a pension identical to that which would be payable on retirement at 65 in the new mature state, 75 years after the system's inception. The evolution of the pensioner and contributor collectives is shown in Figure 1.

The graph shows the evolution of contributors and pensioners in both systems, with base retirement only (Cr, Pr) and with both contingencies separated

TABLE 2

PENSION SYSTEM WITH TWO CONTINGENCIES: SOME SELECTED VALUES. COMPARISON WITH THE BASE SCENARIO OBTAINED BY VIDAL-MELIÁ AND BOADO-PENAS (2013).

Items	Base* Retirement	Retirement + Disability: Retirement	Retirement + Disability: Disability	Retirement + Disability: System
θ	0.165	0.125	0.053	0.178
fr	0.743	0.752	0.581	0.691
dr	0.222	0.166	0.091	0.257
A_r (years)	73.316	73.316	64.890	70.802
A_c (years)	45.724	44.954	44.954	44.954
TD_t (years)	27.592	28.362	19.936	25.849
$\bar{x}_t$ (years)	64.000	64.000	54.614	61.200
pt_c (years)	18.276	19.046	9.661	16.247
pt_r (years)	9.316	9.316	10.276	9.602

Base scenario with $G = (1.01)(1.02)\text{-}1 = 0.0302$

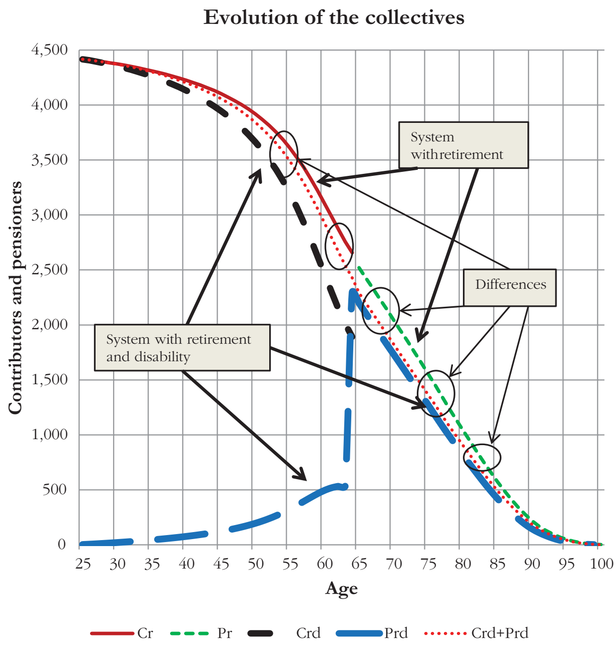

FIGURE 1: Evolution of the collectives. (Color online)

(*Crd, Prd*). The two separate contingencies are also shown combined (*Crd* + *Prd*) so that the result can be compared with the base retirement model.

It can be seen that in the new system there are two types of beneficiary, disability pensioners and retirement pensioners, and that the collective as a whole is smaller than that of the base system because of two effects: disabled people

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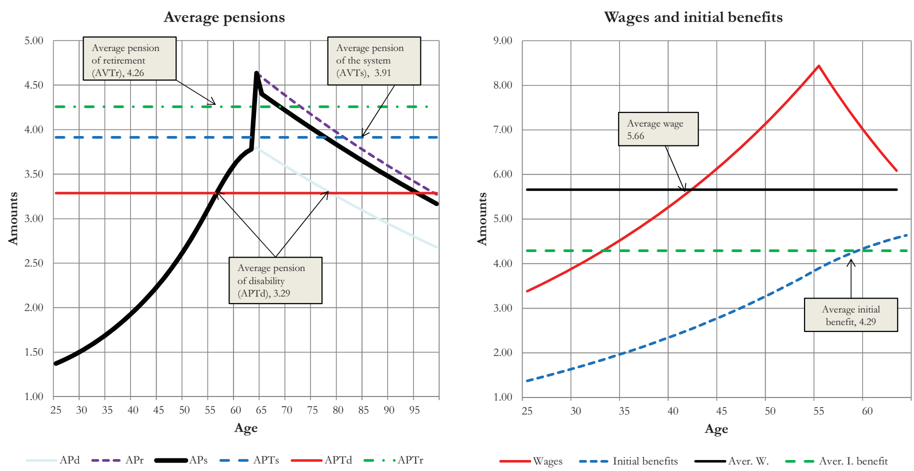

FIGURE 2: Average pensions, wages and initial pensions. (Color online)

do not live as long and population growth does not affect the two systems in the same way — a large proportion of the disabled group consists of survivors and is therefore unaffected by all the increases in population.

Differences by age are shown in the graph by ellipses and reach their maximum at age 65, after which they are decreasing. The two collectives would only coincide under the additional assumption of equal longevity for both disabled and non-disabled (active or retired) and when zero population growth is assumed. If population growth has a positive value, then given the way in which disability is determined, the growth rate of the disabled is lower than that of the contributing population. Therefore, if both collectives are compared, there are always fewer members for all ages in the $(Crd + Prd)$ collective. The greatest difference occurs at about age 65. If there is a decrease in the population the opposite occurs.

Figure 2 shows the evolution of average pensions, wages and initial pensions by age, and also average pensions by contingency, the total for the system, the total average wage and the system's average initial pension. The average disability pension (APd) by age is growing, given that a higher pension is awarded when more contributions have been made, while the average retirement pension (APr) and disability pension (APd) strictly for the retirement period is decreasing because once the pension is awarded it remains constant in real terms.

The main values making up the new system's equilibrium and their comparison with the previous situation are shown in Table 2.

Our attention is drawn to two aspects in particular:

1. The slight increase in the contribution rate for the system as a whole when compared to the base system, despite the fact that there is a new contingency. This is mainly due to two reasons. Firstly there is a transfer of beneficiaries who were previously considered retired but who, in the new system,

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despite being of retirement age, originate in disability. Secondly, as mentioned earlier, disabled people have a lower life expectancy, which lowers the cost of the contingency.

According to ISSA (2012), in many countries when a pensioner reaches statutory retirement age, his or her disability benefits are classified as retirement benefits, a phenomenon known as "pension reclassification". If we were to consider those disabled people who reach retirement age as retirement pensioners, the apparent cost of retirement would increase noticeably. Indeed, if it were supposed that those disabled people who reach or pass normal retirement age were reclassified as retirees, the contribution rate assigned to retirement would increase from 0.125 to 0.152, while the rate for disability would go down from 0.053 to 0.026. The image of the system as a whole would not change from 0.178, but there would be some not very transparent transfers between contingencies because of the change in the average TD for each contingency.

2. The slight variation in the base system's TD along with that of the retirement contingency in the integrated system, which is brought about by the slight change in the average age of the contributors after considering decrements through disability. The system's TD does change more noticeably due to the effect of the disability contingency, which makes the weighted average age at which the last contribution is made almost ten years earlier than for the retirement contingency.

It can also be shown that the system's TD is a weighted average of the TDs for the contingencies, the weighting element being the contribution rate per contingency. This is due to the fact that the annual income from contributions coincides with the annual spending on pensions and in turn corresponds to the new pensions awarded during the year.

Our example is not far from reality because the resulting values for the turnover duration — around 28 years for the retirement contingency — do not differ to any great extent from those calculated by Settergren and Mikula (2007) for a large group of countries (32.7 years). The discrepancy in value stems mainly from the population structure by ages and the age of entry to the labour market.

As regards the liabilities that the system takes on with contributors and pensioners for both contingencies and their relationship with the contribution asset, the profiles by age seen from various perspectives are shown in Figures 3 and 4.

The first part of Figure 3 — the system's assets and liabilities by contingency — which corresponds to the retirement contingency, shows a profile in line with the initial assumptions that the system's total commitments increase with the age of the contributor, given that contributions accumulate until the age at which one becomes entitled to receive retirement pension. From that moment on, due to the fact that pensions are decreasing with age because they were awarded in earlier periods and because the number of pensioners is also decreasing, they gradually become smaller.

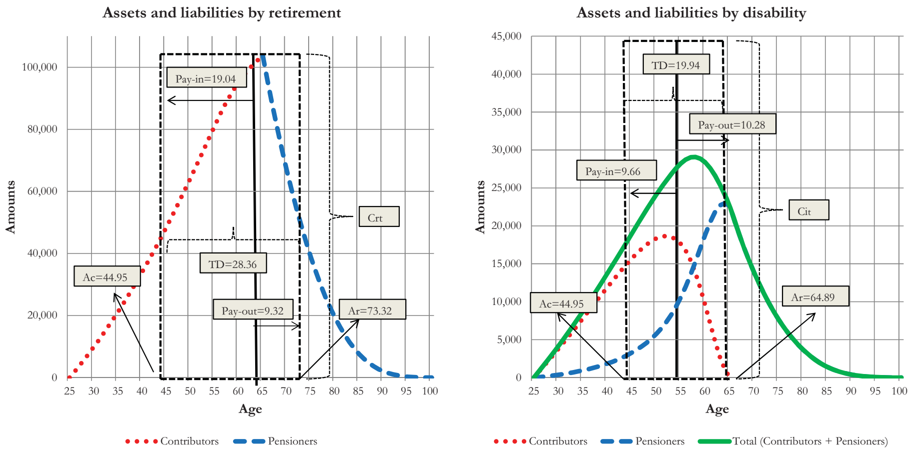

FIGURE 3: The system's assets and liabilities by contingency. (Color online)

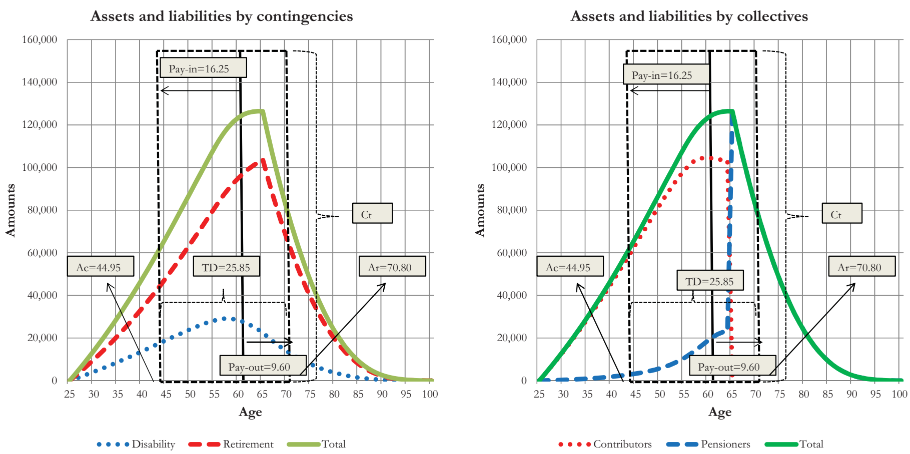

FIGURE 4: The system's assets and liabilities by collective. (Color online)

The liabilities for retirement perfectly match the CA for retirement. The liabilities for retirement is the area beneath the curve for contributors and pensioners, while the CA for retirement is the area represented by the base rectangle, the difference between the weighted average ages of pensioners and contributors. The height is the amount of the contributions made per contingency.[9]

The second part of Figure 3 is for the disability contingency. The system's total commitments for this contingency, in which contributors and pensioners are superimposed, is the result of aggregating the commitments with pensioners and contributors which present a different dynamic.

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TABLE 3

THE ABS OF A BALANCED PAYG SYSTEM. NUMERICAL EXAMPLE.

ASSETS			LIABILITIES		
Items	Amount (monetary units)	%	Items	Amount	%
CA_t^D	879,191.64	23.005	${}^{D}V_t^r$	453,158.65	11.858
			${}^{D}V_t^c$	426,032.99	11.148
CA_t^R	2,942,507.93	76.995	${}^{R}V_t^r$	966,488.69	25.289
			${}^{R}V_t^c$	1,976,019.25	51.705
CA_t^S	3,821,699.57	100.000	V_t^S	3,821,699.57	100.000
Base scenario with $G = (1.01)(1.02)\text{-}1 = 0.0302$					

As far as contributors are concerned, and unlike in the case of retirement, the profile for the system's disability commitments follows an outline typical of risk contingencies, an increase up to a maximum at a particular age, and then a decrease until it disappears. The explanation is obvious. The obligation to contribute comes to an end and the system's commitment with the contributor is extinguished because disability can no longer come about.

In the case of disability pensioners, the commitments increase with age until they reach a maximum at age 64, from which time no more disability pensions can be awarded. From here on, due to the fact that pensions are decreasing with age, the commitments gradually become smaller because the pensions were awarded in earlier periods and because the number of pensioners is also decreasing.

The total liabilities for disability match perfectly with the contribution asset for disability. The total liability for disability is the area below the total curve. The CA for disability is the area represented by the base rectangle, the difference between the weighted average ages of the disability pensioners and contributors, while the height is the amount of contributions paid for the contingency.[10]

Figure 4 shows the perspective from the system's point of view. The system's liabilities is the aggregation of the liabilities by contingency or collective, while the contribution asset derives from the system's turnover duration, which is a weighted average of the TDs for each contingency multiplied by the spending on pensions for each contingency. The profile for the system's total liabilities mainly follows the outline for the main contingency.

Everything shown in Figures 3 and 4 is quantified and included in the ABS presented in Table 3, which shows the values for each of the items that make up the balance and in which it is possible to have a numerical view of the "matching" of the system's different capital amounts that go to determine a solvency indicator equal to the unit.

The picture that the same system would provide with pension reclassification, Table 4, would have noticeable effects on the structure of the ABS by contingency, although the final outcome as regards assets and liabilities is identical

TABLE 4

THE ABS OF A BALANCED PAYG SYSTEM. NUMERICAL EXAMPLE WITH PENSION RECLASSIFICATION.

ASSETS			LIABILITIES		
Items	Amount	%	Items	Amount	%
CA_t^D	250,580.11	6.557	${}^DV_t^r$	43,013.95	1.126
			${}^DV_t^c$	207,566.16	5.431
CA_t^R	3.571,119.46	93.443	${}^RV_t^r$	1,164,380.68	30.468
			${}^RV_t^c$	2,406,738.77	62.976
CA_t^S	3,821,699.57	100.000	V_t^S	3,821,699.57	100.000
Base scenario with G = (1.01)(1.02)-1 = 0.0302					

to the system without reclassification. The so-called true and fair view of the system would be distorted.

It can be said that the reclassification of pensions, which is normal practice in some public SSAs, leads to distortions when assigning both assets and liabilities. Although this has no consequences in overall terms when the system is balanced, it may indeed have consequences and very serious ones when a real unbalanced system is studied. In order to avoid distorting the system's real status and obtain accurate actuarial results by contingency, it would be a good idea for SSAs to provide further pension information, i.e. a breakdown of the sources of old-age pensions in the case of pension reclassification.

It should be pointed out that private capitalization pension systems that cover retirement and disability contingencies do not reclassify pensions once they are in payment as this would prevent them from correctly determining the actuarial result by contingency.

4. THE ABS FOR AN ALREADY-FUNCTIONING DB PAYG SYSTEM

When compiling an ABS for an already-functioning DB PAYG system, other elements may be involved, as shown in Table 5. These include financial assets resulting from an accumulation of treasury surpluses, financial liabilities resulting from an accumulation of treasury deficits, actuarial deficits resulting from an accumulation of actuarial losses, and actuarial surpluses resulting from an accumulation of actuarial profits. The system's actuarial profit or loss — which should not be confused with the treasury surplus or deficit (the difference between income from contributions and expenditure on pensions) — is determined by comparing the system's assets and liabilities in two consecutive periods, while the real solvency index must consider these elements in order to provide a true and fair view of the pension system.

TABLE 5

THE ABS OF AN ALREADY FUNCTIONING DB PAYG SYSTEM AT YEAR t.

ASSETS	LIABILITIES
Financial assets	Financial liabilities (explicit debt, to finance treasury deficits)
Contribution asset for disability	Liability to pensioners for disability
	Liability to contributors for disability
Contribution asset for retirement	Liability to pensioners for retirement
	Liability to contributors for retirement
Accumulated deficit	Accumulated surplus
Total assets	Total liabilities

Therefore, when compiling an actuarial balance for an already-functioning DB pension system, at least four options can be considered when choosing the value of G:

1. An estimated value of G based on the most recently observed data (the previous 3 or 5 years), which is in keeping with the principle that assets and liabilities are valued mainly on the basis of events and transactions that are verifiable at the time of valuation. In our model, we have to consider g and γ to determine G. It is not unlikely that further revisions will have to be made because official data are published with a certain delay and/or corrections are frequently made due to failures in the quality of the information originally supplied. Clearly the position of solvency that the ABS shows will vary depending on which choice is made, and we will only have a single deterministic value for the solvency index.
2. A projected value of G based on official macroeconomic projections. Like in the previous case, we will only have a single deterministic value for the solvency index, but it will be based on projections instead of events and transactions that are verifiable at the time of valuation.
3. Three estimated values of G based on alternative macroeconomic projections, IAA (2010), also known as scenario testing. This methodology, the best alternative to stochastic models, examines the outcome of a projection under alternative sets of assumptions: Alternative I, a low-cost or optimistic forecast; Alternative II, the intermediate or "best estimate" forecast; and Alternative III, a high-cost or pessimistic forecast.[11] The extent of the divergence between these scenarios can provide valuable information concerning the range of possible outcomes for the system's solvency at the valuation date, but in the absence of stochastic model projections, no probability measure can be assigned to the three scenarios. Based on the paper by Lee and Tuljapurkar (1994), Buffin (2002, 2007) suggests for the US AB that it is possible to develop a virtual stochastic model for the solvency index in a practical way using the three projections. In this case, the "best estimate" result is at approximately the 50% percentile of a probability distribution, while the high-cost and low-cost results are represented by a percentile at

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the tails of the probability distribution, e.g. the 95^{th} and 5^{th} or the 99^{th} and 1^{st} percentiles.

4. A value of G based on stochastic modelling techniques. The primary purpose of a stochastic model is to simulate a distribution of possible outcomes — in our case for the solvency index — that reflect the random variations in the inputs, e.g. by determining a range of reasonably possible values of G (independent variable) and assigning a probability to each value. There are several ways to go about this:

One approach, AAA (2005), bases the probability distributions of G on empirical studies. The value chosen for each year of the projection period (in our case the TD period) is independent of the values chosen for the other years.

Another approach, that used by US Social Security actuaries (in their stochastic projections, BOT (2013)), bases the value of an independent variable for each year on the values for previous years, along with some random yearly fluctuation. More specifically, the fluctuation of each variable over time is simulated using historical data and standard time-series techniques. Generally speaking, each variable is modelled using an equation that (a) captures a relationship between current and previous years' values of the variable, and (b) introduces year-by-year random variation as observed in the historical period. For some variables, the equations also reflect relationships with other variables. The equations contain parameters that are estimated using historical data for periods between 25 and 110 years, depending on the nature and quality of the available data. Each time-series equation is designed so that, in the absence of random variation over time, the value of the variable for each year equals its value under the intermediate assumptions (Alternative II). This approach would need to be adapted to the DB actuarial balance sheet, so considerable research would be needed to put the model into practice. Another approach could be mentioned here, Iyer (2008), which uses an analytical model in which projections of demographic and financial variables are made based on the assumption that they evolve according to stochastic processes.

A third approach, also based on stochastic processes, is presented by Boado-Penas *et al.* (2007). This uses a stochastic additive Brownian process to model the value of G, where past information is incorporated as well as a future estimate based on official macroeconomic projections. The model is NDC and would also therefore need to be adapted to the DB ABS. The way in which uncertainty is introduced is simple but encompasses both past experience and simple extrapolation of macroeconomic performance to take care of possible divergences.

Certainly, as the IAA (2012b) warns, when using stochastic models the results should be interpreted with caution and with an understanding of the limitations. Results are very sensitive to equation specifications and the historical periods used for the estimates. For some variables, recent historical variation may not provide a realistic representation of the potential variation for the future. Model risk is also a significant issue.

5. Concluding Remarks and Future Research

Concern about the financial health of public pension systems in all its various designations — solvency, sustainability, viability and equilibrium as affected by population ageing, reduced economic growth and bad practices in system management — occupies a very prominent place on the agenda of many governments and international organizations such as the World Bank, the OECD and the ILO, and can therefore be said to be a matter of world importance. It is no exaggeration to say that the problems of pension systems are a recurring theme in economic policy and are of permanent topical interest for many citizens in various countries.

A basic element for improving pension system management and bringing the planning horizons of the authority in charge of the system and the contributors and pensioners closer together is full information. As Regúlez-Castillo and Vidal-Meliá (2012) point out, the aim is to show the situation of the pension system by providing an indicator of financial solvency or sustainability, the most vital goal being to convey to contributors and pensioners the message that their pensions depend on two things: the individual effort deriving from their actions — amounts contributed, contribution history, retirement age — and the collective situation, i.e. the system's ability to fulfil all its acquired obligations.

The instrument from which the overall indicators are derived is the one known as the actuarial balance sheet, the main examples of which are the "US" and "Swedish" models. The biggest drawback of the Swedish model, from the perspective of applying it to defined benefit systems, is that its theoretical base was only developed for use with the retirement contingency.

In this paper, we have developed a theoretical base for applying a Swedish-type ABS to both retirement and disability contingencies in a DB PAYG system, thereby taking a step towards filling the large gap in the literature in this area. Also, this model starts to make it possible to assess the degree of solvency from the integrated perspective of both retirement and disability contingencies, which are linked together and represent a very high proportion of spending on pensions in DB systems.

The basic element that enables the ABS to be compiled is what is known as the system's contribution asset, which, in the model developed and in line with what the authors already believed intuitively, is a weighted average of the contribution assets of the two contingencies which make up the system and which depend on the economic-demographic structures of the system's collectives, contributors and pensioners, in the so-called "mature" state.

The model makes it possible to obtain an actuarial income statement by contingency, thereby providing richer information about the sources from which future financial imbalances could appear and making it easy to set the contribution rates that should be applied for each contingency.

On the practical side, the numerical example developed enables a debate to be opened regarding the appropriateness of a generalized practice carried out by many public SSAs: pension reclassification. This practice involves considering as disability pensions those pensions being paid to disabled people who reach the normal age of retirement. This widespread practice can mask the system's real solvency situation and makes it more difficult to obtain accurate actuarial results by contingency unless further pension information becomes available. It also makes it more difficult to make projections of the pensioner collective by mixing two collectives (retirement pensioners and disability pensioners of retirement age) with different mortality rates. It would be best for SSAs to break down the source of old-age pensions in cases of pension reclassification with the aim of minimizing the potential negative effects on actuarial reports.

Our model is clearly linked to actuarial practice in social security and gives partial support to the practical adaptation carried out by OSFI (2012) when it applied the concept of the contribution asset to the Canadian Pension Plan (CPP) balance sheet, which includes disability and survivor contingencies. However, further research needs to be done to confirm its suitability for survivor contingencies.

The model developed has many other practical implications which could be of interest not only to DB systems but also to notional defined contribution schemes (NDCs), social security actuaries, public finance economists and policy-makers. For example, as regards the current pension system in Sweden, which is an NDC model covering only the retirement contingency, this could be extended to cover disability now that the relationship between both contingencies is clear. The ABS could be compiled for both contingencies, which would thus notably increase its representativeness as it would include a higher proportion of total spending on pensions. The legitimacy of applying an ABM would also be strengthened as the action would be based on a more reliable solvency indicator. This would be one of the points where this research could most naturally be extended, by having to integrate one of the peculiarities of the Swedish NDC model, Boado-Penas and Vidal-Meliá (2014), the so-called survivor dividend.

Finally, we would like to emphasize that our model is not intended as a replacement for the US AB model. Instead it should be seen as an alternative way of measuring the financial status of DB PAYG systems because ABS results are relatively easy to explain and the concept is widely used for pension plans outside social security systems.

Further, work could be carried out on the model developed here, with future research extending into at least three additional areas:

1. Considering different degrees of disability and/or the possibility of a return to active life. In practice, there are usually various degrees of disability recognized and these have a direct effect on the amount of benefit paid and the likelihood or not of returning to active life. The most natural way to do this would be to extend the states shown in Diagram 1, which would obviously involve a considerable increase in the complexity of the formulas to be obtained.

2. Extending the ABS by incorporating widows' and/or survivor contingencies, which would enable virtually all spending on pensions in DB systems to be included.
3. Extending the ABS by incorporating long-term care (LTC) as a contributory contingency, as has been offered in the German contributory pension system, Rothgang (2010), on a PAYG basis with income-related contributions since the mid-1990s. Given the accelerated ageing process taking place in developed countries, LTC is an area of considerable interest that needs to be specifically taken into account as a cost with fundamental links to retirement and certain degrees of disability.

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Notes

1. An ABM is a set of predetermined measures established by law to be applied immediately as required according to the solvency indicator. For more details, see the papers by Barr & Diamond (2011), Vidal-Meliá *et al* (2009) and Börsch-Supan (2007).

2. See the paper by Boado-Penas & Vidal-Meliá (2012) for an in-depth study of the main differences and similarities.

3. The OASDI program in the United States provides a basic level of monthly income when insured workers become eligible for retirement, and also in cases of death or disability. See the papers by Barr & Diamond (2010), DeWitt (2010) and Diamond & Orszag (2005), and also Appendix 1.

4. The CPP is an earnings-related program. Disability and survivor benefits are important features of the CPP. See the papers by Ménard. (2010) and Billig & Ménard. (2013) and also Appendix 1 for more details.

5. A notional defined contribution scheme (NDC) is a pay-as-you-go scheme that deliberately mimics a financial defined contribution (FDC) scheme by paying an income stream whose present value over a person's expected remaining lifetime equals his or her accumulation at retirement. By doing this it has many features of an FDC scheme. For more information about NDCs, see for example the papers by Lindbeck & Persson (2003), Holzmann & Palmer (2006 and 2012) and Whitehouse (2010).

6. Papers on the Swedish pension system include those by Sunden (2006) and Chłoń-Domińczak *et al* (2012), with more information in Pensionsmyndigheten (2013) and Appendix 1

7. According to information provided by BOT (2013), spending on retirement pensions in the USA accounted for 63.19% of the total, with disability pensions accounting for 16.40%, together totalling 79.59%.

8. In $k = 1$ the disabled are always newly disabled as they come from age x_e in $t-1$, and therefore $I(x_e+1, t) = I^N(x_e+1, t)$.

9. This is equivalent to the present value of benefits awarded during the period, as can be seen in Appendix 2.

10. As demonstrated in Appendix 2 and like in the case of the retirement contingency, this is equivalent to the present value of the disability benefits awarded during the period.

11. This is the approach used by Vidal-Melia (2013) to compile the ABS for the Spanish public retirement pension system at 31–12–2010.

12. The weighted average age for receiving the first disability benefit would be one year later given the hypotheses we considered regarding prepayment of contributions and pensions.

13. See Pensionsmyndigheten (2013), Appendix B. Mathematical Description of the Balance Ratio, formula 2.0.

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APPENDIX 1: BRIEF DESCRIPTION OF THE PENSION SYSTEMS OF CANADA, SWEDEN AND THE USA (RETIREMENT AND PERMANENT DISABILITY)

Items	Canada	Sweden	USA
1. Type of program	An universal flat-rate benefit, which can be topped up with an income-tested benefit, and the earnings-related public schemes Canada Pension Plan (CPP)/Quebec Pension Plan (QPP). A set of major changes introduced in 1998 modified in particular the financing approach, changing it from a PAYG basis to a hybrid of PAYG financing and full funding called "steady-state funding".	There is an old earnings-related pension (social insurance, SI). The new earnings-related part is based on notional accounts (NDC) and there is a small mandatory contribution to individual (MIA), defined-contribution funded pensions. There is also a pension-income-tested top-up. Occupational pension plans – with defined-benefit and defined-contribution elements – have broad coverage.	The OASDI program consists of two separate parts that pay benefits to workers and their families - Old-Age and Survivors Insurance (OASI) and Disability Insurance (DI). The benefit formula is progressive. There is also a means-tested top-up payment available for low-income pensioners.
2. Coverage	CPP/QPP: Employees and self-employed persons working in Canada. Exclusions: Casual workers (annual earnings less than C$ 3,500) and seasonal agricultural workers.	SI: All employed and self-employed persons born in 1937 or earlier, but there is a gradual transition from the SI system to the NDC and MIA systems for persons born from 1938 to 1953. NDC and MIA: All employed and self-employed persons born in 1954 or later.	Gainfully employed persons, including the self-employed. Exclusions: Casual agricultural, household and election employees; some categories of self-employed persons (when annual net income is below $ 400); and certain federal employees hired before 1 January 1984. Voluntary coverage for state and local government employees (mandatory coverage for state and local government employees not covered by a retirement system, effective 1 July 1991) and clergy. Special systems for railroad employees, certain federal employees and many state and local government employees.

Items	Canada	Sweden	USA
3. Qualifying conditions: a) retirement b) permanent disability	a) CPP/QPP: Age 65 (full pension) or aged 60 to 64 (reduced pension) with at least one valid contribution. Before 2012: If the pension is awarded before age 65, the insured person must have fully or substantially ceased employment. If the pension is awarded at age 65 or older, retirement from employment is not necessary. Pensioners who return to work can no longer contribute towards their retirement pension. Beginning in 2012: Pensioners who continue to work contribute to the CPP Post-Retirement Benefit. Contributions on pensionable employment income is mandatory for those aged 60 to 64 and voluntary for those aged 65 to 70. The pension may be deferred. b) CPP/QPP: The insured person must be assessed with a severe and prolonged disability and have contributions in four of the last six years, or three of the last six years for those with 25 or more years of contributions who are assessed with a disability no earlier than 31 December 2006. QPP normally requires contributions in half the years in which contributions could have been made; the contribution period is two of the last three years.	a) SI: Age 65 with 30 years of coverage. The insured person must have years with annual earnings of at least 44,900 kronor. A reduced pension is paid with at least three years of coverage. A reduced pension may be paid from age 61 to 64, and the pension may be deferred until age 70. NDC and MIA: The retirement age is flexible, beginning at age 61. The pension is based on lifetime earnings. The insured person must have years with annual earnings of at least 18,612 kronor. b) The insured person must have at least a 25% assessed loss of work capacity and be covered when the disability began. The disability pension consists of a guarantee (GDP) and an earnings-related pension (ERP). GDP: The insured must have at least three years of coverage. The pension is based on residence. ERP: The insured must have at least one year of income in Sweden within a given period. There is a constant-attendance supplement paid if the insured person requires the constant attendance of others to perform daily functions.	a) Age 66 (rising to age 67 by 2027) with at least 40 quarters of coverage. A reduced pension is paid from age 62, and the pension may be deferred up to age 70. b) The insured person must be assessed as incapable of substantial gainful activity as the result of a physical or mental impairment that is expected to last at least a year or result in death. The insured person must have a quarter of coverage for each year since age 21 up to the year the disability began, up to 40 quarters of coverage. The insured person must also have 20 quarters of coverage in the 10-year period before the disability began. The qualifying conditions for the young and the blind are less strict. The primary insurance amount (PIA) is derived from the insured person's covered lifetime earnings and is the basis for determining benefit amounts for the insured person and their family members.

Items	Canada	Sweden	USA
4. Benefits: a) retirement	a) CPP/QPP: The full pension is paid at age 65 and represents about 25% of the insured person's average monthly pensionable earnings during the contributory period. The pension is reduced by 0.5% a month (rising gradually to 0.6% from 2012 to 2016) for each month under age 65 that the pension is taken. The pension is increased by 0.57% a month (rising gradually to 0.7% by 2013) for each month between age 65 and the start of the pension. No adjustment is made after age 70. The maximum monthly pension taken at age 65 is C$ 960. Pension credits accumulated by spouses or common-law partners (same sex or opposite sex) during marriage or cohabitation may be divided equally in cases of divorce or separation. Recorded earnings are adjusted according to changes in national average wages.	a) SI: The pension is 60% of the insured person's average income above 44,900 kronor in the 15 best years. For years with earnings below 44,900 kronor, 96% if single; 78.5% if married. The average income level used to calculate benefits varies from year to year. The pension is reduced proportionately for less than 30 years of coverage. The pension is reduced by 0.5% for each month the pension is taken before age 65, and is increased by 0.7% for each month the pension is deferred from age 65 to age 70. NDC: At retirement, the accumulated notional capital will be converted into an annuity. This calculation will use a coefficient depending on individual retirement age and contemporaneous life expectancy. A real discount rate of 1.6% a year will be assumed in this calculation MIA: The pension is based on contributions plus net returns converted into an individual, joint, fixed or variable annuity. At retirement, people have a choice over the way benefits are withdrawn.	a) The pension is based on the average of the insured person's 35 highest years of earnings indexed for past wage inflation, up to age 62. The first (bend point) $ 791 a month of relevant earnings attracts a 90% replacement rate. The band of earnings between $ 791 and $ 4,768 a month is replaced at 32%. A replacement rate of 15% applies between the latter threshold and the earnings ceiling. The pension is reduced (increased) for each month of receipt before (after) the full retirement age (FRA). The increment amount depends on the year the insured person reached age 62. There is no minimum pension for insured persons reaching age 62 after 1981. The maximum monthly pension for workers retiring in 2013 at the FRA is $ 2,533. Dependent's allowance: 50% of the insured person's PIA is paid to a wife or husband: 1.-At the FRA (reduced from age 62 up to the FRA) 2.-At any age caring for a child under age 16 or disabled; to each child (or dependent grandchild) under age 18 or aged 18 to 19 and attending elementary or secondary school full time (no age limit if disability began before age 22). The maximum family pension ranges from 150% to 188% of the insured person's PIA.

Items	Canada	Sweden	USA
4. Benefits: b) permanent disability	b) CPP/QPP: A basic monthly pension of C$ 433.37 plus 75% of the earnings-related retirement pension. The maximum monthly pension is C$ 1,153.37. The disability pension is replaced by a retirement pension at age 65. Child's supplement is paid for each child under age 18 (age 25 if a student)	b) Benefit adjustment: Benefits are adjusted annually according to changes in prices. GDP: Full benefit is paid with at least 40 years of residence and no earnings-related benefit. The pension is reduced by 2.5% for each year of residence less than 40. A reduced pension is paid at 75%, 50% or 25% of the full pension according to the assessed degree of disability. ERP: The pension is 64% of the insured person 's assumed future annual income. The maximum annual income used to calculate benefits is 330,000 kronor. Assumed future income is based on the average of the three best income years within a given period immediately before the year of the claim. The maximum annual benefit is 211,200 kronor.	b) The pension is based on the insured person's average covered earnings (indexed for past wage inflation) from age 21 up to the onset of disability, excluding up to five years of the lowest earnings. There is no minimum pension for insured persons disabled after 1981. The maximum monthly pension for insured persons disabled at age 50 in 2010 is $ 2,485. The maximum pension for insured persons disabled at any other age is computed based on that age. Dependent's allowances are also applied. The maximum family benefit ranges from 100% to 150% of the insured person's primary insurance amount.

Source: OECD (2012 and 2011), ISSA (2012) and
http://www.ssa.gov/
http://www.servicecanada.gc.ca/eng/sc/cpp/retirement/canadapension.shtml
http://www.pensionsmyndigheten.se

APPENDIX 2: THE PROCESS FOR OBTAINING THE ANALYTICAL EXPRESSIONS FOR THE SYSTEM'S TURNOVER DURATION (TD) AND CONTRIBUTION ASSET (CA)

A. DESCRIPTION OF THE SYSTEM AND DETERMINATION OF THE YEAR IN WHICH IT REACHES A MATURE STATE

In the scenario described in Section 2, the stability of the total contribution rate ($\theta^D + \theta^R$) that ensures equality between contribution revenue and pension expenditure depends on the stability of the dependency ratios of both contingencies. For the retirement contingency, the contribution rate from year "w-x_e-A", counting from the system's inception, can be considered constant from the actuarial point of view because from that moment the ratio between retirement pensioners (R) and contributors (C), ($\frac{R}{C} = dr^R$), stabilizes

$$dr_t = \frac{\overbrace{\sum_{k=1}^{A} I_{(x_e+k,t)}}^{\text{disabled (working ages) } (D_t^d)} + \overbrace{\sum_{k=1}^{w-x_e-A-1} I_{(x_e+A+k,t)}}^{\text{disabled (retirement ages) } (D_t^r)} + \overbrace{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)\cdot(1+\gamma)^{t-1-k}}}^{\text{retirement pensioners} = R_t}}{\underbrace{(1+\gamma)^{t-1}\cdot\sum_{k=0}^{A-1} N_{(x_e+k,1)}}_{\text{contributors} = C_t}}$$

$$= \frac{\sum_{k=1}^{A}\sum_{s=1}^{k} I_{(x_e+s,1)\cdot(1+\gamma)^{-k+s}} \cdot {}_{k-s}p_{x_e+s}^d + \sum_{k=1}^{w-x_e-A-1}\left(\sum_{s=1}^{A} I_{(x_e+s,1)} \cdot (1+\gamma)^{-A-k+s} \cdot {}_{A+k-s}p_{x_e+s}^d\right)}{\sum_{k=0}^{A-1} N_{(x_e+k,1)}}$$

$$+ \frac{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)\cdot(1+\gamma)^{-k}}}{\sum_{k=0}^{A-1} N_{(x_e+k,1)}} = dr_t = \ldots\ldots = dr = \frac{D+R}{C} = \frac{D}{C} + \frac{R}{C} = dr^D + dr^R. \qquad (18)$$

The same moment "$w - x_e - A$" can be considered for disability pensioners from retirement age onwards, but for continuing disability pensioners these pensions end up being dependent on pensions from before retirement age. Hence the ratio between disability pensioners (D) and contributors (C), ($\frac{D}{C} = dr^D$), does not stabilize until "$w - x_e - 1$". Given that it is clear that $w - x_e - 1 > w - x_e - A$ and it is assumed that $t >= w - x_e - 1$, the contributor/pensioner ratio must be stable because all three collectives — disability pensioners, retirement pensioners and contributors — evolve (growing or shrinking) at a rate exactly the same as γ.

From that year onwards the system is "mature" and the expressions for the contribution rates for both contingencies (retirement/R and disability/D), which can be separated, are:

$$\theta_t^R = \frac{\overbrace{\beta \cdot Y_{C,0} \cdot \sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot (1+G)^{t-k} \cdot (1+\lambda)^k}^{\text{Expenditure on retirement benefits}}}{\underbrace{(1+G)^t \cdot \sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}_{\text{Aggregate contribution base}}}$$

$$= \frac{P^r_{(x_e+A,1)} \cdot \sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}} = \theta_{t+1}^R = \ldots = \theta^R \quad (19)$$

and also like in the case of disability, the retirement pension can be expressed as:

$$P^r_{(x_e+A,t)} = P^r_{(x_e+A,1)} \cdot (1+g)^{t-1} \cdot = \beta \cdot Y_{C,0} \cdot (1+g)^{t-1} \quad (20)$$

while the contribution rate for the disability contingency is:

$$\theta_t^D = \frac{\overbrace{\sum_{k=1}^{A} \sum_{s=1}^{k} P^d_{(x_e+s,1)} \cdot I_{(x_e+s,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{k-s} \cdot {}_{k-s}p^d_{x_e+s}}^{\text{Expenditure on disability benefits (working ages)}} + \overbrace{\sum_{k=1}^{w-x_e-A-1} \left(\sum_{s=1}^{A} P^d_{(x_e+s,1)} \cdot I_{(x_e+s,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{A+k-s} \cdot {}_{A+k-s}p^d_{x_e+s}\right)}^{\text{Expenditure on disability benefits (retirement ages)}}}{\underbrace{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}_{\text{Aggregate contribution base}}} = \ldots = \theta^D. \quad (21)$$

If the system's average disability pension is considered to be:

$$\overline{P}_t^D = \frac{\overbrace{(1+g)^{t-1} \cdot \left(\sum_{k=1}^{A} \left(\sum_{s=1}^{k} P^d_{(x_e+s,1)} \cdot I_{(x_e+s,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{k-s} \cdot {}_{k-s}p^d_{x_e+s}\right)\right)}^{\text{Expenditure on disability benefits (working ages)}}}{\underbrace{\sum_{k=1}^{A} \sum_{s=1}^{k} I_{(x_e+s,1)} \cdot (1+\gamma)^{-k+s} \cdot {}_{k-s}p^d_{x_e+s} + \sum_{k=1}^{w-x_e-A-1} \left(\sum_{s=1}^{A} I_{(x_e+s,1)} \cdot (1+\gamma)^{-A-k+s} \cdot {}_{A+k-s}p^d_{x_e+s}\right)}_{\text{Disability pensioners}}}$$

$$+ \frac{\overbrace{(1+g)^{t-1} \cdot \left(\sum_{k=1}^{w-x_e-A-1} \left(\sum_{s=1}^{A} P^d_{(x_e+s,1)} \cdot I_{(x_e+s,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{A+k-s} {}_{A+k-s}p^d_{x_e+s}\right)\right)}^{\text{Expenditure on disability benefits (retirement ages)}}}{\underbrace{\sum_{k=1}^{A} \sum_{s=1}^{k} I_{(x_e+s,1)} \cdot (1+\gamma)^{-k+s} \cdot {}_{k-s}p^d_{x_e+s} + \sum_{k=1}^{w-x_e-A-1} \left(\sum_{s=1}^{A} I_{(x_e+s,1)} \cdot (1+\gamma)^{-A-k+s} \cdot {}_{A+k-s}p^d_{x_e+s}\right)}_{\text{Disability pensioners}}} \quad (22)$$

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then the system's average retirement pension, taking into account (20), can be expressed as:

$$\overline{P}_t^R = \frac{\overbrace{\beta \cdot Y_{C,0} \cdot \sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot (1+g)^{t-1-k} \cdot (1+\gamma)^{t-1-k} \cdot (1+\lambda)^k}^{\text{Expenditure on retirement pensions}}}{\underbrace{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot (1+\gamma)^{t-1-k}}_{\text{Retirement pensioners}=R_t}}$$

$$= \frac{P^r_{(x_e+A,t)} \cdot \sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot (1+\gamma)^{-k}} \tag{23}$$

with the average contribution base being:

$$\bar{W}_t = \frac{\sum_{k=0}^{A-1} y_{(x_e+k,t)} \cdot N_{(x_e+k,t)}}{\sum_{k=0}^{A-1} N_{(x_e+k,t)}} = \frac{\overbrace{(1+G)^{t-1} \cdot \sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}^{\text{Aggregate contribution base}}}{\underbrace{(1+\gamma)^{t-1} \cdot \sum_{k=0}^{A-1} N_{(x_e+k,1)}}_{\text{Contributors}=C_t}}$$

$$= \frac{(1+g)^{t-1} \cdot \left(\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}\right)}{\sum_{k=0}^{A-1} N_{(x_e+k,1)}} \tag{24}$$

In the "mature" state reached, the average pension/average contribution base quotient is already constant for both contingencies because the numerator and denominator evolve at the rate of variation in wages:

$$\frac{\bar{P}_t^D}{\bar{W}_t} = \frac{\bar{P}_{t+1}^D}{\bar{W}_{t+1}} = \ldots\ldots = \frac{\bar{P}^D}{\bar{W}} = fr^D, \quad \frac{\bar{P}_t^R}{\bar{W}_t} = \frac{\bar{P}_{t+1}^R}{\bar{W}_{t+1}} = \ldots\ldots = \frac{\bar{P}^R}{\bar{W}} = fr^R \tag{25}$$

Therefore the contribution rate that ensures equality between revenue and expenditure is the product of the demographic dependency ratio and the financial ratio:

$$\{\theta^D, \theta^R\} = \{fr^D \cdot dr^D, fr^R \cdot dr^R\} = \left\{\frac{\bar{P}^D}{\bar{W}}\frac{D}{C}, \frac{\bar{P}^R}{\bar{W}}\frac{R}{C}\right\} \tag{26}$$

B. Obtaining the analytical expressions for the system's liabilities from the actuarial point of view

Once the contribution rate has been determined for both contingencies, it is time to calculate the system's liabilities with contributors and pensioners to enable us to continue the process of obtaining the system's average turnover duration and the contribution asset. The formulas were shown in section 2.

C. Obtaining the analytical expression for the system's TD in the form of pay-out and pay-in duration

To obtain the TD in a financially sustainable PAYG system that includes both contingencies, like in the process described by Settergren and Mikula (2005) and Boado-Penas *et al.* (2008) which only considered the retirement contingency, the total liabilities are divided by the annual contribution flow. Also, in line with Gronchi and Nisticò (2008), the interest rate for discounting future pensions and contributions is taken to be the internal rate of return (IRR), i.e. the growth of the wage bill in the mature system. Therefore, the TD_t^D for the disability contingency is:

$$
\begin{aligned}
TD_t^D = \frac{{}^D V_t^T}{C_t^D} = {} & \frac{\overbrace{\sum_{k=1}^{A}\left(\sum_{h=1}^{k} P^d_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{k-h} \cdot {}_{k-h}p^d_{x_e+h}\right) \cdot {}^d\ddot{a}^{\lambda}_{x_e+k}}^{\text{present value of disability pensions (working ages)}}}{\theta^D \cdot \left(\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}\right)} \\
& + \frac{\overbrace{\sum_{k=1}^{w-x_e-A-1}\left(\sum_{h=1}^{A} P^d_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{A+k-h} \cdot {}_{A+k-h}p^d_{x_e+h}\right) \cdot {}^d\ddot{a}^{\lambda}_{x_e+A+k}}^{\text{present value of disability pensions (retirement ages)}}}{\theta^D \cdot \left(\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}\right)} \\
& + \frac{\overbrace{\sum_{k=1}^{A}\sum_{h=k}^{A} P^d_{(x_e+h,1)} \cdot I^N_{(x_e+h,1)} \cdot {}^d\ddot{a}^{\lambda}_{x_e+h} \cdot \left[\frac{1+G}{1+d}\right]^{h}}^{\text{present value of future disability pensions}}}{\theta^D \cdot \left(\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}\right)} \\
& - \frac{\overbrace{\sum_{k=0}^{A-1}\sum_{h=0}^{k} N_{(x_e+k,1)} \cdot y_{(x_e+k,1)} \cdot \left[\frac{1+G}{1+d}\right]^{h}}^{\text{present value of future contributions}}}{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}
\end{aligned} \tag{27}
$$

By substituting formula (21) into equation (27), the TD_t^D can be expressed as:

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$$TD_t^D = \overbrace{\frac{\sum_{k=1}^{A} {}^{d}\ddot{a}_{x_e+k}^{\lambda} \cdot \left(\sum_{h=1}^{k} P_{(x_e+h,1)}^{d} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h} \cdot {}_{k-h}p_{x_e+h}^{d}\right)}{\sum_{k=1}^{A}\left(\sum_{h=1}^{k} P_{(x_e+h,1)}^{d} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h} \cdot {}_{k-h}p_{x_e+h}^{d}\right) + \sum_{k=1}^{w-x_e-A-1}\left(\sum_{h=1}^{A} P_{(x_e+h,1)}^{d} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h} \cdot {}_{A+k-h}p_{x_e+h}^{d}\right)}}^{1}$$

$$+ \overbrace{\frac{\sum_{k=1}^{w-x_e-A-1} {}^{d}\ddot{a}_{x_e+A+k}^{\lambda} \cdot \left(\sum_{h=1}^{A} P_{(x_e+h,1)}^{d} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h} \cdot {}_{A+k-h}p_{x_e+h}^{d}\right)}{\sum_{k=1}^{A}\left(\sum_{h=1}^{k} P_{(x_e+h,1)}^{d} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h} \cdot {}_{k-h}p_{x_e+h}^{d}\right) + \sum_{k=1}^{w-x_e-A-1}\left(\sum_{h=1}^{A} P_{(x_e+h,1)}^{d} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h} \cdot {}_{A+k-h}p_{x_e+h}^{d}\right)}}^{2}$$

$$+ \overbrace{\frac{\sum_{k=1}^{A} P_{(x_e+k,1)}^{d} \cdot I_{(x_e+k,1)}^{N} \cdot {}^{d}\ddot{a}_{x_e+k}^{\lambda} \cdot \left(\sum_{h=1}^{k}\left[\frac{1+G}{1+d}\right]^{h}\right)}{\sum_{k=1}^{A}\left(\sum_{h=1}^{k} P_{(x_e+h,1)}^{d} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h} \cdot {}_{k-h}p_{x_e+h}^{d}\right) + \sum_{k=1}^{w-x_e-A-1}\left(\sum_{h=1}^{A} P_{(x_e+h,1)}^{d} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h} \cdot {}_{A+k-h}p_{x_e+h}^{d}\right)}}^{3}$$

$$- \overbrace{\frac{\sum_{k=0}^{A-1}\sum_{h=0}^{k} N_{(x_e+k,1)} \cdot y_{(x_e+k,1)} \cdot \left[\frac{1+G}{1+d}\right]^{h}}{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}}^{4}. \tag{28}$$

If we assume that $(1+g)\cdot(1+\gamma)-1 = d = G$, the numerator of the third term of (28), after some transformations, is equal to:

$$\sum_{k=1}^{A} P_{(x_e+k,1)}^{d} \cdot I_{(x_e+k,1)}^{N} \cdot {}^{d}\ddot{a}_{x_e+k}^{\lambda}\left(\sum_{h=1}^{k}\left[\frac{1+G}{1+d}\right]^{h}\right) = \sum_{k=1}^{A} P_{(x_e+k,1)}^{d} \cdot I_{(x_e+k,1)}^{N} \cdot {}^{d}\ddot{a}_{x_e+k}^{\lambda} \cdot k \tag{29}$$

and if we consider that the denominator of the first 3 terms of expression (28) — the present value of disability benefits awarded in year t - is equivalent to the year's disability contributions, i.e. expenditure on disability pensions in year t,

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then

$$\sum_{k=1}^{A}\left(\sum_{h=1}^{k} P^{d}_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h} \cdot {}_{k-h}p^{d}_{x_e+h}\right) + \sum_{k=1}^{w-1-(x_e+A)}\left(\sum_{h=1}^{A} P^{d}_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h} \cdot {}_{A+k-h}p^{d}_{x_e+h}\right) = \sum_{k=1}^{A} P^{d}_{(x_e+k,1)} \cdot I^{N}_{(x_e+k,1)} \cdot {}^{d}\ddot{a}^{\lambda}_{x_e+k}. \quad (30)$$

The same result was obtained in the paper by Vidal-Meliá and Boado-Penas (2013) for the case of a system with a retirement contingency. Therefore it can be said that the system's income from contributions is equivalent to the present actuarial value of the pensions awarded in that year (commitments the system takes on with pensioners who have just retired).

Then, after algebraically manipulating the numerator of the fourth term of formula (28), TD^{D}_{t} works out as:

$$TD^{D}_{t} = \left[\overbrace{\frac{\sum_{k=1}^{A} {}^{d}\ddot{a}^{\lambda}_{x_e+k} \cdot \left(\sum_{h=1}^{k} P^{d}_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h} \cdot {}_{k-h}p^{d}_{x_e+h}\right)}{\sum_{k=1}^{A} P^{d}_{(x_e+k,1)} \cdot I^{N}_{(x_e+k,1)} \cdot {}^{I}\ddot{a}^{\lambda}_{x_e+k}}}^{1=pt_r^{D1}} + \overbrace{\frac{\sum_{k=1}^{w-x_e-A-1} {}^{d}\ddot{a}^{\lambda}_{x_e+A+k} \cdot \left(\sum_{h=1}^{A} P^{d}_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h} \cdot {}_{A+k-h}p^{d}_{x_e+h}\right)}{\sum_{k=1}^{A} P^{d}_{(x_e+k,1)} \cdot I^{N}_{(x_e+k,1)} \cdot {}^{d}\ddot{a}^{\lambda}_{x_e+k}}}^{2=pt_r^{D2}}\right] = pt^{D}_{r}$$

$$+ \left[\underbrace{\overbrace{\frac{\sum_{k=1}^{A} P^{d}_{(x_e+k,1)} \cdot I^{N}_{(x_e+k,1)} \cdot {}^{d}\ddot{a}^{\lambda}_{x_e+k} \cdot k}{\sum_{k=1}^{A} P^{d}_{(x_e+k,1)} \cdot I^{N}_{(x_e+k,1)} \cdot {}^{d}\ddot{a}^{\lambda}_{x_e+k}}}^{3}}_{\bar{k}^{D}_{t}} - \overbrace{\frac{\sum_{k=0}^{A-1} N_{(x_e+k,1)} \cdot y_{(x_e+k,1)} \cdot (k+1)}{\sum_{k=0}^{A-1} N_{(x_e+k,1)} \cdot y_{(x_e+k,1)}}}^{4}\right] = pt^{D}_{c}. \quad (31)$$

The third addend of the expression is a weighted average of years contributed until entry into the state of disability starting at age $x_e + 1$ for current contributors, $\bar{k}^{D}_{t} \in [1, A]$. Also, as happened in the case of retirement, the average TD is clearly disaggregated into two sub-periods termed pay-in, pt^{D}_{c}, and pay-out, pt^{D}_{r}, which correspond to the time that one monetary unit contributed to the disability contingency forms part of the liabilities to contributors and pensioners respectively. The pay-out could be broken down further into sub-periods, one part deriving from the disability age band in which there are contributors, pt_r^{D1}, and the other deriving from the disability age band in which there are retirement pensioners, pt_r^{D2}.

According to Vidal-Meliá and Boado-Penas (2013), the TD_t^R for the retirement contingency is:

$$TD_t^R = \frac{\sum_{k=0}^{w-xe-A-1} N_{(x_e+A+k,1)} \cdot \ddot{a}^{\lambda}_{x_e+A+k} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^k}{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)\cdot\left[\frac{(1+\lambda)}{(1+G)}\right]^k}} + \overbrace{\frac{N_{(x_e+A,1)} \cdot \ddot{a}^{\lambda}_{x_e+A} \cdot \sum_{h=1}^{A} \left[\frac{(1+G)}{(1+d)}\right]^h}{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)\cdot\left[\frac{(1+\lambda)}{(1+G)}\right]^k}}}^{A} - \frac{\sum_{k=0}^{A-1} \sum_{h=0}^{k} N_{(x_e+k,1)} \cdot y_{(x_e+k,1)} \cdot \left[\frac{(1+G)}{(1+d)}\right]^h}{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}. \quad (32)$$

After some algebraic manipulations and taking into account that the second term of (32) is equal to A, the formula for the generations of contributors coexisting at each moment in time can be expressed as:

$$TD_t^R = \underbrace{\overbrace{\frac{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot \ddot{a}^{\lambda}_{x_e+A+k} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}{N_{(x_e+A,1)} \cdot \ddot{a}^{\lambda}_{x_e+A}}}^{\text{Pay out duration}}}_{pt_r^R} + \underbrace{\overbrace{A - \frac{\sum_{k=0}^{A-1} N_{(x_e+k,1)} \cdot y_{(x_e+k,1)} \cdot (k+1)}{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}}^{pt_c^R}}_{\text{Pay in duration}}. \quad (33)$$

D. Obtaining the expression for the TD as the difference in the weighted average ages of the pensioners and contributors

The expressions obtained so far are the basis for determining the TD according to the ages of the contributor and pensioner collectives. This will make it possible to calculate representative values for the items forming part of the system's contribution asset and, by comparing them with the liabilities, obtain solvency indicators.

The weighted average age at which contributions cease to be made to the disability contingency, $\bar{x}_t^D$, would be.[12]

$$\bar{x}_t^D = x_e - 1 + \bar{k}_t^D = \frac{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k} \cdot (x_e + k - 1)}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k}}. \tag{34}$$

It is important to bear in mind that for the retirement contingency in this model, determining the average age of entry into retirement needs no further calculation because it is assumed that there is just a single retirement age, $x_e + A$, and contributions for this contingency cease one year earlier. However, formula (34) is similar if not identical in structure to the formula used by the Swedish authorities for the NDC system which only includes the retirement contingency.[13]

If we take the expression for the TD_t^D determined by formula (31) and add to it and subtract from it the weighted average age at which disability contingency contributions cease, $\bar{x}_t^D$, the TD can be expressed as the difference between the weighted average age of the disability pensioners, A_r^D, and the weighted average age of the contributors, $A_c^R = A_c^D$:

$$\begin{aligned} TD_t^{\mathrm{D}} &= x_e + \bar{k}_t^D - 1 \\ &+ \underbrace{\frac{\sum_{k=1}^{A} {}^I\ddot{a}^\lambda_{x_e+k} \left(\sum_{h=1}^{k} P^d_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h} \cdot {}_{k-h}p^d_{x_e+h} \right)}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k}}}_{pt_r^{D1}} \\ &+ \underbrace{\frac{\sum_{k=1}^{w-x_e-A-1} {}^d\ddot{a}^\lambda_{x_e+A+k} \cdot \left(\sum_{h=1}^{A} P^d_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h} \cdot {}_{A+k-h}p^d_{x_e+h} \right)}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k}}}_{pt_r^{D2}} \\ &- \underbrace{\left(x_e + \bar{k}_t^D - 1 \underbrace{- \bar{k}_t^D + \frac{\sum_{k=0}^{A-1} N_{(x_e+k,1)} \cdot y_{(x_e+k,1)} \cdot (k+1)}{\sum_{k=0}^{A-1} N_{(x_e+k,1)} \cdot y_{(x_e+k,1)}}}_{-\,pt_c^D} \right)}_{A_c^D}. \end{aligned} \tag{35}$$

Note that, unlike what happens in the retirement contingency, the pay-in can have a negative value in the disability contingency if the weighted average age

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at which contributions to the disability contingency cease is lower than the weighted average age of the contributors. In fact it is difficult for this situation to come about, but it could happen if the probabilities of becoming disabled were decreasing with the age of the contributors and the system's structure had many more younger contributors than older ones.

If the first term (the weighted average age at which contributions to the disability contingency cease) is added to the second and third addends (pay-out) and it is considered that total spending on disability pensions for beneficiaries aged $x_e + k$ years and $x_e + A + k$ years respectively can be expressed by:

$$PT^D_{(x_e+k,1)} = \sum_{h=1}^{k} P^d_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{k-h} \cdot {}_{k-h}p^d_{x_e+h} \tag{36}$$

$$PT^D_{(x_e+A+k,1)} = \sum_{h=1}^{A} P^d_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{A+k-h} \cdot {}_{A+k-h}p^d_{x_e+h} \tag{37}$$

and then after some algebra we get:

$$pt^D_r = \frac{\overbrace{\sum_{k=1}^{A} \left({}^d\ddot{a}^{\lambda}_{x_e+k} + x_e + \bar{k}^D_t - 1\right) \cdot \left(PT^D_{(x_e+k,1)}\right)}^{1}}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^{\lambda}_{x_e+k}} + \frac{\overbrace{\sum_{k=1}^{w-x_e-A-1} \left({}^d\ddot{a}^{\lambda}_{x_e+A+k} + x_e + \bar{k}^D_t - 1\right) \cdot \left(PT^D_{(x_e+A+k,1)}\right)}^{2}}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^{\lambda}_{x_e+k}}. \tag{38}$$

Once this has been developed as necessary, the numerator of the first addend (1) of expression (38) can be expressed as:

$$\sum_{k=1}^{A} \left({}^d\ddot{a}^{\lambda}_{x_e+k} + x_e + \bar{k}^D_t - 1\right) \cdot PT^D_{(x_e+k,1)} = \sum_{k=1}^{A} PT^D_{(x_e+k,1)} \cdot \left(x_e + \bar{k}^D_t + k - 1\right) + A \cdot \left(\sum_{k=1}^{w-x_e-A-1} PT^D_{(x_e+A+k,1)}\right). \tag{39}$$

Continuing along similar lines with the numerator of the second summand (2) of expression (38), we get:

$$\sum_{k=1}^{w-x_e-A-1} \left({}^{d}\ddot{a}^{\lambda}_{x_e+A+k} + x_e + \bar{k}^{D}_{t} - 1\right) \cdot PT^{D}_{(x_e+A+k,1)} = \sum_{k=1}^{w-x_e-A-1} PT^{D}_{(x_e+A+k,1)} \cdot \left(x_e + \bar{k}^{D}_{t} + k - 1\right). \tag{40}$$

If the results of (39) and (40) are added, we get:

$$\begin{aligned}
&\sum_{k=1}^{A} \left({}^{d}\ddot{a}^{\lambda}_{x_e+k} + x_e + \bar{k}^{D}_{t} - 1\right) \cdot PT^{D}_{(x_e+k,1)} \\
&\quad + \sum_{k=1}^{w-x_e-A-1} \left({}^{a}\ddot{a}^{\lambda}_{x_e+A+k} + x_e + \bar{k}^{D}_{t} - 1\right) \cdot PT^{D}_{(x_e+A+k,1)} \\
&= \sum_{k=1}^{A} PT^{D}_{(x_e+k,1)} \cdot \left(x_e + \bar{k}^{D}_{t} + k - 1\right) + A \cdot \left(\sum_{k=1}^{w-x_e-A-1} PT^{D}_{(x_e+A+k,1)}\right) \\
&\quad + \sum_{k=1}^{w-x_e-A-1} PT^{D}_{(x_e+A+k,1)} \cdot \left(x_e + \bar{k}^{D}_{t} + k - 1\right) \\
&= \overbrace{\sum_{k=1}^{A} PT^{D}_{(x_e+k,1)} \cdot \left(x_e + \bar{k}^{D}_{t} + k - 1\right)}^{1} + \overbrace{\sum_{k=1}^{w-x_e-A-1} PT^{D}_{(x_e+A+k,1)} \cdot \left(x_e + \bar{k}^{D}_{t} + A + k - 1\right)}^{2} \\
&= \overbrace{\sum_{k=1}^{A} \left(x_e + \bar{k}^{D}_{t} + k - 1\right) \cdot \left(\sum_{h=1}^{k} P^{D}_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{k-h} \cdot {}_{k-h}p^{d}_{x_e+h}\right)}^{1} \\
&\quad + \overbrace{\sum_{k=1}^{w-x_e-A-1} \left(x_e + \bar{k}^{D}_{t} + A + k - 1\right) \cdot \left(\sum_{h=1}^{k} P^{d}_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{A+k-h} \cdot {}_{A+k-h}p^{d}_{x_e+h}\right)}^{2}.
\end{aligned} \tag{41}$$

If the values for (1) and (2) shown in formula (41) are substituted into equation (38), the expression for the TD for disability can be formulated according to

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the difference between the average ages of those receiving disability benefits (by aggregating the first two addends) and the average age of the contributors:

$$TD_t^D = A_r^D - A_c^D = \left.\frac{\left(\begin{array}{l}\sum_{k=1}^{A}\left(x_e+\bar{k}_t^D-1+k\right)\cdot\left(\sum_{h=1}^{k}P_{(x_e+h,1)}^d\cdot I_{(x_e+h,1)}\cdot\left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h}\cdot {}_{k-h}p_{x_e+h}^d\right)\\ \quad+\sum_{k=1}^{w-x_e-A-1}\left(x_e+\bar{k}_t^D-1+A+k\right)\\ \quad\times\left(\sum_{h=1}^{A}P_{(x_e+h,1)}^d\cdot I_{(x_e+h,1)}\cdot\left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h}\cdot {}_{A+k-h}p_{x_e+h}^d\right)\end{array}\right)}{\sum_{k=1}^{A}P_{(x_e+k,1)}^d\cdot I_{(x_e+k,1)}^N\cdot {}^d\ddot{a}_{x_e+k}^{\lambda}}\right\} = A_r^D$$

$$-\left(\frac{\sum_{k=0}^{A-1}N_{(x_e+k,1)}\cdot y_{(x_e+k,1)}\cdot(x_e+k)}{\sum_{k=0}^{A-1}N_{(x_e+k,1)}\cdot y_{(x_e+k,1)}}\right) = A_c^D. \tag{42}$$

An alternative formula is:

$$TD_t^D = \left(x_e+\bar{k}_t^D-1\right)$$

$$\left.+\frac{\left(\begin{array}{l}\sum_{k=1}^{A}k\cdot\left(\sum_{h=1}^{k}P_{(x_e+h,1)}^d\cdot I_{(x_e+h,1)}\cdot\left[\frac{(1+\lambda)}{(1+G)}\right]^{k-h}\cdot {}_{k-h}p_{x_e+h}^d\right)\\ \quad+\sum_{k=1}^{w-x_e-A-1}(A+k)\cdot\\ \quad\times\left(\sum_{h=1}^{A}P_{(x_e+h,1)}^d\cdot I_{(x_e+h,1)}\cdot\left[\frac{(1+\lambda)}{(1+G)}\right]^{A+k-h}\cdot {}_{A+k-h}p_{x_e+h}^d\right)\end{array}\right)}{\sum_{k=1}^{A}P_{(x_e+k,1)}^d\cdot I_{(x_e+k,1)}^N\cdot {}^d\ddot{a}_{x_e+k}^{\lambda}}\right\} = A_r^D$$

$$-\left(\frac{\sum_{k=0}^{A-1}N_{(x_e+k,1)}\cdot y_{(x_e+k,1)}\cdot(x_e+k)}{\sum_{k=0}^{A-1}N_{(x_e+k,1)}\cdot y_{(x_e+k,1)}}\right) = A_c^D. \tag{43}$$

The second addend of A_r^D in (42) is just a weighted average of the years that the disabled people in age bands $[x_e+1, x_e+A]$ and $[x_e+A+1, w-1]$ have been receiving disability benefits.

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Vidal-Meliá and Boado-Penas (2013) obtained the equivalent expressions to (42) and (43) for the retirement contingency:

$$\mathrm{T}D_t^R = \overbrace{(x_e + A - 1) + pt_r^R}^{\text{weighted average age for the retirement pensioners}} - \underbrace{(x_e + A - 1 - pt_c^R)}_{\text{weighted average age for the retirement contributors}}$$

$$= \overbrace{(x_e + A - 1) + \frac{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot \ddot{a}^{\lambda}_{x_e+A+k} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}}^{\text{weighted average age for the retirement pensioners}}$$

$$- \underbrace{\left(x_e + A - 1 - A + \frac{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)} \cdot (k+1)}{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}\right)}_{\text{weighted average age for the retirement contributors}}.$$

$$= \underbrace{\frac{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot (x_e + A + k) \cdot \left[\frac{1+\lambda}{1+G}\right]^k}{\sum_{k=0}^{w-x_e-A-1} N_{(x_e+A+k,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}}_{A_r^R}$$

$$- \underbrace{\frac{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)} \cdot (x_e + k)}{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}}_{A_c^R} = A_r^R - A_c^R \tag{44}$$

E. Obtaining the system's TD and CA as weighting for the TDs and CAs for each contingency

Once the TD for each contingency has been determined, it is time to formulate the TD for the system, TD_t^S, which derives from the weighting of the various contingencies the system contains. The starting point for obtaining the expression is the value of the system's commitments with contributors and pensioners for the two contingencies:

$$TD_t^S = \frac{V_t^S}{C_t^S} = \frac{{}^D V_t^T + {}^R V_t^T}{C_t^R + C_t^D}$$

$$= \frac{{}^D V_t^r + {}^D V_t^c + {}^R V_t^r + {}^R V_t^c}{\theta^D \cdot \left((1+G)^{t-1} \cdot \sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}\right) + \theta^R \cdot \left((1+G)^{t-1} \cdot \sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}\right)}$$

$$= \underbrace{\frac{\overbrace{{}^{D}V_t^r + {}^{R}V_t^r}^{\text{liability to pensioners (both contingencies)}}}{(\theta^D + \theta^R)\cdot\left((1+G)^{t-1}\cdot\sum\limits_{k=0}^{A-1} y_{(x_e+k,1)}\cdot N_{(x_e+k,1)}\right)}}_{pt_r^S} + \underbrace{\frac{\overbrace{{}^{D}V_t^c + {}^{R}V_t^c}^{\text{liability to contributors (both contingencies)}}}{(\theta^D + \theta^R)\cdot\left((1+G)^{t-1}\cdot\sum\limits_{k=0}^{A-1} y_{(x_e+k,1)}\cdot N_{(x_e+k,1)}\right)}}_{pt_c^S}. \tag{45}$$

If we develop the second term of this expression, the pay-in for the whole system, pt_c^S, we get:

$$pt_c^S = \left\{\frac{\overbrace{\sum_{k=1}^{A}\sum_{h=k}^{A} P^d_{(x_e+h,t)}\cdot I^N_{(x_e+h,t)}\cdot {}^{d}\ddot{a}^{\lambda}_{x_e+h}\cdot\left[\frac{(1+G)}{(1+d)}\right]^h}^{\text{present value of future pensions (retirement and disability)}} + P^r_{(x_e+A,t)}\cdot N_{(xe+A,t)}\cdot\ddot{a}^{\lambda}_{xe+A}\cdot\sum_{h=1}^{A}\left[\frac{1+G}{1+d}\right]^h}{(\theta^D+\theta^R)\cdot\left((1+G)^{t-1}\cdot\sum_{k=0}^{A-1} y_{(x_e+k,1)}\cdot N_{(x_e+k,1)}\right)}\right\}(=1)$$

$$-\left\{\frac{\overbrace{(\theta^D+\theta^R)\cdot\left(\sum_{k=0}^{A-1}\sum_{h=0}^{k} N_{(xe+k,t)}\cdot y_{(xe+k,t)}\cdot\left[\frac{1+G}{1+d}\right]^h\right)}^{\text{present value of future system's contributions}}}{(\theta^D+\theta^R)\cdot\left((1+G)^{t-1}\cdot\sum_{k=0}^{A-1} y_{(x_e+k,1)}\cdot N_{(x_e+k,1)}\right)}\right\}(=2) \tag{46}$$

substituting $\{\theta^D, \theta^R\}$ with their developed expressions and

$$P^r_{(x_e+A,1)}\cdot N_{(xe+A,1)}\cdot\ddot{a}^{\lambda}_{xe+A}\cdot\sum_{k=1}^{A}\left[\frac{1+G}{1+d}\right]^k = P^r_{(x_e+A,1)}\cdot N_{(xe+A,1)}\cdot\ddot{a}^{\lambda}_{xe+A}\cdot A \tag{47}$$

can be substituted in the numerator, and given that (30) and

$$P^r_{(x_e+A,1)}\cdot\sum_{k=0}^{w-xe-A-1} N_{(x_e+A+k,1)}\cdot\left[\frac{1+\lambda}{1+G}\right]^k = P^r_{(x_e+A,1)}\cdot N_{(x_e+A,1)}\cdot\ddot{a}^{\lambda}_{x_e+A} \tag{48}$$

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can be substituted in the denominator, the minuend of expression (46) turns out to be a weighted average of $\bar{k}_t^D$and of A, with the weightings being the respective present actuarial values of the pensions in payment for each contingency, which is equivalent to pension spending for each contingency. In other words, it is a weighted average of the number of years until entry into the pensioner state beginning from age $x_e + 1$ for current contributors, $\bar{k}_t^S \in [1,\ A]$:

$$\bar{k}_t^S = \frac{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k} \cdot k + P^r_{(x_e+A,1)} \cdot N_{(xe+A,1)} \cdot \ddot{a}^\lambda_{xe+A} \cdot A}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)}{}^d\ddot{a}^\lambda_{x_e+k} + P^r_{(x_e+A,1)} \cdot N_{(x_e+A,1)}\, \ddot{a}^\lambda_{x_e+A}}. \quad (49)$$

The weighted average age at which contributions cease to be paid for both of the system's contingencies, $\bar{x}_t^S$, is a weighted average of $\bar{x}_t^D$ (the weighted average age at which contributions cease to be paid for the disability contingency) and "$x_e + A - 1$" years (the weighted average age at which contributions cease to be paid for the retirement contingency) for the spending on pensions for each contingency. Its expression is:

$$\bar{x}_t^S = x_e + \bar{k}_t^S - 1$$
$$= \frac{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k} \cdot (x_e + k - 1) + P^r_{(x_e+A,1)} \cdot N_{(xe+A,1)} \cdot \ddot{a}^\lambda_{xe+A} \cdot (x_e + A - 1)}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k} + P^r_{(x_e+A,1)} \cdot N_{(x_e+A,1)} \cdot \ddot{a}^\lambda_{x_e+A}}. \quad (50)$$

If we work out the second term, 2, of formula (46), which expresses total future contributions, then the system's pay-in total, pt_c^S, is notably simplified:

$$pt_c^S = \bar{k}_t^S - \frac{\sum_{k=0}^{A-1} N_{(xe+k,1)} \cdot y_{(xe+k,1)} \cdot (k+1)}{\sum_{k=0}^{A-1} y_{(x_e+k,1)} \cdot N_{(x_e+k,1)}}. \quad (51)$$

Returning to the first term, the system's total pay-out, pt_r^S, of formula (46), after substituting $\{\theta^D, \theta^R\}$ by their values in (19) and (21), we get:

$$pt_r^S = \frac{\sum_{k=1}^{A} {}^d\ddot{a}^\lambda_{x_e+k} \cdot PT^D_{(x_e+k,1)} + \sum_{k=1}^{w-x_e-A-1} {}^d\ddot{a}^\lambda_{x_e+A+k} \cdot PT^D_{(x_e+A+k,1)}}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k} + P^r_{(x_e+A,1)} \cdot \sum_{k=0}^{w-xe-A-1} N_{(x_e+A+k,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}$$
$$+ \frac{P^r_{(x_e+A,1)} \cdot \sum_{k=0}^{w-xe-A-1} N_{(x_e+A+k,1)} \cdot \ddot{a}^\lambda_{x_e+A+k} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}{\sum_{k=1}^{A} P^d_{(x_e+k,1)} \cdot I^N_{(x_e+k,1)} \cdot {}^d\ddot{a}^\lambda_{x_e+k} + P^r_{(x_e+A,1)} \cdot \sum_{k=0}^{w-xe-A-1} N_{(x_e+A+k,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^k}, \quad (52)$$

where

$$PT^D_{(x_e+k,1)} = \sum_{h=1}^{k} P^d_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{k-h} \cdot {}_{k-h}p^d_{x_e+h} \quad (53)$$

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$$PT^D_{(x_e+A+k,1)} = \sum_{h=1}^{w-x_e-A-1} P^d_{(x_e+h,1)} \cdot I_{(x_e+h,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^{A+k-h} \cdot {}_{A+k-h}p^d_{x_e+h}. \quad (54)$$

And, if we consider the following expressions for simplifying the weighted formulas:

$$PT^D = \sum_{k=1}^{A} PT^D_{(x_e+k,1)} + \sum_{k=1}^{w-x_e-A-1} PT^D_{(x_e+A+k,1)} \quad (55)$$

$$PT^R = P^r_{(x_e+A,1)} \cdot \sum_{k=0}^{w-xe-A-1} N_{(x_e+A+k,1)} \cdot \left[\frac{1+\lambda}{1+G}\right]^k \quad (56)$$

the denominator for the system's TD, its total spending on pensions, can be expressed by:

$$PT^S_t = PT^D_t + PT^R_t. \quad (57)$$

If the TDs for the disability and retirement contingencies $\{TD^D_t, TD^R_t\}$ are weighted by their respective total spending on pensions as part of the system's total spending on pensions, and given that the denominators of $\{TD^D_t, TD^R_t\}$ are respectively$\{PT^D, PT^R\}$, we get:

$$TD^S_t = \frac{PT^D_t \cdot TD^D_t + PT^R_t \cdot TD^R_t}{PT^S_t} = \frac{PT^D_t \cdot \frac{NTD^D_t}{PT^D_t} + PT^R_t \cdot \frac{NTD^R_T}{PT^R_t}}{PT^S_t}$$
$$= \frac{NTD^D_t + NTD^R_t}{PT^S_t} \quad (58)$$

an expression in which the numerator, NTD^S_t, is the sum of the numerators, $\{NTD^D_t, NTD^R_t\}$, of the TDs for disability and retirement, the same as in (46),

$$NTD^S_t = NTD^D_t + NTD^R_t \quad (59)$$

and the denominator is the system's total spending on pensions, PT^S_t, in the equilibrium of our model (the mature state), like to the value of contributions, C^S_t.

Thus, given that the numerator, NTD^S, and the denominator, PT^S, are the same as in (45), the expression coincides with the definition of the system's TD and we can therefore conclude that it can be calculated as a weighted average of the TDs for both contingencies, the weighting being the spending on pensions by contingency as a part of total spending.

Just like what happens with the TDs for the contingencies, the system's total TD can also be calculated according to the difference between the average ages of all the beneficiaries for both contingencies and the average age of the contributors.

$$TD_t^S = \frac{PT_t^D \cdot \left[A_r^D - A_c^D\right] + PT_t^R \cdot \left[A_r^R - A_c^R\right]}{PT_t^S}$$
$$= \frac{PT_t^D \cdot A_r^D + PT_t^R \cdot A_r^R}{PT_t^S} - \frac{PT_t^S \cdot A_c}{PT_t^S}$$
$$= \frac{PT_t^D \cdot A_r^D + PT_t^R \cdot A_r^R}{PT_t^S} - A_c. \quad (60)$$

To put it a different way, the TD_t^S can be obtained as the difference between the weighted average of the average ages of disability and retirement, the weightings being the spending on pensions per contingency as part of total spending, and the average age of the contributors.