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Stability and bifurcation analysis of a free boundary problem modelling multi-layer tumours with Gibbs–Thomson relation

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Of concern is the stability and bifurcation analysis of a free boundary problem modelling the growth of multi-layer tumours. A remarkable feature of this problem lies in that the free boundary is imposed with nonlinear boundary conditions, where a Gibbs–Thomson relation is taken into account. By employing a functional approach, analytic semigroup theory and bifurcation theory, we prove that there exists a positive threshold value γ_* of surface tension coefficient γ such that if $\gamma > \gamma_*$ then the unique flat stationary solution is asymptotically stable under non-flat perturbations, while for $\gamma < \gamma_*$ this unique flat stationary solution is unstable and there exists a series of non-flat stationary solutions bifurcating from it. The result indicates a significant phenomenon that a smaller value of surface tension coefficient γ may make tumours more aggressive.

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1 Introduction

During the past few decades, considerable attention has been attracted to mathematical models for the growth of various tumours *in vivo* and *in vitro* (cf. [5–7, 21–24, 26, 27]). Accordingly, a lot of illuminative results have been obtained for rigorous analysis of such tumour models (see, e.g., [3, 4, 8, 10–13, 17–20, 29–31] and the references cited therein). These studies provide us with very useful information on tumour growth and suggest a strategy for tumour treatment. Rigorous mathematical analysis of tumour growth models has been shown to be a prosperous subject of research and it contains many challenging problems.

This paper studies the following multi-dimensional free boundary problem modelling tumour growth

$$\begin{cases} \Delta\sigma = \lambda\sigma & \text{in } \Omega_{\rho(t)}, t \geqslant 0, \\ \Delta p = -\mu(\sigma - \tilde{\sigma}) & \text{in } \Omega_{\rho(t)}, t \geqslant 0, \\ \partial_y \sigma = 0, \quad \partial_y p = 0 & \text{on } \Gamma_0, t \geqslant 0, \\ \sigma = \bar{\sigma}(1 - 2\gamma\kappa), \quad p = \bar{p} & \text{on } \Gamma_{\rho(t)}, t \geqslant 0, \\ V = -\partial_{\vec{\nu}} p & \text{on } \Gamma_{\rho(t)}, t > 0, \\ \rho(0, \cdot) = \rho_0 & \text{at } t = 0, \end{cases} \tag{1.1}$$

where $\sigma = \sigma(t, x, y)$ and $p = p(t, x, y)$ are unknown functions defined on the time-space manifold $\cup_{t\geqslant 0}(\{t\} \times \Omega_{\rho(t)})$, and ρ_0 is the given initial data. Here $\Omega_{\rho(t)}$ is an *a priori* unknown strip-like domain

$$\Omega_{\rho(t)} := \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 < y < \rho(t, x), \; x \in \mathbb{R}^{n-1}\},$$

with the lower boundary $\Gamma_0 := \{y = 0\}$ and the upper boundary $\Gamma_{\rho(t)} := \{y = \rho(t, x)\}$ which is moving and has to be determined together with σ and p. In this model, Δ represents the Laplacian in the (x, y)-variables, V and $\vec{\nu}$ denote the normal velocity and the outward normal of the free boundary $\Gamma_{\rho(t)}$, respectively, and $\lambda, \mu, \tilde{\sigma}, \bar{\sigma}, \gamma$ and $\bar{p}$ are positive constants.

The problem (1.1) is a mathematical model for the growth of multi-layer tumour – a cluster of tumour cells cultivated on an impermeable support membrane in a laboratory by using a new tissue culture technique [22, 23, 26]. It is similar to multicellular tumour spheroids in its biological properties, but it consists of many layers of tumour cells and forms a strip-like structure. In this model, σ represents the scaled nutrient concentration, p stands for the scaled internal pressure, and $\tilde{\sigma}$ is the scaled threshold value for apoptosis of tumour cells. The conditions $\partial_y\sigma = 0, \partial_y p = 0$ on Γ_0 mean that none of the nutrient and the tumour cells can pass through the lower boundary. The relation $p = \bar{p}$ on $\Gamma_{\rho(t)}$ indicates that the pressure is continuous across the upper boundary, where $\bar{p}$ is the external pressure. The equation $V = -\partial_{\vec{\nu}} p$ follows from the continuity condition on the free boundary. The model is based on the hypothesis that energy is expended in maintaining the cell–cell bonds on the tumour surface and the nutrient acts as such energy and satisfies the Gibbs–Thomson relation, i.e., $\sigma = \bar{\sigma}(1 - 2\gamma\kappa)$, where $\bar{\sigma}$ is the external nutrient concentration, γ and κ are, respectively the surface tension coefficient and the mean curvature of $\Gamma_{\rho(t)}$. This relation states that on the free boundary σ is less than $\bar{\sigma}$ by a factor $2\gamma\kappa$, this being the energy needed to maintain the inner-cellular bonds existing on this boundary (cf. [5–7]). The Gibbs–Thomson relation in the nutrient boundary condition is a remarkable feature of this model.

The problem (1.1), which is usually called *model with Gibbs–Thomson relation*, has attracted significant mathematical attention. In fact, for the special case $\gamma = 0$, Byrne and Chaplain studied the model for a spheroid domain and proved that the radially symmetric steady state is unstable to small asymmetric perturbations [6]. Shortly after, Byrne did further analysis for this case by using a weakly nonlinear stability analysis in [5]. If the boundary conditions $\sigma = \bar{\sigma}(1-2\gamma\kappa), p = \bar{p}$ on $\Gamma_{\rho(t)}$ are replaced by the form $\sigma = \bar{\sigma}, p = \gamma\kappa$, which means that the tumour receives constant nutrient supply from the tumour surface and the pressure on the tumour surface is proportional to the mean curvature to maintain the cell-to-cell adhesiveness of the tumour, the corresponding problem is called *model with surface tension effect* and has been well studied. In more detail, Cui and Escher studied the model with surface tension effect and proved that if the surface tension coefficient γ is larger than a threshold value γ_* then the unique flat equilibrium is asymptotically stable, and if $\gamma < \gamma_*$ then this flat equilibrium is unstable (cf. [11]). Here a solution is called a flat solution if all the components of this solution are independent of the x-variable. In the situation that a flat stationary solution is unstable, Zhou, Cui and Escher proved in [32] that a series of non-flat stationary solutions bifurcates from this flat stationary solution. Later on, the corresponding analysis was extended to the inhibitor-present situation [33]. While for multicellular tumour spheroid models with surface tension effect, there has also been great progress, e.g., [10, 11, 13, 17–20, 29–31] and references cited therein.

In this paper, we carry out a systematic analysis of the model with Gibbs–Thomson relation (1.1). First, it is expected to figure out stability of flat stationary solutions of equation (1.1) under non-flat perturbations. We find that equation (1.1) has a unique flat stationary solution if and only if $\bar{\sigma} > \tilde{\sigma}$. Moreover, by employing a functional approach, a delicate spectrum analysis and geometric theory for parabolic differential equations in Banach spaces, we prove that there exists a threshold value $\gamma_* > 0$ of the surface tension coefficient γ such that if $\gamma > \gamma_*$ then the unique flat stationary solution of equation (1.1) is asymptotically stable under small non-flat perturbations, while for $\gamma < \gamma_*$ it is unstable. The crucial point in this process is to establish the well-posedness theory, which is handled by showing that the linearized operator of the transformed fully nonlinear equation generates an analytic semigroup. Secondly, we are concerned with existence of non-flat stationary solutions of equation (1.1). Similarly to [4, 10, 19], by treating the problem (1.1) as a bifurcation problem and applying the Crandall–Rabinowitz bifurcation theorem, we prove that for $\gamma < \gamma_*$ there exists a series of non-flat stationary solutions bifurcating from the unique flat stationary solution. It is worth noticing that our results indicate an interesting phenomenon that a smaller value of surface tension coefficient γ leads to smaller energy on the free boundary, which makes tumours more aggressive. Another new character is that the proliferation rate μ does not affect the stability of a tumour, which is in contrast with the widely studied models with surface tension effect where increasing the proliferation rate μ may lower a tumour's stability (cf. [11, 32]).

To give a precise statement of our main result, we need some notation. For the sake of simplicity, we impose the additional condition that $\rho(t,x), \sigma(t,x,y)$ and $p(t,x,y)$ are 2π-periodic in every component of x. Moreover, it is not an essential restriction to consider the case $n = 2$, because higher-dimensional periodic cases can be treated similarly. Thus we impose the following condition:

$$\rho(t,x),\ \sigma(t,x,y),\ p(t,x,y) \ \text{are } 2\pi\text{-periodic in } x \in \mathbb{R}. \tag{1.2}$$

In addition, we identify 2π-periodic functions with functions over the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. Accordingly, we identify the function space $C_{per}(\mathbb{R})$ of periodic functions on $\mathbb{R}$ with the corresponding function space $C(\mathbb{S}^1)$ on the circle $\mathbb{S}^1$. Let $\Omega = \mathbb{S}^1 \times (0,1)$ be fixed, which will be used as the reference domain. Denote the upper boundary $\mathbb{S}^1 \times \{1\}$ of Ω by Γ_1, the lower boundary $\mathbb{S}^1 \times \{0\}$ by Γ_0. Given $m \in \mathbb{N}$ and $\alpha \in (0,1)$, we denote by $h^{m+\alpha}(\mathbb{S}^1)$

the so-called little Hölder spaces on $\mathbb{S}^1$, i.e., the closure of $C^\infty(\mathbb{S}^1)$ in the corresponding usual Hölder spaces $C^{m+\alpha}(\mathbb{S}^1)$. The advantage of working with little Hölder spaces is that continuous injection and density hold between two spaces with different index. Let $h_+^{m+\alpha}(\mathbb{S}^1)$ stand for the cone of all positive functions in $h^{m+\alpha}(\mathbb{S}^1)$. Hereafter, we fix $\alpha \in (0,1)$.

To state the stability result of flat stationary solutions of equation (1.1), we define $\gamma_* := \sup\{\gamma_k |\, k = 1, 2, \ldots\}$, where

$$\gamma_k := \frac{(\bar{\sigma} - \tilde{\sigma})\lambda\rho_*}{2\bar{\sigma}k^2(\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*))} - \frac{\tilde{\sigma}\lambda\rho_*}{2\bar{\sigma}k^2}, \quad k = 1, 2, \ldots.$$

The first main result of this paper is formulated below.

Theorem 1.1

(i) *The problem* (1.1) *has a unique flat stationary solution* (ρ_*, σ_*, p_*) *if and only if* $\bar{\sigma} > \tilde{\sigma}$.

(ii) *If* $\bar{\sigma} > \tilde{\sigma}$ *and* $\gamma > \gamma_*$, *then the unique flat stationary solution* (ρ_*, σ_*, p_*) *is asymptotically stable in the following sense: There exists a constant* $\delta > 0$ *such that if* $\rho_0 \in h_+^{4+\alpha}(\mathbb{S}^1)$ *and* $\|\rho_0 - \rho_*\|_{h^{4+\alpha}(\mathbb{S}^1)} < \delta$ *then the solution* (ρ, σ, p) *of equation (1.1) exists for all* $t \geqslant 0$ *and converges to* (ρ_*, σ_*, p_*) *exponentially as* $t \to \infty$, *i.e., there are positive constants* M *and* ω *such that*

$$\|\rho(t,\cdot) - \rho_*\|_{h^{4+\alpha}(\mathbb{S}^1)} + \|\sigma(t,\cdot) - \sigma_*\|_{h^{2+\alpha}(\bar{\Omega})} + \|p(t,\cdot) - p_*\|_{h^{4+\alpha}(\bar{\Omega})} \leqslant Me^{-\omega t}, \quad t \geqslant 0.$$

If $\bar{\sigma} > \tilde{\sigma}$ *and* $0 < \gamma < \gamma_*$, *then* (ρ_*, σ_*, p_*) *is unstable.*

The proof of Theorem 1.1, which is given in Section 3, is proved in three steps. First, we transform (1.1) into a system of equations defined on a fixed domain, and then reduce the system into a fully nonlinear equation. Secondly, we show that the linearized operator generates a strongly continuous analytic semigroup. Finally, the result follows from the well-posedness and geometric theory for parabolic differential equations in Banach spaces.

To investigate existence of non-flat stationary solutions of equation (1.1), we regard equation (1.1) as a bifurcation problem with bifurcation parameter γ. The second main result is stated as follows.

Theorem 1.2 *Suppose* $\bar{\sigma} > \tilde{\sigma}$. *Then there exists a positive integer* k_* *such that for every* $k \geqslant k_*$, *the parameter* γ_k *is a bifurcation point of the flat stationary solution* (ρ_*, σ_*, p_*). *More precisely, in a suitable neighbourhood of* $(\gamma_k, \rho_*, \sigma_*, p_*)$ *there exists a series of bifurcation solutions* $(\gamma^\varepsilon, \rho^\varepsilon, \sigma^\varepsilon, p^\varepsilon)$ $(0 < \varepsilon \ll 1)$ *possessing the following asymptotic expansion:*

$$\gamma^\varepsilon = \gamma_k + O(\varepsilon), \quad \rho^\varepsilon(x) = \rho_* + \varepsilon \cos kx + O(\varepsilon^2),$$
$$\sigma^\varepsilon(x,y) = \sigma_*(y) + \varepsilon D_k(y) \cos kx + O(\varepsilon^2), \quad p^\varepsilon(x,y) = p_*(y) + \varepsilon E_k(y) \cos kx + O(\varepsilon^2),$$

where $D_k(y)$ *and* $E_k(y)$ *are elementary functions given in equation (4.21).*

We prove Theorem 1.2 by first restricting the reduced problem onto an invariant subspace of Hölder spaces such that each eigenspace of the linearized equation is one-dimensional, and then employing the Crandall–Rabinowitz bifurcation theorem.

Remark 1.3 *It is important to investigate how Gibbs–Thomson relation affects a tumour's ability invading into the surrounding tissue. As pointed out by Friedman and Hu in [18,19], the bifurcation solutions with free boundary $\rho_\varepsilon(x) = \rho_* + \varepsilon \cos kx + O(\varepsilon^2)$ are the protrusions associated with the invasion of a tumour. It follows from our result that bifurcation occurs only for $\gamma < \gamma_*$. Related to the Gibbs–Thomson relation, we know that $2\gamma\kappa$ is the energy required to maintain the inter-cellular bonds existing on the free boundary. Consequently, a smaller value of surface tension coefficient γ leads to smaller energy on the free boundary, which further makes the protrusions generate more easily and tumours more aggressive.*

Remark 1.4 *It is also interesting to compare the model with Gibbs–Thomson relation and the model with surface tension effect well studied in [11,32]. From the expressions of γ_k, we see that for a tumour with Gibbs–Thomson relation the proliferation rate μ does not affect the generation of protrusions. This is in contrast with a tumour with surface tension effect, where a larger value of the proliferation rate μ makes the protrusions generate more easily. Also, by denoting $\tilde{\gamma}_*$ the corresponding threshold value for the model with surface tension effect, from Lemma 2.6 we know that $\gamma_* > \tilde{\gamma}_*$ for small μ, which indicates that a tumour with Gibbs–Thomson relation is more hazardous than a tumour with surface tension effect for small value of the proliferation rate μ, while for large μ the situation is the opposite.*

Remark 1.5 *The above results are connected with the situation $\bar{\sigma} > \tilde{\sigma}$, where a unique flat equilibrium exists. It is worth mentioning the case $\bar{\sigma} < \tilde{\sigma}$, where the asymptotic behaviour of solutions follows readily. To show this, let (ρ, σ, p) be a global smooth solution of equation (1.1). By setting volume of a tumour $Vol(t) := \int_0^{2\pi} \rho(t,x)dx$ for $t \geqslant 0$, we see from the maximum principle and Green's formula that*

$$\frac{d}{dt} Vol(t) \leqslant -\mu(\tilde{\sigma} - \bar{\sigma}) Vol(t).$$

Thus $\lim_{t\to\infty} Vol(t) = 0$ for $\bar{\sigma} < \tilde{\sigma}$, which means that the tumour will eventually vanish. In modelling, this indicates that insufficient nutrient supply cannot sustain the tumour's survival.

The structure of this paper is arranged as follows. In the next section, we convert the free boundary problem (1.1) to a fully nonlinear equation on a fixed reference domain and study its linearization at the flat equilibrium. In Section 3, we prove that the linearized operator generates an analytic semigroup and give the stability analysis. Section 4 aims at investigating bifurcation phenomenon of equation (1.1) and gives the proof of Theorem 1.2. In the last section, we give some conclusions and interesting biological implications of our study.

2 Transformation and linearization

In this section, we convert the free boundary problem (1.1) to a nonlinear differential equation in a Banach space defined on a fixed domain and study its linearization at the flat equilibrium.

We first transform equation (1.1) into a new system on the fixed reference domain Ω. Recall that Ω is defined by $\Omega = \mathbb{S}^1 \times (0, 1)$, with the lower and upper boundary denoted by $\Gamma_0 = \mathbb{S}^1 \times \{0\}$ and $\Gamma_1 = \mathbb{S}^1 \times \{1\}$, respectively. In the following, we identify Γ_0 and Γ_1 with $\mathbb{S}^1$. Given $\rho = \rho(x) \in C^2_+(\mathbb{S}^1)$, define a mapping θ_ρ by

$$\theta_\rho : \Omega \to \Omega_\rho, \quad (x, y) \mapsto (x, y\rho(x)), \tag{2.1}$$

and write $\Omega_\rho := \theta_\rho(\Omega)$. Obviously, Ω_ρ is a strip-like domain with the lower boundary Γ_0. We use Γ_ρ to denote the upper boundary of Ω_ρ, i.e., $\Gamma_\rho := \theta_\rho(\Gamma_1) = \{(x, y) : \ x \in \mathbb{S}^1, y = \rho(x)\}$. It can be easily verified that θ_ρ is a C^2-diffeomorphism. The corresponding pull-back and push-forward operators induced by θ_ρ are denoted by θ^*_ρ and θ^ρ_*, respectively, i.e.,

$$\theta^*_\rho u := u \circ \theta_\rho \ \text{ for } u \in C(\bar{\Omega}_\rho), \qquad \theta^\rho_* v := v \circ \theta^{-1}_\rho \ \text{ for } v \in C(\bar{\Omega}).$$

Given $\rho \in C^2_+(\mathbb{S}^1)$ and $v \in C^2(\bar{\Omega})$, we define the following transformed operators:

$$\mathscr{A}(\rho)v := \theta^*_\rho \Delta(\theta^\rho_* v), \quad \mathscr{D}_0(\rho)v := \theta^*_\rho \langle \Upsilon_0 \nabla(\theta^\rho_* v), n_0\rangle, \quad \mathscr{D}_1(\rho)v := \theta^*_\rho \langle \Upsilon_\rho \nabla(\theta^\rho_* v), n_1\rangle,$$

where ∇ is the gradient operator, Υ_0 and Υ_ρ stand for the trace operators on Γ_0 and Γ_ρ, respectively, $n_0 = (0, -1)$ and $n_1 = (-\partial_x \rho, 1)$ represent the outward normal on Γ_0 and Γ_ρ, respectively, and $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^2$. We introduce the transformed curvature operator

$$\mathscr{N} : C^2_+(\mathbb{S}^1) \to C(\mathbb{S}^1), \qquad \rho \mapsto \mathscr{N}(\rho) := \theta^*_\rho \kappa(\rho),$$

where $\kappa(\rho)$ denotes the mean curvature of the boundary Γ_ρ.

For given $\rho \in C^2_+(\mathbb{S}^1)$, an elementary calculation shows that

$$\mathscr{A}(\rho) = \sum_{j,k=1}^{2} a_{jk}(\rho)\partial_j\partial_k + a_2(\rho)\partial_2, \quad \mathscr{D}_i(\rho) = \sum_{j=1}^{2} b_{ji}(\rho)\Upsilon_i\partial_j, \quad i = 0, 1, \tag{2.2}$$

where

$$\begin{aligned}
&a_{11}(\rho) := 1, \quad a_{12}(\rho) := a_{21}(\rho) := -y\rho^{-1}\rho_x, \quad a_{22}(\rho) := \rho^{-2}\big(1 + y^2\rho_x^2\big),\\
&a_2(\rho) := \rho^{-2}\big(2y\rho_x^2 - y\rho\rho_{xx}\big), \quad b_{10}(\rho) := 0, \quad b_{20}(\rho) := -\rho^{-1},\\
&b_{11}(\rho) := -\rho_x, \quad b_{21}(\rho) := \rho^{-1}\big(1 + y\rho_x^2\big).
\end{aligned}$$

It is not difficult to show that

$$(\mathscr{A}, \mathscr{D}_i) \in C^\infty(h^{m+2+\alpha}_+(\mathbb{S}^1), \mathscr{L}(h^{m+2+\alpha}(\bar{\Omega}), h^{m+\alpha}(\bar{\Omega}) \times h^{m+1+\alpha}(\mathbb{S}^1))), \quad i = 0, 1, m \in \mathbb{N}, \tag{2.3}$$

where $\mathscr{L}(Z_1, Z_0)$ denotes the Banach space of all linear continuous mappings from Z_1 to

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Z_0. Moreover, the transformed curvature has the expression

$$\mathcal{N}(\rho) = -\left(1+\rho_x^2\right)^{-\frac{3}{2}}\rho_{xx}, \tag{2.4}$$

which implies

$$\mathcal{N} \in C^\infty(h_+^{m+2+\alpha}(\mathbb{S}^1), h^{m+\alpha}(\mathbb{S}^1)), \quad m \in \mathbb{N}. \tag{2.5}$$

Let $T > 0$ be given and consider a function $\rho \in C([0,T], h_+^{4+\alpha}(\mathbb{S}^1)) \cap C^1([0,T], h^{3+\alpha}(\mathbb{S}^1))$. We abbreviate the operator families $t \to \mathscr{A}(\rho(t))$ and $t \to \mathscr{D}_i(\rho(t))$ for $t \in [0,T]$ to be $\mathscr{A}(\rho)$ and $\mathscr{D}_i(\rho)$ $(i = 0, 1)$, respectively. Later on we identify a function $\rho : [0,T] \to h(\mathbb{S}^1)$ with the corresponding function on $\mathbb{S}^1 \times [0,T]$ defined by $\rho(t,x) = \rho(t)(x)$ for $t \in [0,T]$ and $x \in \mathbb{S}^1$. Writing

$$u(t) := \theta_\rho^* \sigma(t,\cdot), \quad v(t) := \theta_\rho^* p(t,\cdot),$$

we see that equation (1.1) is transformed into the following initial-boundary value problem: Given $\rho_0 \in h_+^{4+\alpha}(\mathbb{S}^1)$, find $(\rho, u, v) \in h_+^{4+\alpha}(\mathbb{S}^1) \times h^{2+\alpha}(\bar{\Omega}) \times h^{4+\alpha}(\bar{\Omega})$ such that

$$\begin{cases} \mathscr{A}(\rho)u = \lambda u & \text{in } \Omega \times (0,T), \\ \mathscr{A}(\rho)v = -\mu(u - \tilde{\sigma}) & \text{in } \Omega \times (0,T), \\ \mathscr{D}_0(\rho)u = 0, \ \mathscr{D}_0(\rho)v = 0 & \text{on } \Gamma_0 \times (0,T), \\ \Upsilon_1 u = \bar{\sigma}(1 - 2\gamma\mathcal{N}(\rho)), \ \Upsilon_1 v = \bar{p} & \text{on } \Gamma_1 \times (0,T), \\ \partial_t \rho + \mathscr{D}_1(\rho)v = 0 & \text{on } \Gamma_1 \times (0,T), \\ \rho(0) = \rho_0 & \text{at } t = 0. \end{cases} \tag{2.6}$$

Concluding the above steps, we get the following result.

Lemma 2.1 *The problem (1.1) is equivalent to the problem (2.6) in the following sense: If* (ρ, u, v) *is a solution of equation (2.6), then by letting*

$$\sigma(t,\cdot) := \theta_*^\rho u(t,\cdot), \ \ p(t,\cdot) := \theta_*^\rho v(t,\cdot),$$

we see that (ρ, σ, p) *is a solution of equation (1.1). Conversely, if* (ρ, σ, p) *is a solution of equation (1.1), then by setting*

$$u(t,\cdot) := \theta_\rho^* \sigma(t,\cdot), \ \ v(t,\cdot) := \theta_\rho^* p(t,\cdot),$$

we have that (ρ, u, v) *forms a solution of equation (2.6).*

In the following, we shall reduce the system (2.6) to a nonlinear differential equation in some Banach space containing the unknown ρ. For this, let $\rho \in h_+^{4+\alpha}(\mathbb{S}^1)$ be given and consider the boundary value problem

$$\mathscr{A}(\rho)u = \lambda u \ \text{ in } \Omega, \quad \mathscr{D}_0(\rho)u = 0 \ \text{ on } \Gamma_0, \quad \Upsilon_1 u = \bar{\sigma}(1 - 2\gamma\mathcal{N}(\rho)) \ \text{ on } \Gamma_1. \tag{2.7}$$

Actually, the boundary condition $\mathscr{D}_0(\rho)v = 0$ is equivalent to the linear form $\Upsilon_0 \partial_y v = 0$. By equation (2.5) and the theory of elliptic equations we know that equation (2.7) has a unique solution $u \in h^{2+\alpha}(\bar{\Omega})$, which we denote by $u = \mathscr{Q}_\gamma(\rho)$. From equation (2.5) and the regularity theory for elliptic equations we get

$$\mathscr{Q}_\gamma \in C^\infty(h_+^{4+\alpha}(\mathbb{S}^1), h^{2+\alpha}(\bar{\Omega})), \tag{2.8}$$

cf. [16]. Then we consider the following boundary value problem

$$\mathscr{A}(\rho)v = -\mu(\mathscr{Q}_\gamma(\rho) - \tilde{\sigma}) \text{ in } \Omega, \quad \mathscr{D}_0(\rho)v = 0 \text{ on } \Gamma_0, \quad \Upsilon_1 v = \bar{p} \text{ on } \Gamma_1, \tag{2.9}$$

where we have replaced u with $\mathscr{Q}_\gamma(\rho)$. Similarly, equation (2.9) has a unique solution $v \in h^{4+\alpha}(\bar{\Omega})$, which is denoted by $v = \mathscr{R}_\gamma(\rho)$ and satisfies

$$\mathscr{R}_\gamma \in C^\infty(h_+^{4+\alpha}(\mathbb{S}^1), h^{4+\alpha}(\bar{\Omega})). \tag{2.10}$$

By introducing the nonlinear mapping

$$\Phi_\gamma(\rho) := \mathscr{D}_1(\rho)\mathscr{R}_\gamma(\rho), \tag{2.11}$$

we see that the system (2.6) is reduced to the following problem

$$\partial_t \rho + \Phi_\gamma(\rho) = 0, \quad \rho(0) = \rho_0. \tag{2.12}$$

Furthermore, it follows from equations (2.3), (2.5), (2.8), (2.10) and (2.11) that

$$\Phi_\gamma \in C^\infty(h_+^{4+\alpha}(\mathbb{S}^1), h^{3+\alpha}(\mathbb{S}^1)). \tag{2.13}$$

Summarizing the above deductions we get

Lemma 2.2 *The problem (2.6) is equivalent to the problem (2.12) in the following sense: If (ρ, u, v) is a solution of equation (2.6), then ρ forms a solution of equation (2.12). Conversely, if ρ is a solution of equation (2.12), then by solving the problems (2.7) and (2.9) we get $u = \mathscr{Q}_\gamma(\rho)$ and $v = \mathscr{R}_\gamma(\rho)$, which combined with the component ρ form a solution (ρ, u, v) of equation (2.6).*

Next, we consider the linearization of the system (2.6) at its flat stationary solution. To find flat stationary solution of equation (2.6), by virtue of Lemma 2.1, we know that the stationary form of equation (2.6) for a general stationary solution $(\rho_s(x), \sigma_s(x, y), p_s(x, y))$ is given by

$$\begin{cases} \Delta\sigma_s = \lambda\sigma_s & \text{in } \Omega_{\rho_s}, \\ \Delta p_s = -\mu(\sigma_s - \tilde{\sigma}) & \text{in } \Omega_{\rho_s}, \\ \partial_y \sigma_s = 0, \quad \partial_y p_s = 0 & \text{on } \Gamma_0, \\ \sigma_s = \bar{\sigma}(1 - 2\gamma\kappa), \quad p_s = \bar{p} & \text{on } \Gamma_{\rho_s}, \\ \partial_{\vec{v}} p_s = 0 & \text{on } \Gamma_{\rho_s}. \end{cases} \tag{2.14}$$

If $\rho_s(x) \equiv \rho_*$, with a positive constant $\rho_* > 0$, then the equations in the first four lines of

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equation (2.14) can be solved as

$$\sigma_*(y) = \bar{\sigma}\frac{\cosh(\sqrt{\lambda}y)}{\cosh(\sqrt{\lambda}\rho_*)}, \quad p_*(y) = \frac{\mu\bar{\sigma}}{\lambda}\left(1 - \frac{\cosh(\sqrt{\lambda}y)}{\cosh(\sqrt{\lambda}\rho_*)}\right) + \frac{\mu\tilde{\sigma}}{2}\left(y^2 - \rho_*^2\right) + \bar{p}. \tag{2.15}$$

Substituting the expression of $p_*(y)$ into the last equation in equation (2.14), we get

$$\frac{\tanh\sqrt{\lambda}\rho_*}{\sqrt{\lambda}\rho_*} = \frac{\tilde{\sigma}}{\bar{\sigma}}. \tag{2.16}$$

It is easy to verify that the function $\dfrac{\tanh r}{r}$ is strictly monotone decreasing and

$$\lim_{r\to 0}\frac{\tanh r}{r} = 1, \qquad \lim_{r\to\infty}\frac{\tanh r}{r} = 0.$$

As a consequence, we get the following result.

Lemma 2.3 *The problem (1.1) has a unique flat stationary solution* (ρ_*, σ_*, p_*) *if and only if* $\bar{\sigma} > \tilde{\sigma}$. *Moreover, this flat stationary solution* (ρ_*, σ_*, p_*) *satisfies equations (2.15) and (2.16).*

In the rest part of this paper, we always assume that $\bar{\sigma} > \tilde{\sigma}$, which ensures that equation (1.1) has a unique flat stationary solution (ρ_*, σ_*, p_*). It follows from Lemma 2.1 that $(\rho_*, \sigma_*(\rho_* y), p_*(\rho_* y))$ forms a flat stationary solution of equation (2.6). A simple calculation shows that

$$\begin{aligned} &[\partial\mathscr{A}(\rho_*)h]v = -2y\rho_*^{-1}h_x\partial_{12}v - 2\rho_*^{-3}h\partial_{22}v - y\rho_*^{-1}h_{xx}\partial_2 v, \quad \partial\mathscr{N}(\rho_*)h = -h_{xx}, \\ &[\partial\mathscr{D}_0(\rho_*)h]v = \rho_*^{-2}h\Upsilon_0\partial_2 v, \quad [\partial\mathscr{D}_1(\rho_*)h]v = -h_x\Upsilon_1\partial_1 v - \rho_*^{-2}h\Upsilon_1\partial_2 v, \end{aligned} \tag{2.17}$$

for $h \in h^{2+\alpha}(\mathbb{S}^1)$ and $v \in h^{2+\alpha}(\bar{\Omega})$. It follows that $[\partial\mathscr{A}(\rho_*)\cdot]v$ and $\partial\mathscr{N}(\rho_*)$ are second-order differential operators, and $[\partial\mathscr{D}_0(\rho_*)\cdot]v$ and $[\partial\mathscr{D}_1(\rho_*)\cdot]v$ are first-order differential operators. To compute the linearization of equation (2.6) at its flat stationary solution $(\rho_*, \sigma_*(\rho_* y), p_*(\rho_* y))$, we set

$$\rho = \rho_* + \varepsilon\xi(t,x), \quad u = \sigma_*(\rho_* y) + \varepsilon\Sigma(t,x,y), \quad v = p_*(\rho_* y) + \varepsilon P(t,x,y),$$

where ξ, Σ, P are new unknowns and ε is a small parameter. Substituting these expressions into equation (2.6) we get

$$\begin{cases} \mathscr{A}(\rho_*)\Sigma = \lambda\Sigma - [\partial\mathscr{A}(\rho_*)\xi]\sigma_*(\rho_* y) & \text{in } \Omega\times(0,T), \\ \mathscr{A}(\rho_*)P = -\mu\Sigma - [\partial\mathscr{A}(\rho_*)\xi]p_*(\rho_* y) & \text{in } \Omega\times(0,T), \\ \mathscr{D}_0(\rho_*)\Sigma = 0, \ \ \mathscr{D}_0(\rho_*)P = 0 & \text{on } \Gamma_0\times(0,T), \\ \Upsilon_1\Sigma = -2\gamma\bar{\sigma}\partial\mathscr{N}(\rho_*)\xi, \ \ \Upsilon_1 P = 0 & \text{on } \Gamma_1\times(0,T), \\ \partial_t\xi + \mathscr{D}_1(\rho_*)P = 0 & \text{on } \Gamma_1\times(0,T), \\ \xi(0) = \xi_0 & \text{at } t = 0, \end{cases} \tag{2.18}$$

where we have used the fact that $[\partial\mathscr{D}_1(\rho_*)\xi]p_*(\rho_* y) = 0$.

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Given $\xi \in h^{4+\alpha}(\mathbb{S}^1)$, similarly to above by solving the linear problem on Σ we get a unique solution $\Sigma \in h^{2+\alpha}(\bar{\Omega})$, which is 2π-periodic in x. Then substituting Σ into the second equation in equations (2.18) and solving the corresponding linear problem on P, we get that $P \in h^{4+\alpha}(\bar{\Omega})$ is also 2π-periodic in x. Now we can define a linear operator L by

$$L\xi := \mathscr{D}_1(\rho_*)P \quad \text{for } \xi \in h^{4+\alpha}(\mathbb{S}^1). \tag{2.19}$$

It can be easily verified that $L \in \mathscr{L}(h^{4+\alpha}(\mathbb{S}^1),\ h^{3+\alpha}(\mathbb{S}^1))$.

In the following, we shall represent the operator L as a multiplier operator. For this, we always employ the natural complexification in connection with spectral theory without distinguishing this in notation. Since $h^{4+\alpha}(\mathbb{S}^1)$ is compactly embedded into $h^{3+\alpha}(\mathbb{S}^1)$, the resolvent $(\lambda I - L)^{-1}$ is a compact operator for every λ in the resolvent set of L. Therefore, the spectrum of L, which we denote by $\sigma(L)$, consists of a sequence of isolated eigenvalues. Write $\partial\Phi_\gamma(\rho_*)$ for the Fréchet derivative of $\Phi_\gamma(\rho)$ at ρ_*. We then have the following.

Lemma 2.4

(i) $\partial\Phi_\gamma(\rho_*)\xi = L\xi$ *for* $\xi \in h^{4+\alpha}(\mathbb{S}^1)$.

(ii) $\sigma(-\partial\Phi_\gamma(\rho_*)) = \{\lambda_k|\ k = 0, 1, 2, \ldots\}$, *where* λ_k *is given by*

$$\lambda_k = \mu(\bar{\sigma} - \tilde{\sigma}) - \left(\mu\tilde{\sigma} + \frac{2\mu\bar{\sigma}\gamma k^2}{\lambda\rho_*}\right)(\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*)). \tag{2.20}$$

Proof

(i) Recall that equation (2.18) is the linearization of equation (2.6), and the latter is equivalent to equation (2.12). It follows that equation (2.19) is equivalent to the linearized operator $\partial\Phi_\gamma(\rho_*)$ of $\Phi_\gamma(\rho)$.

(ii) To calculate each eigenvalue of $\partial\Phi_\gamma(\rho_*)$, we consider Fourier expansions of ξ, Σ and P

$$\xi(t,x) = \sum_{k=0}^{\infty} a_k(t)e^{ikx}, \quad \Sigma(t,x,y) = \sum_{k=0}^{\infty} B_k(t,y)e^{ikx}, \quad P(t,x,y) = \sum_{k=0}^{\infty} C_k(t,y)e^{ikx}.$$

Substituting the expansion of $\Sigma(t,x,y)$ into equation (2.18) and comparing coefficients of e^{ikx} for every k, we get the following boundary value problems for $B_k(t,y)$

$$\begin{cases} -k^2 B_k(t,y) + \rho_*^{-2}\dfrac{\partial^2 B_k}{\partial y^2}(t,y) = \lambda B_k(t,y) + a_k(t)c_k(y), \\ \dfrac{\partial B_k}{\partial y}(t,0) = 0, \quad B_k(t,1) = -2a_k(t)\gamma\bar{\sigma}k^2, \quad k = 0,1,2,\ldots, \end{cases}$$

where

$$c_k(y) := 2\rho_*^{-1}\bar{\sigma}\lambda\frac{\cosh(\sqrt{\lambda}\rho_* y)}{\cosh(\sqrt{\lambda}\rho_*)} - k^2\bar{\sigma}\sqrt{\lambda}\frac{y\sinh(\sqrt{\lambda}\rho_* y)}{\cosh(\sqrt{\lambda}\rho_*)}.$$

One can easily verify that the unique solution of this problem is given by

$$B_k(t,y) = -a_k(t)\bar{\sigma}(\sqrt{\lambda}\tanh(\sqrt{\lambda}\rho_*) + 2\gamma k^2)\frac{\cosh(\sqrt{k^2+\lambda}\rho_* y)}{\cosh(\sqrt{k^2+\lambda}\rho_*)} + a_k(t)\bar{\sigma}\sqrt{\lambda}\frac{y\sinh(\sqrt{\lambda}\rho_* y)}{\cosh(\sqrt{\lambda}\rho_*)}. \tag{2.21}$$

Next, substituting the expansion of $P(x,y)$ into equation (2.18) we get

$$\begin{cases} -k^2 C_k(t,y) + \rho_*^{-2}\dfrac{\partial^2 C_k}{\partial y^2}(t,y) = -\mu B_k(t,y) + a_k(t)d_k(y), \\ \dfrac{\partial C_k}{\partial y}(t,0) = 0, \quad C_k(t,1) = 0, \quad k = 0,1,2,\ldots, \end{cases}$$

where

$$d_k(y) := -2\rho_*^{-1}\mu\bar{\sigma}\frac{\cosh(\sqrt{\lambda}\rho_* y)}{\cosh(\sqrt{\lambda}\rho_*)} + \frac{k^2\mu\bar{\sigma}}{\sqrt{\lambda}}\frac{y\sinh(\sqrt{\lambda}\rho_* y)}{\cosh(\sqrt{\lambda}\rho_*)} + 2\rho_*^{-1}\mu\tilde{\sigma} - y^2k^2\mu\bar{\sigma}\rho_*.$$

Solving this problem one obtains

$$\begin{aligned} C_k(t,y) = \quad & a_k(t)\mu\tilde{\sigma}\rho_* y^2 - a_k(t)\mu\bar{\sigma}\frac{y\sinh(\sqrt{\lambda}\rho_* y)}{\sqrt{\lambda}\cosh(\sqrt{\lambda_1}\rho_*)} - a_k(t)\mu(\tilde{\sigma}\rho_*\lambda + 2k^2\gamma\bar{\sigma})\frac{\cosh(k\rho_* y)}{\lambda\cosh(k\rho_*)} \\ & + a_k(t)\mu\bar{\sigma}(2k^2\gamma + \sqrt{\lambda}\tanh(\sqrt{\lambda}\rho_*))\frac{\cosh(\sqrt{k^2+\lambda}\rho_* y)}{\lambda\cosh(\sqrt{k^2+\lambda}\rho_*)}. \end{aligned} \tag{2.22}$$

Finally, we substitute the expansions of ξ and $P(t,x,y)$ into equation (2.19) and get

$$L\sum_{k=0}^{\infty} a_k(t)e^{ikx} = -\sum_{k=0}^{\infty} \lambda_k a_k(t)e^{ikx}, \tag{2.23}$$

where λ_k is given in equations (2.20) and (2.16) is used here. This completes the proof. □

Moreover, we have the following spectral analysis.

Lemma 2.5

(i) $\lambda_k \leqslant \mu(\bar{\sigma} - \tilde{\sigma})$ *for all* $\gamma > 0$ *and* $k = 0,1,2,\ldots$.

(ii) *There exists a constant* $\gamma_* > 0$ *depending only on* $\mu, \tilde{\sigma}, \bar{\sigma}, \lambda$ *and* ρ_* *such that for every* $\gamma > \gamma_*$,

$$\lambda_k \leqslant C(\gamma) < 0 \quad \textit{for all } k = 0,1,2,\ldots,$$

where $C(\gamma)$ *is a positive constant depending on* γ. *Also for every* $0 < \gamma < \gamma_*$ *there exists a positive integer* $k_0 = k_0(\gamma)$ *such that* $\lambda_{k_0} > 0$.

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Proof

(i) We split each eigenvalue λ_k as

$$\lambda_k = \lambda_k^0 - \frac{2\mu\bar{\sigma}\gamma k^2}{\lambda\rho_*}(\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*)), \quad k = 0, 1, 2, \ldots, \tag{2.24}$$

where

$$\lambda_k^0 := \mu(\bar{\sigma} - \tilde{\sigma}) - \mu\tilde{\sigma}(\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*)).$$

A standard limitation argument shows that

$$\begin{aligned} &\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*) > 0, \\ &\lim_{k\to\infty} 2k(\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*)) = \lambda\rho_*. \end{aligned} \tag{2.25}$$

It follows that

$$\lim_{k\to\infty} \lambda_k^0 = \mu(\bar{\sigma} - \tilde{\sigma}) > 0. \tag{2.26}$$

It is easy to prove

$$x \tanh x + 1 - \frac{x}{\tanh x} > 0 \quad \text{for all } x > 0,$$

which implies

$$\lambda_0 = \lambda_0^0 = -\mu\tilde{\sigma}\left(\sqrt{\lambda}\rho_* \tanh\sqrt{\lambda}\rho_* + 1 - \frac{\sqrt{\lambda}\rho_*}{\tanh\sqrt{\lambda}\rho_*}\right) < 0, \tag{2.27}$$

where we have used the relation (2.16) again. It is not difficult to prove that λ_k^0 is strictly monotone increasing in k, that is,

$$\lambda_0^0 < \lambda_1^0 < \cdots < \lambda_k^0 < \lambda_{k+1}^0 < \cdots. \tag{2.28}$$

Combining equations (2.26)–(2.28) we get

$$\lambda_k \leqslant \lambda_k^0 \leqslant \mu(\bar{\sigma} - \tilde{\sigma}), \quad k = 0, 1, 2, \ldots. \tag{2.29}$$

(ii) We define

$$\gamma_k := \frac{(\bar{\sigma} - \tilde{\sigma})\lambda\rho_*}{2\bar{\sigma}k^2(\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*))} - \frac{\tilde{\sigma}\lambda\rho_*}{2\bar{\sigma}k^2}, \quad k = 1, 2, \ldots. \tag{2.30}$$

Then λ_k can be re-expressed as

$$\lambda_k = -\frac{2\mu\bar{\sigma}k^2}{\lambda\rho_*}(\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*))(\gamma - \gamma_k), \quad k = 1, 2, \ldots. \tag{2.31}$$

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By virtue of equation (2.25),

$$\gamma_k = \left(1 - \frac{\tilde{\sigma}}{\bar{\sigma}}\right)\frac{1}{k} + O\left(\frac{1}{k^2}\right) \quad \text{as } k \to \infty. \tag{2.32}$$

Thus we can define

$$\gamma_* := \sup\{\gamma_k | \, k = 1, 2, \ldots\}. \tag{2.33}$$

It is obvious that $0 < \gamma_* < \infty$. It follows from equation (2.31) that if $\gamma > \gamma_*$ then $\lambda_k < 0$ for all $k = 1, 2, \ldots$, while for $0 < \gamma < \gamma_*$ there exists $k_0 \in \mathbb{N}$ such that $\lambda_{k_0} > 0$. Combining this with equation (2.27), we get the desired assertion in (*ii*). This completes the proof. □

It is necessary to compare the threshold value γ_* with the corresponding threshold value $\tilde{\gamma}_*$ obtained in [11, 32] for the model with surface tension effect. Recall that $\tilde{\gamma}_*$ can be rewritten as

$$\tilde{\gamma}_* := \sup\{\tilde{\gamma}_k | \, k = 1, 2, \ldots\}, \tag{2.34}$$

where

$$\tilde{\gamma}_k := \mu \left[\frac{\bar{\sigma} - \tilde{\sigma} - \tilde{\sigma}(\sqrt{k^2+\lambda}\rho_* \tanh(\sqrt{k^2+\lambda}\rho_*) - k\rho_* \tanh(k\rho_*))}{k^3 \tanh(k\rho_*)}\right], \quad k = 1, 2, \ldots, \tag{2.35}$$

cf. [11, 32]. It follows from equation (2.25) that

$$\frac{\tilde{\gamma}_k}{\mu} = \frac{\bar{\sigma} - \tilde{\sigma}}{k^3} + O(\frac{1}{k^4}) \quad \text{as } k \to \infty.$$

Thus we can define $q_* := \sup\{\tilde{\gamma}_k/\mu | \, k = 1, 2, \ldots\}$, a positive finite value independent of μ. It follows that $\tilde{\gamma}_* = \mu q_*$ and the following conclusion holds.

Lemma 2.6 $\tilde{\gamma}_* < \gamma_*$ *if and only if* $\mu < \frac{\gamma_*}{q_*}$.

3 Stability

In this section, we investigate stability of the unique flat stationary solution (ρ_*, σ_*, p_*) of equation (1.1) under non-flat perturbations and give the proof of Theorem 1.1.

We first establish well-posedness of equation (1.1) by employing analytic semigroup theory. Let E_0 and E_1 be Banach spaces such that E_1 is densely injected in E_0, and let $\mathscr{H}(E_1, E_0)$ denote the set of all $A \in \mathscr{L}(E_1, E_0)$ such that $-A$ generates a strongly continuous analytic semigroup on E_0. We use $\mathscr{L}_{is}(E_1, E_0)$ to represent the set of all bounded isomorphisms from E_1 onto E_0. In the following, we want to prove that $\partial\Phi_\gamma(\rho_*) \in \mathscr{H}(h^{4+\alpha}(\mathbb{S}^1), h^{3+\alpha}(\mathbb{S}^1))$. Due to the theory of Amann [1], it is suffice to prove that there exist constants $\bar{\omega} > 0$ and $\beta \geqslant 1$ such that for all $\mathrm{Re}\lambda \geqslant \bar{\omega}$

$$\lambda + \partial\Phi_\gamma(\rho_*) \in \mathscr{L}_{is}(h^{4+\alpha}(\mathbb{S}^1), h^{3+\alpha}(\mathbb{S}^1)), \tag{3.1}$$

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and

$$|\lambda| \cdot \|(\lambda + \partial \Phi_\gamma(\rho_*))^{-1}\|_{\mathscr{L}(h^{3+\alpha}(\mathbb{S}^1))} \leqslant \beta. \tag{3.2}$$

For this, we refer to [14, 15] and define the Sobolev space

$$H^s(\mathbb{S}^1) := \{f \in L^2(\mathbb{S}^1) : \sum_{n\in\mathbb{Z}} (n^2+1)^s |f_n|^2 < \infty\},$$

endowed with the scalar product

$$< f, g > := \sum_{n\in\mathbb{Z}} (n^2+1)^s f_n \overline{g_n},$$

where f_n denotes the nth Fourier coefficient of f. The Sobolev embedding theorem and the fact that smooth functions are dense in $H^s(\mathbb{S}^1)$ imply that $H^{k+s}(\mathbb{S}^1) \hookrightarrow C^k(\mathbb{S}^1)$ for all $k \in \mathbb{N}$ and $s > \frac{1}{2}$. Thus

$$H^{k+s}(\mathbb{S}^1) \overset{d}{\hookrightarrow} h^{k+\alpha}(\mathbb{S}^1),$$

for all $k \in \mathbb{N}, \alpha \in (0,1)$ and $s > \frac{3}{2}$. Hereafter we fix the positive constant $\bar{\omega} := 2\mu(\bar{\sigma} - \tilde{\sigma})$.

Lemma 3.1 *Given $s \geqslant 0$ and $Re\lambda \geqslant \bar{\omega}$,*

$$\lambda + \partial \Phi_\gamma(\rho_*) \in \mathscr{L}_{is}(H^{s+1}(\mathbb{S}^1), H^s(\mathbb{S}^1)).$$

Proof From the expression (2.20) of λ_k and the limit in equation (2.25) we know

$$\lim_{k\to\infty} \frac{\lambda_k}{k} = -\mu\bar{\sigma}\gamma,$$

which implies that there exists $C_1, C_2 > 0$ such that

$$C_1(k^2+1)^{\frac{1}{2}} \leqslant |\lambda_k| \leqslant C_2(k^2+1)^{\frac{1}{2}}.$$

Given $Re\lambda \geqslant \bar{\omega}$ and $h = \sum_{k\in\mathbb{Z}} h_k e^{ikx} \in H^{s+1}(\mathbb{S}^1)$, one can find

$$\begin{aligned}\|\partial\Phi_\gamma(\rho_*)h\|_{H^s(\mathbb{S}^1)} &= \left\|\sum_{k\in\mathbb{Z}} \lambda_k h_k e^{ikx}\right\|_{H^s(\mathbb{S}^1)} = \sum_{k\in\mathbb{Z}} (k^2+1)^s |\lambda_k h_k|^2 \\ &\leqslant C_2 \sum_{k\in\mathbb{Z}} (k^2+1)^{s+1} |h_k|^2 = C_2 \|h\|_{H^{s+1}(\mathbb{S}^1)}.\end{aligned}$$

Then $\partial\Phi_\gamma(\rho_*) \in \mathscr{L}(H^{s+1}(\mathbb{S}^1), H^s(\mathbb{S}^1))$, which combined with the fact that $|\lambda - \lambda_k| \geqslant \mu(\bar{\sigma} - \tilde{\sigma}) > 0$ for all $k \in \mathbb{Z}$, yield that $\lambda + \partial\Phi_\gamma(\rho_*)$ is injective. On the other hand, given

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$Re\lambda \geqslant \bar{\omega}$ and $h = \sum_{k\in\mathbb{Z}} h_k e^{ikx} \in H^s(\mathbb{S}^1)$, we have

$$\begin{aligned}\left\|\sum_{k\in\mathbb{Z}}(\lambda-\lambda_k)^{-1}h_k e^{ikx}\right\|_{H^{s+1}(\mathbb{S}^1)} &= \sum_{k\in\mathbb{Z}}(k^2+1)^{s+1}|(\lambda-\lambda_k)^{-1}h_k|^2 \leqslant C_\lambda^{-1}\sum_{k\in\mathbb{Z}}(k^2+1)^s|h_k|^2\\ &= C_\lambda^{-1}\left\|\sum_{k\in\mathbb{Z}} h_k e^{ikx}\right\|_{H^s(\mathbb{S}^1)},\end{aligned}$$

where we have used the estimate $|\lambda - \lambda_k| \geqslant C_\lambda(k^2+1)^{\frac{1}{2}}$ for some C_λ and all $k \in \mathbb{Z}$. As a consequence, the inverse of $\lambda + \partial\Phi_\gamma(\rho_*)$, which is defined by

$$(\lambda + \partial\Phi_\gamma(\rho_*))^{-1}h := \sum_{k\in\mathbb{Z}}(\lambda-\lambda_k)^{-1}h_k e^{ikx} \quad \text{for } h = \sum_{k\in\mathbb{Z}} h_k e^{ikx} \in H^s(\mathbb{S}^1),$$

satisfies $(\lambda + \partial\Phi_\gamma(\rho_*))^{-1} \in \mathscr{L}(H^s(\mathbb{S}^1), H^{s+1}(\mathbb{S}^1))$. Thus $\lambda + \partial\Phi_\gamma(\rho_*)$ is surjective and we complete the proof. □

Lemma 3.2 $\{\lambda \in \mathbb{C} |\ Re\lambda \geqslant \bar{\omega}\} \subset \rho(-\partial\Phi_\gamma(\rho_*))$.

Proof Let $Re(\lambda) \geqslant \bar{\omega}$ be given. We first prove that

$$(\lambda + \partial\Phi_\gamma(\rho_*))^{-1} \in \mathscr{L}(C^{3+\alpha}(\mathbb{S}^1), C^{4+\alpha}(\mathbb{S}^1)). \tag{3.3}$$

Recall that the multiplier operator $(\lambda + \partial\Phi_\gamma(\rho_*))^{-1}$ is defined by

$$(\lambda + \partial\Phi(\rho_*))^{-1}h = \sum_{k\in\mathbb{Z}}(\lambda-\lambda_k)^{-1}h_k e^{ikx} \quad \text{for } h = \sum_{k\in\mathbb{Z}} h_k e^{ikx}.$$

It is easy to see that

$$\lim_{k\to\infty}\frac{k}{\lambda-\lambda_k} = \frac{1}{\mu\bar{\sigma}\gamma}, \tag{3.4}$$

which indicates

$$\sup_{k\in\mathbb{Z}}|k|\left|\frac{1}{\lambda-\lambda_k}\right| < \infty. \tag{3.5}$$

A simple calculation yields that $\lim_{k\to\infty}(\lambda_{k+1}-\lambda_k) = -\mu\bar{\sigma}\gamma$, which combined with equation (3.4) lead to

$$\sup_{k\in\mathbb{Z}}|k|^2\left|\frac{1}{\lambda-\lambda_{k+1}} - \frac{1}{\lambda-\lambda_k}\right| = \sup_{k\in\mathbb{Z}}\left|\frac{k}{\lambda-\lambda_{k+1}}\frac{k}{\lambda-\lambda_k}\right| \cdot |\lambda_{k+1}-\lambda_k| < \infty. \tag{3.6}$$

Moreover, a standard limitation argument shows that

$$\lim_{k\to\infty}(\lambda_{k+2}-2\lambda_{k+1}+\lambda_k) = 0, \quad \lim_{k\to\infty}(\lambda_{k+2}(\lambda_{k+1}-\lambda_k) - \lambda_k(\lambda_{k+2}-\lambda_{k+1})) = 2\mu^2\bar{\sigma}^2\gamma^2. \tag{3.7}$$

Thus

$$\begin{aligned}
&\sup_{k\in\mathbb{Z}}|k|^3\left|\frac{1}{\lambda-\lambda_{k+2}}-\frac{2}{\lambda-\lambda_{k+1}}+\frac{1}{\lambda-\lambda_k}\right| \\
&\quad=\sup_{k\in\mathbb{Z}}\left|\frac{k}{\lambda-\lambda_{k+2}}\frac{k}{\lambda-\lambda_{k+1}}\frac{k}{\lambda-\lambda_k}\right||\lambda(\lambda_{k+2}-2\lambda_{k+1}+\lambda_k)+\lambda_{k+2}(\lambda_{k+1}-\lambda_k) \\
&\quad-\lambda_k(\lambda_{k+2}-\lambda_{k+1})| \\
&\quad<\infty.
\end{aligned} \tag{3.8}$$

Combining equations (3.5), (3.6) and (3.8) and using Theorem 4.5 of [2] (see also [14, 15, 28]), we get equation (3.3).

Noticing the dense embedding $H^{k+s}(\mathbb{S}^1)\overset{d}{\hookrightarrow}h^{k+\alpha}(\mathbb{S}^1)$ and the fact that $h^{4+\alpha}(\mathbb{S}^1)$ is just the closure of $H^s(\mathbb{S}^1)$ in $C^{4+\alpha}(\mathbb{S}^1)$ $(s>\frac{11}{2})$, we derive from Lemma 3.1 and equation (3.3) that

$$(\lambda+\partial\Phi_\gamma(\rho_*))^{-1}\in\mathscr{L}(h^{3+\alpha}(\mathbb{S}^1),h^{4+\alpha}(\mathbb{S}^1)). \tag{3.9}$$

This completes the proof. □

Lemma 3.3 *There exists $\beta\geqslant 1$ such that for all $Re\lambda\geqslant\bar{\omega}$,*

$$|\lambda|\cdot\|(\lambda+\partial\Phi_\gamma(\rho_*))^{-1}\|_{\mathscr{L}(h^{3+\alpha}(\mathbb{S}^1))}\leqslant\beta.$$

Proof Given $h=\sum_{k\in\mathbb{Z}}h_ke^{ikx}\in H^s(\mathbb{S}^1)$, we have

$$|\lambda|(\lambda+\partial\Phi(\rho_*))^{-1}h=\sum_{k\in\mathbb{Z}}|\lambda|(\lambda-\lambda_k)^{-1}h_ke^{ikx},$$

for all $Re\lambda\geqslant\bar{\omega}$. Noticing that $\lambda_k\leqslant\mu(\bar{\sigma}-\tilde{\sigma})$ and $Re(\lambda-\lambda_k)\geqslant\mu(\bar{\sigma}-\tilde{\sigma})$ for all $Re\lambda\geqslant\bar{\omega}$ and $k\in\mathbb{Z}$, we can prove that

$$\left|\frac{\lambda}{\lambda-\lambda_k}\right|\leqslant\frac{|\lambda-\lambda_k|+|\lambda_k|}{|\lambda-\lambda_k|}\leqslant 2,$$

for $-\mu(\bar{\sigma}-\tilde{\sigma})\leqslant\lambda_k\leqslant\mu(\bar{\sigma}-\tilde{\sigma})$ and $\lambda_k\leqslant-\mu(\bar{\sigma}-\tilde{\sigma})$, respectively. Thus for all $Re\lambda\geqslant\bar{\omega}$ and $k\in\mathbb{Z}$ we get

$$\sup_{k\in\mathbb{Z}}\left|\frac{\lambda}{\lambda-\lambda_k}\right|\leqslant 2. \tag{3.10}$$

From equation (3.6) and the above estimate we know that there exists a constant $C_1>0$ such that

$$\sup_{k\in\mathbb{Z}}|k|\left|\frac{|\lambda|}{\lambda-\lambda_{k+1}}-\frac{|\lambda|}{\lambda-\lambda_k}\right|=\sup_{k\in\mathbb{Z}}\frac{|\lambda|}{|\lambda-\lambda_{k+1}|}\frac{|k|}{|\lambda-\lambda_k|}|\lambda_{k+1}-\lambda_k|\leqslant C_1. \tag{3.11}$$

Using equations (3.5), (3.7) and (3.10), one can find that there exists constant $C_2>0$

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such that

$$\sup_{k\in\mathbb{Z}}|k|^2\left|\frac{|\lambda|}{\lambda-\lambda_{k+2}}-\frac{2|\lambda|}{\lambda-\lambda_{k+1}}+\frac{|\lambda|}{\lambda-\lambda_k}\right|$$
$$=\sup_{k\in\mathbb{Z}}\frac{|\lambda|}{|\lambda-\lambda_{k+2}|}\frac{|k|}{|\lambda-\lambda_{k+1}|}\frac{|k|}{|\lambda-\lambda_k|}|\lambda(\lambda_{k+2}-2\lambda_{k+1}+\lambda_k)$$
$$+\lambda_{k+2}(\lambda_{k+1}-\lambda_k)-\lambda_k(\lambda_{k+2}-\lambda_{k+1})|$$
$$\leqslant C_2. \tag{3.12}$$

Combining equations (3.10)–(3.12) and Theorem 4.5 of [2], we get the desired result. □

Define

$$\mathcal{O}_\delta^{m+\alpha}(\mathbb{S}^1):=\{\rho\in h_+^{m+\alpha}(\mathbb{S}^1)|\ \|\rho-\rho_*\|_{C^1(\mathbb{S}^1)}<\delta\}\quad\text{for } m\in\mathbb{N}\text{ and } m\geqslant 4, \tag{3.13}$$

where δ is a small constant to be fixed later on. We have the following result.

Theorem 3.4 *Let $\rho\in\mathcal{O}_\delta^{4+\alpha}(\mathbb{S}^1)$ be given. If δ is sufficiently small, then*

$$\partial\Phi_\gamma(\rho)\in\mathscr{H}(h^{4+\alpha}(\mathbb{S}^1),h^{3+\alpha}(\mathbb{S}^1)),$$

for all $\gamma>0$.

Proof Combining Lemma 3.1–3.3 and employing the theory of Amann [1], we get

$$\partial\Phi_\gamma(\rho_*)\in\mathscr{H}(h^{4+\alpha}(\mathbb{S}^1),h^{3+\alpha}(\mathbb{S}^1)),$$

for all $\gamma>0$. Given $\rho\in\mathcal{O}_\delta^{4+\alpha}(\mathbb{S}^1)$, the result follows readily from the well-known perturbation result of generators (cf. Section 2.4 of [25]). □

We can state the following local well-posedness for the problem (2.12).

Theorem 3.5 *Given $\rho_0\in\mathcal{O}_\delta^{4+\alpha}(\mathbb{S}^1)$, there exist $t^+:=t^+(\rho_0)>0$ and a unique maximal solution*

$$\rho\in C([0,t^+),h_+^{4+\alpha}(\mathbb{S}^1))\cap C^1([0,t^+),h^{3+\alpha}(\mathbb{S}^1))\cap C^\infty((0,t^+)\times\mathbb{S}^1)$$

of the problem (2.12). The map $(t,\rho_0)\mapsto\rho(t,\rho_0)$ defines a smooth semiflow on $h_+^{4+\alpha}(\mathbb{S}^1)$.

Proof It follows from Theorem 8.1.1 of [25] that a unique maximal solution exists and equation (2.12) generates a semiflow on $h^{4+\alpha}(\mathbb{S}^1)$. The fact that ρ is smooth is based on a bootstrapping argument in the scale of $h^{m+\alpha}(\mathbb{S}^1)$. □

Now we can give the proof of Theorem 1.1.

Proof of Theorem 1.1 We prove this by employing geometric theory for parabolic differential equations in Banach spaces. For this, we set $\Psi(\rho):=\Phi_\gamma(\rho+\rho_*)-\partial\Phi_\gamma(\rho_*)\rho$. It follows from equation (2.13) that $\Psi\in C^\infty(h_+^{4+\alpha}(\mathbb{S}^1),h^{3+\alpha}(\mathbb{S}^1))$. It is obvious that

$$\Psi(0)=\Phi_\gamma(\rho_*)=0,\quad \partial\Psi(0)=0. \tag{3.14}$$

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Consequently, the problem (2.12) is equivalent to the following problem

$$\partial_t \rho + \partial \Phi_\gamma(\rho_*)\rho + \Psi(\rho) = 0, \quad \rho(0) = \rho_0. \tag{3.15}$$

Let $\gamma^* > 0$ be the threshold value given in equation (2.33) and assume that $\gamma > \gamma^*$. Noticing equations (3.14) and (3.15) and invoking Lemma 2.4, Theorems 3.4 and 3.5, we see that all the assumptions in Theorem 9.1.2 of [25] are satisfied. This means that there are positive constants ω, ε and M_1 such that if $\rho_0 \in \mathcal{O}_\delta^{4+\alpha}(\mathbb{S}^1)$ for sufficiently small δ, then the solution $\rho(t,\cdot)$ of equation (2.12) exists globally and satisfies

$$\|\rho(t,\cdot) - \rho_*\|_{h^{4+\alpha}(\mathbb{S}^1)} \leqslant M_1 e^{-\omega t}\|\rho_0 - \rho_*\|_{h^{4+\alpha}(\mathbb{S}^1)} \quad \text{for all } t \geqslant 0.$$

From the construction we have $\sigma_* = \mathcal{Q}_\gamma(\rho_*)$, which combined with the mean value theorem implies that there exists a constant C such that

$$\|\sigma(t,\cdot)-\sigma_*\|_{h^{2+\alpha}(\bar{\Omega})} = \|\mathcal{Q}_\gamma(\rho(t))-\mathcal{Q}_\gamma(\rho_*)\|_{h^{2+\alpha}(\bar{\Omega})} \leqslant C\|\rho(t)-\rho_*\|_{h^{4+\alpha}(\mathbb{S}^1)} \leqslant Me^{-\omega t} \quad \text{for all } t \geqslant 0.$$

The corresponding estimate for $p(t,\cdot)$ can be obtained similarly.

If $\gamma < \gamma^*$, then Theorem 9.1.3 of [25] and Lemma 2.4 imply that the equilibrium ρ_* of equation (2.12) is unstable. By using Lemmas 2.1 and 2.2 and returning to the problem (1.1), we complete the proof of Theorem 1.1. □

Remark 3.6 *Concerning the effect of the nutrient on a tumour's growth, it follows from equations (2.30) and (2.33) that the threshold value γ_* is a monotone increasing function of the nutrient supply $\bar{\sigma}$. This implies that an increasing nutrient supply lowers a tumour's stability.*

4 Bifurcation

In this section, we investigate existence of non-flat stationary solutions of equation (1.1) and give the proof of Theorem 1.2.

It follows from Lemma 2.1 that the free boundary problem (1.1) is equivalent to the transformed problem (2.6), and the latter has the following stationary form

$$\begin{cases} \mathscr{A}(\rho)u = \lambda u & \text{in } \Omega, \\ \mathscr{A}(\rho)v = -\mu(u - \tilde{\sigma}) & \text{in } \Omega, \\ \mathscr{D}_0(\rho)u = 0, \quad \mathscr{D}_0(\rho)v = 0 & \text{on } \Gamma_0, \\ \Upsilon_1 u = \bar{\sigma}(1 - 2\gamma\mathscr{N}(\rho)), \quad v = \bar{p} & \text{on } \Gamma_1, \\ \mathscr{D}_1(\rho)v = 0 & \text{on } \Gamma_1. \end{cases} \tag{4.1}$$

It should be observed that the transformed stationary problem (4.1) is equivalent to the problem (2.14) by virtue of Lemma 2.1. From Section 2, we know that the linearization of equation (4.1) at the flat equilibrium $(\rho_*, \sigma_*(\rho_* y), p_*(\rho_* y))$ is just the stationary form of

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equation (2.18), that is,

$$\begin{cases} \mathscr{A}(\rho_*)\Sigma = \lambda\Sigma - [\partial\mathscr{A}(\rho_*)\xi]\sigma_*(\rho_* y) & \text{in } \Omega, \\ \mathscr{A}(\rho_*)P = -\mu\Sigma - [\partial\mathscr{A}(\rho_*)\xi]p_*(\rho_* y) & \text{in } \Omega, \\ \mathscr{D}_0(\rho_*)\Sigma = 0, \ \ \mathscr{D}_0(\rho_*)P = 0 & \text{on } \Gamma_0, \\ \Upsilon_1\Sigma = -2\gamma\bar{\sigma}\partial\mathscr{N}(\rho_*)\xi, \ \ \Upsilon_1 P = 0 & \text{on } \Gamma_1, \\ \mathscr{D}_1(\rho_*)P = 0 & \text{on } \Gamma_1. \end{cases} \tag{4.2}$$

In the following, we first study existence of nontrivial solutions of the linearized problem (4.2). For this, as in Section 2, we consider Fourier expansions of ξ, Σ and P

$$\xi(x) = \sum_{k=0}^{\infty} a_k e^{ikx}, \quad \Sigma(x,y) = \sum_{k=0}^{\infty} B_k(y)e^{ikx}, \quad P(x,y) = \sum_{k=0}^{\infty} C_k(y)e^{ikx}.$$

Following the same procedure as in Section 2, we get $B_k(y)$ and $C_k(y)$ as in equations (2.21) and (2.22), where $a_k(t)$ is replaced with a_k. Then substituting the expressions of ξ and P into the last equation in equation (4.2) we have

$$L\xi(x) = -\sum_{k=0}^{\infty} \lambda_k a_k e^{ikx} = 0, \tag{4.3}$$

where L is defined in equation (2.19) and λ_k is given by equation (2.20) or (2.31).

Lemma 4.1

(i) *The linearization (4.2) has a nontrivial solution* (ξ, Σ, P) *if and only if* $\gamma = \gamma_k$, *where* γ_k *is given by equation (2.30). Moreover, this nontrivial solution* (ξ, Σ, P) *has the form*

$$\xi(x) = a_k e^{ikx}, \quad \Sigma(x,y) = B_k(y)a_k e^{ikx}, \quad P(x,y) = C_k(y)a_k e^{ikx},$$

where a_k *is an arbitrary constant, and* $B_k(y)$ *and* $C_k(y)$ *are given by equations (2.21) and (2.22) with* $a_k(t)$ *replaced with* a_k.

(ii) $\lim_{k\to\infty} \gamma_k = 0$, *and there exists a positive integer* k_* *such that* $\gamma_k > 0$ *and* γ_k *is strictly monotone decreasing for all* $k \geqslant k_*$.

Proof The above deductions indicate that the first statement (*i*) holds. Then (*ii*) follows readily from equation (2.32). □

Similarly in Section 2, we can reduce the transformed stationary problem (4.1) into a differential equation $\Phi_\gamma(\rho) = 0$ in the space $h^{m+\alpha}(\mathbb{S}^1)$. Defining

$$F(\rho, \gamma) := \Phi_\gamma(\rho), \tag{4.4}$$

we see from equation (2.13) and the definitions of $Q_\gamma(\rho), R_\gamma(\rho)$ and $\Phi_\gamma(\rho)$ that

$$F \in C^\infty(\mathcal{O}_\delta^{m+\alpha}(\mathbb{S}^1) \times \mathbb{R}^+, \ h^{m-1+\alpha}(\mathbb{S}^1)), \tag{4.5}$$

where $\mathcal{O}_\delta^{m+\alpha}(\mathbb{S}^1)$ is defined in equation (3.13). Then we have the following conclusion.

Lemma 4.2 *The transformed stationary problem (4.1) is equivalent to the following bifurcation problem with bifurcation parameter γ: Find $\rho \in \mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1)$ and $\gamma \in \mathbb{R}^+$ satisfying*

$$F(\rho, \gamma) = 0. \tag{4.6}$$

Later on we shall use the Crandall–Rabinowitz bifurcation theorem to solve equation (4.6). For this, we must overcome a basic difficulty: By Lemma 4.1, we know that the linearized problem (4.2) has an infinite number of eigenvalues, with the eigenspace of each of them being of two dimensions. Since equation (4.1) is equivalent to equation (4.6), its linearization (4.2) must be equivalent to the linearization of equation (4.6). Hence all the eigenspaces of the linearization of equation (4.6) are also two-dimensional. This implies that the Crandall–Rabinowitz bifurcation theorem cannot be used directly. Fortunately, as we shall see below, we can restrict equation (4.6) onto certain subspace of $C^{m+\alpha}(\mathbb{S}^1)$ such that the restricted problem has eigenspaces of one dimension. To do this, for any given integers $k \geqslant 1$ and $m \geqslant 4$ we introduce

$$U_k^{m+\alpha}(\mathbb{S}^1) := \text{ the closure of the span } \{\cos jkx : \ j = 0, 1, 2, \ldots\} \text{ in } C^{m+\alpha}(\mathbb{S}^1),$$
$$V_k^{m+\alpha}(\bar{\Omega}) := \text{ the closure of } \ C^\infty([0,1], U_k^\infty(\mathbb{S}^1)) \cap C^\infty(\bar{\Omega}) \text{ in } C^{m+\alpha}(\bar{\Omega}),$$

where $U_k^\infty(\mathbb{S}^1)) := \bigcap_{m=0}^\infty U_k^{m+\alpha}(\mathbb{S}^1)$. It is easy to see that $U_k^{m+\alpha}(\mathbb{S}^1)$ and $V_k^{m+\alpha}(\bar{\Omega})$ are subspaces of the little Hölder spaces $h^{m+\alpha}(\mathbb{S}^1)$ and $h^{m+\alpha}(\bar{\Omega})$. We have the following result.

Lemma 4.3 *For any given integers $k \geqslant 1$ and $m \geqslant 4$, there holds*

$$F \in C^\infty(\mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^{m+\alpha}(\mathbb{S}^1) \times \mathbb{R}^+, \ U_k^{m-1+\alpha}(\mathbb{S}^1)). \tag{4.7}$$

Proof For any $\rho \in \mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^{m+\alpha}(\mathbb{S}^1)$ and $\gamma \in \mathbb{R}^+$, we can find a sequence $\{\rho_j\}_{j=1}^\infty \in \mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^\infty(\mathbb{S}^1)$ such that $\rho_j \to \rho$ in $C^{m+\alpha}(\mathbb{S}^1)$. By the continuity of $F : \mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1) \times \mathbb{R}^+ \to C^{m-1+\alpha}(\mathbb{S}^1)$, we have $F(\rho_j, \gamma) \to F(\rho, \gamma)$ in $C^{m-1+\alpha}(\mathbb{S}^1)$. If we can prove $F(\rho_j, \gamma) \in U_k^\infty(\mathbb{S}^1)$, then by the definitions of $U_k^{m-1+\alpha}(\mathbb{S}^1)$ and $U_k^\infty(\mathbb{S}^1)$, we obtain $F(\rho, \gamma) \in U_k^{m-1+\alpha}(\mathbb{S}^1)$. Hence, it suffices to prove that if $\rho \in \mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^\infty(\mathbb{S}^1)$ and $\gamma \in \mathbb{R}^+$, then $F(\rho, \gamma) \in U_k^\infty(\mathbb{S}^1)$.

Let $\rho \in \mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^\infty(\mathbb{S}^1)$ and $\gamma \in \mathbb{R}^+$ be given. We first prove that

$$\mathscr{Q}_\gamma(\rho) \in C^\infty([0,1], U_k^\infty(\mathbb{S}^1)) \cap C^\infty(\bar{\Omega}). \tag{4.8}$$

Indeed, for any given $\rho \in \mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^\infty(\mathbb{S}^1)$, from equations (2.2) and (2.3) we know that the problem (2.7) is equivalent to the following problem:

$$\mathscr{A}(\rho)u = \lambda u \ \text{ in } \Omega, \qquad \partial_y u = 0 \ \text{ on } \Gamma_0, \qquad \Upsilon_1 u = u_\infty \ \text{ on } \Gamma_1, \tag{4.9}$$

where $u_\infty = u_\infty(x) := \bar{\sigma}(1 - 2\gamma\mathscr{N}(\rho(x)))$ is known for given ρ. It is clear that $\mathscr{N} \in C^\infty(\mathscr{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^{m+\alpha}(\mathbb{S}^1), U_k^{m-2+\alpha}(\mathbb{S}^1))$, which indicates $u_\infty \in U_k^\infty(\mathbb{S}^1))$. Since $\rho \in C^\infty(\mathbb{S}^1)$, by the well-known regularity theory for elliptic equations we see that $u = \mathscr{Q}_\gamma(\rho) \in C^\infty(\bar{\Omega})$.

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Hence $u(x, y)$ has the Fourier expansion

$$u(x, y) = a_0(y) + \sum_{j=1}^{\infty}(a_j(y)\cos jx + b_j(y)\sin jx),$$

where $a_0, a_j, b_j \in C^\infty[0,1]$. We shall prove that each $b_j(y)$ is zero, and if j is not proportional to k then every $a_j(y)$ is also zero. Let $H^1(\Omega)$ and $H_0^1(\Omega)$ be the usual H^1 and H_0^1 Sobolev spaces on Ω, respectively, and define

$$W_k(\Omega) := \text{ the closure of } V_k^\infty(\bar{\Omega}) \text{ in } H^1(\Omega).$$

Observe that given $w \in W_k(\Omega)$, there exist a_0, a_{jk} in $C^\infty([0,1])$ such that

$$w(x, y) = \sum_{j=0}^{\infty} a_{jk}(y)\cos jkx \quad \text{for } (x, y) \in \Omega.$$

It is obvious that $H^1(\Omega) = W_k(\Omega) \oplus (W_k(\Omega))^\perp$. We now consider the functional J on $W_k(\Omega) \cap H_0^1(\Omega)$ defined by

$$\begin{aligned} J(h) := &\frac{1}{2}\int_\Omega \left[(\partial_x h)^2 + \frac{1}{\rho^2}(\partial_y h)^2\right]\rho dxdy \\ &+ \int_\Omega \left[\frac{\lambda}{2}(h + u_\infty)^2 - (u_\infty)_{xx}h\right]\rho dxdy, \quad h \in W_k(\Omega) \cap H_0^1(\Omega). \end{aligned}$$

From the definition of u_∞, we know that $u_\infty, (u_\infty)_{xx} \in W_k(\Omega)$. A standard argument indicates that J has a unique local minimum in $W_k(\Omega) \cap H_0^1(\Omega)$, which we denote by u_0. Note that since $\rho \in C^\infty(\mathbb{S}^1)$, we actually have $u_0 \in C^\infty([0,1], U_k^\infty(\mathbb{S}^1)) \cap C^\infty(\bar{\Omega})$. In the following, we show that $u = u_0 + u_\infty$.

Since u_0 is the minimum point of J,

$$0 = J'(u_0)v = \int_\Omega \left[\partial_x u_0 \partial_x v + \frac{1}{\rho^2}\partial_y u_0 \partial_y v\right]\rho dxdy + \int_\Omega [\lambda(u_0 + u_\infty) - (u_\infty)_{xx}]v\rho dxdy \quad (4.10)$$

for any $v \in W_k(\Omega) \cap H_0^1(\Omega)$. Noticing $\rho \in \mathcal{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^\infty(\mathbb{S}^1), u_0 \in C^\infty([0,1], U_k^\infty(\mathbb{S}^1)) \cap C^\infty(\bar{\Omega})$ and $u_\infty, (u_\infty)_{xx} \in U_k^\infty(\mathbb{S}^1)$, by careful calculation we can prove that

$$\int_\Omega \left[\partial_x u_0 \partial_x v + \frac{1}{\rho^2}\partial_y u_0 \partial_y v\right]\rho dxdy + \int_\Omega [\lambda(u_0 + u_\infty) - (u_\infty)_{xx}]v\rho dxdy = 0, \quad (4.11)$$

for any $v \in (W_k(\Omega))^\perp \cap H_0^1(\Omega)$. Then equations (4.10) and (4.11) imply that equation (4.11) holds for all $v \in H_0^1(\Omega)$. It follows by classical variation theory that $u_0 + u_\infty$ is a solution of equation (4.9). By uniqueness we have $u = u_0 + u_\infty$. Hence, $u \in C^\infty([0,1], U_k^\infty(\mathbb{S}^1)) \cap C^\infty(\bar{\Omega})$ and equation (4.8) holds.

Substituting $u = \mathcal{Q}_\gamma(\rho)$ into equation (4.1) and following a similar deduction, we can show that $p = \mathcal{R}_\gamma(\rho) \in C^\infty([0,1], U_k^\infty(\mathbb{S}^1)) \cap C^\infty(\bar{\Omega})$. Hence we get

$$F(\rho, \gamma) = \mathcal{D}_1(\rho)\mathcal{R}_\gamma(\rho) = -\rho_x \Upsilon_1 \partial_x \mathcal{R}_\gamma(\rho) + \rho^{-1}(\rho_x^2 + 1)\Upsilon_1 \partial_y \mathcal{R}_\gamma(\rho) \in U_k^\infty(\mathbb{S}^1).$$

This completes the proof. □

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Let F_k be the restriction of F on $\mathcal{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^{m+\alpha}(\mathbb{S}^1) \times \mathbb{R}^+$. Instead of equation (4.6), hereafter, we consider the following problem: Find $\rho \in \mathcal{O}_\delta^{m+\alpha}(\mathbb{S}^1) \cap U_k^{m+\alpha}(\mathbb{S}^1)$ and $\gamma \in \mathbb{R}^+$ such that

$$F_k(\rho, \gamma) = 0. \tag{4.12}$$

Obviously, equation (4.12) is not equivalent to equation (4.6), but a solution of equation (4.12) is certainly a solution of equation (4.6).

It is time to give the proof of Theorem 1.2.

Proof of Theorem 1.2 We shall prove Theorem 1.2 by employing the well-known Crandall–Rabinowitz bifurcation theorem. For this purpose, we need to verify that the map $F_k(\rho, \gamma)$ satisfies all the assumptions in that theorem.

Noticing that ρ_* is an equilibrium of equation (4.12) for all $\gamma \in \mathbb{R}^+$, we have

$$F_k(\rho_*, \gamma) = 0 \quad \text{for all } \gamma \in \mathbb{R}^+, \tag{4.13}$$

$$\partial_\gamma F_k(\rho_*, \gamma) = 0, \quad \partial_{\gamma\gamma} F_k(\rho_*, \gamma) = 0. \tag{4.14}$$

Since equation (4.12) is the restriction of equation (4.6), and the latter is equivalent to equation (4.1), deducing similarly as in Section 3, we see that the linearized equation of (4.12) at ρ_*, which is denoted by

$$\partial_\rho F_k(\rho_*, \gamma)\xi = 0 \quad \text{for } \xi \in U_k^{m+\alpha}(\mathbb{S}^1),$$

is equivalent to the linearized problem (4.2) with the unknown (ξ, Σ, P) restricted to

$$U_k^{m+\alpha}(\mathbb{S}^1) \times V_k^{m-2+\alpha}(\bar{\Omega}_*) \times V_k^{m+\alpha}(\bar{\Omega}_*).$$

Then Lemma 4.1 indicates that equation (4.12) has nontrivial solutions if and only if $\gamma = \gamma_k$ $(k \geqslant k_*)$, and in this situation all solutions of equation (4.12) are given by

$$\xi(x) = C \cos kx,$$

where C is an arbitrary constant. It follows that

$$\mathrm{Ker}\partial_\rho F_k(\rho_*, \gamma_k) \text{ is of one dimension, spanned by } \xi_k = \cos kx. \tag{4.15}$$

Moreover, by Lemma 2.4, equations (2.27), (2.31) and (4.4) we have

$$\partial_\rho F_k(\rho_*, \gamma)1 = -\lambda_0 \neq 0,$$

$$\partial_\rho F_k(\rho_*, \gamma) \cos jkx = c_{jk}(\gamma - \gamma_{jk}) \cos jkx, \quad j = 1, 2, \ldots,$$

where

$$c_{jk} = \left[2\mu\bar{\sigma} j^2 k^2 (\sqrt{j^2k^2 + \lambda}\rho_* \tanh(\sqrt{j^2k^2 + \lambda}\rho_*) - jk\rho_* \tanh(jk\rho_*))\right] / \lambda\rho_*,$$

and γ_{jk} has the expression in equation (2.30) with k replaced with jk. Hence, we have

$$\mathrm{codim}\ \mathrm{Im}\partial_\rho F_k(\rho_*, \gamma_k) = 1. \tag{4.16}$$

From the above analysis we can easily deduce that

$$\partial_\gamma \partial_\rho F_k(\rho_*, \gamma) \cos kx|_{\gamma=\gamma_k} = c_k \cos kx, \tag{4.17}$$

where c_k has the same form as the above c_{jk} with jk replaced with k. Moreover, equations (4.15) and (4.16) imply that

$$\cos kx \text{ is orthogonal to } \partial_\rho F_k(\rho_*, \gamma_k)h \ \text{ for any } h = \sum_{j=0}^{\infty} \cos jkx. \tag{4.18}$$

It follows from equations (4.17) and (4.18) that

$$\partial_\gamma \partial_\rho F_k(\rho_*, \gamma)\xi_k|_{\gamma=\gamma_k} \notin \mathrm{Im}\partial_\rho F_k(\rho_*, \gamma_k). \tag{4.19}$$

Combining equations (4.13)–(4.16) and (4.19), we see that all the assumptions in Theorem 1.7 of [9] are satisfied by the equation (4.12). Thus using that theorem we infer that in a neighbourhood of (ρ_*, γ_k) the set of solutions of equation (4.12) consists of two C^{m-2} smooth curves Γ_1 and Γ_2, which intersect only at the point (ρ_*, γ_k). Actually, Γ_1 is the curve (ρ_*, γ) and Γ_2 can be parameterized as

$$\Gamma_2 : \quad (\rho^\varepsilon, \gamma^\varepsilon), \quad |\varepsilon| \text{ is small}, \quad (\rho^0, \gamma^0) = (\rho_*, \gamma_k), \quad \rho'(0) = 0.$$

Returning to the original problem (1.1) and using equations (2.21) and (2.22), we see that the corresponding bifurcation solution $(\gamma^\varepsilon, \rho^\varepsilon, \sigma^\varepsilon, p^\varepsilon)$ of equation (1.1) has the asymptotic expansion

$$\begin{aligned} &\gamma^\varepsilon = \gamma_k + O(\varepsilon), \quad \rho^\varepsilon(x) = \rho_* + \varepsilon \cos kx + O(\varepsilon^2), \\ &\sigma^\varepsilon(x,y) = \sigma_*(y) + \varepsilon D_k(y) \cos kx + O(\varepsilon^2), \quad p^\varepsilon(x,y) = p^*(y) + \varepsilon E_k(y) \cos kx + O(\varepsilon^2), \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} D_k(y) &= -\bar\sigma(\sqrt{\lambda}\tanh(\sqrt{\lambda}\rho_*) + 2\gamma k^2)\frac{\cosh(\sqrt{k^2+\lambda}\rho_* y)}{\cosh(\sqrt{k^2+\lambda}\rho_*)} + \bar\sigma\sqrt{\lambda}\frac{y\sinh(\sqrt{\lambda}\rho_* y)}{\cosh(\sqrt{\lambda}\rho_*)}, \\ E_k(y) &= \mu\tilde\sigma\rho_* y^2 - \mu\bar\sigma\frac{y\sinh(\sqrt{\lambda}\rho_* y)}{\sqrt{\lambda}\cosh(\sqrt{\lambda_1}\rho_*)} - \mu(\tilde\sigma\rho_*\lambda + 2k^2\gamma\bar\sigma)\frac{\cosh(k\rho_* y)}{\lambda\cosh(k\rho_*)} \\ &\quad + \mu\bar\sigma(2k^2\gamma + \sqrt{\lambda}\tanh(\sqrt{\lambda}\rho_*))\frac{\cosh(\sqrt{k^2+\lambda}\rho_* y)}{\lambda\cosh(\sqrt{k^2+\lambda}\rho_*)}. \end{aligned} \tag{4.21}$$

This completes the proof of Theorem 1.2. □

5 Conclusions and biological implications

In this paper, we have studied a free boundary problem modelling the growth of multi-layer tumours with Gibbs–Thomson relation. By reducing it to a nonlinear differential equation on Banach spaces and then using analytic semigroup theory and bifurcation analysis, we prove that there exists a positive threshold value γ_* of surface tension coefficient such that if $\gamma > \gamma_*$ then the unique flat stationary solution is asymptotically

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stable, while for $\gamma < \gamma_*$ it is unstable and there exist infinite number of non-flat stationary bifurcation solutions with free boundary $\rho^\varepsilon(x) = \rho_* + \varepsilon \cos kx + O(\varepsilon^2)$.

Instability of a tumour implies that the tumour may grow with a variety of shapes, in particular, generate protrusions, or "fingers" in the form of bifurcation solutions, which are associated with invasion by the tumour of its surrounding stroma (cf. [18, 19]]). Concerning Gibbs–Thomson relation, $2\gamma\kappa$ is the energy required to maintain the intercellular bonds existing on the free boundary. As a consequence, our results indicate that a smaller value of surface tension coefficient γ leads to smaller energy on the free boundary, which further makes the protrusions generate more easily and tumours more aggressive.

It is also interesting to compare the model with Gibbs–Thomson relation and the model with surface tension effect which was well studied in [11, 32], with a similar result for a corresponding threshold value $\tilde{\gamma}_*$ (see equation (2.34)). In the model with Gibbs–Thomson relation, by the definition (2.33) we see that γ_* is independent of the proliferation rate μ, which means that the proliferation rate has no effect on the tumour's stability. However for the model with surface tension effect, $\tilde{\gamma}_*$ depends linearly on μ. Moreover, Lemma 2.6 says that $\gamma_* < \tilde{\gamma}_*$ for large proliferation rate μ, which implies that a tumour with surface tension effect is more hazardous and aggressive than a tumour with Gibbs–Thomson relation. However for small proliferation rate the situation is the opposite.

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