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ALMOST GENTLE ALGEBRAS AND THEIR TRIVIAL EXTENSIONS

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Abstract In this paper we define almost gentle algebras, which are monomial special multiserial algebras generalizing gentle algebras. We show that the trivial extension of an almost gentle algebra by its minimal injective co-generator is a symmetric special multiserial algebra and hence a Brauer configuration algebra. Conversely, we show that any almost gentle algebra is an admissible cut of a unique Brauer configuration algebra and, as a consequence, we obtain that every Brauer configuration algebra with multiplicity function identically one is the trivial extension of an almost gentle algebra. We show that a hypergraph is associated with every almost gentle algebra A, and that this hypergraph induces the Brauer configuration of the trivial extension of A. Among other things, this gives a combinatorial criterion to decide when two almost gentle algebras have isomorphic trivial extensions.

Keywords: gentle algebra; special biserial algebra; symmetric special multiserial algebra; Brauer configuration algebra

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1. Introduction

In this paper we introduce a new class of multiserial algebras called almost gentle algebras. These algebras are monomial quadratic algebras which generalize gentle algebras. Namely, an algebra KQ/I is almost gentle if it is special multiserial and if I is generated by paths of length 2. It is clear from the definition that every gentle algebra is almost gentle. While gentle algebras are of tame representation type, almost gentle algebras are wild in general. However, there are many examples of almost gentle algebras of finite and tame representation type that are not gentle.

Gentle algebras are one of the most studied classes of algebras, as they appear in many different contexts such as Jacobian algebras of unpunctured surfaces in cluster theory [**2**, **25**] and algebras associated with dimer models [**5**, **6**] or in the context of the study of the enveloping algebra of Lie algebra [**22**]. Their representation theory comes with a

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strong combinatorial structure. They are string algebras and as such their indecomposable modules are given by string and band modules, and their Auslander–Reiten quiver is completely determined by the string combinatorics [**8**]. Maps between string and band modules have been given in [**11**, **24**], respectively. They are a class of algebras closed under derived equivalence [**29**] and they are derived tame. The indecomposable objects in the derived category of a gentle algebra have been determined in [**3**]; they are given by homotopy strings and bands. In [**1**] the maps between homotopy strings and bands have been explicitly described. The singularity category of a gentle algebra has been described in [**23**]. Recently, in [**7**, **9**, **10**], a basis of the extensions between string and band modules has been given.

In [**30**], a ribbon graph is associated with every gentle algebra A. It is shown that this ribbon graph gives rise to the Brauer graph of the trivial extension of A, which is a Brauer graph algebra. As a consequence, the gentle algebra and the associated Brauer graph algebra (corresponding to the trivial extension) have many properties in common. Moreover, the associated ribbon graph gives rise to an oriented bounded surface Σ_A. This surface gives a geometric model for the derived category of the gentle algebra A, which in turn is equivalent to the partially wrapped Fukaya category of Σ_A [**21**, **26**, **27**].

Almost gentle algebras do not have the underlying string combinatorics that gentle algebras have. However, the strong similarity in their structure makes this an interesting new class of algebras to consider. It contains many examples of well-studied algebras, such as hereditary algebras arising from many orientations of all Dynkin and extended Dynkin quivers.

As is the case for gentle algebras, almost gentle algebras can be of finite or infinite global dimension. They are of infinite global dimension if and only if the quiver contains an oriented cycle in which every subpath of length 2 is a relation. While gentle algebras are Gorenstein [**17**], this is not necessarily true for almost gentle algebras; see the example in § 2.

In §§ 2 and 3, we give a closed formula for the dimensions of almost gentle algebras and their trivial extensions in terms of maximal paths in the almost gentle algebras.

In § 4 we show that the trivial extension $T(A) = A \ltimes \mathrm{Hom}_k(A, k)$ of an almost gentle algebra is special multiserial. We note, however, that the converse remains an open question: that is, it is not known whether the trivial extension of an algebra A being special multiserial implies that A is almost gentle.

Another class of examples of algebras arising as trivial extensions of almost gentle algebras is given by symmetric algebras with radical cube zero, which have been extensively studied; see for example [**4**, **12**, **13**, **18**]. It follows from the results in this paper and in [**18**] that an algebra is a symmetric algebra with radical cube zero if and only if it is a trivial extension of an almost gentle algebra where the paths in the quiver of the almost gentle algebra are all of length at most one.

In § 5 we show that an admissible cut, as defined in [**30**] and based on the definition of admissible cuts in [**14**, **15**] (see § 5 for the definition), of a symmetric special biserial algebra gives rise to an almost gentle algebra. In the other direction we show that every symmetric special multiserial algebra with no powers in the relations, or, equivalently, every Brauer configuration algebra with multiplicity function equal to one, is the trivial extension of an almost gentle algebra (see [**19**] for the definition of Brauer configuration algebras). We note that this almost gentle algebra is not unique. In fact, our construction

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gives a whole family of almost gentle algebras that have isomorphic trivial extensions. While all of these gentle algebras have the same number of simple modules, they can have very different homological properties. For example, some might have finite global dimension while others might have infinite global dimension. Furthermore, it is straightforward to see that these algebras are not derived equivalent in general. We leave it as an open question to the reader to determine the relationship between all the gentle algebras that have the same trivial extension.

In § 6, we give a construction of the Brauer configuration of the trivial extension of an almost gentle algebra. The construction is based on the notion of an algebra defined by cycles. A Brauer configuration is a vertex-decorated hypergraph with an orientation (or hypergraph ribbon structure). Based on this observation, in § 7 we associate with every almost gentle algebra a decorated hypergraph with orientation, and show that this hypergraph is precisely the Brauer configuration of the trivial extension of the almost gentle algebra. That is, in the terminology of § 6, it is exactly the Brauer configuration of the algebra defined by cycles isomorphic to the trivial extension of the almost gentle algebra. It follows that two almost gentle algebras have the same trivial extensions if and only if they have the same associated hypergraph.

2. Almost gentle algebras

In this section we define almost gentle algebras, generalizing the class of gentle algebras.

First we fix some notation. Let K be a field. All algebras are assumed to be indecomposable K-algebras. Unless otherwise stated, an algebra given by quiver and relations KQ/I is assumed to be finite dimensional and the ideal I is assumed to be admissible. For a quiver Q, we denote by Q_0 the set of vertices in Q and by Q_1 the set of arrows in Q. We set e_v to be the trivial path at a vertex $v \in Q_0$. Furthermore, for $a, b \in Q_1$, we write ab for the path a followed by b. We let $s(a)$ be the vertex at which the arrow a starts and let $t(a)$ be the vertex at which a ends. For a path $p = a_1 \cdots a_n$ in Q, we set $s(p) = s(a_1)$ and $t(p) = t(a_n)$. Given a finite-dimensional algebra Λ, let $\Lambda^e \simeq \Lambda \otimes_k \Lambda^{op}$.

An algebra A is *gentle* if it is Morita equivalent to an algebra KQ/I such that:

(S0) I is generated by paths of length 2;

(S1) for every arrow $a \in Q_1$, there exists at most one arrow b such that $ab \notin I$ and at most one arrow c such that $ca \notin I$;

(S2) for every arrow $a \in Q_1$, there exists at most one arrow b such that $ab \in I$ and at most one arrow c such that $ca \in I$;

(S3) for every vertex $v \in Q_0$, there are at most two arrows ending at v and at most two arrows starting at v.

Recall from [**18**] that an algebra is *special multiserial* if it is Morita equivalent to an algebra KQ/I satisfying condition (S1).

Definition 2.1. We say that an algebra is *almost gentle* if it is Morita equivalent to an algebra KQ/I such that:

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(S0) I is generated by paths of length 2;

(S1) for every arrow $a \in Q_1$, there exists at most one arrow b such that $ab \notin I$ and at most one arrow c such that $ca \notin I$.

So an algebra is almost gentle if it is Morita equivalent to a special multiserial algebra KQ/I, where I is generated by monomial relations of length 2.

Remark 2.2. Every gentle algebra is almost gentle.

We state some basic facts about almost gentle algebras. An almost gentle algebra KQ/I is of infinite global dimension if there is an oriented cycle in Q such that every path of length 2 in that cycle is in I. If no such cycle exists then KQ/I is of finite global dimension. Since the ideal I can be generated by paths of length 2, every almost gentle algebra is a Koszul algebra. The only almost gentle algebras that are self-injective are $K[x]/(x^2)$ and the oriented cycle with all paths of length 2 being relations.

Gentle algebras are Gorenstein [17]. The same does not hold for almost gentle algebras. Consider, for example, the algebra with quiver $\bullet \underset{b}{\overset{a}{\rightleftarrows}} \bullet \xrightarrow{c} \bullet$ and where the ideal of relations is generated by all paths of length 2; then, the resulting algebra is almost gentle but not Gorenstein.

In the following lemmas we collect some obvious properties of almost gentle algebras.

Lemma 2.3. *Let $A = KQ/I$ be an almost gentle algebra and let C be an oriented cycle in Q. Then there exists a path of length 2 in C that is in I.*

Let p be a path in Q. Then we say that p is a *maximal path of* $A = KQ/I$ if $p \notin I$ and for every arrow a in Q we have $ap \in I$ and $pa \in I$. We denote the set of maximal paths of A by $\mathcal{M}$.

Lemma 2.4. *Let $A = KQ/I$ be an almost gentle algebra and let v be a vertex in Q. Then v lies in a unique maximal path of A if and only if one of the following conditions holds:*

(i) *v is a sink with a unique arrow ending at v;*

(ii) *v is a source with a unique arrow starting at v;*

(iii) *there is a unique arrow a ending at v and a unique arrow b starting at v and $ab \notin I$.*

Lemma 2.5. *Let $A = KQ/I$ be an almost gentle algebra. Then the following hold.*

(i) *Every arrow $a \in Q_1$ lies in exactly one maximal path of A.*

(ii) *Let $m \in \mathcal{M}$. Then m has no repeated arrows.*

We introduce two functions associated with an almost gentle algebra A, which will be used later in the paper. Let $\diamond$ be some element not in Q_1 and set $\mathcal{A} = Q_1 \cup \{\diamond\}$. Define

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$\sigma\colon Q_1 \to \mathcal{A}$ and $\tau\colon Q_1 \to \mathcal{A}$ by

$$\sigma(a) = \begin{cases} b & \text{if } ab \notin I \\ \diamond & \text{if } ab \in I \text{ for all } b \in Q_1 \end{cases}$$

and

$$\tau(a) = \begin{cases} c & \text{if } ca \notin I \\ \diamond & \text{if } ca \notin I \text{ for all } c \in Q_1 \end{cases}$$

where a, b, $c \in Q_1$. From the definition of special multiserial, we see that these functions are well defined. Since A is finite dimensional, for every $a \in Q_1$, there are smallest non-negative integers M_a and N_a such that $\sigma^{M_a}(a) = \diamond$ and $\tau^{N_a}(a) = \diamond$. It follows that the unique maximal path of A containing the arrow a is $\tau^{N_a-1}(a)\tau^{N_a-2}(a)\cdots\tau(a)a\sigma(a)\cdots\sigma^{M_a-1}(a)$, which is of length $M_a + N_a - 1$. Since a maximal path of A has no repeated arrows and since every arrow is in a unique maximal path of A, it is easy to see that a maximal path of A is the unique maximal path of A of any of its arrows, and that the position at which that arrow occurs in the path is uniquely determined.

If $A = KQ/I$ and $\pi\colon KQ \to A$ is the canonical surjection, then for $x \in KQ$ we will denote $\pi(x)$ by $\bar{x}$. If $a \in Q_1$, we let U_a be the right A-module $\overline{a}A$ generated by $\overline{a}$. If A is an almost gentle algebra, then the U_a are uniserial A-modules. Note that this holds more generally if A is a special multiserial algebra; see [**18**].

Proposition 2.6. *Let $A = KQ/I$ be an almost gentle algebra. Then $\text{rad}(A) = \bigoplus_{a\in Q_1} U_a$. A K-basis for A is the set of $\bar{p}$, where $\bar{p}$ is a subpath of length ≥ 1 of a maximal path of A, together with the trivial subpaths $\overline{e_v}$, for $v \in Q_0$.*

Proof. Since A is a monomial algebra, A has a K-basis $\{\bar{p} \mid p \text{ is a path in } Q \text{ and } p \notin I\}$. Any such p either has length ≥ 1 or $p = e_v$ for some vertex v. This proves the basis part of the result. If $p = a_1 \cdots a_n$ is a path in Q then $\bar{p} \in U_{a_1}$. It also follows that $\sum_{a\in Q_1} U_a = \text{rad}(A)$, since the images of the arrows generate $\text{rad}(A)$. Using that A is a monomial algebra, we see that $\bigoplus_{a\in Q_1} U_a = \text{rad}(A)$. □

If p is a path in Q, we let $\ell(p)$ denote the length of p.

Corollary 2.7. *Let A be an almost gentle algebra. Then*

$$\dim_K(A) = \mid Q_0 \mid + \sum_{m\in\mathcal{M}} \ell(U_m)(\ell(U_m) + 1)/2,$$

where U_m is the right uniserial A-module generated by the first arrow in m.

3. The symmetric special multiserial algebra associated with an almost gentle algebra

In [**20**], given a special multiserial algebra A, we constructed a symmetric special multiserial algebra A^* such that A is a quotient of A^*. Recall from [**18**] that the class of symmetric special multiserial algebras and the class of Brauer configuration algebras

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coincide. We refer the reader to [19] for the definition of a Brauer configuration algebra. We slightly modify that construction below in the case of an almost gentle algebra. In this section, $A = KQ/I$ will denote an almost gentle algebra, where I is an ideal generated by quadratic elements. Recall that $\mathcal{M}$ is the set of maximal paths of A.

We begin by defining a new quiver Q^*. The vertices of Q^* are the same as those of Q. For each $m \in \mathcal{M}$, let a_m denote an arrow (not in Q_1) from the end vertex of m to the start vertex of m. The arrow set of Q^* is $Q_1 \cup \{a_m \mid m \in \mathcal{M}\}$. Since Q is a subquiver of Q^*, we freely view paths in Q as paths in Q^*. For each $m \in \mathcal{M}$, we obtain a cycle $C_m = ma_m$ in Q^*. We let $\mathcal{S}$ denote the set cycles C^* such that C^* is a cyclic permutation of C_m for some $m \in \mathcal{M}$. Let $\mu\colon \mathcal{S} \to \mathbb{Z}_{>0}$ be defined by $\mu(C^*) = 1$, for all $C^* \in \mathcal{S}$.

We say a cycle in Q^* is *simple* if the cycle has no repeated arrows. Following [20], we say a pair $(\mathcal{T}, \nu)$ is a *defining pair in* Q if $\mathcal{T}$ is a set of simple cycles in Q and $\nu\colon \mathcal{T} \to \mathbb{Z}_{>0}$ which satisfy the following conditions:

D0 if C is a loop at a vertex v and $C \in \mathcal{T}$, then $\nu(C) > 1$;

D1 if a simple cycle is in $\mathcal{T}$, every cyclic permutation of the cycle is in $\mathcal{T}$;

D2 if $C \in \mathcal{T}$ and C' is a cyclic permutation of C then $\nu(C) = \nu(C')$;

D3 every arrow occurs in some simple cycle in $\mathcal{T}$;

D4 if an arrow occurs in two cycles in $\mathcal{T}$, the cycles are cyclic permutations of each other.

Proposition 3.1. *The pair* $(\mathcal{S}, \mu)$ *defined above is a defining pair.*

Proof. Since for $m \in \mathcal{M}$, m has no repeated arrows, the cycles ma_m and their cyclic permutations are simple cycles. If a is a loop in Q, and hence in Q^*, then since A is finite dimensional and I can be generated by paths of length 2, we see that $a^2 \in I$. Let m be the unique maximal path in which a occurs. Then a occurs in ma_m, which is not a loop. That is, $\mathcal{S}$ contains no loops and hence D0 vacuously holds. By construction, D1 holds. Since $\mu \equiv 1$, D2 holds. Since every arrow in Q occurs in some maximal path m, every arrow in Q occurs in some cycle $C^* \in \mathcal{S}$. Each new arrow a_m occurs in $ma_m \in \mathcal{S}$ and we see that D3 holds. Since an arrow in Q occurs in a unique maximal path in Q, D4 holds. □

Following [20], the defining pair $(\mathcal{S}, \mu)$ in Q^* gives rise to a K-algebra with quiver Q^* and ideal of relations generated by all relations of the following three types.

Type 1 $C^{\mu(C)} - C'^{\mu(C')}$, if C and C' are cycles in $\mathcal{S}$ at some vertex $v \in Q_0$.

Type 2 Ca, if $C \in \mathcal{S}$ and a is the first arrow in C.

Type 3 ab, if $a, b \in Q_1$ and ab does not lie on any $C \in \mathcal{S}$.

In [20], an algebra KQ/I given by a defining pair $(\mathcal{T}, \nu)$ in Q such that I is generated by all relations of Types 1, 2 and 3 is called *the algebra defined by cycles* $(\mathcal{T}, \nu)$.

The next result follows from [20].

Theorem 3.2. *Let A be an almost gentle algebra and let A^* be the algebra defined by cycles $(\mathcal{S}, \mu)$ as defined above. Then A^* is a symmetric special multiserial algebra and thus it is a Brauer configuration algebra.*

We call the algebra $A^* = KQ^*/I^*$ above the *symmetric special multiserial algebra associated with A.*

The next result determines the dimension of A^*.

Proposition 3.3. *Let A be an almost gentle algebra and let A^* be the symmetric special multiserial algebra associated with A. Then*

$$\dim_K(A^*) = 2|Q_0| + \sum_{m\in\mathcal{M}} \ell(m)\cdot(\ell(m)+1).$$

In particular, $\dim_K(A^*) = 2\dim_K(A)$.

Proof. The quiver of Q^* of A^* has $|Q_0|$ vertices, and so there are $|Q_0|$ paths of length 0, the e_v, for $v \in Q_0$. Since A^* is a symmetric algebra, the socle of A^* has dimension $|Q_0|$. We now find the dimension of $\operatorname{rad}(A^*)/\operatorname{soc}(A^*)$. Consider $m \in \mathcal{M}$. The cycle $ma_m \in \mathcal{S}$ has length $\ell(m)+1$. If a is an arrow in ma_m then aA^* is a uniserial module of length $\ell(m)+1$. Then $aA^*/(aA^* \cap \operatorname{soc}(A^*))$ has dimension $\ell(m)$ and there are $\ell(m)+1$ choices for a. Therefore, we see that $\dim_K(\operatorname{rad}(A^*)/\operatorname{soc}(A^*)) = \sum_{m\in\mathcal{M}} \ell(m)(\ell(m)+1)$.

The last part follows from Corollary 2.7. □

4. Trivial extension of an almost gentle algebra

The trivial extension of a ring by a bimodule allows one to consider the bimodule as an ideal in the trivial extension. A canonical bimodule, other than the algebra itself, is the dual of the algebra. For this bimodule, the trivial extension always is a symmetric algebra, which is a natural generalization of a group ring. Understanding trivial extensions (by the dual of the algebra) for a class of algebras leads to a deeper understanding of the algebras and their properties.

Let $A = KQ/I$ be a finite-dimensional algebra and let $D(A) = \operatorname{Hom}_K(A, K)$ be its K-linear dual. Recall that the trivial extension $T(A) = A \ltimes D(A)$ is a symmetric algebra defined as the vector space $A \oplus D(A)$ and with multiplication given by $(a, f)(b, g) = (ab, ag + fb)$, for any $a, b \in A$ and $f, g \in D(A)$. Note that $D(A)$ is an A-A-bimodule via the following. If $a, b \in A$ and $f \in D(A)$, then $afb\colon A \to K$ by $(afb)(x) = f(bxa)$. We keep the convention that if $x \in KQ$ and $\pi\colon KQ \to A$ is the canonical surjection, then we denote $\pi(x)$ by $\bar{x}$.

Let $\mathcal{B}$ be the set of finite directed paths in Q and suppose that I is generated by paths; that is, KQ/I is a monomial algebra. Consider the set $\bar{\mathcal{B}} = \{p \in \mathcal{B} \mid p \notin I\}$. The set $\{\bar{p} \mid p \in \bar{\mathcal{B}}\}$ is a K-basis of A. We abuse the notation and view $\bar{\mathcal{B}}$ as a K-basis of A. Then the set $\mathcal{M}$ of maximal paths of A is a subset of $\bar{\mathcal{B}}$ and forms a K-basis of $\operatorname{soc}_{A^e}(A)$.

The dual basis $\bar{\mathcal{B}}^\vee = \{p^\vee \mid p \in \bar{\mathcal{B}}\}$ is a K-basis of $D(A)$ where, if $p \in \bar{\mathcal{B}}$, $p^\vee \in D(A)$ is the element in $D(A)$ defined by $p^\vee(q) = \delta_{p,q}$ for $q \in \bar{\mathcal{B}}$, where $\delta_{p,q}$ is the Kronecker delta.

Lemma 4.1. *Let A be a finite-dimensional monomial algebra with K-basis $\bar{\mathcal{B}}$ as above. Then, for p, q, $r \in \bar{\mathcal{B}}$, the following hold in $T(A)$.*

(1) $(p, 0)(0, r^\vee) = \begin{cases} (0, s^\vee) & \text{if there is some } s \in \bar{\mathcal{B}} \text{ with } sp = r \\ 0 & \text{otherwise.} \end{cases}$

(2) $(0, r^\vee)(q, 0) = \begin{cases} (0, s^\vee) & \text{if there is some } s \in \bar{\mathcal{B}} \text{ with } qs = r \\ 0 & \text{otherwise.} \end{cases}$

(3) $(0, p^\vee)(q, 0)(0, r^\vee) = 0$.

(4) *If $prq \in \bar{\mathcal{B}}$ for some p, q, $r \in \bar{\mathcal{B}}$ then $(q, 0)(0, (prq)^\vee)(p, 0) = (0, r^\vee)$.*

Proof. Parts (1) and (2) are immediate consequences of the multiplication in $T(A)$. Part (3) follows from parts (1) and (2) and that $(0, x^\vee)(0, y^\vee) = 0$ for all x, $y \in \bar{\mathcal{B}}$. Part (4) follows from parts (1) and (2). □

Proposition 4.2. *Let A be a finite-dimensional monomial algebra. Then $T(A)$ is generated by $\{(a, 0) \mid a \in Q_1\} \cup \{(0, m^\vee) \mid m \in \mathcal{M}\}$.*

Proof. Let $\bar{\mathcal{B}}$ be a K-basis of A as defined above. Since $\{(p, 0) \mid p \in \bar{\mathcal{B}}\} \cup \{(0, p^\vee) \mid p \in \bar{\mathcal{B}}\}$ is a K-basis of $T(A)$, we need only show that if $p \in \bar{\mathcal{B}}$, then $(p, 0)$ and $(0, p^\vee)$ are in the two-sided ideal in $T(A)$ generated by $\{(a, 0) \mid a \in Q_1\} \cup \{(0, m^\vee) \mid m \in \mathcal{M}\}$. Let $p \in \bar{\mathcal{B}}$. Since p is a product of arrows, $(p, 0)$ is a product of elements of the form $(a, 0)$, where a is an arrow in Q. Now consider $(0, p^\vee)$. There are paths r and s such that $rps \in \mathcal{M}$. But then $(0, p^\vee) = (s, 0)(0, (rps)^\vee)(r, 0)$ by Lemma 4.1 part (4) and we are done. □

We now prove the main result of this section.

Theorem 4.3. *Let $A = KQ/I$ be an almost gentle algebra, A^* the symmetric special multiserial algebra associated with A, and $T(A)$ the trivial extension of A by $D(A)$. Then A^* is isomorphic to $T(A)$.*

Proof. Let Q^* be the quiver of A^* which is defined in § 3. We begin by defining a ring surjection φ from KQ^* to $T(A)$. Since the vertices of Q^* are the same as the vertices of Q and Q is a subquiver of Q_T, the quiver of $T(A)$, we send a vertex v in Q^* to $\bar{v}$, the image of v in $T(A)$ under the canonical surjection $KQ_T \to T(A)$. We define φ on arrows as follows. If a is an arrow in $Q \subseteq Q^*$, let $\varphi(a) = (a, 0)$. If $m \in \mathcal{M}$, then $\varphi(a_m) = (0, m^\vee)$. Note that a_m is an arrow from $t(m)$ to $s(m)$ and that $(0, m^\vee) = (e_{t(m)}, 0)(0, m^\vee)(e_{s(m)}, 0)$. By the universal mapping property of a path algebra, we obtain a K-algebra homomorphism $\varphi\colon KQ^* \to T(A)$. By Proposition 4.2, φ is a surjection.

Next we show that I^* (defined in § 3) is contained in $\ker(\varphi)$. For this, we prove that relations of Types 1, 2 and 3 are in $\ker(\varphi)$. Recall that $\mathcal{S}$ is defined to be the set of simple cycles in Q^* that are cyclic permutations of the cycles ma_m, for some $m \in \mathcal{M}$. We begin with a Type 1 relation. Let C, $C' \in \mathcal{S}$ be cycles in $\mathcal{S}$ at a vertex $v \in Q^*$. We need

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to show that $\varphi(C - C') = 0$. Let m, $m'\mathcal{M}$, p, q, p', $q' \in \bar{\mathcal{B}}$ such that $pq = m$, $p'q' = a_{m'}$, $C = qa_m p$ and $C' = q'a_{m'}p'$. Then $\varphi(C) = (p, 0)(0, m^\vee)(q, 0)$. Since C is a cycle at v, by Lemma 4.1, $(p, 0)(0, m^\vee)(q, 0) = (0, qm^\vee p) = (0, r^\vee)$ where $prq = m$. It follows that $r = e_v$ since $pe_v q = m$. Thus, we have shown that $\varphi(C) = (0, e_v^\vee)$. By a similar argument, $\varphi(C') = (0, e_v^\vee)$ and we conclude that $\varphi(C - C') = 0$.

Next we show that Type 2 relations are sent to 0 by φ. Let $C \in \mathcal{S}$ be a cycle at v, with first arrow b. Either b is an arrow in Q or $b = a_m$ for some $m \in \mathcal{M}$. Then $\varphi(Cb) = (0, e_v^\vee)\varphi(b)$. If b is an arrow in Q, then $(0, e_v^\vee)(b, 0) = (0, be_v^\vee)$. If $(0, be_v^\vee) \neq 0$ then $(0, be_v^\vee) = (0, r^\vee)$ where $rb = e_v$, which is not possible since b is an arrow. If $b = a_m$, for some $m \in \mathcal{M}$, then $\varphi(Cb) = (0, e_v^\vee)(0, m^\vee) = 0$ by Lemma 4.1. Hence we have shown that Type 2 relations are sent to 0 under φ.

Finally, let ab be a Type 3 relation. We want to show that $\varphi(ab) = 0$. There are four cases: both a and b are arrows in Q, a is an arrow in Q and $b = a_m$ for some $m \in \mathcal{M}$, b is an arrow in Q and $a = a_m$ for some $m \in \mathcal{M}$, and $a = a_m$, $b = a_{m'}$ for some m, $m' \in \mathcal{M}$. If both a and b are arrows in Q, then, since ab is a relation in A^*, $ab = 0$. Next suppose that a is an arrow in Q and $b = a_m$ for some $m \in \mathcal{M}$. Then $\varphi(aa_m) = (a, 0)(0, m^\vee) = (0, am^\vee)$. If $(0, am^\vee) \neq 0$, then $(0, am^\vee) = (0, r^\vee)$ where $ra = m$. But then a is the last arrow in m and aa_m is not a Type 2 relation. The case where $a = a_m$ for some $m \in \mathcal{M}$ and b is an arrow is handled in a similar fashion to the last case. The final case is when $a = a_m$ and $b = a_{m'}$ for some m, $m' \in \mathcal{M}$. Then $\varphi(ab) = (0, m^\vee)(0, m'^\vee) = 0$ by 4.1(4). This completes the proof that $\varphi(I^*) = 0$.

Since $\varphi\colon KQ^* \to T(A)$ is a surjection and $\varphi(I^*) = 0$, φ induces a surjection $\psi\colon KQ^*/I^* \to T(A)$. Now $A^* = KQ^*/I^*$ and, by Proposition 3.3, $\dim_K(A^*) = 2\dim_K(A)$. Clearly, $\dim_K(T(A) = 2\dim_K(A)$. Hence $\psi\colon A^* \to T(A)$ is an isomorphism. □

Corollary 4.4. *Let $A = KQ/I$ be an almost gentle algebra. Then $T(A)$ is symmetric special multiserial; that is, $T(A)$ is a Brauer configuration algebra.*

We end this section with an open question.

Question 4.5. *Is it true that if a trivial extension $T(A)$ of a finite-dimensional K-algebra is special multiserial then A is almost gentle?*

5. Admissible cuts

Let $\Lambda = KQ_\Lambda/I_\Lambda$ be the symmetric special multiserial algebra given by the defining pair $(\mathcal{S}, \mu)$. There is an equivalence relation on $\mathcal{S}$ as follows: two special cycles are equivalent if one is a cyclic permutation of the other. Let $\{C_1, \ldots, C_t\}$ be a set of equivalence class representatives.

Definition 5.1. An *admissible cut* D of Q_Λ is a subset of arrows in Q_Λ consisting of exactly one arrow in each special cycle corresponding to an equivalence class representative C_i, for $i = 1, \ldots, t$. We call $kQ_\Lambda/\langle I_\Lambda \cup D\rangle$ the *cut algebra associated with D* where $\langle I_\Lambda \cup D\rangle$ is the ideal generated by $I_\Lambda \cup D$.

Recall that any symmetric special multiserial algebra is defined by cycles. We show the following theorem.

Theorem 5.2. *Let $\Lambda = KQ_\Lambda/I_\Lambda$ be a symmetric special multiserial algebra defined by a defining pair $(\mathcal{S}, \mu)$ and let D be an admissible cut of Q_Λ. Set Q to be the quiver given by $Q_0 = (Q_\Lambda)_0$ and $Q_1 = (Q_\Lambda)_1 \setminus D$. Then the cut algebra, $KQ_\Lambda/\langle I_\Lambda \cup D\rangle$ associated with D is isomorphic to $KQ/(I_\Lambda \cap KQ)$.*

Moreover, $KQ/(I_\Lambda \cap KQ)$ is an almost gentle algebra.

Proof. The inclusion of quivers $Q \subset Q_\Lambda$ induces a K-algebra homomorphism $f : KQ \to KQ_\Lambda/\langle I_\Lambda \cup D\rangle$. We show that f is surjective. Let $\sum_p \lambda_p p$ be an element in KQ_Λ; that is, $\lambda_p \in K$, with almost all $\lambda_p = 0$ and p a path in Q_Λ. Then $\sum_p \lambda_p p = \sum_q \lambda_q q + \sum_r \lambda_r r$ where the first sum runs over all paths q such that no arrow of D occurs in q, and the second sum runs over all paths r in Q such that there is at least one arrow of D in r. Then $\sum \lambda_r r$ is in the ideal $\langle I_\Lambda \cup D\rangle$ and $\sum \lambda_q q$ is in the image of KQ in KQ_Λ. It follows that f is surjective.

We now show that $\ker f = I_\Lambda \cap KQ$. Clearly $I_\Lambda \cap KQ \subset \ker f$. Now suppose that $f(\sum \lambda_p p) = 0$. Then $\sum \lambda_p p$ is in $\langle I_\Lambda \cup D\rangle$. Thus

$$\sum \lambda_p p = \sum \lambda_{r,s} r(C^{\mu(C)} - (C'^{\mu(C')}))s + \sum \lambda_{r',s'} r' C^{\mu(C)} a s' + \sum \lambda_{r'',s''} r'' ab s' + \sum \lambda_{r''',s'''} r''' a_d s''',$$

where $\lambda_{*,*}$ are elements of K; r, r', r'', r''', s, s', s'', s''' are paths; $C^{\mu(C)} - (C'^{\mu(C')})$ are Type 1 relations; $C^{\mu(C)}a$ are Type 2 relations, ab are Type 3 relations; and a_d are arrows in D. Since the left-hand side is a K-linear combination of paths in Q, the sum of all paths having at least one arrow in D on the right-hand side must equal 0. Each $C \in \mathcal{S}$ has an arrow in D, so we conclude that

$$\sum \lambda_p p = \sum \lambda_{r'',s''} r'' ab s'',$$

where ab is a Type 3 relation and no arrow in D occurs in any $r''abs''$. Noting that such ab are in $I_\Lambda \cap KQ$, we conclude that f is an isomorphism.

It also follows from the above that the relations in $I_\Lambda \cap KQ$ are monomial quadratic. Suppose $ab \notin I_\Lambda \cap KQ$ and $ab' \notin I_\Lambda \cap KQ$ for $a, b, b' \in Q_1$. Then $ab \notin I_\Lambda$ and $ab' \notin I_\Lambda$, which is a contradiction since by [18] KQ_Λ/I_Λ is special multiserial. Similarly, we see that given an arrow in Q, there is at most one arrow $c \in Q_1$ such that $ca \notin I_\Lambda \cap KQ$. Hence, $KQ/(I_\Lambda \cap KQ)$ is a special multiserial algebra and we have shown that $KQ/(I_\Lambda \cap KQ)$ is an almost gentle algebra. □

The next result shows that if one starts with an almost gentle algebra and takes the appropriate admissible cut in the trivial extension of the almost gentle algebra, then the almost gentle algebra is isomorphic to the cut algebra.

Theorem 5.3. *Let $A = KQ/I$ be an almost gentle algebra with set of maximal paths $\mathcal{M}$ and let $T(A) = Q_{T(A)}$ be the trivial extension of A by $D(A)$ where the set of new arrows of $Q_{T(A)}$ is given by $D = \{\beta_m, m \in \mathcal{M}\}$. Then D is an admissible cut of $Q_{T(A)}$ and the cut algebra associated with D is isomorphic to A.*

Proof. It follows from the construction of $T(A)$ that there exists exactly one arrow from D in any special cycle. Hence D is an admissible cut of $T(A)$. The constructions in Theorem 5.2 give the result. □

The next result shows that if one starts with a symmetric special multiserial algebra defined by a defining pair $(\mathcal{S}, \mu)$ and $\mu \equiv 1$ and an admissible cut D, then the algebra associated with D, trivially extended by its dual, is isomorphic to the original symmetric special multiserial algebra.

Theorem 5.4. *Let $\Lambda = KQ_\Lambda/I_\Lambda$ be a symmetric special multiserial algebra defined by the defining pair $(\mathcal{S}, \mu)$ and assume that $\mu \equiv 1$. Let D be an admissible cut of Q_Λ. Denote by $A = KQ/I$ the cut algebra associated with D. Then $T(A)$ is isomorphic to Λ.*

Proof. The special cycles in Q_Λ are of the form $C = p_1\beta p_2$, for $\beta \in D$ and where $p_1 = a_1 \dots a_r$ and $p_2 = a_{r+1} \dots a_s$. We now show that p_2p_1 is a maximal path in A. Since C is a special cycle, we have special cycles $p_2p_1\beta$ and βp_2p_1. Thus, $p_2p_1 \notin I_\Lambda$ and hence $p_2p_1 \notin I_\Lambda \cap KQ$. Since Λ is a special multiserial algebra, and both $a_r\beta$ and βa_{r+1} are not in I_Λ, we see that a_rb and ba_{r+1} are in I_Λ for all arrows $b \in Q$. Thus A is an almost gentle algebra since $I = I_\Lambda \cap KQ$ is generated by quadratic monomials and is special multiserial. It is now easy to see that $T(A)$ is isomorphic to Λ. □

Consider the set of pairs (Λ, D) such that $\Lambda = KQ_\Lambda/I_\Lambda$ is a symmetric special multiserial K-algebra and D is an admissible cut in Q_Λ. We say that (Λ, D) and (Λ', D') are *equivalent* if there is a K-algebra isomorphism from Λ to Λ' sending D to D' and let $\mathcal{X}$ denote the equivalent classes. The next result is an immediate consequence of the previous two theorems.

Corollary 5.5. *There is a bijection $\varphi : \mathcal{A} \longrightarrow \mathcal{X}$ from the set $\mathcal{A}$ of isomorphism classes of almost gentle algebras to the set $\mathcal{X}$ of equivalence classes of pairs consisting of a symmetric special multiserial algebra and a cut as defined above. The isomorphism is given, for $A \in \mathcal{A}$, by $\varphi(A) = (T(A), D)$ where $D = \{\beta_m \mid m$ a maximal path in $A\}$. Moreover, for $(\Lambda, D) \in \mathcal{X}$, we have $\varphi^{-1}(\Lambda, D) = A$ where A is the isomorphism class of the algebras associated with the cut D.*

Remark 5.6.

(1) Given a symmetric special multiserial algebra $\Lambda = KQ_\Lambda/I_\Lambda$, two distinct admissible cuts of Q_Λ yield, in general, non-isomorphic, non-derived equivalent cut algebras A and A'. We note that A and A' have the same number of simple modules and $\dim_K A = \dim_K A'$. But there are examples where $\mathrm{gldim} A < \infty$ and $\mathrm{gldim} A' = \infty$.

(2) If $\Lambda = KQ_\Lambda/I_\Lambda$ is of finite (respectively tame) representation type then any cut algebra associated with a cut of Q_Λ is of finite (respectively tame) representation type. To see this, suppose that A is the cut algebra of Λ associated with an admissible cut. Then Λ is isomorphic to $T(A)$ and there is a full faithful embedding of the category of finitely generated A-modules into the category of finitely generated Λ-modules.

Let $\Lambda = KQ_\Lambda/I_\Lambda$ be a symmetric special multiserial algebra and let $(\mathcal{S}, \mu)$ be a defining pair in Q_Λ so that Λ is defined by $(\mathcal{S}, \mu)$. If μ is identically equal to 1, we say that Λ *has multiplicity function identically equal to 1.* Note that if one views Λ as a Brauer configuration algebra with multiplicity function ν, this corresponds to ν being identically equal to one.

Corollary 5.7. *Every symmetric special multiserial algebra with multiplicity function identically equal to one in its defining pair is a trivial extension of an almost gentle algebra.*

Equivalently, we have that every Brauer configuration algebra with multiplicity function identically equal to one is the trivial extension of an almost gentle algebra.

6. The Brauer configuration algebra associated with an almost gentle algebra

We have seen that for every almost gentle algebra $A = KQ/I$, the trivial extension algebra $T(A)$ is a symmetric special multiserial algebra. In [**18**] we saw that a symmetric special multiserial algebra is a Brauer configuration algebra. In this section we show how to construct the Brauer configuration of the Brauer configuration algebra $T(A)$ from an almost gentle algebra A.

Given an almost gentle algebra A, we saw in § 3 that there is a defining pair $(\mathcal{S}, \mu)$ obtained from A and that the algebra associated with $(\mathcal{S}, \mu)$ is isomorphic to $T(A)$. In this section, we show, more generally, how to construct a Brauer configuration from a defining pair $(\mathcal{T}, \mu)$ so that the associated Brauer configuration algebra is isomorphic to the algebra associated with $(\mathcal{T}, \mu)$.

Let $(\mathcal{T}, \mu)$ be a defining pair for the quiver Q. There is an equivalence relation on $\mathcal{T}$: two cycles in $\mathcal{T}$ are equivalent if one is a cyclic permutation of the other. Suppose there are m equivalence classes of elements of $\mathcal{T}$ and let $c_1, \ldots, c_m$ be a full set of representatives of the equivalence classes.

Recall from [**19**] that a *Brauer configuration* is a 4-tuple, $\Gamma = (\Gamma_0, \Gamma_1, \nu, \mathfrak{o})$, where Γ_0 is a set of vertices, Γ_1 is a set of polygons which are multisets of vertices, $\nu\colon \Gamma_0 \to \mathbb{Z}_{\geq 0}$, and $\mathfrak{o}$ is an orientation. More precisely, the orientation corresponds to the following: at each vertex $\alpha \in \Gamma_0$, there is a cyclic ordering of the set of polygons containing vertex α; the polygon V appears k times in the cyclic order at α if V contains k copies of α. We begin by setting $\Gamma_0 = \{\alpha_1, \ldots, \alpha_m\}$ where m is the number of equivalence classes of elements of $\mathcal{T}$. If $Q_0 = \{v_1, \ldots, v_n\}$ then $\Gamma_1 = \{V_1, \ldots, V_n\}$, where α_j occurs k-times in the multiset V_i if v_i occurs as a vertex k-times in the cycle c_j. The function ν is defined by $\nu(\alpha_i) = \mu(c_i)$. Finally, the orientation at vertex α_i is $V_{i_1} < V_{i_2} < \cdots < V_{i_{\ell(c_i)}} (< V_{i_1})$ if the sequence of vertices in the cycle c_i is $v_{i_1}, v_{i_2}, \ldots, v_{\ell(c_i)}, v_{i_1}$.

It is straightforward to check that the Brauer configuration algebra associated with the Brauer graph $(\Gamma_0, \Gamma_1, \nu, \mathfrak{o})$ defined above is isomorphic to the algebra associated with the defining pair $(\mathcal{T}, \mu)$.

7. The hypergraph of an almost gentle algebra

This section builds on the observation that every Brauer configuration Γ is an oriented hypergraph $\mathcal{H}$ with a vertex decoration, where the decoration on $\mathcal{H}$ corresponds to the multiplicity function on Γ and where the orientation on $\mathcal{H}$ is induced by the orientation on Γ. Given an almost gentle algebra, we will give an alternative direct construction of its associated oriented hypergraph (i.e. without passing to the trivial extension). This construction gives, for example, an easy criterion to determine whether two almost gentle algebras have isomorphic trivial extensions.

A *hypergraph* is a generalization of a graph in which an edge can contain more than two vertices. That is, a hypergraph $\mathcal{H}$ is a tuple $(\mathcal{H}_0, \mathcal{H}_1)$ where $\mathcal{H}_0$ is a finite set of vertices and $\mathcal{H}_1$ is a finite set of hyperedges given by multisets of elements of $\mathcal{H}_0$, with the convention that each multiset contains at least two elements. A *hypergraph with orientation* is a hypergraph $\mathcal{H} = (\mathcal{H}_0, \mathcal{H}_1)$ together with an orientation σ such that for every vertex $x \in \mathcal{H}_0$, the set of hyperedges containing x are cyclically ordered (counting repeats). The orientation can be seen as a ribbon graph-like structure on the hypergraph.

Remark 7.1. In the context of Brauer configurations, we also could adopt the convention of allowing hyperedges with one element. These would correspond to the truncated edges of the Brauer configuration.

Let $A = KQ/I$ be an almost gentle algebra. Recall from § 2 that $\mathcal{M}$ is the set of maximal paths in KQ/I. Let V_0 be the subset of Q_0 containing vertices v such that one of the following holds:

(1) v is the source of exactly one arrow and there is no arrow ending at v;

(2) v is the target of exactly one arrow and there is no arrow starting at v;

(3) v is the target of exactly one arrow a and the source of exactly one arrow b and $ab \notin I$.

Set $\overline{\mathcal{M}} = \mathcal{M} \cup \{e_v | v \in V_0\}$. We say that a vertex $v \in Q_0$ *lies in* $\mathcal{M}$ if there exists $p \in \overline{\mathcal{M}}$ with $p = qe_vr$ where q, r are possibly trivial paths in Q.

The next result follows directly from the definition of $\overline{\mathcal{M}}$ and from Lemma 2.4.

Lemma 7.2. *Every vertex in Q_0 lies in at least two elements of $\overline{\mathcal{M}}$.*

Construction of the hypergraph $\mathcal{H}_A$ with orientation of A: Let $A = KQ/I$ be an almost gentle algebra. Define $\mathcal{H}_A = (\mathcal{H}_0, \mathcal{H}_1)$ as follows.

- The vertices $\mathcal{H}_0$ are in correspondence with the elements in $\overline{\mathcal{M}}$.
- The hyperedges in $\mathcal{H}_1$ correspond to the vertices in Q_0; namely, the hyperedge corresponding to a vertex $v \in Q_0$ is given by all elements $p \in \overline{\mathcal{M}}$ such that v lies in p.
- The orientation is induced by the maximal paths in $\mathcal{M}$. Let x be a vertex in $\mathcal{H}_0$ and let $V_1, V_2, \ldots, V_n$ be the hyperedges corresponding respectively to the vertices $v_1, v_2, \ldots, v_n$ in Q_0 such that $v_1, v_2, \ldots, v_n$ lie in (the maximal path) p corresponding to x. Suppose, without loss of generality, that $p = e_{v_1}a_1e_{v_2}a_2e_{v_3} \cdots a_{n-1}e_{v_n}$ with $a_i \in Q_1$; then, the cyclic ordering at x is given by $V_1 < V_2 < \cdots < V_n < V_1$.

Note that if Γ is the Brauer configuration corresponding to $\mathcal{H}$ then the multiplicity function of Γ is identically equal to one and by our results the associated Brauer configuration algebra Λ_Γ is isomorphic to the trivial extension $T(A)$.

Remark 7.3.

(1) If A is gentle then the construction of the oriented hypergraph gives exactly the ribbon graph constructed in [**30**]. We note that this is the general construction underlying the graphs in [**16**, **28**].

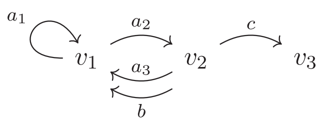
(2) In the case of a gentle algebra associated with a surface triangulation (respectively angulation of a surface), the ribbon graph associated with the gentle algebra A gives rise to the underlying surface and its triangulation (respectively angulation).

The hypergraph of an almost gentle algebra A uniquely determines a Brauer configuration which uniquely determines a Brauer configuration algebra. It follows from §§ 4 and 6 that this Brauer configuration algebra is isomorphic to the trivial extension of A. Hence, we immediately see the following.

Theorem 7.4. *Two almost gentle algebras A and B have the same associated hypergraph with orientation if and only if $T(A) \simeq T(B)$.*

Example 7.5.

(1) Let $A_1 = KQ_1/I_1$ be the almost gentle algebra given by

Q_1: 

and $I = \langle a_2b,\ a_2c,\ a_3a_1,\ ba_1,\ ba_2 \rangle$.

Then

$$\overline{\mathcal{M}} = \{a_1a_2a_3, b, c\} \cup \{e_{v_3}\}$$

Therefore, $\mathcal{H}_A = (\mathcal{H}_0, \mathcal{H}_1)$ is such that $\mathcal{H}_0 = \{1, 2, 3, 4\}$, where

1 corresponds to $a_1a_2a_3$

2 corresponds to b

3 corresponds to c

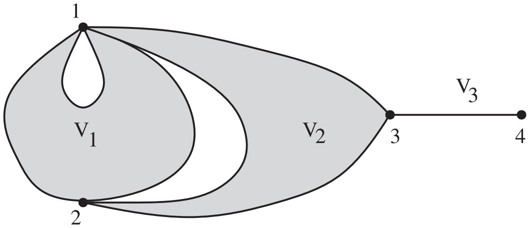
4 corresponds to e_{v_3};

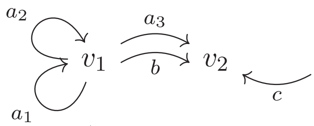
and $\mathcal{H}_1 = \{V_1, V_2, V_3\}$, where

$$V_1 = \{1, 1, 2\}$$
$$V_2 = \{1, 2, 3\}$$
$$V_3 = \{3, 4\}.$$

Finally, the orientation is induced by the order of the vertices in the maximal paths: $e_{v_1}a_1e_{v_1}a_2e_{v_2}a_3e_{v_1}$, $e_{v_2}be_{v_1}$, $e_{v_2}ce_{v_3}$. So the cyclic ordering of the polygons at vertex

1 is given by $V_1 < V_1 < V_2 < V_1 < V_1$, at vertex 2 it is $V_2 < V_1 < V_2$, at vertex 3 it is $V_2 < V_3 < V_2$ and at vertex 4 it is V_3. The hypergraph $\mathcal{H}$ has a geometric realization as follows.



(2) Let Q_2: 

and $I_2 = \langle a_1a_3, a_1b, a_2a_1, a_2b \rangle$. Set $A_2 = KQ_2/I_2$. Then A_2 is almost gentle. Note that A_2 has the same associated hypergraph as A_1, that is, $\mathcal{H}_{A_2} = \mathcal{H}_{A_1}$. Therefore, by Theorem 4.3 and the construction in § 6, the algebras $T(A_1)$ and $T(A_3)$ are isomorphic to each other and isomorphic to the Brauer configuration algebra with Brauer configuration given by $\mathcal{H}$ with multiplicity function identically equal to one.

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