Math. Struct. in Comp. Science (2007), *vol.* 17, *pp.* 957–1027. © 2007 Cambridge University Press
doi:10.1017/S0960129507006287 Printed in the United Kingdom

On categorical models of classical logic and the Geometry of Interaction

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Received 19 January 2005; revised 21 October 2006; first published online 21 September 2007

It is well known that weakening and contraction cause naive categorical models of the classical sequent calculus to collapse to Boolean lattices. In previous work, summarised briefly herein, we have provided a class of models called *classical categories* that is sound and complete and avoids this collapse by interpreting cut reduction by a poset enrichment. Examples of classical categories include boolean lattices and the category of sets and relations, where both conjunction and disjunction are modelled by the set-theoretic product. In this article, which is self-contained, we present an improved axiomatisation of classical categories, together with a deep exploration of their structural theory. Observing that the collapse already happens in the absence of negation, we start with negation-free models called *Dummett categories*. Examples of these include, besides the classical categories mentioned above, the category of sets and relations, where both conjunction and disjunction are modelled by the disjoint union. We prove that Dummett categories are MIX, and that the partial order can be derived from hom-semilattices, which have a straightforward proof-theoretic definition. Moreover, we show that the Geometry-of-Interaction construction can be extended from multiplicative linear logic to classical logic by applying it to obtain a classical category from a Dummett category.
Along the way, we gain detailed insights into the changes that proofs undergo during cut elimination in the presence of weakening and contraction.

1. Introduction

It is notoriously hard to find a decent denotational semantics for the classical sequent calculus, let alone an algorithmic interpretation. This problem is related to the *non-deterministic* behaviour of cut elimination. To see this, consider the sequent proof

$$\Lambda \;=\; \cfrac{\cfrac{\cfrac{\begin{array}{c}\Phi_1\\ \vdots\\ \Gamma \vdash \Delta\end{array}}{\Gamma \vdash A, \Delta}\text{weakening} \qquad \cfrac{\begin{array}{c}\Phi_2\\ \vdots\\ \Gamma \vdash \Delta\end{array}}{\Gamma, A \vdash \Delta}\text{weakening}}{\Gamma, \Gamma \vdash \Delta, \Delta}\text{Cut}}{\cfrac{\text{contractions}}{\Gamma \vdash \Delta}}$$

[§] This work was partially supported by the UK EPSRC.

where Φ_1 and Φ_2 are arbitrary proofs of the sequent $\Gamma \vdash \Delta$. We call this the 'Lafont proof', because it is a variant of an example credited to Lafont (*cf.* Girard *et al.* (1989, page 151)). The sub-proof Φ_1 is weakened on the right and the sub-proof Φ_2 is weakened on the left. There is then a cut, where the cut formula is the formula A introduced by the weakenings. Finally, the double occurrences of Γ and Δ are removed by left and right contractions. (Clearly, the two contractions are supposed to commute with each other, so we do not need to be specific about the order in which they are applied.) The proof Λ reduces to

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\Phi_1\\ \vdots\\ \Gamma \vdash \Delta\end{matrix}}{\text{weakenings}}}{\Gamma, \Gamma \vdash \Delta, \Delta}}{\dfrac{\text{contractions}}{\Gamma \vdash \Delta}} \qquad \text{or to} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\Phi_2\\ \vdots\\ \Gamma \vdash \Delta\end{matrix}}{\text{weakenings}}}{\Gamma, \Gamma \vdash \Delta, \Delta}}{\dfrac{\text{contractions}}{\Gamma \vdash \Delta}}$$

However, it is clear that the weakenings followed by the contractions are essentially nothing (*cf.* Girard *et al.* (1989, page 152)). So Φ_1 and Φ_2 are obtained by reducing the same proof. Thus, the denotations of Φ_1 and Φ_2 must be equal for any semantics that *admits cut reduction* in the sense that the reduct is denotationally equal to the redex. Summarising, any denotational semantics that admits cut reduction must identify all proofs of a sequent $\Gamma \vdash \Delta$. Note that this argument does not rely on negation!

There are various ways to escape from this denotational collapse. First, we might simply abandon classical logic and adopt, for example, intuitionistic logic or linear logic instead. As explained in Gentzen's seminal article (Gentzen 1934), intuitionistic logic can be obtained by restricting the classical sequent calculus in such a way that the succedent Δ contains at most one formula. As is widely known, intuitionistic logic can be modelled by cartesian-closed categories. Models of linear logic also abound. However, both intuitionistic logic and linear logic differ from classical logic with respect to provable sequents, and we do not wish to depart from classical provability.

A second possibility, which enables us to keep with classical logic, might be to move to 'classical natural deduction' systems (Prawitz 1965), where proofs may be represented as terms of the $\lambda\mu\nu$-calculus (Parigot 1992; Pym and Ritter 2001). However, such systems do not admit all cut reductions: it turns out that the call-by-name version of $\lambda\mu\nu$ admits only the reduction to Φ_2, while the call-by-value version only admits the reduction to Φ_1. Each version corresponds to a different choice of $\neg\neg$-translation (also known as 'continuation-passing-style transformations' in programming-language jargon) of classical logic into intuitionistic logic (Troelstra and Schwichtenberg 1996; Plotkin 1975). Models of $\lambda\mu\nu$ can be obtained in fibrations over a base category of structural maps in which each fibre is a model of intuitionistic natural deduction and in which dualising negation is interpreted as certain maps between the fibres (Ong 1996; Pym and Ritter 2001). Alternative models are given by control categories and co-control categories (Selinger 2001).

In our companion paper (Führmann and Pym 2006), we presented a solution that, unlike classical natural deduction, models *all* cut reductions: we introduced a kind of poset-enriched category called a *classical category*, whose objects model types and whose morphisms model proofs of the classical sequent calculus; whenever a proof of Φ can be reduced to another proof Ψ, we only require $\mathbf{C}\lfloor\Phi\rfloor \leqslant \mathbf{C}\lfloor\Psi\rfloor$ (as opposed to $\mathbf{C}\lfloor\Phi\rfloor = \mathbf{C}\lfloor\Psi\rfloor$), where $\mathbf{C}\lfloor\Phi\rfloor$ and $\mathbf{C}\lfloor\Psi\rfloor$ are the morphisms denoted by Φ and Ψ in the classical category $\mathbf{C}$. Classical categories are a special case of symmetric linearly distributive categories (Cockett and Seely 1997b): they have symmetric monoidal products $\otimes$ and $\oplus$ for modelling conjunction and disjunction, respectively. To model contraction and weakening on the right, every object A is endowed with a symmetric monoid ($\blacktriangledown_A : A \oplus A \longrightarrow A, []_A : \bot \longrightarrow A$): the multiplication $\blacktriangledown$ models contraction, and the unit $[]$ models weakening. The dual construction is used to model contraction and weakening on the left. (It is worth mentioning here that symmetric linearly distributive categories *with negation* are equivalent to $*$-autonomous categories; however, the former provide a better choice of primitives for achieving our goals.)

In Führmann and Pym (2006), we proved that classical categories are sound and complete for the classical sequent calculus. More precisely, we introduced a notion of theory with judgments of the form

$$\begin{array}{c} \Phi \\ \vdots \\ \Gamma \vdash \Delta \end{array} \quad \preccurlyeq \quad \begin{array}{c} \Psi \\ \vdots \\ \Gamma \vdash \Delta \end{array}$$

where the $\preccurlyeq$ is a preorder that contains all reductions required for cut elimination. The soundness theorem in Führmann and Pym (2006) states, essentially, that $\Phi \preccurlyeq \Psi$ implies $\mathbf{C}\lfloor\Phi\rfloor \leqslant \mathbf{C}\lfloor\Psi\rfloor$ for every classical category $\mathbf{C}$. The completeness theorem states, essentially, that $\Phi \preccurlyeq \Phi$ is a theorem of a theory $\mathscr{T}$ whenever $\mathbf{C}\lfloor\Phi\rfloor \leqslant \mathbf{C}\lfloor\Psi\rfloor$ holds for every model $\mathbf{C}\lfloor-\rfloor$ of $\mathscr{T}$. Its proof uses a category built from Robinson's proof nets for classical logic (Robinson 2003), which build on ideas due to Girard and correspond directly to the classical sequent calculus. (We shall discuss these nets briefly in Section 2.2.) A morphism of that category is an equivalence class of proof nets with respect to the preorder $\preccurlyeq$. For morphisms $f, g : A \longrightarrow B$ with representing nets N_f and N_g, that category has $f \leqslant g$ if and only if $N_f \preccurlyeq N_g$. (This explains why $\leqslant$ is a partial order even though the preorder $\preccurlyeq$ is not generally antisymmetric.)

In Führmann and Pym (2006), we gave the following concrete examples of classical categories:

— An initial model built from proof nets;

— The category **Rel** of sets and relation, where both $\otimes$ and $\oplus$ are defined to be the evident functor that takes two sets to their cartesian product, and $\preccurlyeq$ is the set-theoretic inclusion of relations;

— Boolean lattices;

— The product of any two classical categories – for example, $\mathbf{Rel} \times \mathbf{B}$ for any Boolean lattice $\mathbf{B}$. This shows that there are models that are non-posetal (that is, there are hom-sets with more than one element) and non-compact (that is, $\otimes \neq \oplus$).

In Führmann and Pym (2004), we found further classical categories that arise from an abstract Geometry-of-Interaction (GoI) construction starting with a quantaloid, and then used those models to study the 'increase' in denotations during cut elimination.

Since we presented Führmann and Pym (2004) in the summer of 2004, we have managed to:

1 considerably advance the axiomatisation and understanding of classical categories, in particular, by proving that they are MIX;
2 strongly generalise the GoI construction we presented in Führmann and Pym (2004).

Many of the new insights were sparked by Masahito Hasegawa in private communications, which is why several propositions in this paper are attributed to him.

This paper gives a comprehensive account of our improved axiomatisation and structural theory of classical categories (Section 3), and of our generalised GoI construction (Section 4). Because of the substantial advances in presentation and axiomatisation, we have chosen to make this paper self-contained so that it requires no previous knowledge of Führmann and Pym (2006) or Führmann and Pym (2004).

On the purely technical side, we have adopted the proof nets in the style of Blute *et al.* (1996); understanding these nets takes a little more effort than understanding Robinson's nets, but they are more efficient for calculations.

1.1. *Outline*

This section gives a detailed overview of this paper.

— In Section 2, we recall some preliminaries: the classical sequent calculus, proof nets and the categorical semantics of multiplicative linear logic (MLL) in symmetric linearly distributive categories.

— In Section 3, we introduce classical categories from the ground up. We proceed in two steps:

  1 We first extend symmetric linearly distributive categories by adding some structure for modelling weakening and contraction. This structure consists of a symmetric monoid and a symmetric comonoid for every object, and a poset enrichment The resulting categories are models of the negation-free fragment of the classical sequent calculus. We call them *Dummett categories* (inspired by Dummett's extensive discussion of multi-succedent intuitionistic sequent calculi given in 'Elements of Intuitionism' (Dummett 1977)).

  2 We then introduce classical categories as Dummett categories with the property of *having negation* in the sense of Cockett and Seely.

  We then establish the close connection between classical categories and the classical sequent calculus by constructing the free classical category from proof nets (Theorem 3.32). (This extends the construction of the free symmetric linearly distributive category from MLL proof nets in the sense of Blute *et al.* (1996).) From a logical point of view, the result means that classical categories are sound and complete (in the order-theoretic sense explained above) with respect to a certain super-relation of cut reduction for the classical sequent calculus.

https://doi.org/10.1017/S0960129507006287 Published online by Cambridge University Press

Our free construction relies on a series of results about the structure of Dummett categories, including:

- the remarkable result (due to Hasegawa) that the monoids or comonoids cause symmetric linearly distributive categories to be MIX (Theorem 3.11);
- the fact that the poset enrichment is not needed as an extra structure, but is induced by hom-semilattices, which are derivable from other primitives (Proposition 3.28).

We conclude Section 3 by presenting an extremely economic axiomatisation of compact Dummett categories (see Proposition 3.42, which is due to Hasegawa), and even more economical axiomatisation of Dummett categories with finite biproducts (see Proposition 3.43, which is also due to Hasegawa).

— In Section 4, we introduce an extended GoI construction that sends a traced Dummett category to a classical category (Theorem 4.4). This shows that GoI works in the presence of weakening and contraction, even with respect to the partial order that models cut reduction. As we shall explain, traced Dummett categories are essentially traced symmetric monoidal categories, plus symmetric monoids and symmetric comonoids on every object that satisfies certain conditions. Our extended GoI construction is an instance of the well-known construction of a compact closed category from a traced symmetric monoidal category. (See the introduction to Section 4 for an overview of the history of GoI leading to that construction.) The key point of our extended construction is that the symmetric monoids and symmetric comonoids, and the conditions required for a Dummett category, 'survive' the extended GoI construction.
In Section 4.4, we study the special case in which the starting point of the extended GoI construction is a traced Dummett category with finite biproducts. In particular, we present a comprehensive characterisation of morphisms in such GoI categories with respect to their behaviour under cut reduction (Proposition 4.5).

— Finally, we suggest some directions for future work in Section 5.

1.2. *Related work*

The article Hyland (2004) introduced a notion of *abstract interpretation of classical proof* as a compact closed category in which every object is equipped with a symmetric monoid and a symmetric comonoid satisfying certain conditions. (This work was foreshadowed in Hyland (2002).) These abstract interpretations are almost the same as our classical categories in the *compact* case where $\otimes = \oplus$. The only difference is that compact classical categories need to satisfy an extra equation (Equation 3 in Section 3.4.2). As we shall show in Section 3.4.2, this equation implies that every compact classical category has hom-semilattices, which yield the partial order we use for modelling cut reduction. So our approach is more general than Hyland's in that it does not require compactness, and more special in that we require certain conditions that lead to the existence of hom-semilattices.

Another overlap with Hyland (2004) occurs when we specialise our GoI construction to categories with finite biproducts. The partial order specific to our models allows a precise analysis of the behaviour of morphisms with respect to cut reduction (which is explained in Section 3.2.1).

The article Bellin *et al.* (2004) contains a semantics of the classical sequent calculus that is finer grained than ours in that it rejects axiom expansions (also called η-rules), that is, the categorical connectives $\otimes$ and $\oplus$ that model conjunction and disjunction do not generally preserve identities. In contrast, our work fits into the existing framework of symmetric linearly distributive categories, in which $\otimes$ and $\oplus$ are functorial. Another difference between our work and Bellin *et al.* (2004) is that we deal with the modelling of cut reduction (using the poset enrichment) but Bellin *et al.* (2004) does not. A notion of 'Boolean category' is introduced in Dosen (1999) and Dosen and Petric (2004). This notion relies on the presence of products and coproducts, which leads to a more 'collapsed' structure than ours, and is closely related to the category of finite sets and relations.

There has also been some interesting work on *confluent* cut elimination in the presence of the MIX rule (Bellin 2003; Lamarche and Straßburger 2004). For example, one can remove the non-determinism of cut reduction by allowing a reduction

$$\dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma\vdash\Delta\end{matrix}}{\Gamma\vdash A,\Delta}\,\mathsf{WR}\qquad \dfrac{\begin{matrix}\Phi'\\ \vdots\\ \Gamma\vdash'\Delta'\end{matrix}}{\Gamma',A\vdash\Delta'}\,\mathsf{WL}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{Cut}\quad\preccurlyeq\quad \dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma\vdash\Delta\end{matrix}\qquad\begin{matrix}\Phi'\\ \vdots\\ \Gamma\vdash'\Delta'\end{matrix}}{\Gamma,\Gamma'\vdash\Delta,\Delta'}\,\mathsf{MIX}$$

The confluent cut elimination procedure in Lamarche and Straßburger (2004) (which is based on proof nets) does this implicitly. Our semantics is compatible with this approach: the MIX rule is denotationally equivalent to a degenerate cut with cut formula $A=\bot$ or $A=\top$ (both choices of A result in the same denotation). So, in our view, this kind of confluent 'cut elimination' is a removal of arbitrary cuts in favour of degenerate cuts (that is, MIXes); a MIX is still non-deterministic – in fact, it is the pure incarnation of proof-theoretic non-determinism, because it is the 'parallel composition' of Φ and Φ' that one might want to reduce to either Φ or Φ'. Our models support this view, because they admit the reduction of MIX to Φ *and* to Φ'. In fact, the hom-semilattices of our models are given by

$$\Phi_1 * \Phi_2 = \dfrac{\dfrac{\dfrac{\begin{matrix}\Phi_1\\ \vdots\\ \Gamma\vdash\Delta\end{matrix}\qquad\begin{matrix}\Phi_2\\ \vdots\\ \Gamma\vdash\Delta\end{matrix}}{\Gamma,\Gamma\vdash\Delta,\Delta}\,\mathsf{MIX}.}{\text{contractions}}}{\Gamma\vdash\Delta}$$

From a technical point of view, this article is based on *symmetric linearly distributive categories*, which were introduced in Cockett and Seely (1997b). In particular, we make heavy use of the proof nets for symmetric linearly distributive categories introduced in Blute *et al.* (1996), because they are very efficient for the calculations required in this article. We also build on the discussions of MIX categories in Blute *et al.* (2000) and Cockett and Seely (1997a), and the notion of traced object in a MIX category presented in Blute *et al.* (2000).

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Table 1. *Inference rules of MLL*

$$\frac{}{A \vdash A}\ \mathsf{Ax}$$

$$\wedge\mathsf{L}\ \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, A \wedge B, \Gamma' \vdash \Delta} \qquad \frac{\Gamma_1 \vdash \Delta_1, A_1, \Delta_1' \quad \Gamma_2 \vdash \Delta_2, A_2, \Delta_2'}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A_1 \wedge A_2, \Delta_1', \Delta_2'}\ \wedge\mathsf{R}$$

$$\vee\mathsf{L}\ \frac{\Gamma_1, A_1, \Gamma_1' \vdash \Delta_1 \quad \Gamma_2, A_2, \Gamma_2' \vdash \Delta_2}{\Gamma_1, \Gamma_2, A_1 \vee A_2, \Gamma_1', \Gamma_2' \vdash \Delta_1, \Delta_2} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, A \vee B, \Delta'}\ \vee\mathsf{R}$$

$$\top\mathsf{L}\ \frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \top, \Gamma' \vdash \Delta} \qquad \frac{}{\vdash \top}\ \top\mathsf{R}$$

$$\bot\mathsf{L}\ \frac{}{\bot \vdash} \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, \bot, \Delta'}\ \bot\mathsf{R}$$

$$\neg\mathsf{L}\ \frac{\Gamma, \Gamma' \vdash \Delta, A, \Delta'}{\Gamma, \neg A, \Gamma' \vdash \Delta, \Delta'} \qquad \frac{\Gamma, A, \Gamma' \vdash \Delta, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \neg A, \Delta'}\ \neg\mathsf{R}$$

$$\mathsf{EL}\ \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\ \mathsf{ER}$$

$$\frac{\Gamma_2 \vdash \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_2 \vdash \Delta_1, \Delta_2, \Delta_3}\ \mathsf{Cut}$$

We also rely on results from the GoI literature; the related work in this area is described in Section 4.

The literature on graphical analyses of classical proofs is too diverse to be summarised here.

2. Preliminaries

2.1. *The sequent calculus*

The version of the sequent calculus we will use consists of the system of multiplicative linear logic (MLL) presented in Table 1, plus the rules for weakening and contraction presented in Table 2. In this way we obtain a calculus that differs from LK (Gentzen 1934) only in its use of the multiplicative form of the introduction rules and in the absence of implication. We consider implication to be derived – that is, $A \Rightarrow B = \neg A \vee B$. A *sequent* has the form $\Gamma \vdash \Delta$ where Γ and Δ are finite lists of formulae. The capital Latin letters range over formulae.

Table 2. *Inference rules for weakening and contraction*

$$\text{WL}\ \frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\ \text{WR}$$

$$\text{CL}\ \frac{\Gamma, A, A, \Gamma' \vdash \Delta}{\Gamma, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\ \text{CR}$$

In the rest of this paper we shall call sequent proofs *derivations*, to avoid any confusion with the notion of 'proof' at the meta-level.

To simplify the presentation of the semantics, we shall introduce a more economic version of the sequent calculus just described: the new version is obtained by replacing the rules $\wedge$R, $\vee$L, $\neg$L, $\neg$R, WL, WR, CL, and CR by axioms. For example, to replace $\wedge$R, we introduce an axiom

$$\frac{}{A, B \vdash A \wedge B}\ \text{Ax} \wedge \text{R}$$

and consider $\wedge$R as an abbreviation for

$$\frac{\Gamma_1 \vdash \Delta_1, A_1, \Delta_1' \qquad \dfrac{\Gamma_2 \vdash \Delta_2, A_2, \Delta_2' \qquad \dfrac{}{A_1, A_2 \vdash A_1 \wedge A_2}\ \text{Ax} \wedge \text{R.}}{A_1, \Gamma_2 \vdash \Delta_2, A_1 \wedge A_2, \Delta_2'}\ \text{Cut}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, A_1 \wedge A_2, \Delta_1', \Delta_2'}\ \text{Cut}$$

The extra axioms lead to the revised version of the sequent calculus described in Tables 3, 4 and 5 (we have put the rules for negation in a separate table because we shall also study the negation-free fragment). This revised version simplifies the presentation of the semantics, because axioms simply denote morphisms, and only seven inference rules remain that are not axioms. However, we shall keep the names $\wedge$R, $\vee$L, $\neg$L, $\neg$R, WL, WR, CL and CR as abbreviations for the evident derivations that involve Ax$\wedge$R, Ax$\vee$L, Ax$\neg$L, Ax$\neg$R, AxWL, AxWR and AxCL, respectively.

For the purposes of categorical logic, we shall consider derivations over any *signature*. A signature Σ consists of a set of atomic formulae and a set of *optional axioms*. The set of *formulae over* Σ is generated in the obvious way from the atomic formulae, using $\wedge$, $\vee$, $\top$, $\bot$ and $\neg$. We say a formula over Σ is *positive* if it is negation free. Optional axioms are of the form

$$\frac{}{\Gamma \vdash \Delta}\ f$$

Typical optional axioms are the ones for weakening and contraction in Table 5.

Definition 2.1. A *derivation* Φ over a signature Σ is a tree generated by the rules in Tables 3 and 4, plus the optional axioms of Σ. We say a derivation over Σ is *positive* if all of its formulae are positive.

Table 3. *Revised inference rules of MLL: negation-free fragment*

$$\frac{}{A \vdash A}\ \mathsf{Ax}$$

$$\wedge\mathsf{L}\ \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, A \wedge B, \Gamma' \vdash \Delta} \qquad \frac{}{A, B \vdash A \wedge B}\ \mathsf{Ax}\wedge\mathsf{R}$$

$$\mathsf{Ax}\vee\mathsf{L}\ \frac{}{A \vee B \vdash A, B} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, A \vee B, \Delta'}\ \vee\mathsf{R}$$

$$\top\mathsf{L}\ \frac{\Gamma, \Gamma' \vdash \Delta}{\Gamma, \top, \Gamma' \vdash \Delta} \qquad \frac{}{\vdash \top}\ \top\mathsf{R}$$

$$\bot\mathsf{L}\ \frac{}{\bot \vdash} \qquad \frac{\Gamma \vdash \Delta, \Delta'}{\Gamma \vdash \Delta, \bot, \Delta'}\ \bot\mathsf{R}$$

$$\mathsf{EL}\ \frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\ \mathsf{ER}$$

$$\frac{\Gamma_2 \vdash \Delta_1, A, \Delta_3 \quad \Gamma_1, A, \Gamma_3 \vdash \Delta_2}{\Gamma_1, \Gamma_2, \Gamma_2 \vdash \Delta_1, \Delta_2, \Delta_3}\ \mathsf{Cut}$$

Table 4. *Revised inference rules of MLL: axioms for negation*

$$\mathsf{Ax}\neg\mathsf{L}\ \frac{}{A, \neg A \vdash} \qquad \frac{}{\vdash \neg A, A}\ \mathsf{Ax}\neg\mathsf{R}$$

Table 5. *Optional axioms for weakening and contraction*

$$\mathsf{AxWL}\ \frac{}{\bot \vdash A} \qquad \frac{}{A \vdash \top}\ \mathsf{AxWR}$$

$$\mathsf{AxCL}\ \frac{}{A \vdash A \wedge A} \qquad \frac{}{A \vee A \vdash A}\ \mathsf{AxCR}$$

2.2. *Proof nets*

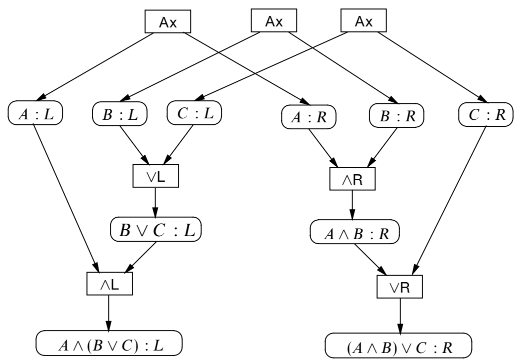
The essence of a sequent proof can be captured by a *proof net*, an idea introduced by Girard (Girard 1987). In this paper, we shall need proof nets (or 'nets' for short) to describe equalities between proofs. The nets we use are, essentially, those from Blute *et al.* (1996), extended to account for the extra structure of classical logic. This marks a departure from Führmann and Pym (2006), where we used the classical proof nets introduced by Robinson (Robinson 2003). Robinson's nets correspond more directly to the sequent calculus than those of Blute *et al.* (1996), but the latter are more convenient for calculations.

Disclaimer. This paper is not about proof nets – it only uses nets *à la* Blute *et al.* (1996) to help presentation and calculations. While these nets are very useful from a mathematical perspective, we make no claims about their logical or philosophical status.

Informally, a net is a graphical skeleton of a derivation. For example, both of the derivations

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathsf{Ax} \qquad \dfrac{}{B \vee C \vdash B, C}\,\mathsf{Ax} \vee \mathsf{L}}{A, (B \vee C) \vdash (A \wedge B), C}\,\wedge\mathsf{R}}{A \wedge (B \vee C) \vdash (A \wedge B), C}\,\wedge\mathsf{L}}{A \wedge (B \vee C) \vdash (A \wedge B) \vee C}\,\vee\mathsf{R} \qquad \text{and} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A, B \vdash A \wedge B}\,\mathsf{Ax} \wedge \mathsf{R} \qquad \dfrac{}{C \vdash C}\,\mathsf{Ax}}{A, (B \vee C) \vdash (A \wedge B), C}\,\vee\mathsf{L}}{A, (B \vee C) \vdash (A \wedge B) \vee C}\,\vee\mathsf{R}}{A \wedge (B \vee C) \vdash (A \wedge B) \vee C}\,\wedge\mathsf{L}$$

have the following proof net:



This net is in the style used in Robinson (2003); in that paper, a *proof structure* is defined to be a bipartite directional graph whose two families of nodes are labelled as follows:

Family 1 labelled by an inference rule of the sequent calculus;
Family 2 labelled by a formula, together with the information Left of Right.

The graph is subject to two additional constraints, which essentially mean that:

1 The incoming (respectively, outgoing) arrows of a rule node uniquely match the hypothesis (respectively, conclusions) of the corresponding rule of the sequent calculus.
2 Each formula node has a unique incoming and at most one outgoing arrow.

Translating derivations into proof structures is straightforward. Not all proof structures are the images of derivations; those that are are called *proof nets*. (When a graph is a proof net can also be characterised by the *switching criterion* introduced in Danos and Regnier (1989), which requires that certain subgraphs of the proof structure be connected

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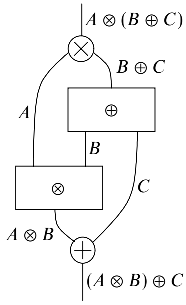
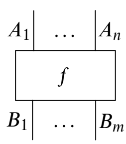
and acyclic.) Robinson's nets, with minor notational changes, were used in Führmann and Pym (2006).

However, in this paper, we have adopted the nets introduced in Blute *et al.* (1996). In that style, the net for the derivations above is



Here, the only nodes are rule nodes. We write $\otimes$ for $\wedge$, $\oplus$ for $\vee$, and $A^{\perp}$ for $\neg A$, because these nets are also used to describe morphisms in symmetric linearly distributive categories, as we shall see in Section 2.4.2. The wires are labelled with *types*, which can be seen either as formulae or as objects of a symmetric linearly distributive category. The left-hand formula of the derived sequent appears at the top of the net, and the right-hand formula at the bottom. The top-to-bottom orientation has advantages over the left-to-right orientation with respect to the alignment of types and wires. It also ensures the nice property that a net is planar if and only if the corresponding derivations are within non-commutative logic, that is, they contain no exchange rules, *cf.* Blute *et al.* (1996). An important difference between nets in the style of Robinson and nets in the style of Blute *et al.* (1996) is that the latter have no axiom links and no cut links. Abandoning these links is possible because a cut and an axiom cancel each other out according to a (poly)categorical neutrality law (Führmann and Pym 2006). Another difference is that Robinson's nets have links for weakening and contraction, while nets in the style of Blute *et al.* (1996) do not. (However, we shall see that such links can be easily added to the latter.) It is a bit harder to make the leap from derivations to nets in the style of Blute *et al.* (1996) than to nets in the style of Robinson. However, the former are better for heavy calculations, because they have no cluttering cut links and axiom links, and because one can drop the type annotations when they are clear from the context. (Just as one sometimes omits type annotations from lambda terms). This is why we have opted for them in this paper.

Now we turn to a formal definition of nets, based on the definition in Blute *et al.* (1996), but not quite as formal. We define the notion of a *typed circuit*. Building a typed circuit requires a set $\mathscr{T}$ of *types* and a set $\mathscr{C}$ of *components*. Each component $f \in \mathscr{C}$ has a list $\alpha = (A_1, \dots, A_n)$ of types describing the *input ports*, and a list $\beta = (B_1, \dots, B_m)$ of types describing the *output ports* of f:



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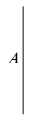
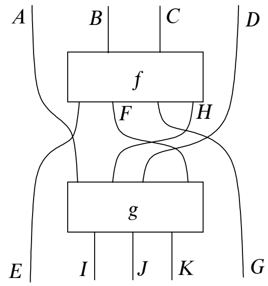
We define the collection of *circuits over $\mathscr{C}$* inductively:

— Every component $f \in \mathscr{C}$ is a circuit.
— The *identity wire*



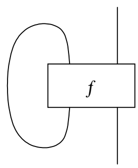
is a circuit, with one input port and one output port, each of type A.
— Given any number of circuits, connecting some output ports with input ports of the same type yields another circuit.

As an example of a circuit, consider



Note that it has two connections (of types F and H) from f to g. As we shall see, this is *not* a net for symmetric linearly distributive categories, because those nets must have exactly one connection between any two components; however, the nets for symmetric monoidal categories that we shall introduce much later in Section 3.4.1 allow such multiple connections.

Remark 2.2. Our definition of circuit is more general than that in Blute *et al.* (1996) in that it allows *feedback*, for example,



which we shall employ in Section 4 only.

A *net* (for symmetric linearly distributive categories), in short, is a circuit built from components that correspond to the introduction rules of the sequent calculus, subject to the condition of *sequentiality*, which means that the circuit must represent a derivation. We shall now spell this out in detail.

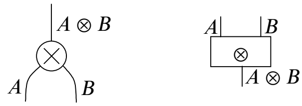
The types for nets are given by the grammar

$$A, B ::= A \otimes B \mid A \oplus B \mid \top \mid \bot \mid A^{\perp} \mid b\,,$$

where b ranges over atomic formulae. We have the following components:

— for conjunction

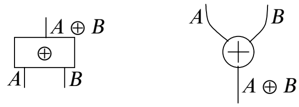
$\otimes L$



$\otimes R$

— for disjunction

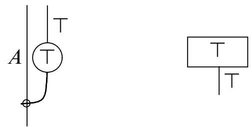
$\oplus L$



$\oplus R$

— to deal with $\top$

$\top L$



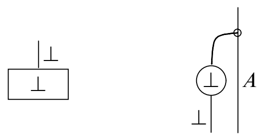
$\top R$

Remark 2.3. A curiosity here is that $\top$L requires the *supporting wire* A. The wire that is directly attached to the supporting wire is called a *thinning link* in Blute *et al.* (1996). Thinning links are needed because of categorical coherence issues: for example, using nets without thinning links would force the identity morphism on $\top \oplus \top$ to be equal to the twist map, which is false in some symmetric linearly distributive categories (see Blute *et al.* (1996)).

Dually, we have components:

— to deal with $\perp$

$\perp L$

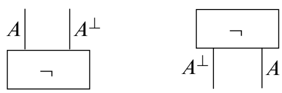


$\perp R$

The components $\top$L and $\perp$R are called *thinning links*.

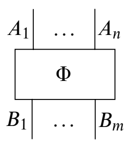
— for when we consider negation

$\neg$L



$\neg$R

Table 6 describes how a derivation Φ of $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is turned into a circuit



The double lines labelled Γ_i or Δ_j stand for bundles of wires, one for every formula contained in Γ_i or Δ_j. In the translations for $\top$L and $\perp$R, any wire in Φ can be used as a supporting wire. (We shall consider any two choices of supporting wire to be equivalent, see Section 2.2.1.)

We call the components $\otimes L$, $\otimes R$, $\oplus L$, $\oplus R$, $\top L$, $\top R$, $\perp L$, $\perp R$, $\neg$L and $\neg$R *links* to distinguish them from arbitrary components. Links depicted by rectangular boxes

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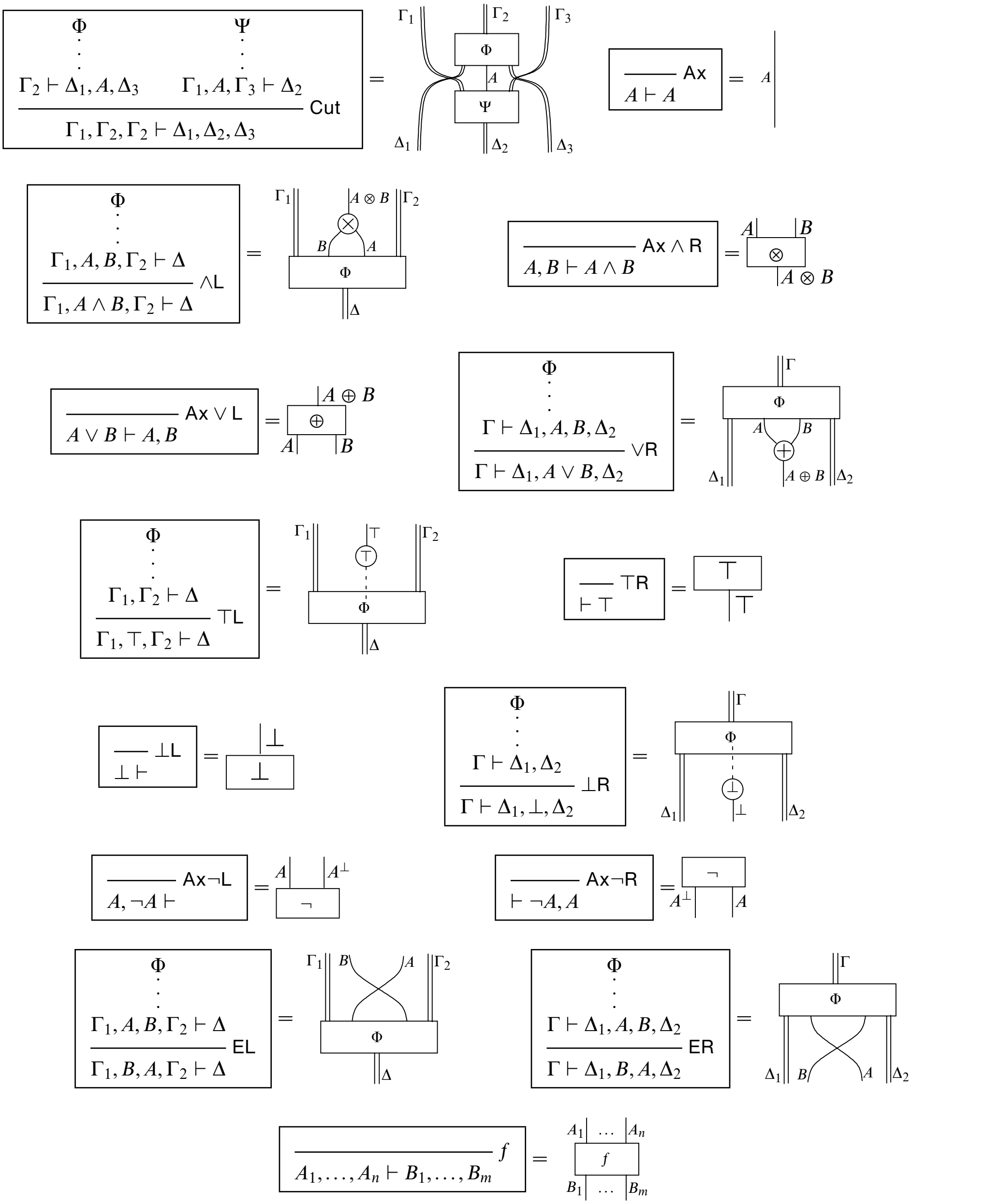
Table 6. *From derivations to nets*



correspond to axioms (for example, Ax $\wedge$ R, Ax $\vee$ L, Ax$\neg$L, Ax$\neg$R, $\bot$L, $\top$R); they are nets. Links with circles correspond to inference rules that have one or more hypotheses; they are used to build nets, but they are not nets. (This is a notational clarification we adopt from Cockett *et al.* (2003).)

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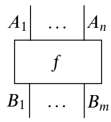
Definition 2.4. A *net over a signature* Σ is a circuit:

1 whose types are the formulae over Σ;
2 whose components are the links $\otimes L$, $\otimes R$, $\oplus L$, $\oplus R$, $\top L$, $\top R$, $\bot L$, $\bot R$, $\neg\mathsf{L}$, $\neg\mathsf{R}$ and components of the form



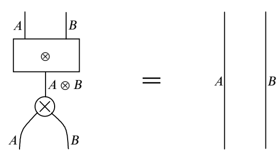
where $\dfrac{}{A_1, \ldots, A_n \vdash B_1, \ldots, B_m}\, f$ is an optional axiom of Σ;
3 which is in the image of translation in Table 6.

We say a net is *positive* if all of its formulae are positive. We write $Net(\Sigma)$ (respectively, $Net^{\neg}(\Sigma)$) for the positive (respectively, arbitrary) nets over Σ. We write $Net(\Sigma)(\Gamma, \Delta)$ (respectively, $Net^{\neg}(\Sigma)(\Gamma, \Delta)$) for the positive (respectively, arbitrary) nets with input ports according to Γ and output ports according to Δ.

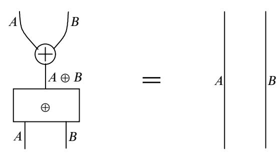
Remark 2.5. We have defined the notion of a net inductively. Evidently, not every circuit is a net (for example, $\oplus R$). In fact, a circuit that satisfies the first two conditions of the preceding definition is a net (that is, in the image of translation in Table 6) if and only if it satisfies the following combinatorial condition due to Girard (Girard 1987), which we present here in the same form as Blute *et al.* (1996): consider the components $\otimes L$ and $\oplus R$ to be 'switchable' in the sense introduced by Girard. This means in the case of $\otimes L$ that at most one of the two bottom wires is to be regarded as being 'connected', although we do not know which switch is set, and similary for the top two wires of $\oplus R$. The criterion for being a net holds if for any choice of switch settings the undirected graph determined by the remaining wires is acyclic and connected.

2.2.1. *Net equivalence* In this section, we shall recall the equivalence between nets introduced in Blute *et al.* (1996). It is defined by a number of rules for rewriting subcircuits. *These rules can only be applied if both the original circuit whose subcircuit is rewritten and the resulting circuit are nets.*

First, we have *reductions* that simulate the cut elimination of MLL:



$(Reduce\otimes)$



$(Reduce\oplus)$

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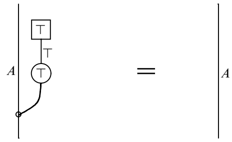
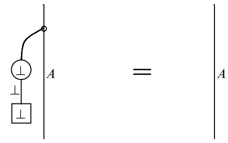
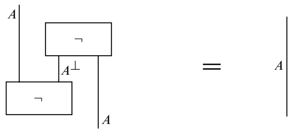


(*Reduce*$\top$)

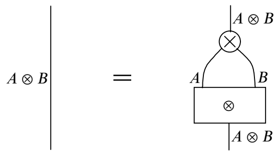


(*Reduce*$\bot$)

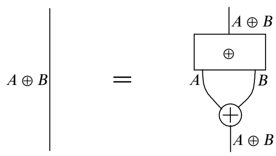


(*Reduce*$\neg$)

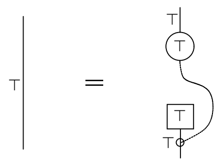
Second, we have *expansions* that allow us to express an axiom on a compound formula in terms of axioms on the subformulae:



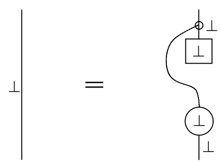
(*Expand*$\otimes$)



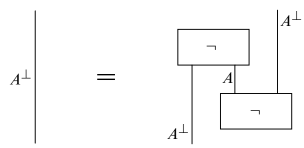
(*Expand*$\oplus$)



(*Expand*$\top$)



(*Expand*$\bot$)



(*Expand*$\neg$)

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Finally, Blute *et al.* (1996) contains a large number of rewriting rules that deal with the manipulation of thinning links. Fortunately, in the case of commutative logic, these rules amount to the *Empire Rewiring* Proposition (Blute *et al.* 1996, Proposition 3.3), which states that the supporting wire can be chosen freely within the empire[†] of the formula introduced by the thinning. This amounts to saying that the supporting wire can be chosen freely within *any* net containing the original supporting wire. For a detailed discussion of rewiring, see Blute *et al.* (1996).

2.3. *Symmetric linearly distributive categories*

Linearly distributive categories, which are due to Cockett and Seely and were initially called 'weakly distributive categories', can be used to model MLL. (This is explained in Cockett and Seely (1997b), but we shall spell out the semantics in Section 2.4.1.) All logical systems we consider in this paper are commutative – that is, they allow unrestricted use of the exchange rule, which allows us to use *symmetric* linearly distributive categories.

A symmetric linearly distributive category (Cockett and Seely 1997b) is a category $\mathbf{C}$ with two symmetric monoidal structures

$$
\begin{array}{rlrl}
\otimes &: \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C} & \oplus &: \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C} \\
\top &\in Ob(\mathbf{C}) & \bot &\in Ob(\mathbf{C}) \\
\alpha_\otimes &: (A \otimes B) \otimes C \cong A \otimes (B \otimes C) & \alpha_\oplus &: (A \oplus B) \oplus C \cong A \oplus (B \oplus C) \\
\lambda_\otimes &: \top \otimes A \cong A & \lambda_\oplus &: \bot \oplus A \cong A \\
\rho_\otimes &: A \otimes \top \cong A & \rho_\oplus &: A \otimes \bot \cong A \\
\sigma_\otimes &: A \otimes B \longrightarrow B \otimes A & \sigma_\oplus &: A \oplus B \longrightarrow B \oplus A
\end{array}
$$

and a natural transformation

$$\delta : A \otimes (B \oplus C) \longrightarrow (A \otimes B) \oplus C$$

called a (*linear*) *distribution*, which must satisfy various coherence conditions. For a description of these conditions, see Cockett and Seely (1997b). The distribution is used to model the cut rule, as we shall explain in Section 2.4.1.

We call $\otimes$ the *tensor* and $\oplus$ the *cotensor* (which is not to be confused with the cotensor product of modules).

A *symmetric linearly distributive category with negation* is a symmetric linearly distributive category together with, for every object A, an object $A^\perp$ and maps

$$\gamma^R : A \otimes A^\perp \longrightarrow \bot \qquad \tau^R : \top \longrightarrow A \oplus A^\perp$$

[†] The *empire* of a formula is the largest subnet containing that formula as an input port or an output port.

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satisfying the following conditions (Cockett and Seely 1997b):

$$\begin{tikzcd}
A \otimes \top \arrow[r, "id \otimes \tau^L"] \arrow[drr, "\rho_\oplus"'] & A \otimes (A^\perp \oplus A) \arrow[r, "\delta"] & (A \otimes A^\perp) \oplus A \arrow[r, "\gamma^R \oplus id"] & \perp \oplus A \arrow[dl, "\lambda_\oplus"] \\
& & A &
\end{tikzcd}$$

$$\begin{tikzcd}
A^\perp \otimes \top \arrow[r, "id \otimes \tau^R"] \arrow[drr, "\rho_\oplus"'] & A^\perp \otimes (A \oplus A^\perp) \arrow[r, "\delta"] & (A^\perp \otimes A) \oplus A^\perp \arrow[r, "\gamma^L \oplus id"] & \perp \oplus A^\perp \arrow[dl, "\lambda_\oplus"] \\
& & A^\perp &
\end{tikzcd}$$

where γ^L and τ^L are the evident maps resulting from γ^R and τ^R by composing with symmetry maps. These maps can be used to model $\mathsf{Ax}\neg\mathsf{L}$ and $\mathsf{Ax}\neg\mathsf{R}$, as we shall explain in Section 2.4.1.

Symmetric linearly distributive categories with negation are equivalent to $*$-autonomous categories (Cockett and Seely 1997b).

Finally, we recall a notion that plays an important role in the GoI construction: a *compact closed* category is a symmetric linearly distributive category $\mathbf{C}$ with negation such that the symmetric monoidal categories $(\mathbf{C}, \otimes, \top)$ and $(\mathbf{C}, \oplus, \perp)$ are identical, and δ is the associativity map.

Remark 2.6. Alternatively, one could define a compact closed category to be a symmetric monoidal category with, for every object A, an assigned left adjoint $A^\perp$ (Kelly and Laplaza 1980). The degenerate versions of the two equational laws for γ and τ are the triangular identities of that adjunction.

2.4. *Categorical semantics of MLL*

In this section, we recall the semantics of MLL in symmetric linearly distributive categories. In Section 2.4.1, we describe the interpretation of derivations as morphisms. In Section 2.4.2, we switch from derivations to nets, because nets allow a smoother presentation. At the end of Section 2.4.2, we state the important result that MLL nets (and therefore also derivations) are in perfect correspondence with symmetric linearly distributive categories (Theorem 2.7).

2.4.1. *The interpretation of sequents*

An *interpretation* for a signature Σ in a symmetric linearly distributive category $\mathbf{C}$ sends every formula A over Σ to an object $\lfloor A \rfloor$ according

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to the rules

$$\begin{aligned} \lfloor A \wedge B \rfloor &= \lfloor A \rfloor \otimes \lfloor B \rfloor \\ \lfloor B \vee B \rfloor &= \lfloor A \rfloor \oplus \lfloor B \rfloor \\ \lfloor \top \rfloor &= \top \\ \lfloor \bot \rfloor &= \bot . \end{aligned}$$

If we consider the scenario with negation, then **C** must be a symmetric linearly distributive category with negation, and we also require

$$\lfloor \neg A \rfloor = \lfloor A \rfloor^{\bot}.$$

A derivation Φ of a sequent $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ is interpreted by a morphism

$$\left\lfloor \begin{array}{c} \Phi \\ \vdots \\ A_1, \ldots, A_n \vdash B_1, \ldots, B_m \end{array} \right\rfloor : \lfloor A_1 \rfloor \otimes \cdots \otimes \lfloor A_n \rfloor \longrightarrow \lfloor B_1 \rfloor \oplus \cdots \oplus \lfloor B_m \rfloor,$$

where $\otimes$ and $\oplus$ are, say, left associative, the tensor for $n = 0$ is $\top$, and the cotensor for $m = 0$ is $\bot$.

1 The rule Ax is interpreted by the identity morphism, as are $\mathsf{Ax} \wedge \mathsf{R}$, $\mathsf{Ax} \vee \mathsf{L}$, $\mathsf{Ax}\top\mathsf{R}$ and $\mathsf{Ax}\bot\mathsf{L}$.

2 The rules $\wedge\mathsf{L}$ and $\top\mathsf{L}$ are interpreted by pre-composing the symmetric monoidal isomorphisms

$$\lfloor \Gamma \rfloor \otimes (\lfloor A \rfloor \otimes \lfloor B \rfloor) \otimes \lfloor \Gamma' \rfloor \cong \lfloor \Gamma \rfloor \otimes \lfloor A \rfloor \otimes \lfloor B \rfloor \otimes \lfloor \Gamma' \rfloor$$

and

$$\lfloor \Gamma \rfloor \otimes \top \otimes \lfloor \Gamma' \rfloor \cong \lfloor \Gamma \rfloor \otimes \lfloor \Gamma' \rfloor ,$$

respectively, and dually for $\vee\mathsf{R}$ and $\bot\mathsf{R}$.

3 The rule EL is interpreted by pre-composing the symmetric-monoidal isomorphism

$$\lfloor \Gamma_1 \rfloor \otimes \lfloor A \rfloor \otimes \lfloor B \rfloor \otimes \lfloor \Gamma_2 \rfloor \cong \lfloor \Gamma_1 \rfloor \otimes \lfloor B \rfloor \otimes \lfloor A \rfloor \otimes \lfloor \Gamma_2 \rfloor ,$$

and dually for ER.

4 The cut rule is interpreted as follows: if the interpretations of the premises are

$$\left\lfloor \begin{array}{c} \Phi \\ \vdots \\ \Gamma_2 \vdash \Delta_1, A, \Delta_3 \end{array} \right\rfloor = f : \lfloor \Gamma_2 \rfloor \longrightarrow \lfloor \Delta_1 \rfloor \oplus \lfloor A \rfloor \oplus \lfloor \Delta_3 \rfloor$$

$$\left\lfloor \begin{array}{c} \Psi \\ \vdots \\ \Gamma_1, A, \Gamma_3 \vdash \Delta_2 \end{array} \right\rfloor = g : \lfloor \Gamma_1 \rfloor \otimes \lfloor A \rfloor \otimes \lfloor \Gamma_3 \rfloor \longrightarrow \lfloor \Delta_2 \rfloor,$$

then the interpretation

$$\left\lfloor \dfrac{\begin{array}{cc} \Phi & \Psi \\ \vdots & \vdots \\ \Gamma_2 \vdash \Delta_1, A, \Delta_3 & \Gamma_1, A, \Gamma_3 \vdash \Delta_2 \end{array}}{\Gamma_1, \Gamma_2, \Gamma_3 \vdash \Delta_1, \Delta_2, \Delta_3} \,\mathsf{Cut} \right\rfloor$$

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of the conclusion is

$$\begin{array}{rcl}\lfloor\Gamma_1\rfloor\otimes\lfloor\Gamma_2\rfloor\otimes\lfloor\Gamma_3\rfloor & \xrightarrow{id\otimes f\otimes id} & \lfloor\Gamma_1\rfloor\otimes(\lfloor\Delta_1\rfloor\oplus\lfloor A\rfloor\oplus\lfloor\Delta_3\rfloor)\otimes\lfloor\Gamma_3\rfloor\\ & \xrightarrow{\delta_1} & \lfloor\Delta_1\rfloor\oplus(\lfloor\Gamma_1\rfloor\otimes\lfloor A\rfloor\otimes\lfloor\Gamma_3\rfloor)\oplus\lfloor\Delta_3\rfloor\\ & \xrightarrow{id\oplus g\oplus id} & \lfloor\Delta_1\rfloor\oplus\lfloor\Delta_2\rfloor\oplus\lfloor\Delta_3\rfloor,\end{array}$$

where δ_1 is obtained by combining the distribution δ and structural isomorphisms of the symmetric monoidal category. (There are different such combinations, but the coherence laws of a symmetric linearly distributive category ensure that they all amount to the same morphism.)

5 If we consider the scenario with negation, Ax¬L and Ax¬R are interpreted by γ^R and τ^L, respectively.

We shall describe the semantics of weakening and contraction later in this paper.

Evidently, an interpretation of a derivation is determined by its action on the optional axioms.

2.4.2. *Nets as symmetric linearly distributive categories* Our goal in this section is to explain the perfect correspondence between MLL and symmetric linearly distributive categories (with negation). To build a term model, we could construct a symmetric linearly distributive category whose morphisms are equivalence classes of derivations. However, the range of required equational laws would be almost unmanagable, because of countless commuting conversions and laws involving the exchange rule. Nets turn out to work much better here, because they deal with commuting conversions and exchange implicitly.

We believe that the transition from derivations to nets is harmless, because translating derivations into nets is almost trivial (essentially, the nets can be drawn into the derivation!).

The aim of this section is to describe how nets can be used to construct free symmetric linearly distributive categories (Theorems 2.7 and 2.9, which are taken from Blute *et al.* (1996)).

Given a set E of equivalences on $Net(\Sigma)$ (where two nets can only be equivalent if they inhabit the same sequent), we can construct a symmetric linearly distributive category $Net_E(\Sigma)$ as follows:

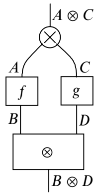
— The objects are the formulae over Σ.

— A morphism from A to B is a net $f \in Net(\Sigma)(A, B)$ modulo the congruence relation generated from E and the reductions, expansions and empire rewiring equations described in Section 2.2.1.

— Composition is defined in the evident way by connecting wires.

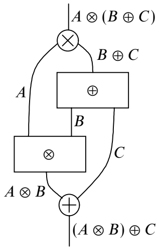
— The identity morphism on A is given by the wire labelled by A.

— Given nets representing morphisms $f : A \longrightarrow B$ and $g : C \longrightarrow D$, the net representing $f \otimes g$ is defined by

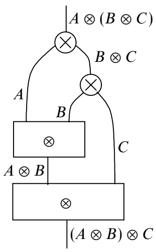
$f \otimes g =$ 

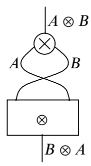
and dually for $\oplus$.

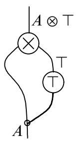
— The distribution is given by

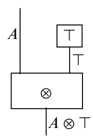
$\delta =$ 

— The symmetric monoidal isomorphisms with respect to $\otimes$ are given by

$\alpha_\otimes =$ 

$\sigma_\otimes =$ 

$\rho_\otimes =$ 

$\rho_\otimes^{-1} =$ 

The remaining isomorphisms ($\alpha_\otimes^{-1}$, $\lambda_\otimes$ and $\lambda_\otimes^{-1}$), and the duals for $\oplus$, are obvious.

We have the following result from Blute *et al.* (1996).

Theorem 2.7. $Net_E(\Sigma)$ is the free symmetric linearly distributive category generated by the signature Σ and the equations E.

Remark 2.8. This theorem implies soundness and completeness when $Net_E(\Sigma)$ is viewed as a theory whose judgments are equalities between nets. Completeness means that a judgment $M = N$ holds in the theory $Net_E(\Sigma)$ whenever it holds in every model; this is true because the theory $Net_E(\Sigma)$ forms a model of itself. Soundness means that every interpretation of the nets over Σ in a symmetric linearly distributive category $\mathbf{C}$ validates

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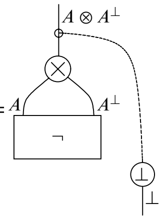
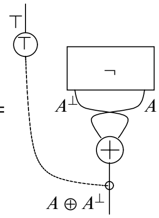
the equations in Section 2.2.1. This is true because the canonical functor from $Net_E(\Sigma)$ to $\mathbf{C}$ is well defined (that is, it sends equivalent nets to the same morphism).

The free construction can be extended with negation, and to do this, we only need to replace $Net(\Sigma)$ by $Net^{\neg}(\Sigma)$, allow the equations $Reduce\neg$ and $Expand\neg$, and define

$\gamma^R =$ 

$\tau^R =$ 

Theorem 2.9. $Net^{\neg}_E(\Sigma)$ is the free symmetric linearly distributive category with negation generated by the signature Σ and the equations E.

3. Modelling weakening and contraction: Dummett categories

In this section, we introduce categories that are in very close correspondence with the classical sequent calculus modulo cut reduction. We proceed by extending the scenario for MLL presented in Section 2.4 with structure for modelling weakening and contraction.

In Section 3.1, we shall start with symmetric linearly distributive categories and add symmetric monoids and symmetric comonoids to model weakening and contraction. In particular, we shall present a remarkable result (explained to us by Hasegawa) that monoids or comonoids force symmetric linearly distributive categories to be MIX (Theorem 3.11).

In Section 3.2, we shall add a poset enrichment to model cut reduction in the presence of weakening and contraction. We call the resulting categories *Dummett categories*. We do not require a Dummett category to have negation; if it does, we call it a *classical category*.

In Section 3.3, we explore the structural properties of Dummett categories. In particular, we show that every hom-set of a Dummett category is a semilattice, in terms of which the poset enrichment can be defined (Proposition 3.28). Moreover, we show that Dummett categories have an axiomatisation in terms of unconditional equalities (Theorem 3.31). Finally, we use this to show that the construction of the free symmetric linearly distributive category can be extended to Dummett categories and classical categories (Theorem 3.32).

In Section 3.4, we study the important case of compact Dummett categories (our extended GoI construction later in the paper involves only compact Dummett categories, and relies heavily on this section). Compactness allows great simplifications of the nets and the axiomatisation. In particular, we shall present an axiomatisation of compact Dummett categories in terms of only one equality (Proposition 3.42). Moreover, we shall show how compact Dummett categories shed light on cut reductions involving contraction (Proposition 3.41).

Finally, we specialise the compact setting to categories with finite biproducts, explain the resulting matrix calculus, and present a single equation that characterises when a category with finite biproducts is a Dummett category (Proposition 3.43).

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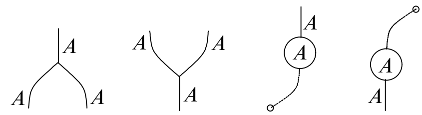
3.1. *Symmetric monoids and comonoids*

To model AxCL, AxCR, AxWL and AxWR in a symmetric linearly distributive category, we introduce maps

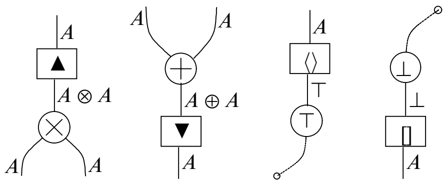
$$
\begin{aligned}
\blacktriangle_A : A &\longrightarrow A \otimes A \\
\blacktriangledown_A : A \oplus A &\longrightarrow A \\
\langle\rangle_A : A &\longrightarrow \top \\
[]_A : \bot &\longrightarrow A
\end{aligned}
$$

for every object A.

Definition 3.1. When we use nets, we shall use the abbreviations



for



We shall require certain conditions to ensure that $\blacktriangledown$, $[]$, $\blacktriangle$ and $\langle\rangle$ are sensibly defined: we require $(A, \blacktriangledown_A, []_A)$ to be a symmetric monoid – that is, the following associativity, neutrality and commutativity laws have to hold:

$$
\begin{tikzcd}
(A \oplus A) \oplus A \arrow[rr, "\alpha_\oplus"] \arrow[d, "id \oplus \blacktriangledown"'] & & A \oplus (A \oplus A) \arrow[d, "\blacktriangledown \oplus id"] \\
A \oplus A \arrow[dr, "\blacktriangledown"'] & & A \oplus A \arrow[dl, "\blacktriangledown"] \\
& A &
\end{tikzcd}
\qquad (\blacktriangledown assoc)
$$

$$
\begin{tikzcd}
A \oplus \bot \arrow[r, "id \oplus []"] \arrow[dr, "\rho_\oplus"'] & A \oplus A \arrow[d, "\blacktriangledown"] & \bot \oplus A \arrow[l, "[] \oplus id"'] \arrow[dl, "\lambda_\oplus"] \\
& A &
\end{tikzcd}
\qquad ([]neutral)
$$

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$$\begin{tikzcd} A \oplus A \arrow[rr, "\sigma_\oplus"] \arrow[dr, "\blacktriangledown"'] & & A \oplus A \arrow[dl, "\blacktriangledown"] \\ & A & \end{tikzcd} \qquad (\blacktriangledown symm)$$

It is easy to see that these laws correspond to the following widely-accepted equalities between sequent proofs:

$$\dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, A, A, \Delta'\end{matrix}}{\text{CR applied to the left } A \text{ and the middle } A}}{\dfrac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\,\mathsf{CR}} = \dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, A, A, \Delta'\end{matrix}}{\text{CR applied to the middle } A \text{ and the right } A}}{\dfrac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\,\mathsf{CR}}$$

$$\dfrac{\dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, \Delta'\end{matrix}}{\text{WR applied in either of the two evident ways}}}{\dfrac{\Gamma \vdash \Delta, A, A, \Delta'}{\Gamma \vdash \Delta, A, \Delta'}\,\mathsf{CR}} = \begin{matrix}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, \Delta'\end{matrix}$$

$$\dfrac{\begin{matrix}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, A, \Delta'\end{matrix}}{\Gamma \vdash \Delta, A, A, \Delta'}\,\mathsf{ER} = \begin{matrix}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, A, \Delta'\end{matrix}$$

The net versions of these laws are presented in Table 7 (symmetry means that we need only one of the two neutrality laws). Moreover, we require for all objects A and B that the monoid on $A \oplus B$ is defined pointwise in terms of the monoids on A and B; that is, we require

$$\begin{tikzcd} A \oplus B \oplus A \oplus B \arrow[rr, "id \oplus \sigma_\oplus \oplus id"] \arrow[dr, "\blacktriangledown_{A\oplus B}"'] & & A \oplus A \oplus B \oplus B \arrow[dl, "\blacktriangledown_A \oplus \blacktriangledown_B"] \\ & A \oplus B & \end{tikzcd} \qquad (\blacktriangledown pointwise)$$

$$\begin{tikzcd} \perp \arrow[rr, phantom, "\cong"] \arrow[dr, "{[]_{A\oplus B}}"'] & & \perp \oplus \perp \arrow[dl, "{[]_A \oplus []_B}"] \\ & A \oplus B & \end{tikzcd} \qquad ([]pointwise)$$

and the nullary cases

$$\blacktriangledown_\perp = \lambda_\oplus : \perp \oplus \perp \longrightarrow \perp$$

$$[]_\perp = id_\perp \qquad ([]trivial)$$

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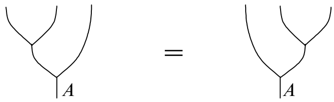
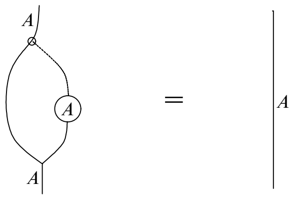
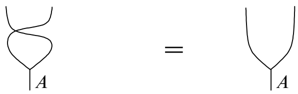
Table 7. *Net equalities for symmetric monoids*



(CR-*assoc*)



(WR-*neutral*)



(CR-*symm*)

It is easy to check that the two nullary laws are interderivable; in the remainder of this paper, we shall stick with $[]$*trivial* and make no further mention of the other law. The laws $\blacktriangledown$*pointwise*, $[]$*pointwise* and $[]$*trivial* correspond to the following equalities between sequent proofs:

$$\dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, B, A, B, \Delta'\end{array}}{\Gamma \vdash \Delta, A\wedge B, A\wedge B, \Delta'}\ \text{two applications of } \wedge\mathsf{R}}{\Gamma \vdash \Delta, A\wedge B, \Delta'}\ \mathsf{CR} \quad = \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A, B, A, B, \Delta'\end{array}}{\Gamma \vdash \Delta, A, A, B, B, \Delta'}\ \mathsf{ER}}{\Gamma \vdash \Delta, A, B, \Delta'}\ \text{two applications of } \mathsf{CR}}{\Gamma \vdash \Delta, A\wedge B, \Delta'}\ \wedge\mathsf{R}$$

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, \Delta'\end{array}}{\Gamma \vdash \Delta, A\wedge B, \Delta'}\ \mathsf{WR} \quad = \quad \dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, \Delta'\end{array}}{\Gamma \vdash \Delta, A, B, \Delta'}\ \text{two applications of } \mathsf{WR}}{\Gamma \vdash \Delta, A\wedge B, \Delta'}\ \mathsf{WR}$$

$$\dfrac{\dfrac{}{\bot \vdash}\ \bot\mathsf{L}}{\bot \vdash \bot}\ \mathsf{WR} \quad = \quad \dfrac{}{\bot \vdash \bot}\ \mathsf{Ax}$$

The net versions of these laws are presented in Table 8.

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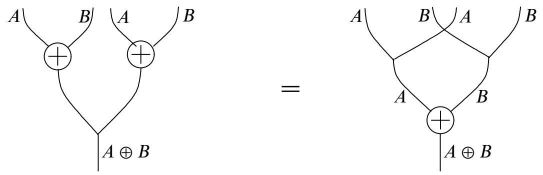
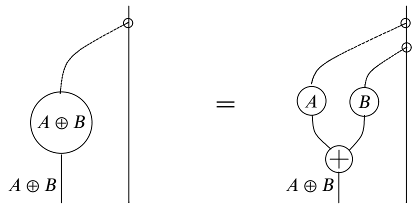
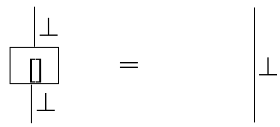
Table 8. *Net equalities for the pointwise definition of the symmetric monoids*



(CR-*pointwise*)



(WR-*pointwise*)



(WR-*trivial*)

Remark 3.2. While we believe that the laws in Table 7 are hard to dismiss (logicians seem to use them implicitly), the laws in Table 8 are perhaps more contentious. We require them because they seem highly plausible and required for numerous propositions and constructions.

Dually, we shall use comonoids ($\blacktriangle_A : A \longrightarrow A \otimes A, \langle\rangle_A : A \longrightarrow \top$) to model left contraction and weakening. The laws for comonoids are called $\blacktriangle$*assoc*, $[]$*neutral*, $\blacktriangle$*symm*, $\blacktriangle$*pointwise*, $\langle\rangle$*pointwise* and $\langle\rangle$*trivial*. Their net versions are called CL-*assoc*, WL-*neutral*, CL-*symm*, CL-*pointwise*, WL-*pointwise* and WL-*trivial*.

Definition 3.3. A symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \oplus, \bot)$ is said to have *symmetric monoids* if every object A has a chosen symmetric monoid $(\blacktriangledown_A, []_A)$, and the laws $\blacktriangledown$*pointwise*, $[]$*pointwise* and $[]$*trivial* hold.

Dually, a symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \otimes, \top)$ is said to have *symmetric comonoids* if every object A has a chosen symmetric comonoid $(\blacktriangle_A, \langle\rangle_A)$, and the laws $\blacktriangle$*pointwise*, $\langle\rangle$*pointwise* and $\langle\rangle$*trivial* hold.

Definition 3.4. A *pre-Dummett category* is a symmetric linearly distributive category $\mathbf{C}$ such that:

1 The symmetric monoidal category $(\mathbf{C}, \oplus, \bot)$ has symmetric monoids.
2 The symmetric monoidal category $(\mathbf{C}, \otimes, \top)$ has symmetric comonoids.

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Remark 3.5. This agrees with Hasegawa's notion of a pre-Dummett category, except that we do not require that the hom-semigroups (defined in Section 3.3) are idempotent.

Example 3.6. *Every distributive lattice* $\mathbf{D}$.
The objects are the elements of $\mathbf{D}$, and there is at most one morphism $A \longrightarrow B$, which exists if and only if $A \leqslant B$. The functor $\otimes$ is the greatest lower bound, and $\oplus$ is the least upper bound. The object $\top$ is the greatest element, and $\perp$ is the least element. The distribution exists because $A \otimes (B \oplus C) = (A \otimes B) \oplus (A \otimes C) \leqslant (A \otimes B) \oplus C$. The monoids and comonoids exist because $A = A \otimes A$, $A \leqslant \top$, $A \oplus A = A$ and $\perp \leqslant A$ for all $A \in \mathbf{D}$.

Example 3.7. *Every symmetric monoidal category* $\mathbf{C} = (\mathbf{C}, \odot, I)$ *with symmetric monoids and symmetric comonoids, if both* $\otimes$ *and* $\oplus$ *are defined to be* $\odot$*, and both* $\top$ *and* $\perp$ *are defined to be* I.
The distribution is the associativity $A \odot (B \odot C) \cong (A \odot B) \odot C$. Examples of such categories include:

— *The category* **Rel** *whose objects are (small) sets and whose morphism* $A \longrightarrow B$ *are subsets of* $A \times B$*, if* $\odot$ *is defined to be the evident functor that sends two sets to their set-theoretic product and* I *is defined to be the singleton set* $\{*\}$.
We have $\blacktriangledown_A = \{((x, x), x) : x \in A\}$ and $[]_A = \{(*, x) : x \in A\}$, and dually for $\blacktriangle_A$ and $\langle\rangle_A$. We write $(\mathbf{Rel}, \times)$ for this pre-Dummett category.

— *Every category with finite biproducts if* $\odot$ *is defined to be the binary biproduct, and* I *is defined to be the zero (that is, initial and terminal) object.*
The comonoids are given by the diagonals and projections of the product structure, and dually for the monoids. Examples include:

  – *The category* **Rel** *if* $\odot$ *is defined to be evident functor that sends two sets to their disjoint union and* I *is defined to be the empty set.*
  We write $(\mathbf{Rel}, \uplus)$ for this pre-Dummett category.

  – *The category* $\mathbf{FDVec}_K$ *of finite-dimensional vector spaces over a field* K*, if* $\odot$ *is defined to be the 'direct sum', which sends two spaces to their set-theoretic product, and* I *is defined to be the one-dimensional space* K.
  We write $(\mathbf{FDVec}_K, \times)$ to distinguish it from the compact closed category based on the tensor product.

The product $\mathbf{C}_1 \times \mathbf{C}_2$ of two pre-Dummett categories is a pre-Dummett category. Letting $\mathbf{C}_1$ be a distributive lattice and $\mathbf{C}_2$ be any of the categories in Example 3.7 shows that there exist pre-Dummett categories with non-trivial hom-sets such that $\otimes \neq \oplus$ and $\top \neq \perp$.

Theorem 3.8. Let Σ be a signature containing AxWL, AxWR, AxCL and AxCR. Let E be a set of equations on $Net(\Sigma)$ and E' be the set of equations for pre-Dummett categories described in Tables 7 and 8, and their duals. Then $Net_{E \cup E'}(\Sigma)$ is the free pre-Dummett category generated by Σ and E.

Proof. The result follows from Theorem 2.7, and the fact that E' characterises pre-Dummett categories. $\square$

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We conclude this section with some definitions. In a pre-Dummett category, the morphisms π_1^{AB} and π_2^{AB} are defined to be

$$A \otimes B \xrightarrow{id \otimes \langle\rangle} A \otimes \top \cong A \qquad \text{and} \qquad A \otimes B \xrightarrow{\langle\rangle \otimes id} \top \otimes B \cong B,$$

respectively. Dually, ι_1^{AB} and ι_2^{AB} are defined to be the evident morphisms $A \longrightarrow A \oplus B$ and $B \longrightarrow A \oplus B$.

3.1.1. *MIX* By the *MIX rule* we mean the following inference rule

$$\frac{\Gamma \vdash \Delta \quad \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{MIX}$$

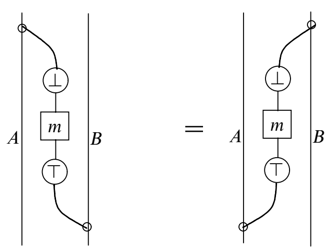
(This is the two-sided version of the MIX rule presented in Girard (1987), not the MIX rule presented in Gentzen (1934).) It is obviously derivable in the classical sequent calculus, for example, as follows:

$$\frac{\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, \perp}\,\mathsf{WR} \qquad \dfrac{\Gamma' \vdash \Delta'}{\perp, \Gamma' \vdash \Delta'}\,\mathsf{WL}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{Cut}$$

A (symmetric) linearly distributive category is called a (*symmetric*) *MIX category* or said to be *MIX* if it satisfies a certain condition (which we shall present below) that ensures that the MIX rule has a canonical semantics.

In this section, we show that every pre-Dummett category is MIX. In fact, we show a stronger result stating that every symmetric linearly distributive category with a monoid on $\perp$ or a comonoid on $\top$ is MIX (Theorem 3.11). The MIX property is interesting from a proof-theoretic point of view; it is also important for the equational characterisation of Dummett categories (Section 3.3) and for our extended GoI construction (Section 4).

A (symmetric) *MIX category* is a (symmetric) linearly distributive category with a morphism $m : \perp \longrightarrow \top$ such that, for all objects A and B, the two evident morphisms $A \otimes B \longrightarrow A \oplus B$ agree (Cockett and Seely 1997a):



We write mix_{AB} for the canonical morphism $A \otimes B \longrightarrow A \oplus B$. The family mix_{AB} of morphisms is easily seen to be a natural in A and B.

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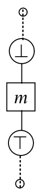
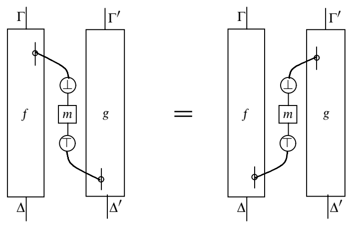
In a symmetric MIX category, the *MIX barbell*

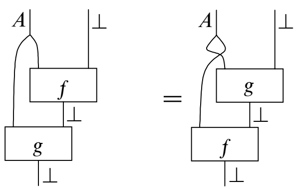


provides a canonical way of gluing together any two nets f and g:



(the supporting wire of the thinning link within each net does not matter owing to the empire rewiring proposition mentioned in Section 2.2.1). So a symmetric MIX category provides a canonical semantics to the MIX rule.

Lemma 3.9. Let $\mathbf{C}$ be a symmetric linearly category $\mathbf{C}$ with a morphism $\blacktriangle_A : A \longrightarrow A \otimes A$. Then for all $f, g : A \otimes \perp \longrightarrow \perp$, we have



(1)

Similarly, when each side of Equation 1 has $n = 0$ or $n \geqslant 2$ copies of A as input (that is, each side has $n = 0$ or $n \geqslant 2$ occurrences of $\blacktriangle_A$).

Proof. See the Appendix. □

Lemma 3.10. A linearly distributive category with a morphism $m : \perp \longrightarrow \top$ is MIX if and only if the following diagram commutes:

$$\begin{tikzcd}
\perp \otimes \perp \arrow[r, "m \otimes id"] \arrow[d, "id \otimes m"'] & \top \otimes \perp \arrow[d, "\cong"] \\
\perp \otimes \top \arrow[r, "\cong"] & \perp
\end{tikzcd} \tag{2}$$

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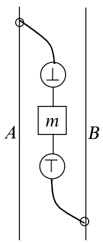
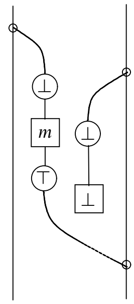
Proof. For the right-to-left direction, suppose that Diagram 2 commutes. Then

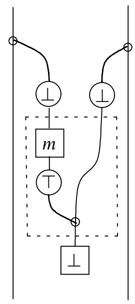


=



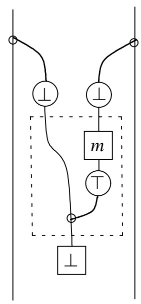
(by *Reduce*$\perp$)

=



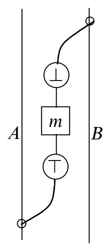
(by rewiring)

=



(by Diagram 2)

=



(by steps similar to the first three)

The left-to-right direction, which plays no role in this paper, follows from simple calculations; we leave the details to the reader. $\square$

Theorem 3.11 (Führmann and Hasegawa). Every symmetric linearly distributive category with a comonoid

$$\blacktriangle_\perp : \perp \longrightarrow \perp \otimes \perp \qquad \langle\rangle_\perp : \perp \longrightarrow \top$$

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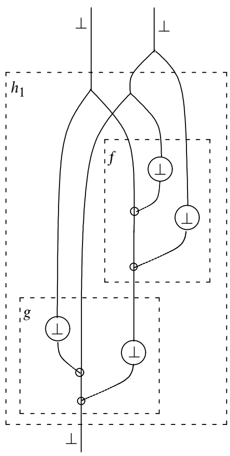
is MIX (with $m = \langle\rangle_\perp$). Dually, every symmetric linearly distributive category with a monoid

$$\blacktriangledown_\top : \top \oplus \top \longrightarrow \top \qquad []_\top : \perp \longrightarrow \top$$

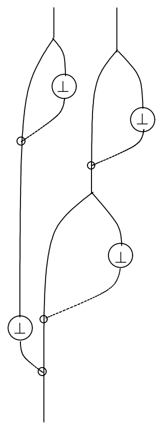
is MIX (with $m = []_\top$).

Proof. We show the comonoid case, with the help of Lemma 3.10. First, we present a net k_1, which denotes the top-right leg of Diagram 2 (with $m = \langle\rangle_\perp$), and another net k_2, which denotes the left-bottom leg. Then, we use Lemma 3.9 to show that k_1 and k_2 are equal. The dashed boxes labelled f, g, h_1 and h_2 denote subnets.

$k_1 =$

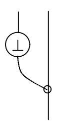


$=$



(by empire rewiring)

$=$



(by WL-*neutral*)

$$= \quad \perp \otimes \perp \xrightarrow{\langle\rangle_\perp \otimes id} \top \otimes \perp \cong \perp\,.$$

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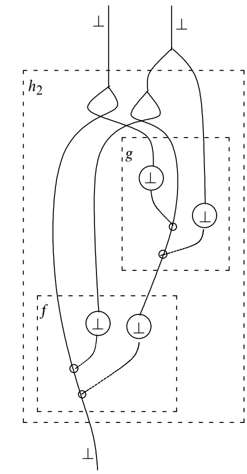
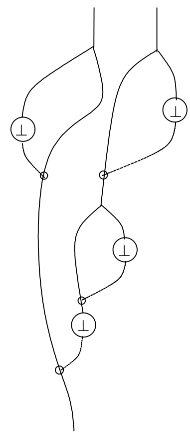
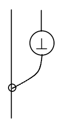
$k_2 =$ 

$=$  (by empire rewiring)

$=$  (by WL-*neutral*)

$$= \quad \perp \otimes \perp \xrightarrow{id \otimes \langle\rangle_\perp} \perp \otimes \top \cong \perp.$$

By Lemma 3.9 with $n = 2$, we have $h_1 = h_2$, and therefore $k_1 = k_2$. □

Corollary 3.12. Every pre-Dummett category is MIX with $m = \langle\rangle_\perp$, and also with $m = []_\top$.

We write $mix^{\langle\rangle}_{AB}$ (respectively, $mix^{[]}_{AB}$) for the natural transformation $A \otimes B \longrightarrow A \oplus B$ built from $\langle\rangle_\perp$ (respectively, $[]_\top$).

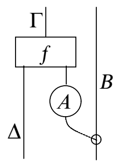
Are pre-Dummett categories canonically MIX? In other words, do we have $\langle\rangle_\perp = []_\top$? We do not know the answer to this question in general, but we shall see (Lemma 3.23) that the answer for Dummett categories is 'yes'.

Table 9. *Cut reductions for weakening and contraction (representative cases)*

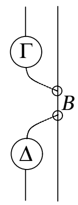
$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A\end{array} \qquad \dfrac{\begin{array}{c}\Psi\\ \vdots\\ \Gamma' \vdash \Delta'\end{array}}{A, \Gamma' \vdash \Delta'}\,\mathsf{WL}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{Cut} \;\leqslant\; \dfrac{\begin{array}{c}\Psi\\ \vdots\\ \Gamma' \vdash \Delta'\end{array}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{WL}, \mathsf{WR} \qquad (\mathit{Reduce}\mathsf{WL})$$

$$\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A\end{array} \qquad \dfrac{\begin{array}{c}\Psi\\ \vdots\\ A, A, \Gamma' \vdash \Delta'\end{array}}{A, \Gamma' \vdash \Delta'}\,\mathsf{CL}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{Cut} \;\leqslant\; \dfrac{\dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A\end{array} \qquad \dfrac{\begin{array}{c}\Phi\\ \vdots\\ \Gamma \vdash \Delta, A\end{array} \qquad \begin{array}{c}\Psi\\ \vdots\\ A, A, \Gamma' \vdash \Delta'\end{array}}{A, \Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{Cut}}{\Gamma, \Gamma, \Gamma' \vdash \Delta, \Delta, \Delta'}\,\mathsf{Cut}}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\,\mathsf{CL}, \mathsf{CR} \qquad (\mathit{Reduce}\mathsf{CL})$$

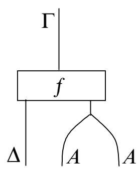
Table 10. *Net version of Table 9*



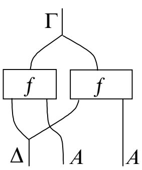
$\leqslant$



$(\mathit{Reduce}\mathsf{WL})$



$\leqslant$



$(\mathit{Reduce}\mathsf{CL})$

3.2. *Poset enrichment*

Our next goal is to model the cut-reduction rules for weakening and contraction – that is, the equations in Table 9 and their duals *Reduce*WR and *Reduce*CR.

We use the symbol $\leqslant$ instead of the equality symbol, because we shall not require that the denotation of redex and reduct be the same: and if we required them to be the same in the rules *Reduce*WL and *Reduce*WR, then any two derivations of $\Gamma \vdash \Delta$ would have the same denotation because of Lafont's example; and if we required them to be the same in the rules *Reduce*CL and *Reduce*CR, we would rule out desirable models, as we shall see in Example 3.15.

Table 10 contains the net versions of the reductions in Table 9. The derivation Φ corresponds to the net f. The net corresponding to Ψ is not needed, because we allow ourselves to rewrite subcircuits. We assume without loss of generality that Γ and Δ consist

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of single formulae; this is possible because we can always bundle a wire labelled with $\Gamma = A_1, \ldots, A_n$ in a single wire labelled with $A_1 \otimes \cdots \otimes A_n$ (by using the two kinds of links for $\otimes$), and a wire labelled with $\Delta = B_1, \ldots, B_m$ in a single wire labelled with $B_1 \oplus \cdots \oplus B_m$ (by using the two kinds of links for $\oplus$).

In our categorical models, $\leqslant$ will be a poset enrichment. Consider the net version of the law *Reduce*CL; if Δ is empty, it corresponds to the categorical law

$$\blacktriangle_A \circ f \leqslant (f \otimes f) \circ \blacktriangle_\Gamma \,, \tag{$\blacktriangle lax$}$$

which states that $\blacktriangle$ is a lax natural transformation. Similarly, the law *Reduce*WL for empty Δ corresponds to the categorical law

$$\langle\rangle_A \circ f \leqslant \langle\rangle_\Gamma \,, \tag{$\langle\rangle lax$}$$

which states that $\langle\rangle$ be a lax natural transformation.

The parametric categorical laws (that is, the versions for non-empty Δ) are very cumbersome: stating them requires multiple uses of the distribution δ; alternatively, one can stick with the non-parametric versions and add four extra inequalities (see Table 1 in Führmann and Pym (2004)). By contrast, the net versions of the laws are elegant; moreover, equations between nets are perfectly suited to describing equalities between morphisms, owing to Theorem 2.7. So we stick with the net versions in this paper.

Definition 3.13. A *Dummett category* is a pre-Dummett category **C** together with a poset enrichment $\leqslant$ such that:

1 The functors $\otimes$ and $\oplus$ are monotonic in both arguments.
2 The laws *Reduce*WL, *Reduce*WR, *Reduce*CL and *Reduce*CR hold.

Example 3.14. *Every distributive lattice* **D** (which we know to be a pre-Dummett category from Example 3.6).
The partial order is trivial, because each hom-set contain at most one element.

Example 3.15. *The pre-Dummett category* $(\mathbf{Rel}, \times)$*, where for relations* $f, f' : A \longrightarrow B$*, we define* $f \leqslant f' \iff f \subseteq f'$*, where* $\subseteq$ *is the set-theoretic inclusion.*
To see that *Reduce*WL holds, let Γ, Δ, A, B be sets, and let f be a relation $\Gamma \longrightarrow \Delta \times A$. The relation denoted by the redex turns out to be

$$\{((g, b), (d, b)) : b \in B \wedge \exists a \in A : (g, (d, a)) \in f\} \,,$$

while the reduct turns out to be

$$\{((g, b), (d, b)) : b \in B \wedge g \in \Gamma \wedge d \in \Delta\} \,.$$

The two are equal if and only if for all $g \in \Gamma$ and $d \in \Delta$, there exists an $a \in A$ such that $(g, (d, a)) \in f$. We call such relations $f : \Gamma \longrightarrow \Delta \times A$ *total*; for empty Δ, this agrees with the usual notion of a total relation. *Reduce*WR holds by the dual argument. To see that *Reduce*CL holds, note that reduct turns out to be

$$\{(g, (d, a, a)) : (g, (d, a)) \in f\} \,,$$

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while the reduct turns out to be

$$\{(g,(d,a_1,a_2)) : (g,(d,a_1)) \in f \wedge (g,(d,a_2)) \in f\}.$$

The two are equal if and only if for all $g \in \Gamma$, $d \in \Delta$, and $a_1, a_2 \in A$, we have $a_1 = a_2$ whenever $(g,(d,a_1)) \in f$ and $(g,(d,a_2)) \in f$. We call such relations $f : \Gamma \longrightarrow \Delta \times A$ *functional*; for empty Δ, this agrees with the usual notion of a functional relation.

Example 3.16. *The pre-Dummett category* $(\mathbf{Rel}, \uplus)$, *where for relations* $f, f' : A \longrightarrow B$, *we define* $f \leqslant f' \iff f' \subseteq f$.
To see that *Reduce*WL holds, let Γ, Δ, A, B be sets, and let f be a relation $\Gamma \longrightarrow \Delta \uplus A$. Then f consists of components $f_{\Gamma\Delta} : \Gamma \longrightarrow \Delta$ and $f_{\Gamma A} : \Gamma \longrightarrow A$. The relation denoted by the redex, when presented as a 2×2-matrix, turns out to be

$$\left(\begin{array}{c|cc} & \Gamma & B \\ \hline \Delta & f_{\Gamma\Delta} & \varnothing \\ B & \varnothing & id_B \end{array}\right)$$

while the reduct is

$$\left(\begin{array}{c|cc} & \Gamma & B \\ \hline \Delta & \varnothing & \varnothing \\ B & \varnothing & id_B \end{array}\right)$$

The two are equal if and only if $f_{\Gamma\Delta} = 0$. *Reduce*WR holds by the dual argument. To see that *Reduce*CL holds, note that both redex and reduct turn out to be given by the 3×1-matrix

$$\left(\begin{array}{c|c} & \Gamma \\ \hline \Delta & f_{\Gamma\Delta} \\ A & f_{\Gamma A} \\ A & f_{\Gamma A} \end{array}\right)$$

*Reduce*CR holds by the dual argument.

As we shall see in Remark 3.26, the pre-Dummett category $(\mathbf{FDVec}_K, \times)$ of finite-dimensional vector spaces over a field K does not form a Dummett category.

As in the case of pre-Dummett categories, the product of two Dummett categories forms a Dummett category. The product of a distributive lattice and $(\mathbf{Rel}, \uplus)$ or $(\mathbf{Rel}, \times)$ shows that there are Dummett categories with non-trivial hom-sets such that $\otimes \neq \oplus$ and $\top \neq \bot$.

Definition 3.17. A *classical category* is a Dummett category with negation (in the sense of Section 2.3).

Example 3.18. *Every boolean lattice, where $A^\perp$ is the complement of A.*

Example 3.19. *The Dummett category* $(\mathbf{Rel}, \times)$, *with* $A^\perp = A$.
The maps γ^L and γ^R are $\{((a,a),*) : a \in A\}$; and similarly for τ^L and τ^R.

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The Dummett category $(\mathbf{Rel}, \uplus)$ does not have negation: because both $\perp$ and $\top$ are the empty set, the maps τ^L, τ^R, γ^L and γ^R could only be the empty relations; so they could not satisfy the required equations. However, we shall see later that every traced Dummett category (for example, $(\mathbf{Rel}, \uplus)$) induces a classical category via an extended Geometry of Interaction construction (Theorem 4.4).

As in the case of Dummett categories, the product of two classical categories forms a classical category. The product of a boolean lattice and $(\mathbf{Rel}, \times)$ shows that there are classical categories with non-trivial hom-sets such that $\otimes \neq \oplus$ and $\top \neq \perp$.

3.2.1. *Homomorphisms* Next, we introduce certain kinds of homomorphisms for studying how morphisms of a Dummett category behave with respect to *Reduce*WL, *Reduce*WR, *Reduce*CL, and *Reduce*CR.

If the law

$$f \circ \blacktriangledown \leqslant \blacktriangledown \circ (f \oplus f) \qquad (\blacktriangledown lax)$$

holds for f as an *equality*, then f is a semigroup homomorphism. If the law

$$f \circ [] \leqslant [] \qquad ([] lax)$$

holds for f as an equality, then f preserves the unit $[]$ of the monoid; in this case, we call f a *pointed homomorphism*. If both laws hold for f as equalities, then f is a monoid homomorphism. Dually, we have notions of cosemigroup homomorphism, copointed homomorphism and comonoid homomorphism. Now we generalise this to the parametric case.

Definition 3.20. A morphism $f : \Gamma \longrightarrow \Delta \oplus A$ of a Dummett category is said to be a:

— *parametrised copointed homomorphism (from* Γ *to* A*)* if the law *Reduce*WL (that is, the parametrised version of the law $\langle\rangle lax$) holds for f as an equality;

— *parametrised cosemigroup homomorphism (from* Γ *to* A*)* if the law *Reduce*CL (that is, the parametrised version of the law $\blacktriangle lax$) holds for f as an equality;

— *parametrised comonoid homomorphism (from* Γ *to* A*)* if f is both of the above.

Parametrised pointed, semigroup and monoid homomorphisms are defined dually.

According to the discussions in Examples 3.15 and 3.16, the situation for $(\mathbf{Rel}, \times)$ and $(\mathbf{Rel}, \uplus)$ is as follows (and dually for semigroup/pointed/monoid homomorphisms):

property	$(\mathbf{Rel}, \times)$	$(\mathbf{Rel}, \uplus)$
$f : \Gamma \longrightarrow \Delta \oplus A$ is a parametrised copointed homomorphism	if f is total	if $f_{\Gamma\Delta} = \varnothing$
$f : \Gamma \longrightarrow \Delta \oplus A$ is a parametrised cosemigroup homomorphism	if f is functional	always
$f : \Gamma \longrightarrow A$ is a copointed homomorphism	if f is total	always
$f : \Gamma \longrightarrow A$ is a cosemigroup homomorphism	if f is functional	always
$f : \Gamma \longrightarrow A$ is a comonoid homomorphism	if f is a total function	always

What prevents $(\mathbf{Rel}, \uplus)$ from identifying both reducts in Lafont's example is that not every morphism is a *parametrised* (co)pointed homomorphism!

As we shall see in Section 3.4.3, the (co)homomorphism analysis for any Dummett category with finite biproducts leads to the same result as we found for $(\mathbf{Rel}, \uplus)$.

3.3. *The structure of Dummett categories*

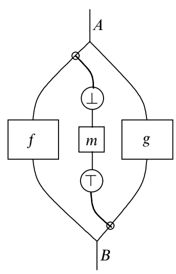
In this section, we show that the partial order of a Dummett category can be expressed in terms of the underlying pre-Dummett category (Proposition 3.28), and we use that result to show that Dummett categories can be axiomatised in terms of unconditional equations (Theorem 3.31). After that, we shall present the construction of the free Dummett category from nets (Theorem 3.32).

The key to the equational axiomatisation is the observation that every hom-set of a pre-Dummett category has a binary operation $*$, which in the case of a Dummett category is a semilattice (in general, without a neutral element), from which the partial order $\leqslant$ can be derived.

Definition 3.21. For two morphisms $f, g : A \longrightarrow B$ of a pre-Dummett category, the morphism $f * g : A \longrightarrow B$ is defined as follows:

$$\begin{tikzcd}
 & & A \oplus A \arrow[dr, "f \otimes g"] & & \\
A \arrow[r, "\blacktriangle"] & A \otimes A \arrow[ur, "mix^{\Diamond}_{A,A}"] \arrow[dr, "f \oplus g"'] & & B \oplus B \arrow[r, "\blacktriangledown"] & B \\
 & & B \otimes B \arrow[ur, "mix^{\Diamond}_{B,B}"'] & &
\end{tikzcd}$$

The corresponding net is



where $m = \langle\rangle_{\top}$. That is, we glue f and g together with a MIX barbell (as discussed in Section 3.1.1) to give a morphism $A \otimes A \longrightarrow B \oplus B$, and then pre-compose $\blacktriangle$ and post-compose $\blacktriangledown$. (Re-attaching the bottom thinning link of the MIX barbell to the wire above g would yield the upper leg $A \longrightarrow B$ of the commuting diagram above, while re-attaching the top thinning link of the MIX barbell to the wire below f would yield the

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bottom leg $A \longrightarrow B$ of the commuting diagram above. Both nets are equal to the net above owing to empire rewiring.)

We shall see soon (Lemma 3.23) that $\langle\rangle_\perp = []_\top$ in the case of a Dummett category, so in that case it does not matter whether we use $mix^{\langle\rangle}$ or $mix^{[]}$ in the definition of the operation $*$.

Example 3.22.

— *In* $(\mathbf{Rel}, \times)$, *where* $mix_{AB} = id_{A\times B}$, $*$ *is the set-theoretic intersection.*
— *In* $(\mathbf{Rel}, \uplus)$, *where* $mix_{AB} = id_{A\uplus B}$, $*$ *is the set-theoretic union.*
— *In* $(\mathbf{FDVec}_K, \times)$, *where* $mix_{AB} = id_{A\times B}$, $*$ *is the usual addition.*

Note that the operation $*$ is associative (owing to CR-*assoc* and CL-*assoc*) and commutative (owing to CR-*symm* and CL-*symm*). Now we turn to proving that in every Dummett category $*$ is idempotent, and therefore a semilattice.

Lemma 3.23. Every Dummett category has a greatest morphism $\perp \longrightarrow \top$, and it is equal to $\langle\rangle_\perp$ and $[]_\top$.

Proof. For every morphism $f : \perp \longrightarrow \top$, we have

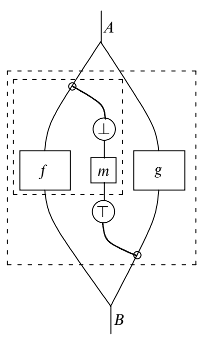
$$\begin{aligned} f &= id_\top \circ f \\ &= \langle\rangle_\top \circ f && \text{(by } \langle\rangle\textit{trivial}) \\ &\leqslant \langle\rangle_\perp\,. && \text{(by } \langle\rangle\textit{lax}) \end{aligned}$$

Dually, we get $f \leqslant []_\top$. □

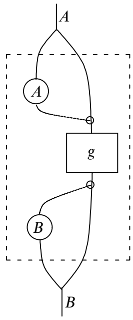
Lemma 3.24. In every Dummett category, we have the following laws:

$$f * g \leqslant f \qquad\qquad f * g \leqslant g\,.$$

Proof. Without loss of generality, we show $f * g \leqslant g$. We have

$f * g =$ 

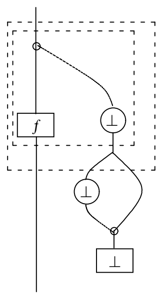
where the dashed boxes are only for visual guidance. By applying *Reduce*WL to the outermost dashed box, the above net is less than or equal to



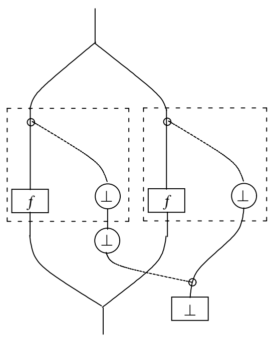
By WR-*neutral* and WL-*neutral*, this is equal to g. □

Lemma 3.25. For every morphism f of a Dummett category, $f * f = f$.

Proof. The inequality $f * f \leqslant f$ follows directly from Lemma 3.24. For the converse, note that f is equal to



owing to WL-*neutral* and *Reduce*⊥. (The dashed boxes are only for visual guidance.) By applying *Reduce*CL to the subnet in the outermost dashed box, it follows that the above net is less than or equal to



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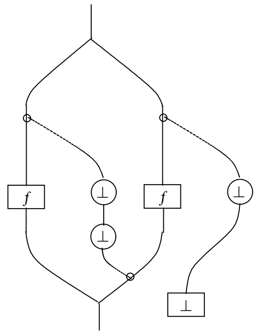
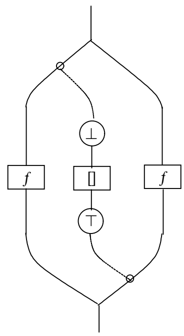
By empire rewiring, this is equal to



Now we apply *Reduce*$\perp$ to remove the 'appendix' and expand the bottom thinning link in the sense of Definition 3.1 to give



By Lemma 3.23, we have $[]_\top = m$, so the above net is equal to $f * f$. □

Remark 3.26. So the pre-Dummett category $(\mathbf{FDVec}_K, \times)$ cannot be a Dummett category because the addition of vectors is not idempotent.

Lemma 3.27. In every Dummett category, $*$ is monotonic in both arguments with respect to $\leqslant$.

Proof. The result follows from the definition of $*$ and the fact that $\oplus$, $\otimes$ and $\circ$ are monotonic with respect to $\leqslant$. □

Proposition 3.28. In every Dummett category, the partial order $\leqslant$ agrees with the one induced by the semilattice structure – that is,

$$f \leqslant g \iff f = f * g.$$

Proof. For the left-to-right implication, suppose that $f \leqslant g$. By Lemma 3.24, we have $f * g \leqslant f$. To see that $f \leqslant f * g$, consider

$$\begin{aligned} f &= f * f && \text{(by Lemma 3.25)} \\ &\leqslant f * g. && \text{(by Lemma 3.27)} \end{aligned}$$

The right-to-left implication holds because $f * g \leqslant g$ by Lemma 3.24. □

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Composition does not generally preserve the semilattice structure – that is, Dummett categories are not generally semilattice-enriched. In fact, even classical categories are not generally semilattice-enriched. To see this, consider the classical category $(\mathbf{Rel}, \times)$. The operation $*$ is the set-theoretic intersection. We have

$$\begin{aligned}(x,z) \in h \circ (f * g) &\iff \exists y : (x,y) \in h \text{ and } (y,z) \in f \text{ and } (y,z) \in g\\ (x,z) \in (h \circ f) * (h \circ g) &\iff \exists y_1, y_2 : (x,y_1) \in h \text{ and } (y_1,z) \in f\\ &\qquad\quad \text{and } (x,y_2) \in h \text{ and } (y_2,z) \in g.\end{aligned}$$

Obviously, the two relations differ for some f, g and h. However, composition preserves $*$ in a lax way, and the same is true for $\otimes$ and $\oplus$.

Lemma 3.29. In every Dummett category, the following laws hold:

$$\begin{aligned}
h \circ (f * g) &\leqslant (h \circ f) * (h \circ g) & (f * g) \circ k &\leqslant (f \circ k) * (g \circ k) && (\circ lax)\\
h \otimes (f * g) &\leqslant (h \otimes f) * (h \otimes g) & (f * g) \otimes k &\leqslant (f \otimes k) * (g \otimes k) && (\otimes lax)\\
h \oplus (f * g) &\leqslant (h \oplus f) * (h \oplus g) & (f * g) \oplus k &\leqslant (f \oplus k) * (g \oplus k). && (\oplus lax)
\end{aligned}$$

Proof. By Lemma 3.24, we have $f*g \leqslant f$ and $f*g \leqslant g$. Because $h\circ(-)$ is monotonic, we have $h\circ(f*g) \leqslant h\circ f$ and $h\circ(f*g) \leqslant h\circ g$. By Proposition 3.28, we get $h\circ(f*g) \leqslant (h\circ f)*(h\circ g)$. The other five inequalities can be proved similarly. □

Lemma 3.30 (Hasegawa). In a pre-Dummett category that satisfies the equation $id_B * id_B = id_B$ for every object B, the laws *Reduce*WL and *Reduce*WR are derivable.

Proof. See the Appendix. □

The following theorem provides a characterisation of Dummett categories in terms of unconditional inequalities (which can be stated as equalities owing to the semilattice structure).

Theorem 3.31. To give a Dummett category is to give a pre-Dummett category satisfying the laws $\langle\rangle_\perp = []_\top$ and $f * f = f$, and, letting $\leqslant$ be the partial order induced by the semilattice $*$, the laws *Reduce*CL, *Reduce*CR, $\circ lax$, $\otimes lax$ and $\oplus lax$.

Proof. Every Dummett category satisfies the law $\langle\rangle_\perp = []_\top$ by Lemma 3.23 and the law $f * f = f$ by Lemma 3.25. By Proposition 3.28, the partial order of the Dummett category agrees with the order induced by the semilattice $*$. So *Reduce*CL and *Reduce*CR hold for the induced partial order because they are required to hold in a Dummett category; the laws $\circ lax$, $\otimes lax$ and $\oplus lax$ hold for the induced partial order because they hold in a Dummett category, owing to Lemma 3.29.

Conversely, let $\mathbf{C}$ be a pre-Dummett category satisfying the equations $\langle\rangle_\perp = []_\top$. By Corollary 3.12, $\mathbf{C}$ is MIX with $m = \langle\rangle_\perp = []_\top$, so $*$ is canonically defined. Now suppose that $f*f = f$ for every morphism f. Let $\leqslant$ be the partial order induced by the semilattice $*$, and suppose that the laws *Reduce*CL, *Reduce*CR, $\circ lax$, $\otimes lax$ and $\oplus lax$ hold. By Lemma 3.30, we have *Reduce*WL and *Reduce*WR. So, to see that we have a Dummett category, we still have to show that $\circ$, $\otimes$, $\oplus$ are monotonic in each argument. We shall show that $h \circ (-)$

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is monotonic for every morphism h; the other cases are similar. So, let $f \leqslant g$, that is, $f = f * g$. Using $\circ lax$, we get $h \circ f = h \circ (f * g) \leqslant (h \circ f) * (h \circ g) \leqslant h \circ g$. □

Theorem 3.32. Let Σ be a signature containing AxWL, AxWR, AxCL and AxCR and let:

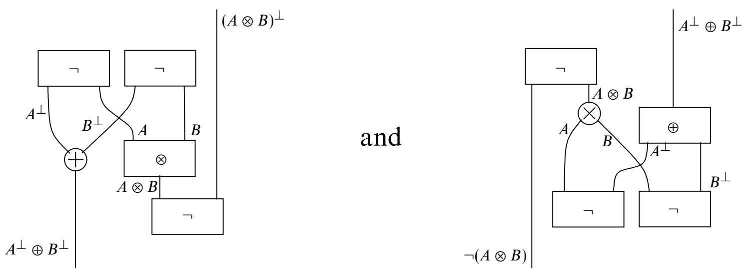
- E be a set of equations on $Net(\Sigma)$;
- E' be the set of equations for pre-Dummett categories described in Tables 7 and 8 and their duals; and
- E'' be the set of equations (between nets) corresponding to the laws $\langle\rangle_\bot = []_\top$, $f * f = f$, *Reduce*CL, *Reduce*CR, $\circ lax$, $\otimes lax$ and $\oplus lax$, where $\leqslant$ is the partial order induced by the semilattice $*$.

Then $Net_{E \cup E' \cup E''}(\Sigma)$ is the free Dummett category generated by Σ and E.

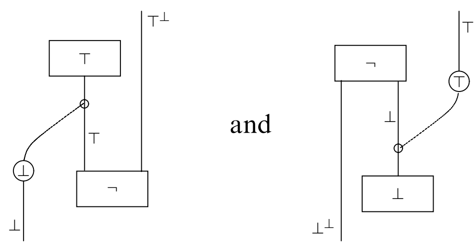
The similar result holds for classical categories.

Proof. The result follows from Theorem 2.7, together with the fact that E' characterises pre-Dummett categories and E'' characterises Dummett categories, as stated in Theorem 3.31. □

3.3.1. *Duality of the monoids and comonoids* In this section, we show that the monoids and comonoids of a classical category are mutually dual via De Morgan isomorphisms. Every symmetric linearly distributive category with negation has, for all objects A and B, a De Morgan isomorphism $(A \otimes B)^\bot \cong A^\bot \oplus B^\bot$. The mutually-inverse morphisms $(A \otimes B)^\bot \longrightarrow A^\bot \oplus B^\bot$ and $(A \otimes B)^\bot \longrightarrow A^\bot \oplus B^\bot$ are given by the nets



Dually, there is an isomorphism $(A \oplus B)^\bot \cong A^\bot \otimes B^\bot$. Also, there is an isomorphism $\top^\bot \cong \bot$, consisting of the mutually-inverse morphisms



Dually, there is an isomorphism $\bot^\bot \cong \top$.

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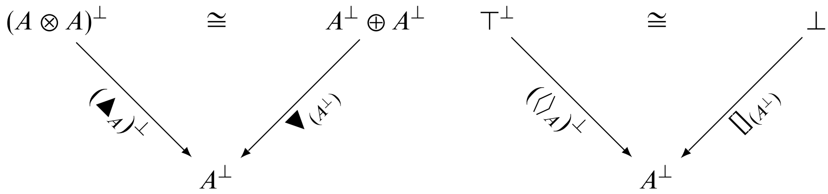
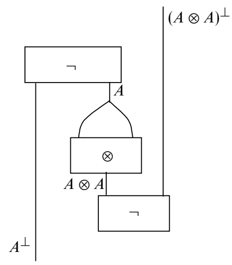
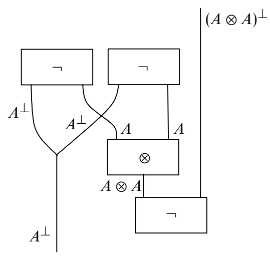
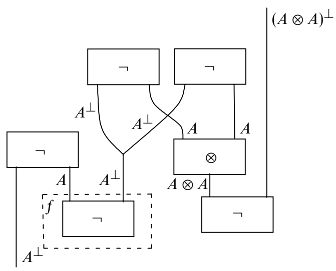
Proposition 3.33. In every classical category, the following diagrams and their dual versions commute:



Proof. Let d be the isomorphism $(A \otimes A)^\perp \cong A^\perp \oplus A^\perp$. We have

$(\blacktriangle_A)^\perp =$  and $\blacktriangledown_{(A^\perp)} \circ d =$ 

The inequality $(\blacktriangle_A)^\perp \leqslant \blacktriangledown_{(A^\perp)} \circ d$ is an instance of *Reduce*CR, where the subnet that gets duplicated is the top-left negation component in the net for $(\blacktriangle_A)^\perp$. Now for the converse. Note that, by *Expand*¬, the net for $\blacktriangledown_{(A^\perp)} \circ d$ is equal to



where the subnet f is marked purely for reference. By *Reduce*CL, where the duplicated subnet is f, followed by elimination of the two resulting logical cuts involving negation, we find that this net is less than or equal to the net for $(\blacktriangle_A)^\perp$.

The proof for the triangle involving the isomorphism $\top^\perp \cong \perp$ is somewhat similar and left to the reader. □

Remark 3.34. Recall that symmetric linearly distributive categories are equivalent to $*$-autonomous categories. Proposition 3.33 implies that classical categories are equivalent to $*$-autonomous categories with symmetric comonoids satisfying certain equations; the monoids can be defined from the comonoids as shown in Proposition 3.33. The term calculus presented in Section 5 takes advantage of this fact.

Remark 3.35. Linearly distributive categories have a notion of *complemented object*: simply speaking, a complemented object of a linearly distributive category is an object whose negated version exists (Cockett and Seely 1999, Appendix). For a symmetric linearly distributive category **C**, the full subcategory **S** given by the complemented objects has

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negation, because complemented objects are closed under $\otimes$, $\oplus$, $\top$ and $\perp$ (Cockett and Seely 1999, Proposition 34).

Moreover, it is easy to see that if **C** is a Dummett category, then **S** is a classical category. So even in a Dummett category, the monoids and comonoids are dual whenever it makes sense to speak of duality.

3.4. *Compact Dummett categories*

We can think of a symmetric monoidal category $\mathbf{C} = (\mathbf{C}, \otimes, \top)$ as a *compact* symmetric linearly distributive category, by which we mean that the two symmetric monoidal structures agree and distribution is the associativity $A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$. If **C** has symmetric monoids and symmetric comonoids

$$\begin{array}{ll} \blacktriangle_A : A \longrightarrow A \otimes A & \qquad \blacktriangledown_A : A \otimes A \longrightarrow A \\ \langle\rangle_A : A \longrightarrow \top & \qquad []_A : \top \longrightarrow A, \end{array}$$

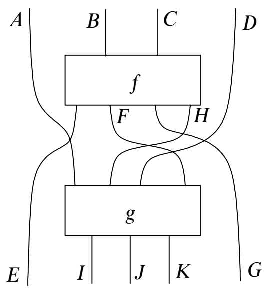
it forms a pre-Dummett category; in this section, we shall study the situation in which **C** is a Dummett category.

In particular, we shall present an axiomatisation of such compact Dummett categories as compact pre-Dummett categories satisfying only one extra equality (Proposition 3.42). Moreover, we will show how compact Dummett categories shed light on cut reductions involving contraction (Proposition 3.41).

Some parts of this section are also required for our extended GoI construction, in Section 4.

3.4.1. *Nets for symmetric monoidal categories*

There are circuits that make sense in a symmetric monoidal category that do not make sense in every symmetric linearly distributive category. For example, the circuit



describes the morphism

$$\begin{array}{c} A \otimes B \otimes C \otimes D \xrightarrow{id \otimes f \otimes id} A \otimes E \otimes F \otimes G \otimes H \otimes D \\ \cong \\ E \otimes A \otimes H \otimes D \otimes F \otimes G \xrightarrow{id \otimes g \otimes id} E \otimes I \otimes J \otimes F \otimes K \otimes G. \end{array}$$

This does not make sense in every symmetric linearly distributive category, because f and g are connected by *two* wires (F and H), which requires us to have a morphism

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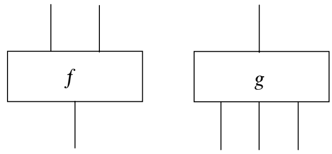
$F \oplus H \longrightarrow F \otimes H$. Also, juxtapositions like



make sense in a symmetric monoidal category (the semantics is $f \otimes g$), but not in every symmetric linearly distributive category. In fact, if f and g are circuits that describe morphisms in a symmetric monoidal category $\mathbf{C}$, and we connect any number of output ports of f to input ports of g (such that the types match), then the resulting circuit also describes a morphism in $\mathbf{C}$. We call this a *symmetric monoidal composition* of circuits. (However, we must not connect an output port with an input port of the *same* circuit, unless the category is traced. We will say more about this in Section 4.1.)

The links we shall use in nets for symmetric monoidal categories are

$\otimes L'$

$\otimes R$

$\top L'$

$\top R$

We have already used $\otimes R$ and $\top R$ in nets for symmetric linearly distributive categories; but not the links $\otimes L'$ and $\top L'$. Both $\otimes L'$ and $\otimes R$ denote $id_{A\otimes B}$, and are useful for bundling multiple wires. Both $\top L'$ and $\top R$ denote $id_\top$, and are useful for removing wires of type $\top$.

When the category is compact closed, we also use the links $\neg\mathsf{L}$ and $\neg\mathsf{R}$.

Definition 3.36. The *symmetric monoidal nets* over a set of atomic types and a set $\mathscr{C}$ of components are the following circuits:

— Types are given by the grammar

$$A, B ::= A \otimes B \mid \top \mid b,$$

where b ranges over atomic types;

— Components are the links $\otimes L'$, $\otimes R$, $\top L'$ and $\top R$, and all elements of $\mathscr{C}$. In the compact closed case, we also have the links $\neg\mathsf{L}$ and $\neg\mathsf{R}$;

— If f and g are symmetric monoidal nets, then so is any symmetric monoidal composition of f and g.

There is an evident translation that sends nets for symmetric linearly distributive categories to symmetric monoidal nets. It translates formulae by sending $\oplus$ to $\otimes$ and $\bot$ to $\top$. It translates circuits by replacing subcircuits according to the rules in Table 11. It is easy to see that this translation preserves the semantics. That is, if f is a net for symmetric linearly distributive categories that denotes a morphism in a symmetric monoidal category, then the symmetric monoidal net that results from the translation denotes the same morphism.

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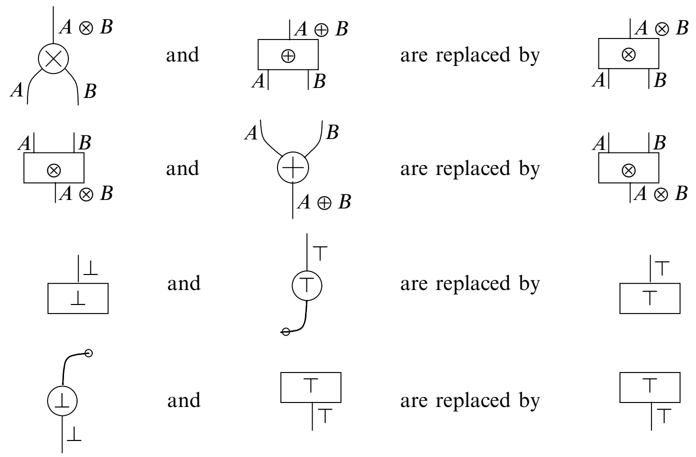
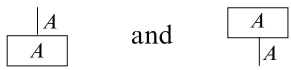
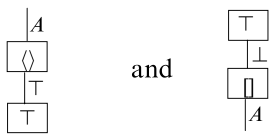
Table 11. *Rules for translating nets for symmetric linearly distributive categories into nets for symmetric monoidal categories*



When symmetric monoids and comonoids are present, we shall write



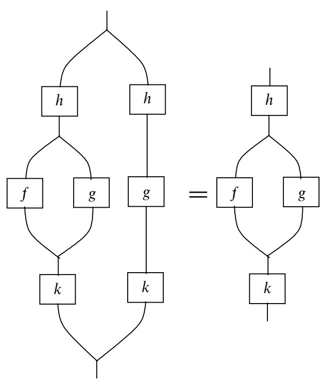
for



and we keep the notation for contractions given in Definition 3.1.

3.4.2. *Characterising compact Dummett categories by one equality* In this section, we show that a symmetric monoidal category with symmetric monoids and symmetric comonoids forms a Dummett category if and only if it satisfies the following law (Proposition 3.42):

$$(k \circ (f * g) \circ h) * (k \circ g \circ h) = k \circ (f * g) \circ h \tag{3}$$



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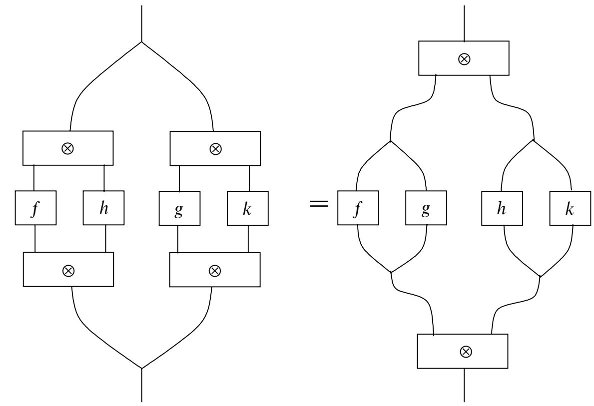
Moreover, we will show how compact Dummett categories can shed light on cut reductions involving contraction (Proposition 3.41).

Lemma 3.37. Every symmetric monoidal category with symmetric monoids and symmetric comonoids satisfies the following laws:

$$f * e = f \qquad \text{where } e = [] \circ \langle\rangle \tag{4}$$
$$(f \otimes h) * (g \otimes k) = (f * g) \otimes (h * k)\,. \tag{5}$$

The net version of Equation 5 is



Proof. Equation 4 holds because

$$\begin{aligned} f * e &= \blacktriangledown \circ (f \otimes ([] \circ \langle\rangle)) \circ \blacktriangle \\ &= \blacktriangledown \circ (id \otimes []) \circ (f \otimes id) \circ (id \otimes \langle\rangle) \circ \blacktriangle \\ &= f\,. && \text{(by } \langle\rangle \textit{neutral} \text{ and } []\textit{neutral}) \end{aligned}$$

Equation 5 holds because of CL*pointwise* and CR*pointwise*. □

Lemma 3.38. In every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, the law $g * g = g$ holds.

Proof. Let $k = id$, $h = id$ and $f = e$ in Equation 3. □

Lemma 3.39. In every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, $\circ$ and $\otimes$ are monotonic in both arguments with respect to the partial order induced by the hom-semilattice $*$.

Proof. The monotonicity of $\circ$ follows from Equation 3: for $f \leqslant g$ (that is, $f * g = id$) we have $k \circ f \circ h \leqslant k \circ g \circ h$ because $(k \circ f \circ h) * (k \circ g \circ h) = (k \circ (f * g) \circ h) * (k \circ g \circ h) = k \circ (f * g) \circ h = k \circ f \circ h$. To see the monotonicity of $\otimes$, suppose that $f \leqslant g$. Then we have $f \otimes h \leqslant g \otimes h$ because

$$\begin{aligned} (f \otimes h) * (g \otimes h) &= (f * g) \otimes (h * h) && \text{(by Equation 5 of Lemma 3.37)} \\ &= (f * g) \otimes h \\ &= f \otimes g\,. \end{aligned}$$

□

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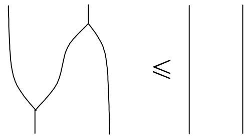
Lemma 3.40. In every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, $(\blacktriangledown \otimes id) \circ (id \otimes \blacktriangle) \leqslant id$.

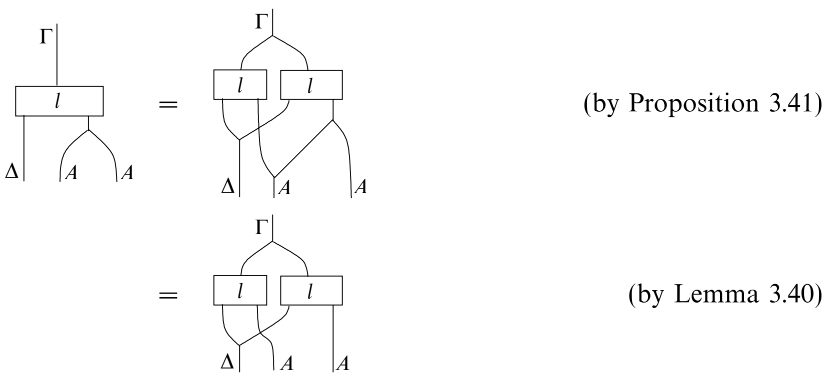


Proof. We have

$$\begin{aligned}(\blacktriangledown \otimes id) \circ (id \otimes \blacktriangle) &= (\blacktriangledown \otimes id) \circ (id \otimes id \otimes id) \circ (id \otimes \blacktriangle)\\ &\leqslant (\blacktriangledown \otimes id) \circ (id \otimes e \otimes id) \circ (id \otimes \blacktriangle) \quad \text{(because, by previous lemmas, } e \text{ is the neutral/greatest element and } \circ \text{ and } \otimes \text{ are monotonic in both arguments).}\\ &= (\blacktriangledown \otimes id) \circ (id \otimes ([] \circ \langle\rangle) \otimes id) \circ (id \otimes \blacktriangle)\\ &= id. \qquad \text{(by } []\mathit{neutral} \text{ and } \langle\rangle\mathit{neutral}\text{)}\end{aligned}$$

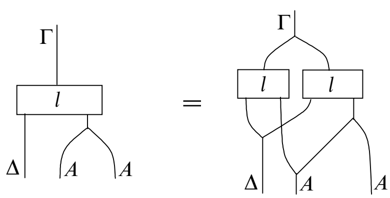
□

Proposition 3.41 below implies that, in every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, the law *Reduce*CL can be split into two steps. The first step is the *equality* stated in the proposition, and the second step is an *inequality* that results from applying Lemma 3.40:



(however, the net on the right-hand side of the first equation is not generally the denotation of a sequent proof).

Proposition 3.41. In every symmetric monoidal category with symmetric monoids and symmetric comonoids that satisfies Equation 3, we have



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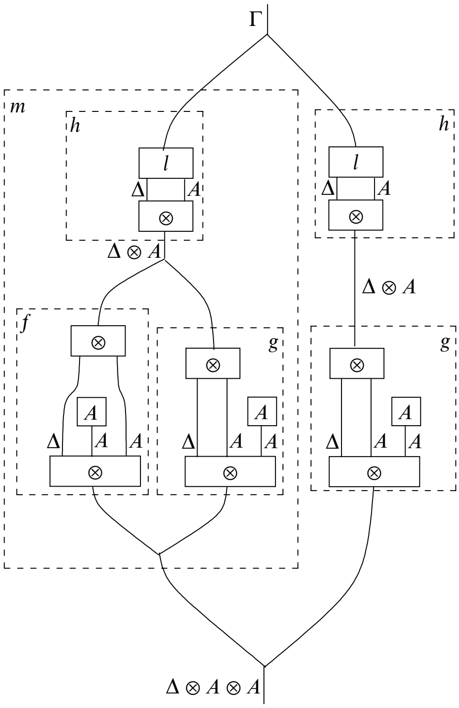
Proof. We use Equation 3 with $k = id$ and f, g, h such that the left-hand side looks as follows:

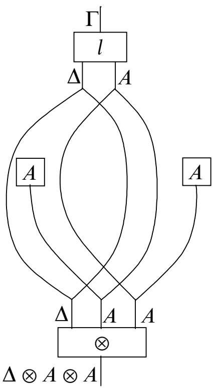
$n \quad =$



(6)

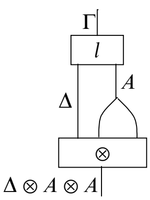
First we focus on the morphism given by the subnet m, which is the right-hand side of Equation 3. We have

$m =$



(by CL-*pointwise*, CR-*pointwise*, and *Reduce*⊗)

$=$



(by WR-*neutral* and, by Lemma 3.38, $\blacktriangledown \circ \blacktriangle = id * id = id$)

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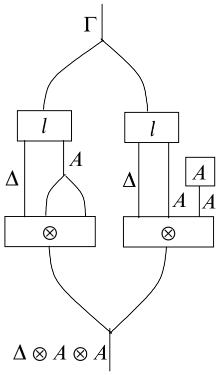
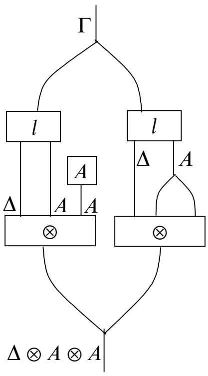
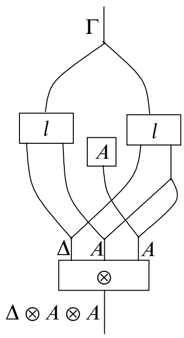
By simplifying m accordingly in the net n, we get

$n =$ 

$=$ 

(by CL-*symm* and CR-*symm*)

$=$ 

(by CR-*pointwise*)

By WR-*neutral*, this is equal to the right-hand side of Equation 6. So the right-hand side of Equation 6 is equal to m, which by Equation 3 is equal to n, which, as we have just shown, is equal to the left-hand side of Equation 6. □

Proposition 3.42 (Hasegawa). A symmetric monoidal category with symmetric monoids and symmetric comonoids forms a Dummett category if and only if Equation 3 holds.

Proof. Let **C** be a symmetric monoidal category with monoids and comonoids. For the 'only if' direction, suppose that **C** forms a Dummett category. The $\leqslant$ direction of Equation 3 is trivial, because the semilattice operation $*$ is the greatest lower bound with

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respect to $\leqslant$. The $\geqslant$ direction holds because

$$\begin{aligned}
k \circ (f * g) \circ h &= k \circ ((f * g) * g) \circ h \\
&= k \circ \blacktriangledown \circ ((f * g) \otimes g) \circ \blacktriangle \circ h \\
&\leqslant \blacktriangledown \circ (k \otimes k) \circ ((f * g) \otimes g) \circ (h \otimes h) \circ \blacktriangle && \text{(by } \blacktriangle lax \text{ and } \blacktriangledown lax\text{)} \\
&= (k \circ (f * g) \circ h) * (k \circ g \circ h).
\end{aligned}$$

For the 'if' direction, suppose **C** satisfies Equation 3. The monotonicity of $\circ$ and $\otimes$ in both arguments follows from Lemma 3.39. The law *Reduce*CL holds because of Proposition 3.41 and Lemma 3.40, and dually for *Reduce*CR. □

3.4.3. *Dummett categories with finite biproducts* In this section, we discuss the special case of compact Dummett categories where the tensor/cotensor is a biproduct. That is, Dummett categories in which $\otimes = \oplus$ is:

— the cartesian product, with diagonals and projections given by $\blacktriangle$ and $\langle\rangle$;
— the cartesian coproduct, with codiagonals and coprojections given by $\blacktriangledown$ and $[]$, and

$\bot = \top$ is the zero (that is, initial and terminal) object.

The following proposition shows that such categories have a very simple axiomatisation.

Proposition 3.43 (Hasegawa). A category with finite biproducts forms a Dummett category (with the biproduct as the tensor/cotensor) if and only if $\blacktriangle \circ \blacktriangledown = id$.

Proof. Let **C** be a category with finite biproducts. If **C** is a Dummett category, we have

$$\begin{aligned}
\blacktriangledown \circ \blacktriangle &= id * id \\
&= id. && \text{(by Lemma 3.25)}
\end{aligned}$$

If **C** satisfies the equation $\blacktriangledown \circ \blacktriangle$, then Equation 3 holds because:

$$\begin{aligned}
(k \circ (f * g) \circ h) * (k \circ g \circ h) &= \blacktriangle \circ ((k \circ (\blacktriangledown \circ (f \times g) \circ \blacktriangle) \circ h) \times (k \circ g \circ h)) \circ \blacktriangledown \\
&= k \circ \blacktriangle \circ (f \times (\blacktriangledown \circ \blacktriangle)) \circ (id \times g) \circ \blacktriangledown \circ h \quad \text{(by calculations} \\
&\qquad\qquad \text{that hold in every category with biproducts)} \\
&= k \circ \blacktriangle \circ (f \times id) \circ (id \times g) \circ \blacktriangledown \circ h \\
&= k \circ (f * g) \circ h.
\end{aligned}$$

□

Remark 3.44. We have already seen in Example 3.16 that $(\mathbf{Rel}, \uplus)$ is a Dummett category with finite biproducts. Proposition 3.43 makes clear that we could have checked this simply by verifying the equation $\blacktriangledown \circ \blacktriangle = id$, which obviously holds.

When dealing with categories with finite biproducts, we follow common practice and write:

— $\oplus$ (not $\otimes$) for the biproduct;
— $\bot$ (not $\top$) for the zero object;
— $+$ instead of $*$; and
— 0 instead of $e = [] \circ \langle\rangle$.

Given objects $A_1, \cdots, A_n$ and $B_1, \cdots, B_m$, and maps $f_{lk} : A_l \longrightarrow B_k$ for $l \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m\}$, we write

$$\begin{pmatrix} f_{11} & \cdots & f_{n1} \\ \vdots & & \vdots \\ f_{1m} & \cdots & f_{nm} \end{pmatrix}$$

for the unique morphism $f : A_1 \oplus \cdots \oplus A_n \longrightarrow B_1 \oplus \cdots \oplus B_m$ such that $\pi_k \circ f \circ \iota_l = f_{lk}$. It is easy to see that composition agrees with matrix multiplication.

The homomorphism analysis carried out for $(\mathbf{Rel}, \uplus)$ in Section 3.2.1 can be generalised to *all* Dummett categories with finite biproducts. Let $f : \Gamma \longrightarrow \Delta \oplus A$ be a homomorphism of a category with finite biproducts. Both the redex and the reduct of *Reduce*CL turn out to be

$$\left(\begin{array}{c|c} & \Gamma \\ \hline \Delta & f_{\Gamma\Delta} \\ A & f_{\Gamma A} \\ A & f_{\Gamma A} \end{array}\right)$$

So every $f : \Gamma \longrightarrow \Delta \oplus A$ is a parametrised cosemigroup homomorphism. The redex and reduct of *Reduce*WL turn out to be given by the matrices

$$\left(\begin{array}{c|cc} & \Gamma & B \\ \hline \Delta & f_{\Gamma\Delta} & 0 \\ A & 0 & id_B \end{array}\right) \text{ and } \left(\begin{array}{c|cc} & \Gamma & B \\ \hline \Delta & 0 & 0 \\ A & 0 & id_B \end{array}\right)$$

respectively. So f is a parametrised copointed homomorphism if and only if $f_{\Gamma\Delta} = 0$.

4. Geometry of interaction in the presence of weakening and contraction

The *Geometry of Interaction* (GoI) was introduced by Girard (Girard 1989; Girard 1990; Girard 1995) in the late 1980s in the context of modelling the dynamics of cut elimination in (classical) linear logic (Girard 1987). The aim was to capture the essential structure of the proof theory of cut elimination while avoiding the semantically inessential aspects of the syntax.

A categorical approach to GoI, based on domain theory and arising from the construction of a categorical model of linear logic, was described in Abramsky and Jagadeesan (1994). Some years later, Abramsky *et al.* presented what can be seen as a general form of the Geometry of Interaction: a compact closed category is constructed from a traced symmetric monoidal category (Abramsky 1996; Abramsky *et al.* 2002). This construction also appeared in Joyal *et al.* (1996).

Many of the ideas contributing to these developments have also been described by Hyland. Beginning in lectures dating from 1992, Hyland has described a range of ideas, from the construction of compact closed categories from what are now called traced monoidal categories, and explaining GoI as a matter of interpreting derivations with cuts in such categories, through to a recent invited paper (Hyland 2004) in which interpretations of contraction and weakening in traced categories with biproducts are also considered.

Recently, Haghverdi and Scott (Haghverdi and Scott 2006) have considered a GoI semantics for multiplicative exponential linear logic based on 'unique decomposition categories'. Their objective is distinct from ours in that they are not concerned with classical logic, and they give an abstract account of Girard's original 'untyped' notion of GoI.

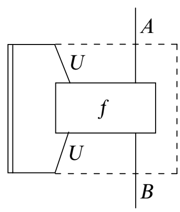
The main contribution of this section is an extended GoI construction that sends a traced Dummett category to a classical category. (This is a generalisation of the GoI construction in Führmann and Pym (2004), where the traced Dummett category had to be a quantaloid with finite biproducts.) This shows that GoI works in the presence of weakening and contraction, even with respect to the partial order that models cut reductions.

In Section 4.1, we introduce traced symmetric monoidal categories as symmetric MIX categories all of whose objects are traced. In Section 4.2, we review the traditional construction of a compact closed category from a traced symmetric monoidal category and present it in terms of nets. In Section 4.3, we extend that construction to traced Dummett categories. In Section 4.4, we study the special case where the starting point of the extended GoI construction is a traced Dummett category with finite biproducts, and carry out a homomorphism analysis in the sense of Section 3.2.1.

4.1. *Traced symmetric MIX categories*

In this section, we recall the notion of a *traced object* in a symmetric MIX category from Blute *et al.* (2000), because our extended GoI construction starts with a Dummett (and therefore symmetric MIX) category $\mathbf{C}$ all of whose objects are traced (in a compatible way). But, as we shall see, a symmetric MIX category in which every object is traced is compact in the sense that all maps $mix_{AB} : A \otimes B \longrightarrow A \oplus B$ and $m : \bot \longrightarrow \top$ are isomorphisms. To make the presentation simpler, we shall assume that these isomorphisms are identities, so $\mathbf{C}$ is simply a traced monoidal category with the extra structure required for a Dummett category. We take this detour via traced objects, as opposed to introducing traced symmetric monoidal categories straight away, to show that assuming compactness in the presence of a trace implies no loss of generality.

An object U of a symmetric MIX category $\mathbf{C}$ is said to have a *trace* if there is a family of functions $tr_U^{AB} : \mathbf{C}(U \otimes A, U \otimes B) \longrightarrow \mathbf{C}(A, B)$ satisfying certain equations that we shall present shortly. Following Blute *et al.* (2000), we write



for the net that represents $tr(f : U \otimes A \longrightarrow U \oplus B)$. We think of the trace of 'feedback along U'. The dashed box indicates the scope of the trace.

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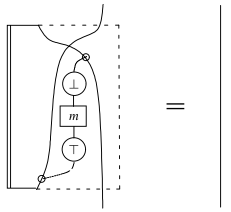
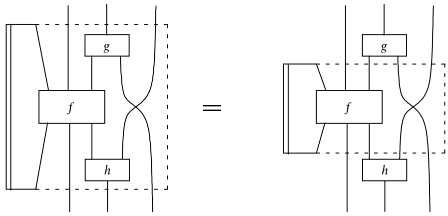
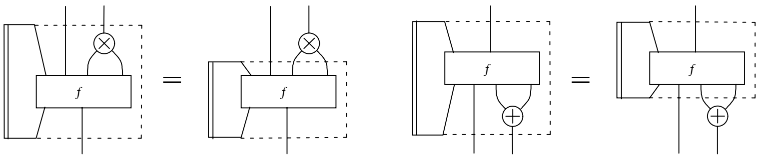
The equational laws, presented in terms of nets, are



(Yanking)



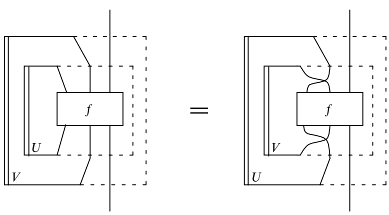
(Tightening)



(Superposing)

The left-hand net in the Yanking law describes a trace over the twisted version of $m : U \otimes U \longrightarrow U \oplus U$, that is, over the map $mix \circ \sigma_{\otimes} = \sigma_{\oplus} \circ mix : U \otimes U \longrightarrow U \oplus U$. The Tightening law lives up to its name and describes how the scope of the trace can be tightened. The Superposing law (called 'Superposing (ii)' in Blute *et al.* (2000)) explains how the scope of the trace can be tightened when the links $\otimes L$ and $\oplus R$ are involved. The categorical-style versions of these laws and a more detailed discussion can be found in Blute *et al.* (2000).

Now we let U and V be objects of a symmetric MIX category **C**, with trace operators tr_U and tr_V, respectively. These traces are called *compatible* if the equation



(Compatibility)

holds for every $f : U \otimes V \otimes A \longrightarrow U \oplus V \oplus B$. Note that compatibility, like Tightening and Superposing, is about manipulating the scopes of traces.

Now let **C** be a symmetric MIX category *some* of whose objects have a trace, and suppose all those traces are compatible. Then it is not hard to show that the laws for Tightening, Superposing and compatibility together imply that the scope of a trace can be extended and contracted arbitrarily (as long as the net stays syntactically correct), so the dashed boxes become unnecessary.

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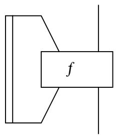
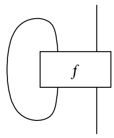
By Blute *et al.* (2000, Proposition 10), every traced object U of a symmetric MIX category $\mathbf{C}$ is in the *core* of $\mathbf{C}$ – that is, for every object A, the map $mix : U \otimes A \longrightarrow U \oplus A$ has an inverse (given by the trace over the distribution $\delta : U \otimes (U \oplus A) \longrightarrow U \oplus (U \otimes A)$). By Blute *et al.* (2000, Proposition 11), if either $\bot$ or $\top$ has a trace, then $m : \bot \longrightarrow \top$ has an inverse (given by the trace over the map $U \otimes \top \cong U \cong U \oplus \bot$, where $U = \bot$ or $U = \top$). Hence, if every object of $\mathbf{C}$ has a trace, as will be the case in our extended GoI construction, then all maps $mix_{AB} : A \otimes B \longrightarrow A \oplus B$ and $m : \bot \longrightarrow \top$ have inverses. To make the presentation simpler, we shall assume that mix and m are identities. Thus, we recover the original notion of a traced monoidal category as a symmetric monoidal category in which every object is traced such that any two traces are compatible. (For a more detailed discussion of this fact, see Blute *et al.* (2000).)

So we shall use the symmetric-monoidal nets described in Section 3.4.1.

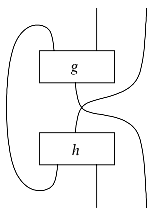
We shall further simplify the notation for the trace by replacing



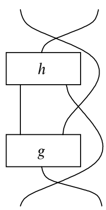
with



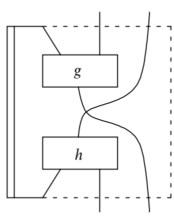
The right-hand circuit introduces no unwanted ambiguity: if the loop is part of a cycle (that is, there is a connection between the loop's entry point into f and the loop's exit point from f), the loop necessarily stands for a trace; otherwise, the loop's entry point is connected with some subnet g of f and the exit point is connected with some subnet h of f such that g and h are not connected:



This can be rewritten as

$k =$ 

The scenario where the loop is a trace, presented in our old notation, is



By Tightening and Yanking, this is equivalent to k.

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So, in the presence of a trace, we can allow wires from an output port to an input port of the *same* circuit. So *every* circuit built from $\otimes L'$, $\otimes R$, $\top L'$ and $\top R$ and other components forms a *traced symmetric monoidal net*. The equations for the trace, even yanking, are built into the syntax of the nets. As in the untraced case, the only equations needed are *Reduce*$\otimes$, *Reduce*$\top$, *Expand*$\otimes$ and *Expand*$\top$.

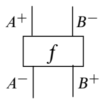
4.2. *The 'traditional' GoI construction*

In this section we describe the traditional construction of a compact closed category from a traced symmetric monoidal category. Our point here is to express this well-known construction in terms of the traced symmetric monoidal nets described in Section 4.1; this helps calculations in Section 4.3.

Given a traced symmetric monoidal category $\mathbf{C}$, the category $\mathscr{G}(\mathbf{C})$ is defined by:

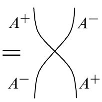
— Objects are pairs (A^+, A^-) of objects of $\mathbf{C}$.

— A morphism $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ of $\mathscr{G}(\mathbf{C})$ is a morphism $f : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ of $\mathbf{C}$:



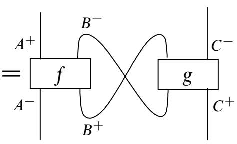
— The identity on (A^+, A^-) is the twist map $A^+ \otimes A^- \cong A^- \otimes A^+$ of $\mathbf{C}$:

$id_{(A^+,A^-)} =$



— The composition of morphisms $(A^+, A^-) \xrightarrow{f} (B^+, B^-) \xrightarrow{g} (C^+, C^-)$ is given by the net

$g \circ f =$



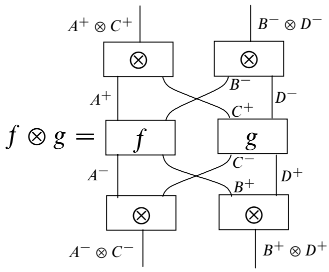
Proposition 4.1. $\mathscr{G}(\mathbf{C})$ is a compact closed category.

The proof of this theorem is well known; however, we shall present our own proof to aid familiarisation with our usage of nets.

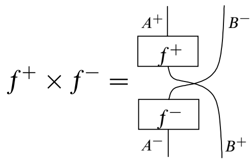
Proof. For objects (A^+, A^-) and (B^+, B^-), we define

$$(A^+, A^-) \otimes (B^+, B^-) = (A^+ \otimes B^+, A^- \otimes B^-).$$

The tensor unit of $\mathscr{G}(\mathbf{C})$ is $(\top, \top)$. For morphisms $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ and $g : (C^+, C^-) \longrightarrow (D^+, D^-)$, we define



The fact that $\otimes : \mathscr{G}(\mathsf{C}) \times \mathscr{G}(\mathsf{C}) \longrightarrow \mathscr{G}(\mathsf{C})$ is a functor follows immediately from the reduction and expansion rules for nets. We now give an auxiliary definition: for morphisms $f^+ : A^+ \longrightarrow B^+$ and $f^- : A^- \longrightarrow B^-$ of $\mathbf{C}$, we define



Note that $\times$ forms a (faithful) functor $\mathbf{C} \times \mathbf{C} \longrightarrow \mathscr{G}(\mathbf{C})$ that preserves the monoidal product, that is

$$(f^+ \otimes g^+) \times (f^- \otimes g^-) = (f^+ \times f^-) \otimes (g^+ \times g^-).$$

The symmetric-monoidal isomorphisms of $\mathscr{G}(\mathbf{C})$ are $\alpha \times \alpha^{-1}$, $\lambda \times \lambda^{-1}$, $\rho \times \rho^{-1}$ and $\sigma \times \sigma^{-1}$. Showing their naturality is straightforward. Their coherence follows immediately from the coherence of the corresponding maps of $\mathbf{C}$ and the fact that $\times$ is a functor that preserves $\otimes$.

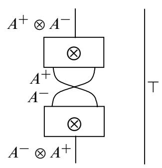
We define

$$(A^+, A^-)^\perp = (A^-, A^+);$$

the map

$$\gamma^R : (A^+, A^-) \otimes (A^+, A^-)^\perp = (A^+ \otimes A^-, A^- \otimes A^+) \longrightarrow (\top, \top) = \top$$

is given by



and dually for τ^R, and symmetrically for γ^L and τ^L. Checking the two equations required for γ and τ is a laborious but routine verification. $\square$

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4.3. *The GoI construction extended to traced Dummett categories*

Our extended GoI construction starts with a Dummett category in which *every* object has a trace such that the traces on any two objects are compatible. As explained in Section 4.1, this causes $m : \bot \longrightarrow \top$ and $mix_{AB} : A \otimes B \longrightarrow A \oplus B$ to be isomorphisms, and to make our lives a little easier, we assume that they are identities. So we have the following definition.

Definition 4.2. A *traced Dummett category* is a traced symmetric monoidal category together with symmetric comonoids and symmetric monoids that satisfies the conditions of a Dummett category.

Example 4.3. $(\mathbf{Rel}, \uplus)$ is a traced Dummett category: the trace of a relation

$$\left(\begin{array}{c|cc} & U & A \\ \hline U & f_{UU} & f_{AU} \\ B & f_{UB} & f_{AB} \end{array}\right) : U \otimes A \longrightarrow U \otimes B$$

is $f_{AB} \cup f_{BU} \circ f^*_{UU} \circ f_{UA} : A \longrightarrow B$, where f^*_{UU} is the reflexive-transitive closure of f_{UU}.

Theorem 4.4. If $\mathbf{C}$ is a traced Dummett category, the compact closed category $\mathscr{G}(\mathbf{C})$ is a classical category.

Proof. The multiplication

$$(A^+ \otimes A^+, A^- \otimes A^-) = (A^+, A^-) \otimes (A^+, A^-) \longrightarrow (A^+, A^-)$$

is $\blacktriangledown_{A^+} \times \blacktriangle_{A^-}$, and the unit

$$(\top, \top) \longrightarrow (A^+, A^-)$$

is $[]_{A^+} \times \langle\rangle_{A^-}$. The laws $\blacktriangledown$*assoc*, $[]$*neutral*, $\blacktriangledown$*symm*, $\blacktriangledown$*pointwise*, $[]$*pointwise* and $[]$*trivial*, result from the corresponding laws for the monoids *and* comonoids of $\mathbf{C}$ and the fact that the functor $\times : \mathbf{C} \times \mathbf{C} \longrightarrow \mathscr{G}(\mathbf{C})$ preserves $\otimes$. We obtain the laws for comonoids on $\mathscr{G}(\mathbf{C})$ dually. So $\mathscr{G}(\mathbf{C})$ is a pre-Dummett category with negation. To turn it into a classical category, we define $f \leqslant g : (A^+, A^-) \longrightarrow (B^+, B^-)$ if and only if $f \leqslant g : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$ holds in $\mathbf{C}$. The monotonicity of $\leqslant$ with respect to $\otimes$ and $\circ$ in $\mathscr{G}(\mathbf{C})$ follows from the same kind of monotonicity of $\leqslant$ in $\mathbf{C}$.

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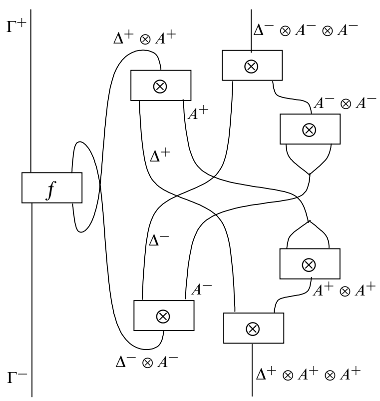
It is easy to check that $\blacktriangledown \circ \blacktriangle = id$ in $\mathscr{G}(\mathbf{C})$, that is, $id * id = id$. Hence, by Lemma 3.30, we have *Reduce*WL and *Reduce*WR.

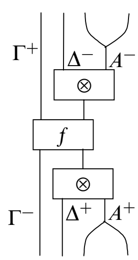
It just remains to check *Reduce*CL and *Reduce*CR. We will check *Reduce*CL. In a compact pre-Dummett category, *Reduce*CL is

$$
\begin{tikzcd}
\Gamma \arrow[r, "\blacktriangle"] \arrow[ddd, "f"'] & \Gamma \otimes \Gamma \arrow[d, "f \otimes f"] \\
& \Delta \otimes A \otimes \Delta \otimes A \arrow[d, "\cong"] \\
\arrow[r, phantom, "\leqslant"] & \Delta \otimes \Delta \otimes A \otimes A \arrow[d, "\blacktriangledown \otimes id_A \otimes id_A"] \\
\Delta \otimes A \arrow[r, "id \otimes \blacktriangle"] & \Delta \otimes A \otimes A
\end{tikzcd}
\tag{7}
$$

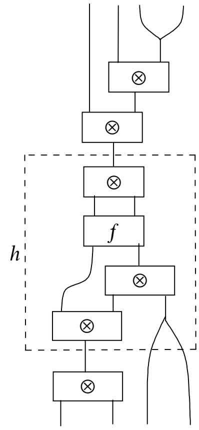
We have directly from the definition of $\mathscr{G}(\mathbf{C})$ that the bottom-left leg is



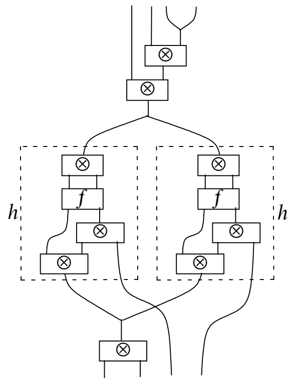
Optimising the layout and removing inessential outermost ⊗-links gives



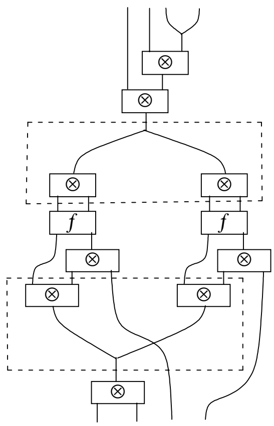
Now we apply two cut reductions for ⊗ backwards and focus on the subnet h:



Applying the law *Reduce*CL to h gives



Now we forget h and focus on two new subnets:



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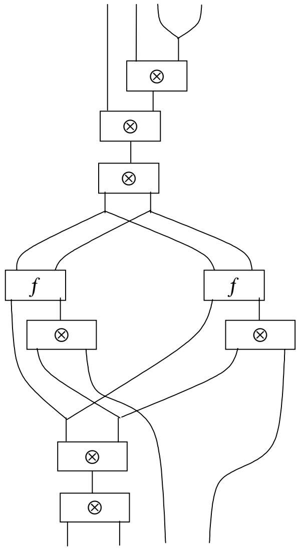
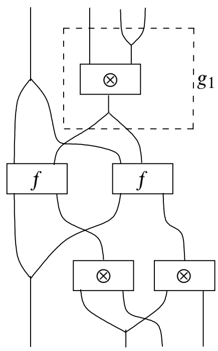
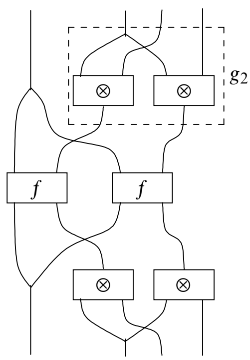
Applying the law CL-*pointwise* to the upper subnet and CR-*pointwise* to the lower subnet gives



After eliminating the two logical cuts, we get



(the subnet g_1 is distinguished purely for later reference).
The definition of $\mathscr{G}(\mathbf{C})$ gives us directly that the top-right leg of Diagram 7 is



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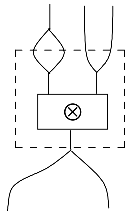
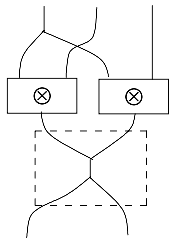
The last two nets differ only in the subnets g_1 and g_2, so it just remains to show that they are equivalent. Because we have $id = id * id = \blacktriangledown \circ \blacktriangle$ in $\mathbf{C}$ (Lemma 3.25), g_1 is equivalent to



Applying the law CR-*pointwise* to the marked subnet gives

$k =$



The subnet in the dashed box is $\blacktriangle \circ \blacktriangledown$. By Lemma A.2 (applied to the compact case where $m_{AB} = id_{AB}$), we have $\blacktriangle \circ \blacktriangledown \leqslant id$. So $k \leqslant g_2$. □

4.4. *GoI for traced categories with finite biproducts*

In this section, we study our extended GoI construction in the case where the traced Dummett category $\mathbf{C}$ is a category with finite biproducts. (Recall that, by Proposition 3.43, a category with finite biproducts is a Dummett category if and only if the equation $\blacktriangledown \circ \blacktriangle = id$ holds.) Using the matrix presentation of morphisms, which is available in the presence of biproducts (recall Section 3.4.3), we obtain a precise characterisation of parametrised (co)pointed homomorphisms and parametrised (co)semigroup homomorphisms (Proposition 4.5). In this way, we gain a complete understanding of the denotational change caused by *Reduce*CL/*Reduce*CR and *Reduce*WL/*Reduce*WR in $\mathscr{G}(\mathbf{C})$. We shall also see that (unparametrised) monoid homomorphism and comonoid homomorphisms are the same in $\mathscr{G}(\mathbf{C})$ (Corollary 4.6), and all denotations of positive (that is, negation-free) derivations or nets are monoid/comonoid homomorphisms (Corollary 4.7).

Without loss of generality, we shall focus on *Reduce*WL and *Reduce*CL. Let $f : \Gamma \longrightarrow \Delta \otimes A$ be a morphism of $\mathscr{G}(\mathbf{C})$, where $\Gamma = (\Gamma^+, \Gamma^-)$, $\Delta = (\Delta^+, \Delta^-)$ and $A = (A^+, A^-)$. We want to characterise when f is a parametrised copointed homomorphism (respectively, parametrised cosemigroup homomorphism), that is, when the laws *Reduce*WL (respectively, *Reduce*CL) hold as equalities. In $\mathbf{C}$, we have $f : \Gamma^+ \otimes \Delta^- \otimes A^- \longrightarrow \Gamma^- \otimes \Delta^+ \otimes A^+$; because of the biproducts, f can be presented as a 3×3-matrix

$$f = \left(\begin{array}{c|ccc} & \Gamma^+ & \Delta^- & A^- \\ \hline \Gamma^- & f_{\Gamma\Gamma} & f_{\Delta\Gamma} & f_{A\Gamma} \\ \Delta^+ & f_{\Gamma\Delta} & f_{\Delta\Delta} & f_{A\Delta} \\ A^+ & f_{\Gamma A} & f_{\Delta A} & f_{AA} \end{array}\right)$$

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Proposition 4.5. Let **C** be a traced category with finite biproducts satisfying the law $\blacktriangledown \circ \blacktriangle = id$. Let

$$f : (\Gamma^+, \Gamma^-) \longrightarrow (\Delta^+, \Delta^-) \otimes (A^+, A^-)$$

be a morphism of $\mathscr{G}(\mathbf{C})$. Then f is

— a parametrised copointed homomorphism if and only if it has the form

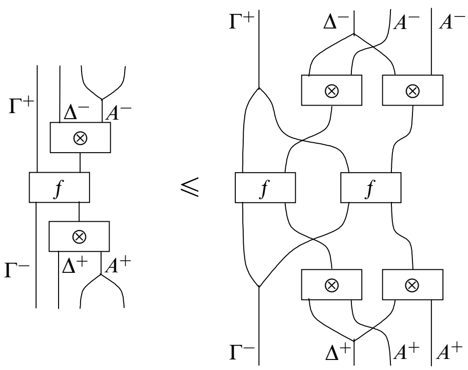
$$f = \begin{array}{cc} & \begin{matrix} \Gamma^+ & \Delta^- & A^- \end{matrix} \\ \begin{matrix} \Gamma^- \\ \Delta^+ \\ A^+ \end{matrix} & \begin{pmatrix} 0 & 0 & f_{A\Gamma} \\ 0 & 0 & f_{A\Delta} \\ f_{\Gamma A} & f_{\Delta A} & f_{AA} \end{pmatrix} \end{array}$$

— a parametrised cosemigroup homomorphism if and only if it has the form

$$f = \begin{array}{cc} & \begin{matrix} \Gamma^+ & \Delta^- & A^- \end{matrix} \\ \begin{matrix} \Gamma^- \\ \Delta^+ \\ A^+ \end{matrix} & \begin{pmatrix} f_{\Gamma\Gamma} & f_{\Delta\Gamma} & f_{A\Gamma} \\ f_{\Gamma\Delta} & f_{\Delta\Delta} & f_{A\Delta} \\ f_{\Gamma A} & f_{\Delta A} & 0 \end{pmatrix} \end{array}$$

The dual statements hold for pointed homomorphisms and semigroup homomorphisms.

Proof. By definition, f is a parametrised cosemigroup homomorphism if it satisfies *Reduce*CL as an equality. As observed in the proof of Theorem 4.4, in $\mathscr{G}(\mathbf{C})$, the law *Reduce*CL boils down to



Translating this into matrix form gives

$$\begin{pmatrix} f_{\Gamma\Gamma} & f_{\Delta\Gamma} & f_{A\Gamma} & f_{A\Gamma} \\ f_{\Gamma\Delta} & f_{\Delta\Delta} & f_{A\Delta} & f_{A\Delta} \\ f_{\Gamma A} & f_{\Delta A} & f_{AA} & f_{AA} \\ f_{\Gamma A} & f_{\Delta A} & f_{AA} & f_{AA} \end{pmatrix} \leqslant \begin{pmatrix} f_{\Gamma\Gamma} & f_{\Delta\Gamma} & f_{A\Gamma} & f_{A\Gamma} \\ f_{\Gamma\Delta} & f_{\Delta\Delta} & f_{A\Delta} & f_{A\Delta} \\ f_{\Gamma A} & f_{\Delta A} & 0 & f_{AA} \\ f_{\Gamma A} & f_{\Delta A} & f_{AA} & 0 \end{pmatrix}$$

This is an equality if and only if $f_{AA} = 0$.

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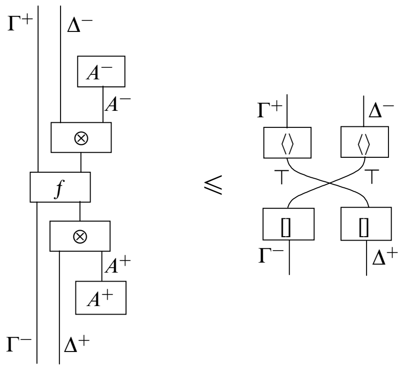
By definition, f is a parametrised copointed homomorphism if it satisfies *Reduce*WL as an equality. It turns out that *Reduce*WL boils down to



in $\mathbf{C}$. Translating this into matrix form gives

$$\left(\begin{array}{c|cc} & \Gamma^+ & \Delta^- \\ \hline \Gamma^- & f_{\Gamma\Gamma} & f_{\Delta\Gamma} \\ \Delta^+ & f_{\Gamma\Delta} & f_{\Delta\Delta} \end{array}\right) \leqslant \left(\begin{array}{c|cc} & \Gamma^+ & \Delta^- \\ \hline \Gamma^- & 0 & 0 \\ \Delta^+ & 0 & 0 \end{array}\right)$$

This is an equality if and only if $f_{\Gamma\Gamma}$, $f_{\Delta\Gamma}$, $f_{\Gamma\Delta}$ and $f_{\Delta\Delta}$ are zero. $\square$

Corollary 4.6. Let $\mathbf{C}$ be a traced category with finite biproducts satisfying the law $\blacktriangledown \circ \blacktriangle = id$. Let $f : (A^+, A^-) \longrightarrow (B^+, B^-)$ be a morphism of $\mathscr{G}(\mathbf{C})$, and let

$$f = \left(\begin{array}{c|cc} & A^+ & B^- \\ \hline A^- & f_{AA} & f_{BA} \\ B^+ & f_{BA} & f_{BB} \end{array}\right) : A^+ \otimes B^- \longrightarrow A^- \otimes B^+$$

be the matrix presentation of f in $\mathbf{C}$. Then the following are equivalent:

— $f_{AA} = 0$;
— f is a copointed homomorphism;
— f is a semigroup homomorphism.

Dually, the following are equivalent:

— $f_{BB} = 0$;
— f is a pointed homomorphism;
— f is a cosemigroup homomorphism.

In particular, f is a monoid homomorphisms if and only if it is a comonoid homomorphism, which is the case if

$$f = \left(\begin{array}{c|cc} & A^+ & B^- \\ \hline A^- & 0 & f_{BA} \\ B^+ & f_{BA} & 0 \end{array}\right)$$

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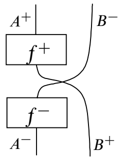
that is, if f is of the form

$$f^+ \times f^- =$$



Corollary 4.7. Let $\mathbf{C}$ be a traced category with finite biproducts that satisfies the equation $\blacktriangledown \circ \blacktriangle = id$. Then all denotations of positive (that is, negation-free) derivations or nets in $\mathscr{G}(\mathbf{C})$ are monoid/comonoid homomorphisms.

Proof. The denotations of all axioms except Ax¬L and Ax¬R are of the form $f^+ \times f^-$, and denotations of the form $f^+ \times f^-$ are closed under ∧L, ∨R, ⊤L, ⊥R, EL, ER and Cut. (Note also that the denotations of Ax¬L (respectively, Ax¬R) are γ^R (respectively, τ^L), as defined in the proof of Proposition 4.1, and it is clear that they are not of the form $f^+ \times f^-$.) □

5. Directions for future work

More non-compact classical categories. We have presented classical categories with non-trivial hom-sets (that is, hom-sets with more than one element) – for example, $(\mathbf{Rel}, \times)$ and $\mathscr{G}(\mathbf{C})$, where $\mathbf{C}$ is a Dummett category (for example, $(\mathbf{Rel}, \uplus)$). However, these models are compact – that is, $\otimes = \oplus$. On the other hand, boolean lattices form classical categories that are not generally compact, but have trivial hom-sets. The product of any two classical categories is a classical category. In particular, $(\mathbf{Rel}, \times) \times \mathbf{B}$, where $\mathbf{B}$ is a boolean lattice, is a non-compact classical category with non-trivial hom-sets. However, what seems to be lacking is a more natural example of a non-compact classical category with non-trivial hom-sets. Categories of games and strategies seem to be natural candidates. Also, the *double gluing* construction (Loader 1994; Tan 1997; Hyland and Schalk 2003) is known to turn compact closed categories into non-compact $*$-autonomous categories (that is, non-compact symmetric linearly distributive categories with negation). It would be interesting to check whether there are circumstances in which the extra structure of a (compact) classical category survives this construction. In other words: can double gluing be extended to classical logic just as we extended GoI to classical logic?

Term calculi and programming It would be interesting to study term calculi for Dummett categories and classical categories.

A classical category is essentially a $*$-autonomous category with symmetric comonoids satisfying certain conditions that result in hom-semilattices. In private communications, Hasegawa has suggested using a modified version of the multiplicative fragment of his lambda calculus DCLL (*Dual Classical Linear Logic*) (Hasegawa 2002). To be precise, his approach is based on the lambda calculus below, which is sound and complete with respect to $*$-autonomous categories with symmetric comonoids:
Types

$$\sigma ::= b \mid \bot \mid \sigma \to \sigma$$

Terms

$$\frac{}{\Gamma_1, x : \sigma, \Gamma_2 \vdash x : \sigma}\,(\mathrm{Ax})$$

$$\frac{\Gamma, x : \sigma_1 \vdash M : \sigma_2}{\Gamma \vdash \lambda x^{\sigma_1}.M : \sigma_1 \to \sigma_2}\,(\to I) \qquad \frac{\Gamma \vdash M : \sigma_1 \to \sigma_2 \quad \Gamma \vdash \sigma_2}{\Gamma \vdash MN : \sigma_2}\,(\to E)$$

$$\frac{}{\Gamma \vdash \mathsf{C}_\sigma : ((\sigma \to \bot) \to \bot) \to \sigma}\,(\neg\neg E)$$

Axioms

$$\begin{array}{llll}
(\beta_{\mathrm{lin}}) & (\lambda x.E[x])N & = E[N] & \\
(\eta) & \lambda x.Mx & = M & (x \notin FV(M)) \\
(\mathsf{C}_1) & L(\mathsf{C}_\sigma M) & = ML & (L : \sigma \to \bot) \\
(\mathsf{C}_2) & \mathsf{C}_\sigma(\lambda k^{\sigma\to\bot}.kM) & = M & (k \notin FV(M)) \\
(\beta_{\mathrm{var}}) & (\lambda x.M)y & = M[y/x]\,. &
\end{array}$$

$E[-]$ stands for a lambda term with a single hole. The laws (C_1) and (C_2) state, essentially, that C_σ is the left and right inverse of the evident lambda term $\sigma \to ((\sigma \to \bot) \to \bot)$. The first four laws characterise $*$-autonomous categories. The law (β_{var}) allows non-linear substitutions, but only if the arguments are variables. This allows us to express the multiplication and unit of the symmetric comonoids as follows.

Derived constructs

$$\begin{aligned}
\top &= \bot \to \bot \\
\sigma_1 \wedge \sigma_2 &= (\sigma_1 \to \sigma_2 \to \bot) \to \bot \\
\sigma_1 \vee \sigma_2 &= (\sigma_1 \to \bot) \to (\sigma_2 \to \bot) \to \bot \\
\langle\rangle_\sigma &= \lambda x^\sigma.\lambda u^\bot.u \\
\blacktriangle_\sigma &= \lambda x^\sigma.\lambda k^{\sigma\to\sigma\to\bot}.kxx \\
&\vdots
\end{aligned}$$

It turns out that the extra axioms required for a classical category can be given as follows:

$$(\sigma) \quad (\lambda x.M)N \preccurlyeq M[N/x]$$

$$\frac{M \preccurlyeq N}{E[M] \preccurlyeq E[N]} \qquad \frac{M \preccurlyeq N \quad N \preccurlyeq M}{M = N}$$

The order $\preccurlyeq$ turns out to be derivable from the hom-semilattice operation

$$\frac{M, N : \sigma \qquad x, k \notin FV(M), FV(N)}{M * N = \mathsf{C}_\sigma(\lambda k^{\sigma\to\bot}.(\lambda x^\bot.kM)(kN)) : \sigma}\,.$$

We believe that it would be interesting to deepen the study of classical categories via this lambda calculus.

However, this calculus can only be used for Dummett categories with negation. Also, its syntax hides the beautiful self-duality of the structure. So it is tempting to devise a self-dual, negation-free term calculus for Dummett categories. Such a calculus might be based

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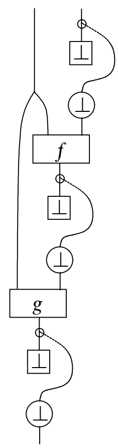
on the *circuit expressions* in Blute *et al.* (1996), on term calculi for the classical sequent calculus along the lines of Curien and Herbelin (2000) and Wadler (2003). Expressions in such calculi can be seen as functional programs with an unspecified evaluation strategy, while MIX introduces an element of parallelism. Lafont's example corresponds to a critical pair that can be resolved by choosing between call-by-value and call-by-name evaluation. In the literature, there seems to be no semantics that models this non-determinism *within one category*. Dummett categories, or something similar, might help here.

Other starting points for a term-language for classical categories might be Filinski's symmetric lambda calculus (Filinski 1989) and the symmetric lambda calculus by Barbanera and Berardi (Barbanera and Berardi 1996).

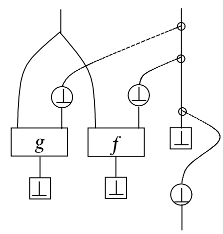
Extending Dummett categories to first-order logic Finally, we should like to mention the possibility of extending our categorical semantics to first-order classical logic. This can be achieved using certain indexed categories whose fibres are classical categories. This idea has been explored in Richard McKinley's doctoral work (McKinley 2006).

Appendix A. Some lemmas and proofs

Proof of Lemma 3.9. Applying $Expand_\perp$ to the left-hand side of Equation 1 yields



By empire rewiring, we get

$h =$ 

By applying similar transformations to the right-hand side of Equation 1, we also get h. (The fact that f and g appear in opposite order is compensated for by the twisted wires in the right-hand side of Equation 1.)

The cases for $n = 0$ and $n \geqslant 2$ are similar. □

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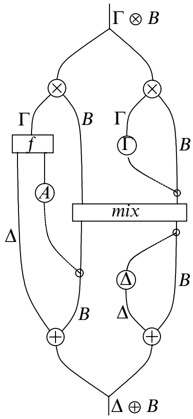
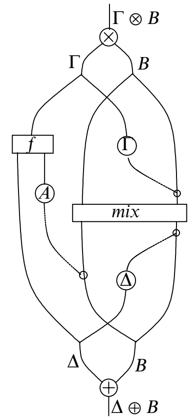
Proof of Lemma 3.30. Let l and r be the left-hand side (respectively, right-hand side) of *Reduce*WL. We show $l * r = r$. We have

$l * r =$

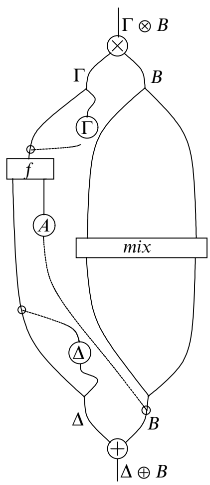


$=$



(by CL-*pointwise* and CR-*pointwise*)

$=$



(by empire rewiring)

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By WL-*neutral* and WR-*neutral*, we can remove the weakenings that introduce Γ and Δ. Because $\blacktriangledown_B \circ mix_{BB} \circ \blacktriangle_B = id_B * id_B = id_B$, we obtain r. $\square$

Lemma A.1. In every pre-Dummett category:

$$(\pi_1^{AB} \otimes \pi_2^{AB}) \circ \blacktriangle_{A\otimes B} = id_{A\otimes B} \tag{8}$$

$$\blacktriangledown_A \circ mix_{AA}^{\langle\rangle} = \pi_1^{AA} * \pi_2^{AA} \tag{9}$$

$$mix_{AB}^{\langle\rangle} = (\iota_1^{AB} \circ \pi_1^{AB}) * (\iota_2^{AB} \circ \pi_2^{AB}) \tag{10}$$

Proof. Equation 8 follows from a routine calculation using the laws $\blacktriangle$*pointwise* and $\langle\rangle$*neutral*. For Equation 9, consider

$$\begin{aligned} \blacktriangledown_A \circ mix_{AA}^{\langle\rangle} &= \blacktriangledown_A \circ mix_{AA}^{\langle\rangle} \circ (\pi_1^{AA} \otimes \pi_2^{AA}) \circ \blacktriangle_{A\otimes A} && \text{(by Equation 8)} \\ &= \pi_1^{AA} * \pi_2^{AA}. && \text{(by definition of } *) \end{aligned}$$

Equation 10 holds because, by the definition of $*$ and the naturality of $mix^{\langle\rangle}$, the morphism $(\iota_1^{AB} \circ \pi_1^{AB}) * (\iota_2^{AB} \circ \pi_2^{AB})$ is equal to

$$\blacktriangledown_{A\oplus B} \circ (\iota_1^{AB} \oplus \iota_2^{AB}) \circ mix_{AB}^{\langle\rangle} \circ (\pi_1^{AB} \otimes \pi_2^{AB}) \circ \blacktriangle_{A\otimes B},$$

which by Equation 8 and its dual is equal to $mix_{AB}^{\langle\rangle}$. $\square$

Lemma A.2. In every Dummett category

$$mix_{AA} \circ \blacktriangle_A \circ \blacktriangledown_A \circ mix_{AA} \leqslant mix_{AA}.$$

Proof. By Equation 9 and its dual, we have

$$mix_{AA} \circ \blacktriangle_A \circ \blacktriangledown_A \circ mix_{AA} = (\iota_1^{AA} * \iota_2^{AA}) \circ (\pi_1^{AA} * \pi_2^{AA}).$$

Because $*$ is the greatest lower bound with respect to $\leqslant$, and because $\circ$ is monotonic in both arguments, we have

$$mix_{AA} \circ \blacktriangle_A \circ \blacktriangledown_A \circ mix_{AA} \leqslant \iota_k^{AA} \circ \pi_k^{AA}$$

for $k \in \{1,2\}$. So

$$mix_{AA} \circ \blacktriangle_A \circ \blacktriangledown_A \circ mix_{AA} \leqslant (\iota_1^{AA} \circ \pi_1^{AA}) * (\iota_2^{AA} \circ \pi_2^{AA}).$$

The claim follows because the right-hand side is equal to mix_{AA} by Equation 10. $\square$

Acknowledgments

We are very grateful to Masahito Hasegawa, who contributed many ideas and results in private communications, but declined to be named as an author. We should also like to thank Robin Cockett, Martin Hyland, François Lamarche, Richard McKinley, Eike Ritter, Phil Scott, Lutz Straßburger, John Power, Edmund Robinson, Robert Seely and Christian Urban for their comments and criticism.

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