

THE p -COHOMOLOGY OF ALGEBRAIC VARIETIES AND SPECIAL VALUES OF ZETA FUNCTIONS

JAMES S. MILNE¹ AND NIRANJAN RAMACHANDRAN²

¹*Mathematics Department, University of Michigan, Ann Arbor, MI 48109, USA*
(jmilne@umich.edu) URL: www.jmilne.org/math/

²*Mathematics Department, University of Maryland, College Park, MD 20742, USA*
(atma@math.umd.edu) URL: www2.math.umd.edu/~atma/

(Received 16 October 2013; revised 4 May 2014; accepted 5 May 2014;
first published online 2 June 2014)

Abstract The p -cohomology of an algebraic variety in characteristic p lies naturally in the category $D_c^b(R)$ of coherent complexes of graded modules over the Raynaud ring (Ekedahl, Illusie, Raynaud). We study homological algebra in this category. When the base field is finite, our results provide relations between the absolute cohomology groups of algebraic varieties, log varieties, algebraic stacks, etc., and the special values of their zeta functions. These results provide compelling evidence that $D_c^b(R)$ is the correct target for p -cohomology in characteristic p .

Keywords: zeta functions; crystalline cohomology

2010 *Mathematics subject classification:* 14G10; 11G20

Contents

1	Coherent complexes of graded R-modules	804
2	The numerical invariants of a coherent complex	809
3	Tensor products in the category $D_c^b(R)$ and internal Homs	814
4	Homological algebra in the category $D_c^b(R)$	815
5	The proof of the main theorem	820
6	Applications to algebraic varieties	828
	References	834

Introduction

Each of the usual cohomology theories $X \rightsquigarrow H^j(X, r)$ on algebraic varieties over a field k arises from a functor $R\Gamma$ taking values in a triangulated category $\mathbf{D}(k)$ equipped with

The second author was partly supported by the NSF and the Graduate Research Board (UMD).

a t -structure and a Tate twist $N \rightsquigarrow N(r)$. The heart of $\mathbf{D}(k)$ has a tensor structure and, in particular, an identity object $\mathbb{1}$. The cohomology theory satisfies

$$H^j(X, r) \simeq H^j(R\Gamma(X)(r)), \tag{1}$$

and there is an absolute cohomology theory

$$H_{\text{abs}}^j(X, r) \simeq \text{Hom}_{\mathbf{D}(k)}(\mathbb{1}, R\Gamma(X)(r)[j]). \tag{2}$$

See, for example, [Deligne \(1994, §3\)](#).

Let k be a perfect base field of characteristic p . For the ℓ -adic étale cohomology, \mathbf{D} is the category of bounded constructible \mathbb{Z}_ℓ -complexes ([Ekedahl, 1990](#)). For the p -cohomology, it is the category $\mathbf{D}_c^b(R)$ of coherent complexes of graded modules over the Raynaud ring. This category was defined in [Illusie and Raynaud \(1983\)](#), and its properties were developed in [Ekedahl \(1984, 1985, 1986\)](#). We study homological algebra in this category and, when k is finite, we prove relations between Exts and zeta functions.

Let $k = \mathbb{F}_q$ with $q = p^a$. The Ext of two objects M, N of $\mathbf{D}_c^b(R)$ is defined by the usual formula

$$\text{Ext}^j(M, N) = \text{Hom}_{\mathbf{D}_c^b(R)}(M, N[j]).$$

Using that k is finite, we construct a canonical complex

$$E(M, N): \quad \dots \rightarrow \text{Ext}^{j-1}(M, N) \rightarrow \text{Ext}^j(M, N) \rightarrow \text{Ext}^{j+1}(M, N) \rightarrow \dots$$

of abelian groups for each pair M, N in $\mathbf{D}_c^b(R)$.

An object P of $\mathbf{D}_c^b(R)$ can be regarded as a double complex of $W_\sigma[F, V]$ -modules. On tensoring P with \mathbb{Q} and forming the associated simple complex, we obtain a bounded complex $sP_{\mathbb{Q}}$ whose cohomology groups $H^j(sP_{\mathbb{Q}})$ are F -isocrystals over k . We define the zeta function $Z(P, t)$ of P to be the alternating product of the characteristic polynomials of F^a acting on these F -isocrystals. It lies in $\mathbb{Q}_p(t)$.

Attached to each P in $\mathbf{D}_c^b(R)$ there is a bounded complex $R_1 \otimes_R^L P$ of graded k -vector spaces whose cohomology groups have finite dimension (see § 1.1 below for the precise definitions of R and R_1). The Hodge numbers $h^{i,j}(P)$ of P are defined to be the dimensions of the k -vector spaces $H^j(R_1 \otimes_R^L P)^i$.

Finally, we let $\underline{R\text{Hom}}(-, -)$ denote the internal Hom in $\mathbf{D}_c^b(R)$ (see § 3 below for the definition of $\underline{R\text{Hom}}(-, -)$).

Theorem 0.1. *Let $M, N \in \mathbf{D}_c^b(R)$, and let $P = \underline{R\text{Hom}}(M, N)$. Let $r \in \mathbb{Z}$, and assume that q^r is not a multiple root of the minimum polynomial of F^a acting on $H^j(sP_{\mathbb{Q}})$ for any integer j .*

- (a) *The groups $\text{Ext}^j(M, N(r))$ are finitely generated \mathbb{Z}_p -modules, and the alternating sum of their ranks is zero.*
- (b) *The zeta function $Z(P, t)$ of P has a pole at $t = q^{-r}$ of order*

$$\rho = \sum_j (-1)^{j+1} \cdot j \cdot \text{rank}_{\mathbb{Z}_p}(\text{Ext}^j(M, N(r))).$$

- (c) The cohomology groups of the complex $E(M, N(r))$ are finite, and the alternating product of their orders $\chi(M, N(r))$ satisfies

$$\left| \lim_{t \rightarrow q^{-r}} Z(P, t) \cdot (1 - q^r t)^\rho \right|_p^{-1} = \chi(M, N(r)) \cdot q^{\chi(P,r)},$$

where

$$\chi(P, r) = \sum_{i,j (i \leq r)} (-1)^{i+j} (r - i) \cdot h^{i,j}(P).$$

Here, $|\cdot|_p$ is the p -adic valuation, normalized so that $|p^r \frac{m}{n}|_p^{-1} = p^r$ if m and n are prime to p .

We identify the identity object of $D_c^b(R)$ with the ring W of Witt vectors. Then $\text{RHom}(W, N) \simeq N$.

Each algebraic variety (or log variety or stack) over k defines several objects in $D_c^b(R)$ (see § 6). Let $M(X)$ be one of the objects of $D_c^b(R)$ attached to an algebraic variety X over k , and define the absolute cohomology of X to be

$$H_{\text{abs}}^j(X, \mathbb{Z}_p(r)) = \text{Hom}_{D_c^b(R)}(W, M(X)(r)[j]).$$

The complex $E(W, M(X)(r))$ becomes

$$E(X, r): \quad \dots \rightarrow H_{\text{abs}}^{j-1}(X, \mathbb{Z}_p(r)) \rightarrow H_{\text{abs}}^j(X, \mathbb{Z}_p(r)) \rightarrow H_{\text{abs}}^{j+1}(X, \mathbb{Z}_p(r)) \rightarrow \dots$$

Theorem 0.2. Assume that q^r is not a multiple root of the minimum polynomial of F^a acting on $H^j(sM(X)_{\mathbb{Q}})$ for any j .

- (a) The groups $H_{\text{abs}}^j(X, \mathbb{Z}_p(r))$ are finitely generated \mathbb{Z}_p -modules, and the alternating sum of their ranks is zero.
- (b) The zeta function $Z(M(X), t)$ of $M(X)$ has a pole at $t = q^{-r}$ of order

$$\rho = \sum_j (-1)^{j+1} \cdot j \cdot \text{rank}_{\mathbb{Z}_p} \left(H_{\text{abs}}^j(X, \mathbb{Z}_p(r)) \right).$$

- (c) The cohomology groups of the complex $E(X, r)$ are finite, and the alternating product of their orders $\chi(X, \mathbb{Z}_p(r))$ satisfies

$$\left| \lim_{t \rightarrow q^{-r}} Z(M(X), t) \cdot (1 - q^r t)^\rho \right|_p^{-1} = \chi(X, \mathbb{Z}_p(r)) \cdot q^{\chi(M(X),r)}.$$

Let X be a smooth projective variety over k , and let $M(X) = R\Gamma(X, W\Omega_X^\bullet)$. Then $H^j(sM(X)_{\mathbb{Q}}) = H_{\text{crys}}^j(X/W)_{\mathbb{Q}}$, and in an earlier article the authors showed that $H_{\text{abs}}^j(X, \mathbb{Z}_p(r))$ is the group $H^j(X, \mathbb{Z}_p(r))$ defined in terms of logarithmic de Rham–Witt differentials (see § 4.1 below). Moreover, the zeta function and the Hodge numbers of $M(X)$ agree with those of X , and so, in this case, Theorem 0.2 becomes the p -part of the main theorem of Milne (1986). See p. 828 below.

Remarks

0.3. Let $\zeta(P, s) = Z(P, q^{-s})$, $s \in \mathbb{C}$. Then ρ is the order of the pole of $\zeta(P, s)$ at $s = r$, and

$$\lim_{t \rightarrow q^{-r}} Z(P, t) \cdot (1 - q^r t)^\rho = \lim_{s \rightarrow r} \zeta(P, s) \cdot (s - r)^\rho \cdot (\log q)^\rho.$$

0.4. We expect that the F -isocrystals $H^j(sP_{\mathbb{Q}})$ are always semisimple (so F^a always acts semisimply) when P arises from algebraic geometry. If this fails, there will be spurious extensions over \mathbb{Q} that will have to be incorporated into the statement of (0.1).

0.5. The statement of Theorem 0.1 depends only on $D_c^b(R)$ as a triangulated category with a dg-lifting.

0.6. We leave it as an (easy) exercise for the reader to prove the analogue of (0.1) for $\ell \neq p$ (the indolent may refer to the article below, namely, to [Milne and Ramachandran \(2013\)](#)).

0.7. In a second article, we apply (0.1) to study the analogous statement in a triangulated category of motivic complexes ([Milne and Ramachandran, 2013](#)).

Outline of the article

In § 1 and § 3, we review some of the basic theory of the category $D_c^b(R)$ (Ekedahl, Illusie, Raynaud), and in § 2 we prove a relation between the numerical invariants of an object of $D_c^b(R)$. In § 4, we begin the study of the homological algebra of $D_c^b(R)$, and in § 5 we take the ground field to be finite and prove Theorem 0.1. In § 6, we study applications of Theorem 0.1 to algebraic varieties.

Notation

Throughout, k is a perfect field of characteristic $p \neq 0$, and W is the ring of Witt vectors over k . As usual, σ denotes the automorphism of W inducing $a \mapsto a^p$ modulo p . We use a bar to denote base change to an algebraic closure \bar{k} of k . For example, \overline{W} denotes the Witt vectors over \bar{k} . We use \simeq to denote a canonical, or specific, isomorphism.

1. Coherent complexes of graded R -modules

In this section, we review some definitions and results of Ekedahl, Illusie, and Raynaud, for which [Illusie \(1983\)](#) is a convenient reference.

1.1. The *Raynaud ring* is the graded W -algebra $R = R^0 \oplus R^1$ generated by F and V in degree 0 and d in degree 1, subject to the relations

$$FV = p = VF, \quad Fa = \sigma a \cdot F, \quad aV = V \cdot \sigma a, \tag{3}$$

$$d^2 = 0, \quad FdV = d, \quad ad = da \quad (a \in W). \tag{4}$$

In other words, R^0 is the Dieudonné ring $W_\sigma[F, V]$, and R is generated as an R^0 -algebra by a single element d of degree 1 satisfying (4). For $m \geq 1$,

$$R_m \stackrel{\text{def}}{=} R/(V^m R + dV^m R). \tag{5}$$

1.2. To give a graded R -module $M = \bigoplus_{i \in \mathbb{Z}} M^i$ is the same as giving a complex

$$M^\bullet: \dots \rightarrow M^{i-1} \xrightarrow{d} M^i \xrightarrow{d} M^{i+1} \rightarrow \dots$$

of W -modules whose components M^i are R^0 -modules and whose differentials d satisfy $FdV = d$. For $n \in \mathbb{Z}$, $M\{n\}$ is the graded R -module deduced from M by a shift of degree,¹ i.e., $M\{n\}^i = M^{n+i}$ and $d_{M\{n\}}^i = (-1)^n d_M^{n+i}$. The graded R -modules and graded homomorphisms of degree 0 form an abelian category $\mathbf{Mod}(R)$ with derived category $\mathbf{D}(R)$. The bifunctor $M, N \rightsquigarrow \text{Hom}(M, N)$ of graded R -modules derives to a bifunctor

$$R \text{Hom}: \mathbf{D}(R)^{\text{opp}} \times \mathbf{D}^+(R) \rightarrow \mathbf{D}(\mathbb{Z}_p)$$

(denoted by $R \text{Hom}_R$ in Illusie (1983, 2.6.2), and Ekedahl (1986, p. 8), and by $R\text{Hom}_R$ in Ekedahl (1985, p. 73)).

1.3. A graded R -module is said to be *elementary* (Illusie, 1983, 2.2.2, p. 30) if it is one of the following two types.

Type I The module is concentrated in degree zero, finitely generated over W , and V is topologically nilpotent on it. In other words, it is a $W_\sigma[F, V]$ -module whose p -torsion submodule has finite length over W , and whose torsion-free quotient is finitely generated and free over W with slopes lying in the interval $[0, 1[$ (we review slopes in 5.3 below).

Type II The module is isomorphic to

$$U_l: \prod_{\substack{n \geq 0 \\ \text{deg } 0}} kV^n \xrightarrow{d} \prod_{\substack{n \geq 1 \\ \text{deg } 1}} kdV^n$$

for some $l \in \mathbb{Z}$. Here F (respectively, V) acts as zero on U_l^0 (respectively, U_l^1), and dV^n should be interpreted as $F^{-n}d$ when $n < 0$. In more detail, U_l^0 is the $W_\sigma[F, V]$ -module $k[[V]]$ with F acting as zero. When $l \geq 0$, U_l^1 consists of the formal sums

$$a_l dV^l + a_{l+1} dV^{l+1} + \dots \quad (a_i \in k),$$

and when $l < 0$, U_l^1 consists of the formal sums

$$a_{-l} F^{-l} d + \dots + a_{-1} F^{-1} d + a_0 d + a_1 dV + a_2 dV^2 + \dots \quad (a_i \in k).$$

1.4. A graded R -module M is said to be *coherent* if it admits a finite filtration $M \supset \dots \supset 0$ whose quotients are degree shifts of elementary modules (i.e., of the form $M\{n\}$ with M elementary and $n \in \mathbb{Z}$). Coherent R -modules need not be noetherian or artinian—the object U_0 is obviously neither.

¹Illusie et al. write $M(n)$ for the degree shift of M , but this conflicts with our notation for Tate twists.

1.5. A complex M of R -modules is said to be *coherent* if it is bounded with coherent cohomology. Let $D_c^b(R)$ denote the full subcategory of $D(R)$ consisting of coherent complexes. Ekedahl has given a criterion for a complex to lie in $D_c^b(R)$, from which it follows that $D_c^b(R)$ is a triangulated subcategory of $D(R)$; in particular, the coherent modules form an abelian subcategory of $\text{Mod}(R)$ closed under extensions (Illusie, 1983, 2.4.8). In more detail (ibid. 2.4), define a graded R_\bullet -module to be a projective system

$$M_\bullet = (M_1 \leftarrow \cdots \leftarrow M_m \leftarrow M_{m+1} \leftarrow \cdots)$$

equipped with maps $F: M_{m+1} \rightarrow M_m$ and $V: M_m \rightarrow M_{m+1}$ of degree zero satisfying (3) and (4); here, M_m is a graded $W_m[d]$ -module. The graded R_\bullet -modules form an abelian category. The functor $M_\bullet \rightsquigarrow \varprojlim M_m: \text{Mod}(R_\bullet) \rightarrow \text{Mod}(R)$ derives to a functor

$$R \varprojlim: D(R_\bullet) \rightarrow D(R).$$

On the other hand, the functor sending a graded R -module M to the R_\bullet -module $(R_m \otimes_R M)_{m \geq 1}$ derives to a functor

$$R_\bullet \otimes_R^L -: D(R) \rightarrow D(R_\bullet).$$

These functors compose to a functor

$$M \rightsquigarrow \widehat{M}: D(R) \rightarrow D(R).$$

For M in $D^-(R)$, there is a natural map $M \rightarrow \widehat{M}$ inducing isomorphisms $R_m \otimes_R^L M \rightarrow R_m \otimes_R^L \widehat{M}$ for all m , and M is said to be *complete* if this map is an isomorphism. Ekedahl’s criterion states: A bounded complex of graded R -modules M lies in $D_c^b(R)$ if and only if M is complete and $R_1 \otimes_R^L M$ is a bounded complex such that $H^i(R_1 \otimes_R^L M)$ is finite dimensional over k for all i .

1.6. Following Illusie (1983, 2.1), we view a complex of graded R -modules

$$M: \cdots \rightarrow M^{\bullet,j} \rightarrow M^{\bullet,j+1} \rightarrow \cdots$$

as a bicomplex $M^{\bullet,\bullet}$ of R^0 -modules in which the first index corresponds to the R -gradation. Thus the j th row $M^{\bullet,j}$ of the bicomplex is a graded R -module and the i th column $M^{i,\bullet}$ is a complex of R^0 -modules:

$$\begin{array}{ccccccc}
 \vdots & & \vdots & & \vdots & & \vdots \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 M^{\bullet,j+1} : & \cdots \longrightarrow & M^{i-1,j+1} & \xrightarrow{d} & M^{i,j+1} & \xrightarrow{d} & M^{i+1,j+1} \longrightarrow \cdots \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 M^{\bullet,j} : & \cdots \longrightarrow & M^{i-1,j} & \xrightarrow{d} & M^{i,j} & \xrightarrow{d} & M^{i+1,j} \longrightarrow \cdots \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \vdots & & \vdots & & \vdots & & \vdots
 \end{array} \tag{6}$$

In this diagram, the squares commute, the vertical differentials commute with F and V , and the horizontal differentials satisfy $FdV = d$. The cohomology modules of M are obtained by passing to the cohomology in the columns:

$$H^j(M): \quad \dots \rightarrow H^j(M^{i-1,\bullet}) \xrightarrow{d} H^j(M^{i,\bullet}) \xrightarrow{d} H^j(M^{i+1,\bullet}) \rightarrow \dots .$$

In other words, for a complex $M = M^{\bullet,\bullet}$ of graded R -modules, $H^j(M)$ is the graded R -module with $H^j(M)^i = H^j(M^{i,\bullet})$.

By definition, $M\{m\}[n]$ is the bicomplex with

$$(M\{m\}[n])^{i,j} = M^{i+m,j+n} \tag{7}$$

and with the appropriate sign changes on the differentials.

1.7. Let \mathcal{T} be the functor of graded R -modules such that $(\mathcal{T}M)^i = M^{i+1}$ and $\mathcal{T}(d) = -d$, i.e., $\mathcal{T}M = M\{1\}$ (degree shift). It is exact and defines a self-equivalence $\mathcal{T}: \mathbf{D}_c^b(R) \rightarrow \mathbf{D}_c^b(R)$. The *Tate twist* of a coherent complex of graded R -modules M is defined as

$$M(r) = \mathcal{T}^r(M)[-r] = M\{r\}[-r];$$

thus $M(r)^{i,j} = M^{i+r,i-r}$ (cf. Milne and Ramachandran, 2005, § 2).

1.8. With any complex M of graded R -modules, there is an associated simple complex sM of W -modules with

$$(sM)^n = \bigoplus_{i+j=n} M^{i,j}, \quad dx^{ij} = d'x^{ij} + (-1)^i d''x^{ij}.$$

The functor s extends to a functor $s: \mathbf{D}^+(R) \rightarrow \mathbf{D}(W)$. If $M \in \mathbf{D}_c^b(M)$, then sM is a perfect complex of W -modules (Illusie, 1983, p. 34).

1.9. For a coherent complex M of graded R -modules, the filtration of sM by the first degree defines a spectral sequence

$$E_1^{ij} = H^j(M)^i \implies H^{i+j}(sM) \tag{8}$$

called the *slope spectral sequence*. The slope spectral sequence degenerates at E_1 modulo torsion and at E_2 modulo W -modules of finite length. In particular, for $r \geq 2$, E_r^{ij} is a finitely generated W -module of rank equal to that of $H^j(M)^i/\text{torsion}$. This was proved by Bloch (1977) and Illusie and Raynaud (1983) for the complex $M = R\Gamma(X, W\Omega_X^\bullet)$ attached to a smooth complete variety X , and by Ekedahl for a general M (see Illusie, 1983, 2.5.4).

1.10. Let $K = W \otimes \mathbb{Q}$ (field of fractions of W). Then $K \otimes_W W_\sigma[F, V] \simeq K_\sigma[F]$. Recall that an F -isocrystal over k is a $K_\sigma[F]$ -module that is finite dimensional as a K -vector space and such that F is bijective. The F -isocrystals form an abelian subcategory of $\text{Mod}(K_\sigma[F])$ closed under extensions, and so the subcategory $\mathbf{D}_{\text{iso}}^b(K_\sigma[F])$ of $\mathbf{D}^b(K_\sigma[F])$ consisting of bounded complexes whose cohomology modules are F -isocrystals is triangulated.

1.11. Let M be a complex of graded R -modules with only nonnegative first degrees, and let F' act on $M^{i,j}$ as $p^i F$. The condition $FdV = d$ implies that $pFd = dF$, and so both differentials in the diagram (6) commute with the action of F' . Therefore $s(M)$ is a complex of $W_\sigma[F']$ -modules. If $M \in D_c^b(R)$, then $s(M)_K$ lies in $D_{\text{iso}}^b(K_\sigma[F'])$. From the degeneration of the slope spectral sequence at E_1 , we get isomorphisms

$$(H^j(M)_K^i, p^i F) \simeq (H^{i+j}(sM)_K, F)_{[i, i+1[} \tag{9}$$

for $M \in D_c^b(R)$. This can also be written²

$$(H^{n-i}(M)_K^i, p^i F) \simeq (H^n(sM)_K, F)_{[i, i+1[}. \tag{10}$$

1.12. A *domino* N is a graded R -module that admits a finite filtration $N \supset \dots \supset 0$ whose quotients are elementary of type II.

Let N be elementary of type II, say $N = U_l$. Then $N^0 = k_\sigma[[V]]$, and so $V: N^0 \rightarrow N^0$ is injective with cokernel $N^0/V = k_\sigma[[V]]/(V) \simeq k$. Similarly, $F: N^1 \rightarrow N^1$ is surjective with kernel $k dV^l$ ($l \geq 0$) or $kF^{-l}d$ ($l < 0$).

Let N be a domino, and suppose that N admits a filtration of length $l(N)$ with elementary quotients. Induction on $l(N)$ shows that

- (a) the map $V: N^0 \rightarrow N^0$ is injective with cokernel of dimension $l(N)$ (as a k -vector space) and $F|N^0$ is nilpotent;
- (b) the map $F: N^1 \rightarrow N^1$ is surjective with kernel of dimension $l(N)$ and $V|N^1$ is nilpotent.

Therefore the number of quotients in such a filtration is independent of the filtration, and equals the common dimension of the k -vector spaces N^0/V and of $\text{Ker}(F: N^1 \rightarrow N^1)$. This number is called the *dimension* of N .

1.13. Let M be a graded R -module. Then $Z^i(M) \stackrel{\text{def}}{=} \text{Ker}(d: M^i \rightarrow M^{i+1})$ is stable under F but not in general under V , whereas $B^i(M) \stackrel{\text{def}}{=} \text{Im}(d: M^{i-1} \rightarrow M^i)$ is stable under V but not in general under F . Instead, one puts

$$V^{-\infty}Z^i(M) = \{x \in M^i \mid V^n x \in Z^i(M) \text{ for all } n\},$$

$$F^\infty B^i(M) = \{x \in M^i \mid x \in F^n B^i(M) \text{ for some } n\}.$$

Then $V^{-\infty}Z^i(M)$ is the largest R^0 -submodule of $Z^i(M)$, and $F^\infty B^i(M)$ is the smallest R^0 -submodule of M^i containing $B^i(M)$:

$$B^i(M) \subset F^\infty B^i(M) \subset V^{-\infty}Z^i(M) \subset Z^i(M). \tag{11}$$

²For each n , we have $H^n(sM)_K = \bigoplus H^j(M)_K^i$, where the sum is over pairs (i, j) with $i + j = n$. Our assumption on M says that $i \geq 0$, and so only $H^n(M)_K^0, H^{n-1}(M)_K^1, \dots, H^0(M)_K^n$ contribute. Each of these (with the map F) is an isocrystal with slopes $[0, 1[$. But with the map $p^i F$, the slopes of $H^{n-i}(M)_K^i$ are in $[i, i + 1[$. The slopes of distinct summands do not overlap. Hence we get (10). Cf. Illusie (1983, p. 64).

The homomorphism of W -modules $d: M^i \rightarrow M^{i+1}$ factors as

$$\begin{array}{ccc}
 M^i & \xrightarrow{d} & M^{i+1} \\
 \downarrow & & \uparrow \\
 M^i / V^{-\infty} Z^i(M) & \xrightarrow{d} & F^\infty B^{i+1}(M)
 \end{array} \tag{12}$$

When M is coherent, the lower row in (12) is an R -module admitting a finite filtration whose quotients are of the form $U_l\{-i\}$; in other words, (lower row) $\{i\}$ is a domino (Illusie, 1983, 2.5.2).

1.14. The *heart* of a graded R -module M is the graded R_0 -module $\heartsuit(M) = \bigoplus \heartsuit^i(M)$ with $\heartsuit^i(M) = V^{-\infty} Z^i(M) / F^\infty B^i(M)$ (see (11)). When M is coherent, $\heartsuit(M)$ is finitely generated as a W -module; moreover, $Z^i(M) / V^{-\infty} Z^i(M)$ and $F^\infty B^i(M) / B^i(M)$ are of finite length, and so

$$\heartsuit^i(M)_K \simeq \left(Z^i(M) / B^i(M) \right)_K$$

(Illusie, 1983, 2.5.3).

Example 1.15. Let X be a smooth variety over a perfect field k . The de Rham–Witt complex

$$W\Omega_X^\bullet: W\mathcal{O}_X \rightarrow \dots \rightarrow W\Omega_X^i \xrightarrow{d} W\Omega_X^{i+1} \rightarrow \dots$$

is a sheaf of graded R -modules on X for the Zariski topology. On applying $R\Gamma$ to this complex, we get a complex $R\Gamma(X, W\Omega_X^\bullet)$ of graded R -modules, which we regard as a bicomplex with (i, j) th term $R\Gamma(X, W\Omega_X^i)^j$. When we replace each vertical complex with its cohomology, the j th row of the bicomplex becomes

$$R^j\Gamma(X, W\Omega_X^\bullet): H^j(X, W\mathcal{O}_X) \rightarrow \dots \rightarrow H^j(X, W\Omega_X^i) \xrightarrow{d} H^j(X, W\Omega_X^{i+1}) \rightarrow \dots$$

The complex $R\Gamma(X, W\Omega_X^\bullet)$ is bounded and complete (Illusie, 1983, 2.4), and becomes $R\Gamma(X, \Omega_X^\bullet)$ when tensored with R_1 , and so $R\Gamma(X, W\Omega_X^\bullet)$ is coherent when X is complete. In this case, $R\Gamma(X/W) \stackrel{\text{def}}{=} s(R\Gamma(X, W\Omega_X^\bullet))$ is a perfect complex of W -modules such that

$$H^j(R\Gamma(X/W)) \simeq H_{\text{crys}}^j(X/W) \quad (\text{isomorphism of } W_\sigma[F]\text{-modules})$$

(ibid. 1.3.5), and the slope spectral sequence (8) becomes

$$E_1^{ij} = H^j(X, W\Omega_X^i) \implies H^{i+j}(X, W\Omega_X^\bullet) \quad (\simeq H_{\text{crys}}^*(X/W)). \tag{13}$$

2. The numerical invariants of a coherent complex

Definition of the invariants

Let M be a coherent graded R -module. The dimension of the domino attached to $d: M^i \rightarrow M^{i+1}$ (see (1.13)) is denoted by $T^i(M)$. It is equal to the number of quotients of the form $U_l\{-i\}$ (varying l) in a filtration of M with elementary quotients.

We let $W[[V]]$ denote the noncommutative power series ring (in which $a \cdot V = V \cdot \sigma a$) and $W((V))$ the ring obtained from $W[[V]]$ by inverting V .

Lemma 2.1. *For a coherent graded R -module M ,*

$$T^i(M) = \text{length}_{W((V))} W((V)) \otimes_{W[[V]]} M^i. \tag{14}$$

Proof. It suffices to prove this for an elementary graded R -module M . If M is elementary of type I, then V is topologically nilpotent on it, and so, when we invert V , M becomes 0; this agrees with $T^0(M) = 0$. If M is elementary of type II, say $M = U_l$, then $W((V)) \otimes M^0 \simeq W((V))$ and $W((V)) \otimes M^1 = 0$, agreeing with $T^0(M) = 1$ and $T^i(M) = 0$ for $i \neq 0$. \square

Let M be an object of $D_c^b(R)$. Ekedahl (1986, p. 14) defines the *slope numbers* of M to be

$$m^{i,j}(M) = \dim_k \frac{H^j(M)^i}{H^j(M)_{p\text{-tors}}^i + V(H^j(M)^i)} + \dim_k \frac{H^{j+1}(M)^{i-1}}{H^{j+1}(M)_{p\text{-tors}}^{i-1} + F(H^{j+1}(M)^{i-1})},$$

where $X_{p\text{-tors}}$ denotes the torsion submodule of X regarded as a W -module. Set

$$T^{i,j}(M) = T^i(H^j(M)).$$

Ekedahl (ibid., p. 85) defines the *Hodge–Witt numbers* of M to be

$$h_W^{i,j}(M) = m^{i,j}(M) + T^{i,j}(M) - 2T^{i-1,j+1}(M) + T^{i-2,j+2}(M)$$

(see also Illusie (1983, 6.3)). Note that the invariants $m^{i,j}(M)$ and $T^{i,j}(M)$ (hence also $h_W^{i,j}(M)$) depend only on the finite sequence $(H^j(M))_{j \in \mathbb{Z}}$ of graded R -modules. It follows from (7) that

$$h_W^{i,j}(M\{m\}[n]) = h_W^{i+m,j+n}(M). \tag{15}$$

In particular (see 1.7),

$$h_W^{i,j}(M(r)) = h_W^{i+r,j-r}(M). \tag{16}$$

Example 2.2. We compute these invariants for certain $M \in D_c^b(R)$.

(a) Suppose that $H^j(M)^i$ has finite length over W for all i, j . Then $H^j(M)^i = H^j(M)_{p\text{-tors}}^i$, and so $m^{i,j}(M)$ is zero for all i, j . Moreover, V is nilpotent on $H^j(M)^i$, and so $T^{i,j}(M) = 0$. It follows that $h_W^{i,j}(M)$ is also zero for all i, j .

(b) Suppose that

$$H^j(M)^i = \begin{cases} R^0/R^0(F^{r-s} - V^s) & \text{if } (i, j) = (i_0, j_0) \\ 0 & \text{otherwise} \end{cases}$$

for some $r > s \geq 0$. Then

$$m^{i,j}(M) = \begin{cases} \dim_k(k_\sigma[F]/(F^{r-s})) = r - s & \text{if } (i, j) = (i_0, j_0) \\ \dim_k(k_\sigma[V]/(V^s)) = s & \text{if } (i, j) = (i_0 + 1, j_0 - 1) \\ 0 & \text{otherwise.} \end{cases}$$

Note that

$$(R^0/R^0(F^{r-s} - V^s)) \otimes K \simeq K_\sigma[F]/(F^r - p^s),$$

which is an F -isocrystal of slope $\lambda = s/r$ with multiplicity $m = r$. As the dominoes attached to the $H^j(M)^i$ are obviously all zero, we see that

$$h_W^{i,j}(M) = m^{i,j}(M) = \begin{cases} m(1 - \lambda) & \text{if } (i, j) = (i_0, j_0) \\ m\lambda & \text{if } (i, j) = (i_0 + 1, j_0 - 1) \\ 0 & \text{otherwise,} \end{cases}$$

where λ is the unique slope of the F -isocrystal $(H^{j_0}(M)_K^{i_0}, F)$ and m is its multiplicity.

(c) Suppose that

$$\left(H^{j_0}(M)^{i_0} \xrightarrow{d} H^{j_0}(M)^{i_0+1} \right) = \left(\prod_{\substack{n \geq 0 \\ \deg i_0}} kV^n \xrightarrow{d} \prod_{\substack{n \geq 1 \\ \deg i_0+1}} kdV^n \right) = U_l\{-i_0\},$$

and that $H^j(M)^i = 0$ for all other values of i and j . Then $H^j(M)^i = H^j(M)_{p\text{-tors}}^i$, and so $m^{i,j}(M)$ is zero for all i, j . The only nonzero T invariant is $T^{i_0, j_0}(M) = 1$. It follows that the only nonzero Hodge–Witt numbers are

$$h_W^{i_0, j_0}(M) = 1, \quad h_W^{i_0+1, j_0-1}(M) = -2, \quad h_W^{i_0+2, j_0-2}(M) = 1.$$

Weighted Hodge–Witt Euler characteristics

Theorem 2.3. For every M in $D_c^b(R)$ and $r \in \mathbb{Z}$,

$$\sum_{i,j \ (i \leq r)} (-1)^{i+j} (r - i) h_W^{i,j}(M) = e_r(M), \tag{17}$$

where

$$e_r(M) = \sum_j (-1)^{j-1} T^{r-1, j-r}(M) + \sum_{i,j,l \ (\lambda_{i,j,l} \leq r-i)} (-1)^{i+j} (r - i - \lambda_{i,j,l}). \tag{18}$$

The sum in (17) is over the pairs of integers (i, j) such that $i \leq r$, and the first sum in (18) is over the integers j . In the second sum in (18), $(\lambda_{i,j,l})_l$ is the family of slopes (with multiplicities) of the F -isocrystal $H^j(M)_K^i$, and the sum is over the triples (i, j, l) such that $\lambda_{i,j,l} \leq r - i$.

Example 2.4. Let M be a graded R -module, regarded as an element of $D_c^b(R)$ concentrated in degree j . Let F' act on M^i as $p^i F$ (assuming that only nonnegative i occur). Then F' is a σ -linear endomorphism of M regarded as a complex of R^0 -modules

$$\begin{array}{ccccccc} \dots & \longrightarrow & M^{i-1} & \xrightarrow{d} & M^i & \xrightarrow{d} & M^{i+1} & \xrightarrow{d} & \dots \\ & & \downarrow p^{i-1}F & & \downarrow p^i F & & \downarrow p^{i+1}F & & \\ \dots & \longrightarrow & M^{i-1} & \xrightarrow{d} & M^i & \xrightarrow{d} & M^{i+1} & \xrightarrow{d} & \dots \end{array},$$

and the second term in (18) equals

$$\sum_{i,l \ (\lambda_{i,l} \leq r)} (-1)^{i+j} (r - \lambda_{i,l}),$$

where $(\lambda_{i,l})_l$ is the family of slopes of the F -isocrystal $(M^i, p^i F)_K$.

Lemma 2.5. *For every distinguished triangle $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ in $D_c^b(R)$,*

$$m^{i,j}(M) = m^{i,j}(M') + m^{i,j}(M'')$$

$$T^{i,j}(M) = T^{i,j}(M') + T^{i,j}(M''),$$

and so

$$h_W^{i,j}(M) = h_W^{i,j}(M') + h_W^{i,j}(M'').$$

Proof. The distinguished triangle gives rise to an exact sequence of graded R -modules

$$\dots \rightarrow H^j(M') \rightarrow H^j(M) \rightarrow H^j(M'') \rightarrow \dots$$

with only finitely many nonzero terms. It suffices to show that m and T are additive on short exact sequences

$$0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0 \tag{19}$$

of coherent graded R -modules. But $m^{ij}(M)$ depends only on $\overline{K} \otimes_W M$, where \overline{K} is the field of fractions of \overline{W} , and the sequence (19) splits when tensored with \overline{K} . The additivity of T follows from the description of T^i in Lemma 2.1. □

Lemma 2.6. *For every distinguished triangle $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ in $D_c^b(R)$,*

$$e_r(M) = e_r(M') + e_r(M''). \tag{20}$$

Proof. The same argument as in the proof of Lemma 2.5 applies. □

Proof of Theorem 2.3

The numbers do not change under extension of the base field, and so we may suppose that k is algebraically closed. First note that, if $M' \rightarrow M \rightarrow M'' \rightarrow M'[1]$ is a distinguished triangle in $D_c^b(R)$ and (17) holds for M' and M'' , then it holds for M (apply 2.5 and 2.6).

A complex M in $D_c^b(R)$ has only finitely many nonzero cohomology groups, and each has a finite filtration whose quotients are elementary graded R -modules. By using induction on the sum of the lengths of the shortest such filtrations, one sees that it suffices to prove the formula for a complex M having only one nonzero cohomology module, which is a degree shift of an elementary graded R -module; i.e., we may assume that $M = H^{j_0}(M) = N\{-i_0\}$, where N is elementary.

Assume that N is elementary of type I. If N is torsion, then both sides are zero. We may suppose that N is a Dieudonne module of slope $\lambda \in [0, 1[$ with multiplicity m (because N is isogenous to a direct sum of such modules — recall that k is algebraically closed). In this case (see 2.2b), the only nonzero Hodge–Witt invariants of M are

$$h_W^{i_0, j_0}(M) = m^{i_0, j_0}(M) = m(1 - \lambda)$$

$$h_W^{i_0+1, j_0-1}(M) = m^{i_0+1, j_0-1}(M) = m\lambda.$$

Both sides of (17) are zero if $r \leq i_0$, and so we may suppose that $r > i_0$. Then the left-hand side (17) is

$$\begin{aligned} & (-1)^{i_0+j_0}(r - i_0)h^{i_0, j_0} + (-1)^{i_0+1+j_0-1}(r - i_0 - 1)h^{i_0+1, j_0-1} \\ &= (-1)^{i_0+j_0}(r - i_0)(1 - \lambda)m + (-1)^{i_0+j_0}(r - i_0 - 1)\lambda m \\ &= (-1)^{i_0+j_0}(r - i_0 - \lambda)m. \end{aligned}$$

On the other hand, the isocrystal $H^j(M)_K^i$ is zero for $(i, j) \neq (i_0, j_0)$, and $H^{j_0}(M)_K^{i_0}$ is an isocrystal with slope λ of multiplicity m , and so

$$e_r(M) = (-1)^{i_0+j_0}(r - i_0 - \lambda)m.$$

If N is of type II, i.e., $H^{j_0}(M) = U_l\{-i_0\}$, then $T^{i_0, j_0} = 1$ is the only nonzero T -invariant (see 2.2c), and so

$$e_r(M) = \begin{cases} (-1)^{i_0+j_0} & \text{if } r = i_0 + 1 \\ 0 & \text{otherwise.} \end{cases}$$

The nonzero h_W -invariants are

$$h_W^{i_0, j_0}(M) = 1 \quad h_W^{i_0+1, j_0-1}(M) = -2 \quad h_W^{i_0+2, j_0-2}(M) = 1,$$

from which (17) follows by an elementary calculation.

Aside 2.7. Here is an alternative proof of Theorem 2.3. Let

$$L(r) = \sum_{i,j} (-1)^{i+j}(r - i) \left(T^{i,j}(M) - 2T^{i-1,j+1}(M) + T^{i-2,j+2}(M) \right).$$

The contribution of T^{i_0, j_0} to this sum is $(-1)^{i_0+j_0}T^{i_0, j_0}$ if $i_0 = r - 1$ and 0 otherwise. Therefore

$$\begin{aligned} L(r) &= \sum_j (-1)^{r-1+j} T^{r-1, j} \\ &= \sum_j (-1)^{j-1} T^{r-1, j-r}. \end{aligned} \tag{21}$$

For an F -crystal P , let $P_{[i, i+1[} = (K \otimes_W P)_{[i, i+1[}$ (part with slopes λ , $i \leq \lambda < i + 1$). From the degeneration of the slope spectral sequence (1.9) at E_1 modulo torsion, we find that

$$H^n(sM)_{[i, i+1[} \simeq (H^{n-i}(M)_K^i, p^i F).$$

From this, it follows that

$$m^{i, n-i}(M) = \sum_{\lambda \in [i, i+1[} (i + 1 - \lambda)h_\lambda^n - \sum_{\lambda \in [i-1, i[} (i - 1 - \lambda)h_\lambda^n,$$

where h_λ^n is the multiplicity of λ as a slope of $H^n(sM)$ (cf. Illusie (1983, 6.2)). Using these two statements, we find that

$$\sum_{i,j} (-1)^{i+j}(r - i)m^{i,j}(M) = \sum_{i,j,l} (-1)^{i+j}(r - i - \lambda_{i,j,l}). \tag{22}$$

On adding (21) and (22), we obtain (17).

Weighted Hodge Euler characteristics

Following Ekedahl (1986, p. 14), we define the *Hodge numbers* of an M in $D_c^b(R)$ to be

$$h^{i,j}(M) = \dim_k(H^j(R_1 \otimes_R^L M)^i).$$

Theorem 2.8. For every M in $D_c^b(R)$ and $i \in \mathbb{Z}$,

$$\sum_j (-1)^j h_W^{i,j}(M) = \sum_j (-1)^j h^{i,j}(M). \tag{23}$$

Proof. As for Theorem 2.3, it suffices to prove this for an elementary R -module, where it can be checked directly. See Ekedahl (1986, IV, Theorem 3.2). \square

For $M = R\Gamma(W\Omega_X^\bullet)$, the formula (23) was found independently by Crew and Milne (cf. *ibid.* p. 86).

Corollary 2.9. For every M in $D_c^b(R)$,

$$\sum_{i,j \ (i \leq r)} (-1)^{i+j} (r-i) h^{i,j}(M) = e_r(M). \tag{24}$$

Proof. We have

$$\begin{aligned} \text{LHS} &= \sum_{i \leq r} (-1)^i (r-i) \left(\sum_j (-1)^j h^{i,j}(M) \right) \\ &\stackrel{(23)}{=} \sum_{i \leq r} (-1)^i (r-i) \left(\sum_j (-1)^j h_W^{i,j}(M) \right) = \text{RHS}. \end{aligned} \quad \square$$

3. Tensor products in the category $D_c^b(R)$ and internal Homs

We review some constructions from Ekedahl (1985).

The internal tensor product

Let M and N be graded R -modules. Ekedahl (1985, p. 69) defines $M * N$ to be the largest quotient of $M \otimes_W N$,

$$x \otimes y \mapsto x * y: M \otimes_W N \rightarrow M * N,$$

in which the following relations hold: $Vx * y = V(x * Fy)$, $x * Vy = V(Fx * y)$, $F(x * y) = Fx * Fy$, and $d(x * y) = dx * y + (-1)^{\deg(x)} x * dy$.

Regard W as a graded R -module concentrated in degree zero with F acting as σ . Then

$$W * M \simeq M \simeq M * W, \tag{25}$$

and so W plays the role of the identity object $\mathbb{1}$.

The bifunctor $(M, N) \rightsquigarrow M * N$ of graded R -modules derives to a bifunctor

$$*^L: D^-(R) \times D^-(R) \rightarrow D^-(R).$$

If M and N are in $D_c^b(R)$, then so also is

$$M \widehat{*} N \stackrel{\text{def}}{=} \widehat{M *^L N}.$$

See Ekedahl (1985, I, 4.8) and Illusie (1983, 2.6.1.10).

The internal Hom

For graded R -modules M, N , we let $\text{Hom}^d(M, N)$ denote the set of graded R -homomorphisms $M \rightarrow N$ of degree d , and we let $\text{Hom}^\bullet(M, N) = \bigoplus_d \text{Hom}^d(M, N)$. Let ${}_R R$ denote the ring R regarded as a graded left R -module. The internal Hom of two graded R -modules M, N is

$$\underline{\text{Hom}}(M, N) \stackrel{\text{def}}{=} \text{Hom}^\bullet({}_R R * M, N).$$

This graded \mathbb{Z}_p -module becomes a graded R -module thanks to the right action of R on ${}_R R$, and $\underline{\text{Hom}}$ derives to a bifunctor

$$R\underline{\text{Hom}}: \mathbf{D}(R)^{\text{opp}} \times \mathbf{D}^+(R) \rightarrow \mathbf{D}(R)$$

(denoted by $R\underline{\text{Hom}}_R$ in Illusie (1983, 2.6.2.6), by $R\underline{\text{Hom}}_R^!$ in Ekedahl (1985, p. 73), and by $R \text{Hom}_R^!$ in Ekedahl (1986, p. 8)).

The functor $R\underline{\text{Hom}}(M, N)$ commutes with extension of the base field. For M in $\mathbf{D}^-(R)$ and N in $\mathbf{D}^+(R)$,

$$R\underline{\text{Hom}}(W, N) \stackrel{(25)}{\simeq} R \text{Hom}^\bullet({}_R R, N) \simeq N \tag{26}$$

$$R \text{Hom}(W, R\underline{\text{Hom}}(M, N)) \simeq R \text{Hom}(M, N) \tag{27}$$

(isomorphisms in $\mathbf{D}_c^b(R)$ and $\mathbf{D}(\mathbb{Z}_p)$ respectively). Ekedahl shows that

$$R_1 \otimes_R^L R\underline{\text{Hom}}(M, N) \simeq R \text{Hom}(R_1 \otimes_R^L M, R_1 \otimes_R^L N)$$

(isomorphism in $\mathbf{D}(k[d])$) and that

$$\widehat{R\underline{\text{Hom}}(M, N)} \simeq R\underline{\text{Hom}}(\widehat{M}, \widehat{N}), \tag{28}$$

and so his criterion (see 1.5) shows that $R\underline{\text{Hom}}(M, N)$ lies in $\mathbf{D}_c^b(R)$ when both M and N do (Illusie, 1983, 2.6.2.9, 2.6.2.10, 2.6.2.11).

4. Homological algebra in the category $\mathbf{D}_c^b(R)$

Throughout this section, $S = \text{Spec } k$, and $\Lambda_m = \mathbb{Z}/p^m\mathbb{Z}$.

The perfect site

An S -scheme U is *perfect* if its absolute Frobenius map $F_{\text{abs}}: U^{(1/p)} \rightarrow U$ is an isomorphism. The *perfection* T^{Pf} of an S -scheme T is the limit of the projective system $T \xleftarrow{F_{\text{abs}}} T^{(1/p)} \xleftarrow{F_{\text{abs}}} \dots$. The scheme T^{Pf} is perfect, and, for any perfect S -scheme U , the canonical map $T^{\text{Pf}} \rightarrow T$ defines an isomorphism

$$\text{Hom}_S(U, T^{\text{Pf}}) \rightarrow \text{Hom}_S(U, T).$$

Let \mathbf{Pf}/S denote the category of perfect affine schemes over S . A *perfect group scheme* over S is a representable functor $\mathbf{Pf}/S \rightarrow \mathbf{Gp}$. For any affine group scheme G over S , the

functor $U \rightsquigarrow G(U): \mathbf{Pf}/S \rightarrow \mathbf{Gp}$ is a perfect group scheme represented by G^{pf} . We say that a perfect group scheme is *algebraic* if it is represented by an algebraic S -scheme.

Let \mathcal{S} denote the category of sheaves of commutative groups on $(\mathbf{Pf}/S)_{\text{et}}$. The commutative perfect algebraic group schemes killed by some power of p form an abelian subcategory \mathcal{G} of \mathcal{S} which is closed under extensions. Let $G \in \mathcal{G}$. The identity component G° of G has a finite composition series whose quotients are isomorphic to \mathbb{G}_a^{pf} , and the quotient G/G° is étale. The dimension of G is the dimension of any algebraic group whose perfection is G° . The category \mathcal{G} is artinian. See [Milne \(1976, §2\)](#), or [Berthelot \(1981, II\)](#).

Example 4.1. Let $f: X \rightarrow S$ be a smooth scheme over S . The functors $U \rightsquigarrow \Gamma(U, W_m \Omega_X^i)$ are sheaves for the étale topology on X . The composite

$$W_{m+1} \Omega_X^i \xrightarrow{F} W_m \Omega_X^i \rightarrow W_m \Omega_X^i / d(W_m \Omega_X^{i-1})$$

factors through the projection $W_{m+1} \Omega_X^i \rightarrow W_m \Omega_X^i$, and so defines a homomorphism

$$F: W_m \Omega_X^i \rightarrow W_m \Omega_X^i / d(W_m \Omega_X^{i-1}).$$

The sheaf $v_m(i)$ on X_{et} is defined to be the kernel of

$$1 - F: W_m \Omega_X^i \rightarrow W_m \Omega_X^i / d(W_m \Omega_X^{i-1})$$

([Milne, 1976, §1](#); [Berthelot, 1981, p. 209](#)). The map $W_{m+1} \Omega_X^i \rightarrow W_m \Omega_X^i$ defines a surjective map $v_{m+1}(i) \rightarrow v_m(i)$ with kernel $v_1(i)$.

Assume that f is proper. The sheaves $R^i f_* v_m(r)$ lie in \mathcal{G} . When $m = 1$, this is proved in [Milne \(1976, 2.7\)](#), and the general case follows by induction on m . See also [Illusie and Raynaud \(1983, IV, 3.2.2\)](#). Following [Milne \(1986, p. 309\)](#), we define

$$\begin{aligned} H^i(X, (\mathbb{Z}/p^m \mathbb{Z})(r)) &= H^{i-r}(X_{\text{et}}, v_m(r)) \\ H^i(X, \mathbb{Z}_p(r)) &= \varprojlim H^i(X, (\mathbb{Z}/p^m \mathbb{Z})(r)). \end{aligned}$$

In the terminology introduced in the Introduction, the main theorem of [Milne and Ramachandran \(2005\)](#) states that there are canonical isomorphisms

$$H^i(X, \mathbb{Z}_p(r)) \simeq H_{\text{abs}}^i(X, \mathbb{Z}_p(r)).$$

The functor $M \rightsquigarrow M^F$

For a complex M of graded R -modules, we define

$$M^F = R \text{Hom}(W, M). \tag{29}$$

Then $M \rightsquigarrow M^F$ is a functor $\mathbf{D}^+(R) \rightarrow \mathbf{D}(\mathbb{Z}_p)$.

Let \widehat{R} denote the completion $\varprojlim R_m$ of R . From

$$W \simeq R^0/R^0(1 - F) \simeq \widehat{R}/\widehat{R}(1 - F),$$

we get an exact sequence

$$0 \rightarrow \widehat{R} \xrightarrow{1-F} \widehat{R} \rightarrow W \rightarrow 0 \tag{30}$$

of graded R -modules (Ekedahl, 1985, III, 1.5.1, p. 90). If M is in $D_c^b(R)$, then, because M is complete,

$$R \operatorname{Hom}(\widehat{R}, M) \simeq R \operatorname{Hom}(R, M) \simeq M^0 \tag{31}$$

(isomorphisms of graded R -modules; ibid. I, 5.9.3ii, p. 78). Now (30) gives a canonical isomorphism

$$M^F \simeq_s (M^0 \xrightarrow{1-F} M^0) \tag{32}$$

(ibid. I, 1.5.4(i), p. 90), which explains the notation. Note that

$$s(M^0 \xrightarrow{1-F} M^0) = \operatorname{Cone}(1 - F: M^0 \rightarrow M^0)[-1]. \tag{33}$$

For M, N in $D_c^b(R)$, we have

$$R \operatorname{Hom}(M, N) \stackrel{(27)}{\simeq} R \operatorname{Hom}(W, R \underline{\operatorname{Hom}}(M, N)) \stackrel{\text{def}}{=} R \underline{\operatorname{Hom}}(M, N)^F \tag{34}$$

in $D(\mathbb{Z}_p)$.

The functor $M \rightsquigarrow \mathcal{M}_\bullet^F$

Let \mathcal{S}_\bullet denote the category of projective systems of sheaves $(P_m)_{m \in \mathbb{N}}$ on $(\mathbf{P}f/S)_{\text{et}}$ with P_m a sheaf of Λ_m -modules, and let \mathcal{G}_\bullet denote the full subcategory of systems $(P_m)_{m \in \mathbb{N}}$ with P_m in \mathcal{G} . Then \mathcal{G}_\bullet is an abelian subcategory of \mathcal{S}_\bullet closed under extensions.

Let M be a graded R -module, and let $M_m = R_m \otimes_R M$. Let \mathcal{M}_m^i denote the sheaf $\operatorname{Spec}(A) \rightsquigarrow M_m^i \otimes_W WA$ on $(\mathbf{P}f/S)_{\text{et}}$, and let \mathcal{M}_\bullet^i denote the projective system $(\mathcal{M}_m^i)_{m \in \mathbb{N}}$. Thus $\mathcal{M}_\bullet^i \in \mathcal{S}_\bullet$. Let F (respectively, V) denote the endomorphism of \mathcal{M}_\bullet^i defined by $F \otimes \sigma$ (respectively, $V \otimes \sigma^{-1}$) on $(M_m^i \otimes_W WA)_m$. In this way, we get an R_\bullet -module

$$\mathcal{M}_\bullet: \dots \rightarrow \mathcal{M}_\bullet^i \xrightarrow{d} \mathcal{M}_\bullet^{i+1} \rightarrow \dots$$

in \mathcal{S}_\bullet . Cf. Illusie and Raynaud (1983, IV, 3.6.3).

Example 4.2. Let $M = M^0$ be an elementary graded R -module of type I. For each m , the map $1 - F: \mathcal{M}_m \rightarrow \mathcal{M}_m$ is surjective with kernel the étale group scheme \mathcal{M}_m^F over k corresponding to the natural representation of $\operatorname{Gal}(\bar{k}/k)$ on $((M/V^m) \otimes_W \bar{W})^{F \otimes \sigma}$. Therefore \mathcal{M}_\bullet^F is a pro-étale group scheme over k with

$$\mathcal{M}_\bullet^F(\bar{k}) \stackrel{\text{def}}{=} \varprojlim \mathcal{M}_m^F(\bar{k}) = (M \otimes_W \bar{W})^{F \otimes \sigma}.$$

Cf. (5.5) below.

Example 4.3. Let M be an elementary graded R -module of type II. Then $1 - F: \mathcal{M}_\bullet^i \rightarrow \mathcal{M}_\bullet^i$ is bijective for $i = 0$, and it is surjective with kernel canonically isomorphic to \mathbb{G}_a^{pf} for $i = 1$ (Illusie and Raynaud, 1983, IV, 3.7, p. 195).

Proposition 4.4. *Let M be a coherent graded R -module. For each i , the map $1 - F: \mathcal{M}_\bullet^i \rightarrow \mathcal{M}_\bullet^i$ is surjective, and its kernel $(\mathcal{M}_\bullet^i)^F$ lies in \mathcal{G}_\bullet . There is an exact sequence*

$$0 \rightarrow U^i \rightarrow (\mathcal{M}_\bullet^i)^F \rightarrow D^i \rightarrow 0$$

with U^i a connected unipotent perfect algebraic group of dimension $T^{i-1}(M)$ and D^i the profinite étale group corresponding to the natural representation of $\text{Gal}(\bar{k}/k)$ on $(\heartsuit^i M \otimes_W \bar{W})^{F \otimes \sigma}$.

Proof. When M is an elementary graded R -module, the proposition is proved in the two examples. The proof can be extended to all coherent graded R -modules by using Illusie and Raynaud (1983, IV 3.10, 3.11, p. 196). □

Corollary 4.5. *Let M be a coherent graded R -module, and let $H^i(M) = Z^i(M)/B^i(M)$. Then $D^i(\bar{k}) \stackrel{\text{def}}{=} \varprojlim_m D_m^i(\bar{k})$ is a finitely generated \mathbb{Z}_p -module, and*

$$D^i(\bar{k}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (H^i(M) \otimes_W \bar{K})^{F \otimes \sigma}.$$

Proof. According to (4.4), $D^i(\bar{k}) \simeq (\heartsuit^i M \otimes_W \bar{W})^{F \otimes \sigma}$. Now the statement follows from (1.14). □

Let $\Gamma(\text{Set}, -)$ denote the functor

$$(M_m)_{m \in \mathbb{N}} \rightsquigarrow \varprojlim \Gamma(\text{Set}, M_m): \mathcal{S}_\bullet \rightarrow \text{Mod}(\mathbb{Z}_p).$$

It derives to a functor $R\Gamma(\text{Set}, -): \mathbf{D}(\mathcal{S}_\bullet) \rightarrow \mathbf{D}(\mathbb{Z}_p)$.

For a coherent graded R -module M , the system \mathcal{M}_\bullet depends only on the projective system $M_\bullet = (M_m)_m$. The functor $M_\bullet \rightsquigarrow \mathcal{M}_\bullet: \text{Mod}(R_\bullet) \rightarrow \mathcal{S}_\bullet$ is exact, and so it defines a functor

$$M_\bullet \rightsquigarrow \mathcal{M}_\bullet: \mathbf{D}(R_\bullet) \rightarrow \mathbf{D}(\mathcal{S}_\bullet).$$

Let

$$\mathcal{M}_\bullet^F = \text{Cone}(\mathcal{M}_\bullet^0 \xrightarrow{1-F} \mathcal{M}_\bullet^0)[-1]. \tag{35}$$

Proposition 4.6. *The following diagram commutes:*

$$\begin{array}{ccccc} \mathbf{D}_c^b(R) & \xrightarrow{M \rightsquigarrow M_\bullet} & \mathbf{D}^b(R_\bullet) & \xrightarrow{(-)^F} & \mathbf{D}^b(\mathcal{S}_\bullet) \\ & \searrow^{M \rightsquigarrow \widehat{M}} & \downarrow R \varprojlim & & \downarrow R\Gamma(\text{Set}, -) \\ & & \mathbf{D}^b(R) & \xrightarrow{(-)^F} & \mathbf{D}^b(\mathbb{Z}_p). \end{array}$$

The functor $(-)^F$ on the top row (respectively, bottom row) is that defined in (35) (respectively, (29)). In other words, for M in $\mathbf{D}_c^b(R)$,

$$R\Gamma(\text{Set}, \mathcal{M}_\bullet^F) \simeq M^F.$$

Proof. This follows directly from the definitions and the isomorphism

$$M^F \simeq \text{Cone}(1 - F: M^0 \rightarrow M^0)[-1]$$

obtained by composing the isomorphisms (32) and (33). □

Proposition 4.7. Let $M \in \mathbf{D}_c^b(R)$, and let $r \in \mathbb{Z}$. For each j , there is an exact sequence

$$0 \rightarrow U^j \rightarrow H^j(\mathcal{M}(r)_\bullet^F) \rightarrow D^j \rightarrow 0$$

with U^j a connected unipotent perfect algebraic group of dimension $T^{r-1} \cdot j - r$ and D^j the profinite étale group corresponding to the natural representation of $\text{Gal}(\bar{k}/k)$ on $(\heartsuit^r(H^j(M)) \otimes \bar{W})^{F \otimes \sigma}$.

The reader should take care to distinguish $H^i(M)$ in (4.5) from the similar looking $H^j(M)$ in (4.7): the first comes from the kernel and image of the element d of the Raynaud ring, whereas the second is cohomology in a complex.

Proof. Apply (4.4) to $H^j(M(r))$ with $i = 0$. □

Corollary 4.8. The \mathbb{Z}_p -module $D^j(\bar{k})$ is finitely generated, and

$$D^j(\bar{k}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq (H^r(H^j(M)) \otimes \bar{K})^{F \otimes \sigma}. \tag{36}$$

Here, $H^r(H^j(M))$ is the $E_2^{r,j}$ term in the slope spectral sequence for M .

Proof. Apply (4.5) to $H^j(M)$. (In order to be able apply (4.5), one needs to know that, for any P in $\mathbf{D}_c^b(R)$, $H^0(H^j(P))$ and $H^j(P)^0$ become isomorphic when tensored with K . This follows from the degeneration of the slope spectral sequence (8).) □

The functors $R\mathcal{H}om$

If M, N in $\mathbf{D}_c^b(R)$, then $P \stackrel{\text{def}}{=} R\mathbf{H}om(M, N)$ lies in $\mathbf{D}_c^b(R)$ (see § 3). Let

$$R\mathcal{H}om(M, N) = \mathcal{P}_\bullet^F \stackrel{\text{def}}{=} \text{Cone}(\mathcal{P}_\bullet^0 \xrightarrow{1-F} \mathcal{P}_\bullet^0)[-1]$$

(see (35)). Then $R\mathcal{H}om$ is a bifunctor

$$R\mathcal{H}om: \mathbf{D}_c^b(R) \times \mathbf{D}_c^b(R) \rightarrow \mathbf{D}_{\mathcal{G}_\bullet}^b(\mathcal{S}_\bullet)$$

(denoted by $R\mathbf{H}om_R$ in Ekedahl (1986, p. 11), except that he allows graded homomorphisms of any degree; see p. 817 for the categories \mathcal{G}_\bullet and \mathcal{S}_\bullet).

Proposition 4.9. For $M, N \in \mathbf{D}_c^b(R)$,

$$R\Gamma(\mathcal{S}_{\text{et}}, R\mathcal{H}om(M, N)) \simeq R\mathbf{H}om(M, N). \tag{37}$$

Proof. From (4.6) with $P = R\mathbf{H}om(M, N)$, we find that

$$R\Gamma(\mathcal{S}_{\text{et}}, R\mathcal{H}om(M, N)) \simeq R\mathbf{H}om(M, N)^F.$$

But $R\mathbf{H}om(M, N)^F \simeq R\mathbf{H}om(M, N)$ (see (34)). □

For M, N in $\mathbf{D}_c^b(R)$, we let

$$\begin{aligned} \text{Ext}^j(M, N) &= H^j(R\mathbf{H}om(M, N)) \\ \underline{\text{Ext}}^j(M, N) &= H^j(R\mathbf{H}om(M, N)) \\ \mathcal{E}xt^j(M, N) &= H^j(R\mathcal{H}om(M, N)). \end{aligned}$$

The first is a \mathbb{Z}_p -module, the second is a coherent graded R -module, and the third is an object of \mathcal{G}_\bullet . From (27) and (37), we get spectral sequences

$$\begin{aligned} \text{Ext}^i(W, \underline{\text{Ext}}^j(M, N)) &\implies \text{Ext}^{i+j}(M, N) \\ R^i \Gamma(\text{Set}, \mathcal{E}xt^j(M, N)) &\implies \text{Ext}^{i+j}(M, N). \end{aligned}$$

For example, it follows from (26) that

$$\begin{aligned} \underline{\text{Ext}}^j(W, M) &= H^j(M) \\ \mathcal{E}xt^j(W, M) &= H^j(\mathcal{M}_\bullet^F), \quad \text{and} \\ T^{i,j}(W, M) &= T^{i,j}(M). \end{aligned}$$

Aside 4.10. If $M \in D_c^b(R)$, then the dual

$$D(M) \stackrel{\text{def}}{=} R\underline{\text{Hom}}(M, W)$$

of M also lies in $D_c^b(R)$. If $M, N \in D_c^b(R)$, then

$$D(M) \widehat{\ast} N \simeq R\underline{\text{Hom}}(M, N)$$

(see Illusie, 1983, 2.6.3.4). In particular,

$$T^{i,j}(M, N) = T^{i,j}(D(M) \widehat{\ast} N).$$

5. The proof of the main theorem

Throughout this section, Γ is a profinite group isomorphic to $\widehat{\mathbb{Z}}$, and γ is a topological generator for Γ . For a Γ -module M , the kernel and cokernel of $1 - \gamma: M \rightarrow M$ are denoted by M^Γ and M_Γ , respectively.

Elementary preliminaries

Let $|S|$ denote the cardinality of a set S . For a homomorphism $f: M \rightarrow N$ of abelian groups, we let

$$z(f) = \frac{[\text{Ker}(f)]}{[\text{Coker}(f)]}$$

when both cardinalities are finite.

Lemma 5.1. *Let M be a finitely generated \mathbb{Z}_p -module with an action of Γ , and let $f: M^\Gamma \rightarrow M_\Gamma$ be the map induced by the identity map on M . Then $z(f)$ is defined if and only if 1 is not a multiple root of the minimum polynomial γ on M , in which case M^Γ has rank equal to the multiplicity of 1 as an eigenvalue of γ on $M_{\mathbb{Q}_p}$, and*

$$z(f) = \left| \prod_{i, a_i \neq 1} (1 - a_i) \right|_p,$$

where $(a_i)_{i \in I}$ is the family of eigenvalues of γ on $M_{\mathbb{Q}_p}$.

Proof. The proof is elementary (Tate, 1966, z.4). □

Lemma 5.2. Consider a commutative diagram

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & C^{j-1} & & C^j & \xrightarrow{f^j} & C^{j+1} \\
 & & \downarrow g^{j-1} & & \uparrow h^j & & \downarrow g^{j+1} \\
 \dots & \longrightarrow & A^{j-1} & \xrightarrow{d^{j-1}} & A^j & \xrightarrow{d^j} & A^{j+1} \longrightarrow \dots \\
 & & \downarrow h^{j-1} & & \uparrow g^j & & \downarrow h^{j+1} \\
 & & B^{j-1} & \xrightarrow{f^{j-1}} & B^j & & B^{j+1} \longrightarrow \dots
 \end{array}$$

in which A^\bullet is a bounded complex of abelian groups and each column is a short exact sequence (in particular, the g are injective and the h are surjective). The cohomology groups $H^j(A^\bullet)$ are all finite if and only if the numbers $z(f^j)$ are all defined, in which case

$$\prod_j [H^j(A^\bullet)]^{(-1)^j} = \prod_j z(f^j)^{(-1)^j}.$$

Proof. Because h^{j-1} is surjective, g^j maps the image of f^{j-1} into the image of d^{j-1} . Because g^{j+1} is injective and h^j is surjective, h^j maps the kernel of d^j onto the kernel of f^j . The snake lemma applied to

$$\begin{array}{ccccccc}
 \text{Im}(f^{j-1}) & \xrightarrow{g^j} & \text{Im}(d^{j-1}) & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & B^j & \xrightarrow{g^j} & \text{Ker}(d^j) & \xrightarrow{h^j} & \text{Ker}(f^j) \longrightarrow 0
 \end{array}$$

gives an exact sequence

$$0 \rightarrow \text{Coker}(f^{j-1}) \rightarrow H^j(A^\bullet) \rightarrow \text{Ker}(f^j) \rightarrow 0.$$

Therefore $H^j(A^\bullet)$ is finite if and only if $\text{Coker}(f^{j-1})$ and $\text{Ker}(f^j)$ are both finite, in which case

$$[H^j(A^\bullet)] = [\text{Coker}(f^{j-1})] \cdot [\text{Ker}(f^j)].$$

On combining these statements for all j , we obtain the lemma. □

Cohomological preliminaries

Let Λ be a finite ring, and let $\Lambda\Gamma$ be the group ring. For a Λ -module M , we let M_* denote the corresponding co-induced module. Thus M_* consists of the locally constant maps $f: \Gamma \rightarrow M$ and $\tau' \in \Gamma$ acts on f according to the rule $(\tau'f)(\tau) = f(\tau\tau')$. When M is a discrete Γ -module, there is an exact sequence

$$0 \rightarrow M \rightarrow M_* \xrightarrow{\alpha_\gamma} M_* \rightarrow 0, \tag{38}$$

in which the first map sends $m \in M$ to the map $\tau \mapsto \tau m$ and the second map sends $f \in M_*$ to $\tau \mapsto f(\tau\gamma) - \gamma f(\tau)$. Let F be the functor $M \rightsquigarrow M^\Gamma: \text{Mod}(\Lambda\Gamma) \rightarrow \text{Mod}(\Lambda)$. The class of co-induced $\Lambda\Gamma$ -modules is F -injective, and so (38) defines isomorphisms

$$RF(M) \simeq F(M_* \xrightarrow{\alpha_\gamma} M_*) \simeq (M \xrightarrow{1-\gamma} M)$$

in $\mathbf{D}^+(\Lambda)$. For the second isomorphism, note that M_*^Γ is the set of constant functions $\Gamma \rightarrow M$, and, if f is the constant function with value m , then $(\alpha_\gamma f)(\tau) = f(\tau\gamma) - \gamma f(\tau) = m - \gamma m$.

Now let $\mathbf{Mod}(\Lambda_\bullet \Gamma)$ denote the category of projective systems $(M_m)_{m \in \mathbb{N}}$ with M_m a discrete Γ -module killed by p^m , and let F be the functor $\mathbf{Mod}(\Lambda_\bullet \Gamma) \rightarrow \mathbf{Mod}(\mathbb{Z}_p)$ sending $(M_m)_m$ to $\varprojlim M_m^\Gamma$. We say that an object $(M_m)_m$ of $\mathbf{Mod}(\Lambda_\bullet \Gamma)$ is co-induced if M_m is co-induced for each m . For every complex $X = (X_m)_m$ of $\Lambda_\bullet \Gamma$ -modules, there is an exact sequence

$$0 \rightarrow X \rightarrow X_* \xrightarrow{\alpha_\gamma} X_* \rightarrow 0 \tag{39}$$

of complexes with $X_*^j = (X_{m*}^j)_m$ for all j, m . The class of co-induced $\Lambda_\bullet \Gamma$ -modules is F -injective, and so (39) defines isomorphisms

$$RF(X) \simeq s(F(X_* \rightarrow X_*)) \simeq s(\tilde{X} \xrightarrow{1-\gamma} \tilde{X}) \tag{40}$$

in $\mathbf{D}^+(\mathbb{Z}_p)$, where $\tilde{X} = (R\varprojlim)(X)$ and $\tilde{X} \xrightarrow{1-\gamma} \tilde{X}$ is a double complex with \tilde{X} as both its zeroth and first column. From (40), we get a long exact sequence

$$\dots \rightarrow H^{j-1}(\tilde{X}) \xrightarrow{1-\gamma} H^{j-1}(\tilde{X}) \rightarrow R^j F(X) \rightarrow H^j(\tilde{X}) \xrightarrow{1-\gamma} H^j(\tilde{X}) \rightarrow \dots \tag{41}$$

If $(M_m)_m$ is a $\Lambda_\bullet \Gamma$ -module satisfying the Mittag-Leffler condition, then

$$R^j F((M_m)_m) \simeq H_{\text{cts}}^j(\Gamma, \varprojlim M_m)$$

(continuous cohomology). Let $\Lambda_\bullet = (\mathbb{Z}/p^m \mathbb{Z})_m$. Then

$$R^1 F(\Lambda_\bullet) \simeq H_{\text{cts}}^1(\Gamma, \mathbb{Z}_p) \simeq \text{Hom}_{\text{cts}}(\Gamma, \mathbb{Z}_p),$$

which has a canonical element θ , namely, that mapping γ to 1. We can regard θ as an element of

$$\text{Ext}^1(\Lambda_\bullet, \Lambda_\bullet) = \text{Hom}_{\mathbf{D}^+(\Lambda_\bullet \Gamma)}(\Lambda_\bullet, \Lambda_\bullet[1]).$$

Thus, for X in $\mathbf{D}^+(\Lambda_\bullet \Gamma)$, we obtain maps

$$\begin{aligned} \theta: X &\rightarrow X[1] \\ R\theta: RF(X) &\rightarrow RF(X)[1]. \end{aligned}$$

The second map is described explicitly by the following map of double complexes:

$$\begin{array}{ccc} RF(X) & & \tilde{X} \xrightarrow{1-\gamma} \tilde{X} \\ \downarrow R\theta & & \downarrow \gamma \\ RF(X)[1] & & \tilde{X} \xrightarrow{1-\gamma} \tilde{X} \\ & -1 & 0 \qquad 1 \end{array}$$

For all j , the following diagram commutes:

$$\begin{array}{ccc} R^j F(X) & \xrightarrow{d^j} & R^{j+1} F(X) \\ \downarrow & & \uparrow \\ H^j(\tilde{X}) & \xrightarrow{\text{id}} & H^j(\tilde{X}) \end{array} \tag{42}$$

where $d^j = H^j(R\theta)$ and the vertical maps are those in (41). The sequence

$$\dots \longrightarrow R^{j-1}F(X) \xrightarrow{d^{j-1}} R^jF(X) \xrightarrow{d^j} R^{j+1}F(X) \longrightarrow \dots \tag{43}$$

is a complex, because $R\theta \circ R\theta = 0$. See Rapoport and Zink (1982, § 1).

Review of F -isocrystals

Let V be an F -isocrystal over k . The \overline{K} -module $\overline{V} \stackrel{\text{def}}{=} \overline{K} \otimes_K V$ becomes an F -isocrystal over \overline{k} with \overline{F} acting as $\sigma \otimes F$ (recall that \overline{K} is the field of fractions of $\overline{W} = W(\overline{k})$).

5.3. Let λ be a nonnegative rational number, and write $\lambda = s/r$ with $r, s \in \mathbb{N}$, $r > 0$, $(r, s) = 1$. Define E^λ to be the F -isocrystal $K_\sigma[F]/(K_\sigma[F](F^r - p^s))$.

When k is algebraically closed, every F -isocrystal is semisimple, and the simple F -isocrystals are exactly the E^λ with $\lambda \in \mathbb{Q}_{\geq 0}$. Therefore an F -isocrystal has a unique (slope) decomposition

$$V = \bigoplus_{\lambda \geq 0} V_\lambda \tag{44}$$

with V_λ a sum of copies of E^λ . See Demazure (1972, IV).

When k is merely perfect, the decomposition (44) of \overline{V} is stable under $\text{Gal}(\overline{k}/k)$, and so arises from a (slope) decomposition of V . In other words, $V = \bigoplus_\lambda V_\lambda$ with $\overline{V}_\lambda = \overline{V}_\lambda$. If $\lambda = r/s$ with r, s as above, then V_λ is the largest K -submodule of V such that $F^r V_\lambda = p^s V_\lambda$. The F -isocrystal V_λ is called the part of V with slope λ , and $\{\lambda \mid V_\lambda \neq 0\}$ is the set of slopes of V .

5.4. Let V be an F -isocrystal over k . The characteristic polynomial

$$P_{V,\alpha}(t) \stackrel{\text{def}}{=} \det(1 - \alpha t | V)$$

of an endomorphism α of V lies in $K[t]$, but commutes with F , and so lies in $\mathbb{Q}_p[t]$. Let $k = \mathbb{F}_q$ with $q = p^a$, so that F^a is an endomorphism of (V, F) , and let

$$P_{V,F^a}(t) = \prod_{i \in I} (1 - a_i t), \quad a_i \in \overline{\mathbb{Q}}_p.$$

According to a theorem of Manin, $(\text{ord}_q(a_i))_{i \in I}$ is the family of slopes of V . Here, ord_q is the p -adic valuation on $\overline{\mathbb{Q}}_p$ normalized so that $\text{ord}_q(q) = 1$. See Demazure (1972, pp. 89–90).

5.5. Let (V, F) be an F -isocrystal over $k = \mathbb{F}_{p^a}$, and let $\lambda \in \mathbb{N}$. Let

$$V_{(\lambda)} = \{v \in \overline{V} \mid \overline{F}v = p^\lambda v\} \quad (\mathbb{Q}_p\text{-subspace of } \overline{V}).$$

Then $V_{(\lambda)}$ is a \mathbb{Q}_p -structure on V_λ . In other words, V_λ has a basis of elements e with the property that $\overline{F}e = p^\lambda e$, and hence

$$(\gamma \otimes F^a)e = \overline{F}^a e = q^\lambda e.$$

Therefore, as c runs over the eigenvalues of F^a on V with $\text{ord}_q(c) = \lambda$, the quotient q^λ/c runs over the eigenvalues of γ on $V_{(\lambda)}$; moreover, c is a multiple root of the minimum polynomial of F^a on V_λ if and only if q^λ/c is a multiple root of the minimum polynomial of γ on $V_{(\lambda)}$. See [Milne \(1986, 5.3\)](#).

5.6. Let (V, F) be an F -isocrystal over $k = \mathbb{F}_{p^a}$. If F^a is a semisimple endomorphism of V (as a K -vector space), then $\text{End}(V, F)$ is semisimple, because it is a \mathbb{Q}_p -form of the centralizer of F^a in $\text{End}(V)$; it follows that (V, F) is semisimple. Conversely, if (V, F) is semisimple, then F^a is semisimple, because it lies in the centre of the semisimple algebra $\text{End}(V, F)$. Let V and V' be nonzero F -isocrystals; then $V \otimes V'$ is semisimple if and only if both V and V' are semisimple.

A preliminary calculation

In this subsection, k is the finite field \mathbb{F}_q with $q = p^a$, and $\Gamma = \text{Gal}(\bar{k}/k)$. We take the Frobenius element $x \mapsto x^q$ to be the generator γ of Γ .

Recall that, for P in $\mathbb{D}_c^b(R)$, $H^j(sP)_K$ is an F -isocrystal over k .

Proposition 5.7. *Let $M, N \in \mathbb{D}_c^b(R)$, let $P = R\text{Hom}(M, N)$, and let $r \in \mathbb{Z}$. For each j , let*

$$f_j : \text{Ext}^j(\bar{M}, \bar{N}(r))^\Gamma \rightarrow \text{Ext}^j(\bar{M}, \bar{N}(r))_\Gamma$$

be the map induced by the identity map. Then $z(f_j)$ is defined if and only if q^r is not a multiple root of the minimum polynomial of F^a on $H^j(sP)_K$, in which case

$$z(f_j) = \left| \prod_{a_{j,l} \neq q^r} \left(1 - \frac{a_{j,l}}{q^r} \right) \right|_p \left| \prod_{\text{ord}_q(a_{j,l}) < r} \frac{q^r}{a_{j,l}} \right|_p q^{T^{r-1, j-r}(P)},$$

where $(a_{j,l})_l$ is the family of eigenvalues of F^a acting on $H^j(sP)_K$.

Proof. (Following the proof of [Milne \(1986, 6.2\)](#).) Let G^j denote the perfect pro-group scheme $\mathcal{E}xt^j(M, N(r)) \stackrel{\text{def}}{=} H^j(\mathcal{P}(r)_\bullet^F)$. There is an exact sequence

$$0 \rightarrow U^j \rightarrow G^j \rightarrow D^j \rightarrow 0$$

in which U^j is a connected unipotent perfect algebraic group of dimension $T^{r-1, j-r}(P)$ and D^j is a pro-étale group such that $D^j(\bar{k})$ is a finitely generated \mathbb{Z}_p -module and

$$D^j(\bar{k}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq H^j(sP(r)_K)_{(0)} \simeq H^j(sP_K)_{(r)} \tag{45}$$

(see [4.7, 4.8](#)). The map $1 - \gamma : U^j(\bar{k}) \rightarrow U^j(\bar{k})$ is surjective, because it is étale and U^j is connected. On applying the snake lemma to

$$\begin{array}{ccccccc} 0 & \longrightarrow & U^j(\bar{k}) & \longrightarrow & G^j(\bar{k}) & \longrightarrow & D^j(\bar{k}) \longrightarrow 0 \\ & & \downarrow 1-\gamma & & \downarrow 1-\gamma & & \downarrow 1-\gamma \\ 0 & \longrightarrow & U^j(\bar{k}) & \longrightarrow & G^j(\bar{k}) & \longrightarrow & D^j(\bar{k}) \longrightarrow 0, \end{array}$$

and using that the first vertical arrow is surjective, we obtain the upper and lower rows of the following exact commutative diagram:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & U^j(\bar{k})^\Gamma & \longrightarrow & G^j(\bar{k})^\Gamma & \longrightarrow & D^j(\bar{k})^\Gamma \longrightarrow 0 \\
 & & \downarrow f'_j & & \downarrow f_j & & \downarrow f''_j \\
 0 & \longrightarrow & 0 & \longrightarrow & G^j(\bar{k})_\Gamma & \longrightarrow & D^j(\bar{k})_\Gamma \longrightarrow 0.
 \end{array} \tag{46}$$

Because U^j has a composition series whose quotients are isomorphic to \mathbb{G}_a^{pf} ,

$$[U^j(k)] = q^{\dim(U^j)} = q^{T^{r-1, j-r}(P)}.$$

On the other hand, it follows from (5.5) that the eigenvalues of γ acting on $D^j(\bar{k})_{\mathbb{Q}_p}$ are the quotients $q^r/a_{j,l}$ with $\text{ord}_q(a_{j,l}) = r$. Therefore, (5.1) and (5.5) show that $z(f''_j)$ is defined if and only if the minimum polynomial of F^a on $H^j(sP)_K$ does not have q^r as a multiple root, in which case

$$z(f''_j) = \left| \prod_l \left(1 - \frac{q^r}{a_{j,l}} \right) \right|_p,$$

where the product is over the $a_{j,l}$ such that $\text{ord}_q(a_{j,l}) = r$ but $a_{j,l} \neq q^r$. Note that

$$\left| 1 - \frac{a_{j,l}}{q^r} \right|_p = \left| 1 - \frac{q^r}{a_{j,l}} \right|_p \quad \text{if } \text{ord}_q(a_{j,l}) = r,$$

and

$$\left| 1 - \frac{a_{j,l}}{q^r} \right|_p = \begin{cases} |a_{j,l}/q^r|_p & \text{if } \text{ord}_q(a_{j,l}) < r \\ 1 & \text{if } \text{ord}_q(a_{j,l}) > r. \end{cases}$$

Therefore,

$$z(f''_j) = \left| \prod_{a_{j,l} \neq q^r} \left(1 - \frac{a_{j,l}}{q^r} \right) \right|_p \left| \prod_{\text{ord}_q(a_{j,l}) < r} \frac{q^r}{a_{j,l}} \right|_p,$$

where both products are over all $a_{j,l}$ satisfying the conditions. The snake lemma applied to (46) shows that $z(f_j)$ is defined if and only if both $z(f'_j)$ and $z(f''_j)$ are defined, in which case $z(f_j) = z(f'_j) \cdot z(f''_j)$. The proposition now follows. □

Definition of the complex $E(M, N(r))$

Recall (4.9) that the bifunctor

$$R \text{Hom} : \mathbf{D}(R)^{\text{opp}} \times \mathbf{D}^+(R) \rightarrow \mathbf{D}(\mathbb{Z}_p)$$

factors canonically through

$$R\Gamma(\mathcal{S}_{\text{et}}, -) : \mathbf{D}^+(\mathcal{S}_\bullet) \rightarrow \mathbf{D}(\mathbb{Z}_p),$$

where $\Gamma(\mathcal{S}_{\text{et}}, -)$ is the functor $(P_m)_m \rightsquigarrow \varprojlim \Gamma(\mathcal{S}_{\text{et}}, P_m)$. Since $R\Gamma(\mathcal{S}_{\text{et}}, -)$ obviously factors through

$$RF : \mathbf{D}^+(\Lambda_\bullet \Gamma) \rightarrow \mathbf{D}(\mathbb{Z}_p), \quad F = \left((M_m)_m \rightsquigarrow \varprojlim M_m^\Gamma \right), \quad \Gamma = \text{Gal}(\bar{k}/k),$$

so also does $R\text{Hom}$. Therefore, for $M, N \in D_c^b(R)$, there exists a well-defined object X in $D^+(\Lambda_\bullet \Gamma)$ such that $RF(X) = R\text{Hom}(M, N(r))$. For an algebraically closed base field k , $RF(X) = \tilde{X}$, and so, for a general k , $\tilde{X} = R\text{Hom}(\overline{M}, \overline{N}(r))$.

Now let k be \mathbb{F}_q with $q = p^a$. With X as in the last paragraph, the sequence (41) gives us short exact sequences

$$0 \rightarrow \text{Ext}^{j-1}(\overline{M}, \overline{N}(r))_\Gamma \rightarrow \text{Ext}^j(M, N(r)) \rightarrow \text{Ext}^j(\overline{M}, \overline{N}(r))^\Gamma \rightarrow 0. \tag{47}$$

Moreover, (43) becomes a complex:

$$E(M, N(r)): \quad \dots \rightarrow \text{Ext}^{j-1}(M, N(r)) \rightarrow \text{Ext}^j(M, N(r)) \rightarrow \text{Ext}^{j+1}(M, N(r)) \rightarrow \dots .$$

This is the unique complex for which the following diagram commutes:

$$\begin{array}{ccccccc}
 & & & \text{Ext}^j(\overline{M}, \overline{N}(r))^\Gamma & \xrightarrow{f^j} & \text{Ext}^j(\overline{M}, \overline{N}(r))_\Gamma & \\
 & & & \uparrow & & \downarrow & \\
 \dots \rightarrow & \text{Ext}^{j-1}(M, N(r)) & \xrightarrow{d^{j-1}} & \text{Ext}^j(M, N(r)) & \xrightarrow{d^j} & \text{Ext}^{j+1}(M, N(r)) & \rightarrow \dots \tag{48} \\
 & \downarrow & & \uparrow & & & \\
 & \text{Ext}^{j-1}(\overline{M}, \overline{N}(r))^\Gamma & \xrightarrow{f^{j-1}} & \text{Ext}^{j-1}(\overline{M}, \overline{N}(r))_\Gamma & & &
 \end{array}$$

(the vertical maps are those in (47) and the maps f^j are induced by the identity map).

Let $P \in D_c^b(R)$. The zeta function $Z(P, t)$ of P is the alternating product of the characteristic polynomials of F^a acting on the isocrystals $H^j(sP)_K$:

$$Z(P, t) = \prod_j \det(1 - F^a t \mid H^j(sP)_K)^{(-1)^{j+1}}.$$

Proof of Theorem 0.1

We first note that the condition on the minimum polynomial of F^a implies that the minimum polynomial of γ on $H^j(sP_{\overline{K}})_{(r)}$ does not have 1 as a multiple root (see 5.5). Let

$$P_j(t) = \det(1 - F^a t \mid H^j(sP)_K) = \prod_l (1 - a_{j,l} t).$$

(a) We have $\text{Ext}^j(\overline{M}, \overline{N}(r)) = \mathcal{E}xt^j(M, N(r))(\overline{k})$, where $\mathcal{E}xt^j(M, N(r))$ is a pro-algebraic group such that the identity component of $\mathcal{E}xt^j(M, N(r))$ is algebraic and the quotient of $\mathcal{E}xt^j(M, N(r))$ by its identity component is a pro-étale group $(D_m^j)_m$ such that $\varprojlim_m D_m^j(\overline{k})$ is a finitely generated \mathbb{Z}_p -module (see 4.7, 4.8). Hence the \mathbb{Z}_p -modules $\text{Ext}^j(\overline{M}, \overline{N}(r))^\Gamma$ and $\text{Ext}^j(\overline{M}, \overline{N}(r))_\Gamma$ are finitely generated. Now

$$\text{rank}_{\mathbb{Z}_p}(\text{Ext}^j(M, N(r))) = \text{rank}_{\mathbb{Z}_p}(\text{Ext}^{j-1}(\overline{M}, \overline{N}(r))_\Gamma) + \text{rank}_{\mathbb{Z}_p}(\text{Ext}^j(\overline{M}, \overline{N}(r))^\Gamma).$$

The hypothesis on the action of the Frobenius element implies that

$$\text{Ext}^j(\overline{M}, \overline{N}(r))^\Gamma \otimes \mathbb{Q} \simeq \text{Ext}^j(\overline{M}, \overline{N}(r))_\Gamma \otimes \mathbb{Q}$$

for all j , and so

$$\text{rank}_{\mathbb{Z}_p}(\text{Ext}^j(M, N(r))) = \text{rank}_{\mathbb{Z}_p}(\text{Ext}^{j-1}(\overline{M}, \overline{N}(r))^\Gamma) + \text{rank}_{\mathbb{Z}_p}(\text{Ext}^j(\overline{M}, \overline{N}(r))^\Gamma).$$

Therefore,

$$\sum_j (-1)^j \text{rank}_{\mathbb{Z}_p}(\text{Ext}^j(M, N(r))) = 0.$$

(b) Let ρ_j be the multiplicity of q^r as an inverse root of P_j . Then

$$\rho_j = \text{rank}_{\mathbb{Z}_p} \text{Ext}^j(\overline{M}, \overline{N}(r))^\Gamma = \text{rank}_{\mathbb{Z}_p} \text{Ext}^j(\overline{M}, \overline{N}(r))_\Gamma,$$

and so

$$\begin{aligned} \sum_j (-1)^{j+1} \cdot j \cdot \text{rank}_{\mathbb{Z}_p}(\text{Ext}^j(M, N(r))) &= \sum_j (-1)^{j+1} \cdot j \cdot (\rho_{j-1} + \rho_j) \\ &= \sum_j (-1)^j \rho_j \\ &= \rho. \end{aligned}$$

(c) From Lemma 5.2 applied to the diagram (48), we find that

$$\chi(M, N(r)) = \prod_j z(f^j)^{(-1)^j}.$$

According to Proposition 5.7,

$$z(f^j) = \left| \prod_{a_{j,l} \neq q^r} \left(1 - \frac{a_{j,l}}{q^r}\right) \right|_p \left| \prod_{\text{ord}_q(a_{j,l}) < r} \frac{q^r}{a_{j,l}} \right|_p q^{T^{r-1, j-r}(P)}.$$

where $(a_{j,l})_l$ is the family of eigenvalues of F^a acting on $H^j(sP(r))_{\mathbb{Q}}$. Note that

$$\prod_{a_{j,l} \neq q^r} \left(1 - \frac{a_{j,l}}{q^r}\right) = \lim_{t \rightarrow q^{-r}} \frac{P_j(t)}{(1 - q^r t)^{\rho_j}}.$$

According to (5.4),

$$\left| \prod_{\text{ord}_q(a_{j,l}) < r} \frac{q^r}{a_{j,l}} \right|_p^{-1} = \sum_{l \ (\lambda_{j,l} < r)} r - \lambda_{j,l},$$

where $(\lambda_{j,l})_l$ is the family of slopes $H^j(sP(r))_{\mathbb{Q}}$. Because of the degeneration of the slope spectral sequence (8) at E_1 modulo torsion, the family of slopes of $H^j(sP(r))_{\mathbb{Q}}$ is the same as the family of slopes of the groups $H^i(P)_{\mathbb{Q}}^j$, used in the definition $e_r(P)$. Using this, we find that

$$\chi(M, N(r)) = \left| \lim_{t \rightarrow q^{-r}} Z(M, N, t) \cdot (1 - q^r t)^\rho \right|_p^{-1} q^{-e_r(P)}.$$

Theorem 2.9 completes the proof.

6. Applications to algebraic varieties

Throughout, $S = \text{Spec}(k)$, where k is a perfect field of characteristic $p > 0$.

Recall that the zeta function of an algebraic variety X over a finite field \mathbb{F}_q is defined to be the formal power series $Z(X, t) \in \mathbb{Q}[[t]]$ such that

$$\log(Z(X, t)) = \sum_{n>0} \frac{N_n t^n}{n}, \quad N_n = \#(X(\mathbb{F}_{q^n})), \tag{49}$$

and that [Dwork \(1960\)](#) proved that $Z(X, t) \in \mathbb{Q}(t)$.

Smooth complete varieties

Let X be a smooth complete variety over k , and let

$$M(X) = R\Gamma(X, W\Omega_X^\bullet) \in \mathbf{D}_c^b(R)$$

(see [1.15](#)). For all $j \geq 0$,

$$H^j(s(M(X))) \simeq H_{\text{crys}}^j(X/W) \tag{50}$$

(isomorphism of F -isocrystals; see [1.15](#)), and so

$$Z(M(X), t) = \prod_j \det(1 - F^a t \mid H_{\text{crys}}^j(X/W)_{\mathbb{Q}}^{(-1)^{j+1}}).$$

That this equals $Z(X, t)$ is proved in [Katz and Messing \(1974\)](#) for X projective, and the complete case can be deduced from the projective case by using de Jong’s theory of alterations ([Nakkajima, 2005](#), Remark 2.2 (4); [Suh, 2012](#); see also [Chiarellotto and Le Stum, 1998](#)). Moreover, $H_{\text{crys}}^j(X/W)_{\mathbb{Q}}$ can be replaced by $H_{\text{rig}}^j(X)$ (see [6.2](#) below). Finally, $H_{\text{abs}}^j(X, \mathbb{Z}_p(r))$ is the group $H^j(X, \mathbb{Z}_p(r))$ defined in [\(4.1\)](#) ([Milne and Ramachandran, 2005](#)), and

$$h^{i,j}(M(X)) = h^{i,j}(X) \stackrel{\text{def}}{=} \dim H^j(X, \Omega_X^i),$$

because $R_1 \otimes_R^L M(X) \simeq R\Gamma(X, \Omega_X^\bullet)$ (see [1.15](#)). Therefore, when X is projective, [Theorem 0.2](#) becomes the p -part of [Theorem 0.1](#) of [Milne \(1986\)](#).

Rigid cohomology

Before considering more general algebraic varieties, we briefly review the theory of rigid cohomology. This was introduced in the 1980s by Pierre Berthelot as a common generalization of crystalline and Washnitzer–Monsky cohomology. The book [Le Stum \(2007\)](#) is a good reference for the foundations. We write $H_{\text{rig}}^i(X)$ (respectively, $H_{\text{rig},c}^i(X)$) for the rigid cohomology (respectively, rigid cohomology with compact support) of a variety X over a perfect field k .

6.1. Both $H_{\text{rig}}^i(X)$ and $H_{\text{rig},c}^i(X)$ are F -isocrystals over k ; in particular, they are finite-dimensional vector spaces over K . Cohomology with compact support is contravariant for proper maps and covariant for open immersions; ordinary cohomology is contravariant for all regular maps. The Künneth theorem is true for both cohomology theories. (See [Berthelot \(1997a,b\)](#), and [Grosse-Klönne \(2002\)](#).)

6.2. When X is smooth complete variety,

$$H_{\text{rig}}^i(X) \simeq H_{\text{crys}}^i(X)_{\mathbb{Q}}$$

(canonical isomorphism of F -isocrystals) (Berthelot, 1986).

6.3. Let U be an open subvariety of X with closed complement Z ; then there is a long exact sequence

$$\cdots \rightarrow H_{\text{rig},c}^i(U) \rightarrow H_{\text{rig},c}^i(X) \rightarrow H_{\text{rig},c}^i(Z) \rightarrow \cdots$$

(Berthelot, 1986, 3.1).

6.4. Rigid cohomology is a Bloch–Ogus theory. In particular, there is a theory of rigid homology and cycle class maps (Petrequin, 2003).

6.5. Rigid cohomology is a mixed-Weil cohomology theory, and hence factors through the triangulated category of complexes of mixed motives (Cisinski and Déglise, 2012a,b).

6.6. Rigid cohomology satisfies proper cohomological descent (Tsuzuki, 2003).

6.7. Rigid cohomology (with compact support) can be described in terms of the logarithmic de Rham–Witt cohomology of smooth simplicial schemes (Lorenzon, Mokrane, Tsuzuki, Shiho, Nakajima). We explain this below.

6.8. When k is finite, say, $k = \mathbb{F}_{p^a}$,

$$Z(X, t) = \det(1 - F^a t \mid H_{\text{rig},c}^j(X))^{(-1)^{j+1}}$$

(Étesse and Le Stum, 1993).

6.9. When k is finite, the F -isocrystals $H_{\text{rig}}^i(X)$ and $H_{\text{rig},c}^i(X)$ are mixed; in particular, the eigenvalues of $\Phi = F^a$ are Weil numbers.

The functors $X \rightsquigarrow H_{\text{rig}}^i(X)$ and $X \rightsquigarrow H_{\text{rig},c}^i(X)$ arise from functors to $\mathbf{D}_{\text{iso}}^b(K_{\sigma}[F'])$, which we denote $h_{\text{rig}}(X)$ and $h_{\text{rig},c}(X)$ respectively.

Varieties with log structure

Endow S with a fine log structure, and let X be a complete log-smooth log variety of Cartier type over S (Kato, 1989). In this situation, Lorenzon (2002, Theorem 3.1) defines a complex $M(X) \stackrel{\text{def}}{=} R\Gamma(X, W\Omega_X^{\bullet})$ of graded R -modules, and proves that it lies in $D_c^b(R)$. Therefore, Theorem 0.2 applies to X .

Smooth varieties

Let $V = X \setminus E$ be the complement of a divisor with normal crossings E in a smooth complete variety X of dimension n , and let m_X be the canonical log structure on X

defined by E :

$$m_X = \{f \in \mathcal{O}_X \mid f \text{ is invertible outside } E\}$$

(Kato, 1989, 1.5). Then (X, m_X) is log-smooth (ibid. § 3).

Define $M(V) \in \mathbf{D}_c^b(R)$ to be the complex of graded R -modules attached to (X, m_X) as above:

$$M(V) = R\Gamma((X, m_X), W\Omega_X^\bullet) = R\Gamma(X, W\Omega_X^\bullet(\log E)).$$

We caution that this definition of $M(V)$ uses the presentation of V as $X \setminus E$; it is not known at present that $M(V)$ depends only on V . However,

$$H^i(s(M(V))_{\mathbb{Q}}) \simeq H_{\text{rig}}^i(V)$$

(Nakkajima, 2012, 1.0.18, p. 13), and so $s(M(V))_{\mathbb{Q}}$ is independent of the compactification X of V .

We define $M_c(V)$ to be the Tate twist of the dual of $M(V)$:

$$M_c(V) = D(M(V))(-n)$$

(see 4.10). From Berthelot’s duality of rigid cohomology (Berthelot, 1997a; Nakkajima and Shiho, 2008, 3.6.0.1), we have the following isomorphism of F -isocrystals:

$$H_{\text{rig},c}^j(V) \simeq \text{Hom}_K(H_{\text{rig}}^{2n-j}(V), K(-n)).$$

It follows that

$$H^j(s(M_c(V))_{\mathbb{Q}}) \simeq H_{\text{rig},c}^j(V). \tag{51}$$

We define

$$H_c^j(V, \mathbb{Z}_p(r)) = \text{Hom}(W, M_c(V)(r)[j]).$$

Now take $k = \mathbb{F}_{p^a}$. It follows from (6.8) and (51) that

$$Z(V, t) = Z(M_c(V), t).$$

Moreover,

$$R_1 \otimes_R^L M(V) \simeq R\Gamma(X, \Omega_X^\bullet(\log E)).$$

(Lorenzon, 2002, 2.17, or Nakkajima and Shiho, 2008, p. 184). Therefore, in this case, Theorem 0.2 becomes the following statement.

Theorem 6.10. *Assume that q^r is not a multiple root of the minimum polynomial of F^a acting on $H_{\text{rig}}^j(V)$ for any j .*

- (a) *The groups $H_c^j(V, \mathbb{Z}_p(r))$ are finitely generated \mathbb{Z}_p -modules, and the alternating sum of their ranks is zero.*
- (b) *The zeta function $Z(V, t)$ of X has a pole at $t = q^{-r}$ of order*

$$\rho = \sum_j (-1)^{j+1} \cdot j \cdot \text{rank}_{\mathbb{Z}_p} \left(H_c^j(V, \mathbb{Z}_p(r)) \right).$$

(c) The cohomology groups of the complex

$$E(V, r): \dots \rightarrow H_c^{j-1}(V, \mathbb{Z}_p(r)) \rightarrow H_c^j(V, \mathbb{Z}_p(r)) \rightarrow H_c^{j+1}(V, \mathbb{Z}_p(r)) \rightarrow \dots$$

are finite, and the alternating product of their orders $\chi(V, \mathbb{Z}_p(r))$ satisfies

$$\left| \lim_{t \rightarrow q^{-r}} Z(V, t) \cdot (1 - q^r t)^\rho \right|_p^{-1} = \chi(V, \mathbb{Z}_p(r)) \cdot q^{\chi(V, r)},$$

where $\chi(V, r) = \sum_{i \leq r, j} (-1)^{i+j} (r-i) h^{i,j}(V)$.

We caution the reader that it is not known that every smooth variety U can be expressed as the complement of a normal crossings divisor in a smooth complete variety.

General varieties

Philosophy. With each variety V over k , there should be associated objects $M(V)$, $M_c(V)$, $M^{BM}(V)$, and $M^h(V)$ in $D_c^b(R)$ arising as the p -adic realizations of the various motives of V . See the discussion in [Voevodsky et al. \(2000, pp. 181–182\)](#).

At present, it does not seem to be known whether there exists a W -linear cohomology theory underlying Berthelot’s rigid cohomology, i.e., a cohomology theory that gives finitely generated W -modules $H_c^j(V)$ stable under F with $\mathbb{Q} \otimes_{\mathbb{Z}} H_c^j(V) = H_{\text{rig},c}^j(V)$ for each variety V .

Deligne’s technique of cohomological descent in Hodge theory has been transplanted to rigid and log-de Rham–Witt theory by the brave efforts of N. Tsuzuki, Y. Nakkajima, and A. Shiho. While their results do not provide the invariants of V above, they are still sufficient for applications to zeta functions. Even though $M_c(V)$ is the only relevant object for zeta values, we consider both $M(V)$ and $M_c(V)$.

The ordinary cohomology object $M(V)$. Let V be a variety of dimension n over k equipped with an embedding $V \hookrightarrow V'$ of V into a proper scheme V' . Then (see [Nakkajima \(2012\)](#), especially 1.0.18, p. 13), there is a simplicial proper hypercovering (U_\bullet, X_\bullet) of (V, V') with X_\bullet a proper smooth simplicial scheme over k and U_\bullet the complement of a simplicial strict divisor with normal crossings E_\bullet on X_\bullet ; moreover,

$$H_{\text{rig}}^i(V) \simeq H^i(X_\bullet, W\Omega_{X_\bullet}^\bullet(\log E_\bullet))_{\mathbb{Q}}.$$

For each $j \geq 0$,

$$R\Gamma(X_j, W\Omega_{X_j}^\bullet(\log E_j)) \in D_c^b(R)$$

([Lorenzon, 2002](#)). As $D_c^b(R)$ is a triangulated subcategory of $D(R)$, this implies that

$$R\Gamma(X_{\leq d}, W\Omega_{X_{\leq d}}^\bullet(\log E_{\leq d})) \in D_c^b(R)$$

for each truncation $X_{\leq d}$ of the simplicial scheme X_\bullet . The inclusion $X_{\leq d} \rightarrow X_\bullet$ induces an isomorphism

$$\tau_{\leq 2n} H^i(X_\bullet, W\Omega_{X_\bullet}^\bullet(\log E_\bullet))_{\mathbb{Q}} \simeq \tau_{\leq 2n} H^i(X_{\leq d}, W\Omega_{X_{\leq d}}^\bullet(\log E_{\leq d}))_{\mathbb{Q}}$$

for all i provided that $d > (n+1)(n+2)$, because both terms are isomorphic to $H_{\text{rig}}^i(V)$. Here, $\tau_{\leq 2n}$ is the usual truncation functor in the derived category. For the left-hand

side, this follows from 1.0.8 or 12.9.1 of [Nakkajima \(2012\)](#). For the right-hand side, we apply Theorem 3.5.4, p. 243, of [Nakkajima and Shiho \(2008\)](#): $H_{\text{rig}}^i(V) = 0$ for $i > 2n$ and the spectral sequence 3.5.4.1 degenerates at E_1 , implying that only finitely many X_j contribute to the rigid cohomology of V . The bound on d comes from the arguments following the isomorphism 3.5.0.4 on p. 242 *ibid.* See also pp. 122–125 of [Nakkajima \(2012\)](#).

We let

$$M(V) = \tau_{\leq 2n} R\Gamma(X_{\leq d}, W\Omega_{X_{\leq d}}^\bullet(\log E_{\leq d}))$$

for any integer $d > (n + 1)(n + 2)$. While $M(V)$ may depend on d , X_\bullet , and the embedding into V' , the object $s(M(V))_{\mathbb{Q}}$ is independent of these choices up to canonical isomorphism, because $H^i(s(M(V))_{\mathbb{Q}}) \simeq H_{\text{rig}}^i(V)$. Recall that $H_{\text{rig}}^i(V) = 0$ for $i > 2n$.

We need to truncate, because it is not clear that the object $R\Gamma(X_\bullet, W\Omega_{X_\bullet}^\bullet(\log E_\bullet))$ lies in $D_c^b(R)$.

The cohomology object with compact support $M_c(V)$. Let $V \hookrightarrow V'$ be as in the last subsection. Let $\iota: Z \hookrightarrow V'$ denote the inclusion of the reduced closed complement Z of V . One can find a proper hypercovering $Y_\bullet \rightarrow Z$ and a morphism $f: Y_\bullet \rightarrow X_\bullet$ lifting ι . Applying the results of the previous subsection to Z and fixing an integer $d > (n + 1)(n + 2)$, we get $M(Z)$ and a map $f^*: M(V') \rightarrow M(Z)$. We define $M_c(V)[1]$ to be the mapping cone of f^* . This is an object of $D_c^b(R)$.

Lemma 6.11. *For all $V \hookrightarrow V'$ as above,*

$$H^i(s(M_c(V))) \simeq H_{\text{rig},c}^i(V).$$

Proof. As the map f lifts ι , the map

$$f^*: H^i(s(M(V')))_{\mathbb{Q}} \rightarrow H^i(s(M(Z)))_{\mathbb{Q}}$$

can be identified with the map $\iota^*: H_{\text{rig}}^i(V') \rightarrow H_{\text{rig}}^i(Z)$. But as V' and Z are proper, rigid cohomology is the same as rigid cohomology with compact support. The lemma now follows from the long exact sequence (6.3). □

Combining the lemma with the result of Etesse and Le Stum above (6.8), we obtain that the zeta function $Z(V, t)$ of V is equal to the zeta function of $M_c(V)$. Therefore, from Theorem 0.1, we obtain Theorem 6.10 for V .

Application of strong resolution of singularities

[Geisser \(2006\)](#) has shown how the assumption of a strong form of resolution of singularities leads to a definition of groups $H_c^i(V, \mathbb{Z}(r))$ for an arbitrary variety V over k , which, when k is finite, are closely connected to special values of zeta functions. His definition involves the eh-topology, where the coverings are generated by étale coverings and abstract blow-ups (*ibid.* 2.1).

We now sketch how his argument provides an object $M_c(V) \in D_c^b(R)$. For a complete V , we define $M(V) = R\Gamma(V_{\text{eh}}, \rho^*W\Omega_V^\bullet)$, where ρ^* denotes pullback from eh-sheaves on the category of smooth varieties over k to eh-sheaves on all varieties over k . We show that

$M(V) \in \mathbf{D}_c^b(R)$ by using induction on the dimension of V . Resolution of singularities gives us a proper map $V' \rightarrow V$ inducing an isomorphism from an open subvariety U' of the smooth variety V' onto an open subvariety U of V . Moreover, U' is the complement in V' of a divisor with normal crossings, and so we can define $M(U') \in \mathbf{D}_c^b(R)$ as above (using the eh-topology). Now $M(V) \in \mathbf{D}_c^b(R)$, because $M(U) \stackrel{\text{def}}{=} M(U') \in \mathbf{D}_c^b(R)$ and $M(V \setminus U) \in \mathbf{D}_c^b(R)$ (by induction).

To define $M_c(V)$ for an arbitrary V , choose a compactification V' of V , and let

$$M_c(V) = \text{Cone}(M(V') \rightarrow M(Z))[-1], \quad Z \stackrel{\text{def}}{=} V' \setminus V.$$

Clearly, $M_c(V) \in \mathbf{D}_c^b(R)$. The eh-topology is crucial for proving that this definition is independent of the compactification (ibid. 3.4). Given $M_c(V)$, we define

$$H_c^i(V, \mathbb{Z}_p(r)) = \text{Hom}_{\mathbf{D}_c^b(R)}(W, M_c(V)(r)[i]).$$

This agrees with Geisser’s group tensored with \mathbb{Z}_p , because the two agree for smooth complete varieties and satisfy the same functorial properties.

Deligne–Mumford Stacks

Olsson (2007, first three chapters) extends the theory of crystalline cohomology to certain algebraic stacks. He also shows (ibid. Chapter 4) that the crystalline definition (Illusie, 1983, 1.1(iv)) of the de Rham–Witt complex can be extended to stacks. Let \mathcal{S}/W be a flat algebraic stack equipped with a lift of the Frobenius endomorphism from \mathcal{S}_0 compatible with the action of σ in W . Let $\mathcal{X} \rightarrow \mathcal{S}$ be a smooth morphism of algebraic stacks with \mathcal{X} a Deligne–Mumford stack. Then $W\Omega_{\mathcal{X}/\mathcal{S}}^\bullet$ is a complex of sheaves of R -modules on \mathcal{X} , and there is a canonical isomorphism

$$H^j(s(R\Gamma(W\Omega_{\mathcal{X}/\mathcal{S}}^\bullet)))_{\mathbb{Q}} \simeq H_{\text{crys}}^j(\mathcal{X}/W)_{\mathbb{Q}} \tag{52}$$

(Olsson, 2007, 4.4.17). Under certain hypotheses on \mathcal{S} and \mathcal{X} (ibid. 4.5.1), Ekedahl’s criterion (see 1.5) can be used to show that $R\Gamma(W\Omega_{\mathcal{X}/\mathcal{S}}^\bullet) \in \mathbf{D}_c^b(R)$ (ibid. 4.5.19) and that (52) is an isomorphism of F -isocrystals.

Now assume that $k = \mathbb{F}_q$, $q = p^a$. The zeta function $Z(\mathcal{X}, t)$ of a stack \mathcal{X} over k is defined by (49), but with

$$N_m = \sum_{x \in [\mathcal{X}(\mathbb{F}_{q^m})]} \frac{1}{\#\text{Aut}_x \mathbb{F}_{q^m}}$$

(see Sun, 2012, p. 49). Assume that \mathcal{X} is a Deligne–Mumford stack over \mathcal{S} satisfying Olsson’s conditions, and let $M(\mathcal{X}) = R\Gamma(W\Omega_{\mathcal{X}/\mathcal{S}}^\bullet) \in \mathbf{D}_c^b(R)$. From (52), we see that

$$Z(M(\mathcal{X}), t) = \prod_j \det(1 - F^a t \mid H_{\text{crys}}^j(\mathcal{X}/W)_{\mathbb{Q}})^{(-1)^{j+1}}.$$

We expect that the two zeta functions agree (see ibid. 1.1 for the ℓ -version of this). Then Theorem 6.10 will hold for \mathcal{X} with

$$H^j(\mathcal{X}, \mathbb{Z}_p(r)) \stackrel{\text{def}}{=} \text{Hom}_{\mathbf{D}_c^b(R)}(W, M(\mathcal{X})(r)[j]).$$

Crystals

Let X be a smooth scheme over S , and let E be a crystal on X . Étéssé (1988a, II, 1.2.5) defines a de Rham–Witt complex $E \otimes W\Omega_{X/S}^\bullet$ on X , and, under some hypotheses on X and E , he proves that $M(X, E) \in \mathcal{D}_c^b(R)$ (ibid., II, 1.2.7) and that there is a canonical isomorphism

$$H^j(R\Gamma(E \otimes W\Omega_{X/S}^\bullet)) \simeq H_{\text{crys}}^j(X/S, E)$$

(ibid. II, 2.7.1). Let

$$M(X, E) = R\Gamma(E \otimes W\Omega_{X/S}^\bullet).$$

When k is finite, Theorem 0.2 for $M(X, E)$ becomes Theorem (0.1)' of Étéssé (1988b).

References

- BERTHELOT, P., Le théorème de dualité plate pour les surfaces (d'après J. S. Milne), in *Algebraic surfaces (Orsay, 1976–78)*, Lecture Notes in Math, Volume 868, pp. 203–237 (Springer, Berlin, 1981).
- BERTHELOT, P., Géométrie rigide et cohomologie des variétés algébriques de caractéristique p , in *Introductions aux cohomologies p -adiques (Luminy, 1984)*, Mém. Soc. Math. Fr. (N.S.), (1986) 3, 7–32.
- BERTHELOT, P., Dualité de Poincaré et formule de Künneth en cohomologie rigide, *C. R. Acad. Sci. Paris Sér. I Math.* **325** (1997a), 493–498.
- BERTHELOT, P., Finitude et pureté cohomologique en cohomologie rigide, *Invent. Math.* **128** (1997b), 329–377.
- BLOCH, S., Algebraic K -theory and crystalline cohomology, *Inst. Hautes Études Sci. Publ. Math.* (1977), 187–268.
- CHIARELLOTTO, B. AND LE STUM, B., Sur la pureté de la cohomologie cristalline, *C. R. Acad. Sci. Paris Sér. I Math.* **326** (1998), 961–963.
- CISINSKI, D.-C. AND DÉGLISE, F., Mixed Weil cohomologies, *Adv. Math.* **230** (2012a), 55–130.
- CISINSKI, D.-C. AND DÉGLISE, F., (2012b), Triangulated categories of mixed motives. [arXiv:0912.2110v3](https://arxiv.org/abs/0912.2110v3).
- DELIGNE, P., A quoi servent les motifs? in *Motives (Seattle, WA, 1991)*, Proc. Sympos. Pure Math, Volume 55, pp. 143–161 (Amer Math Soc, Providence, RI, 1994).
- DEMAZURE, M., Lectures on p -divisible groups, *Lecture Notes in Mathematics, Volume 302* (Springer, Berlin, 1972).
- DWORK, B., On the rationality of the zeta function of an algebraic variety, *Amer. J. Math.* **82** (1960), 631–648.
- EKEDAHL, T., On the multiplicative properties of the de Rham–Witt complex I, *Ark. Mat.* **22** (1984), 185–239.
- EKEDAHL, T., On the multiplicative properties of the de Rham–Witt complex II, *Ark. Mat.* **23** (1985), 53–102.
- EKEDAHL, T., *Diagonal complexes and F -gauge structures* (Travaux en Cours Hermann, Paris, 1986).
- EKEDAHL, T., On the adic formalism, in *The Grothendieck Festschrift, Vol II*, Progr Math, Volume 87, pp. 197–218 (Birkhäuser Boston, Boston, MA, 1990).
- ÉTESSE, J.-Y., Complexe de de Rham–Witt à coefficients dans un cristal, *Compositio Math.* **66** (1988a), 57–120.
- ÉTESSE, J.-Y., Rationalité et valeurs de fonctions L en cohomologie cristalline, *Ann. Inst. Fourier (Grenoble)* **38** (1988b), 33–92.

- ÉTESSE, J.-Y. AND LE STUM, B., Fonctions L associées aux F -isocristaux surconvergentes I Interprétation cohomologique, *Math. Ann.* **296** (1993), 557–576.
- GEISSER, T., Arithmetic cohomology over finite fields and special values of ζ -functions, *Duke Math. J.* **133** (2006), 27–57.
- GROSSE-KLÖNNE, E., Finiteness of de Rham cohomology in rigid analysis, *Duke Math. J.* **113** (2002), 57–91.
- ILLUSIE, L., Finiteness, duality, and Künneth theorems in the cohomology of the de Rham–Witt complex, in *Algebraic geometry (Tokyo/Kyoto, 1982)*, Lecture Notes in Math, Volume 1016, pp. 20–72 (Springer, Berlin, 1983).
- ILLUSIE, L. AND RAYNAUD, M., Les suites spectrales associées au complexe de de Rham–Witt, *Inst. Hautes Études Sci. Publ. Math.* (1983), 73–212.
- KATO, K., Logarithmic structures of Fontaine-Illusie, in *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pp. 191–224 (Johns Hopkins Univ. Press, Baltimore, MD, 1989).
- KATZ, N. M. AND MESSING, W., Some consequences of the Riemann hypothesis for varieties over finite fields, *Invent. Math.* **23** (1974), 73–77.
- LE STUM, B., Rigid cohomology, in *Cambridge Tracts in Mathematics, Volume 172* (Cambridge University Press, Cambridge, 2007).
- LORENZON, P., Logarithmic Hodge–Witt forms and Hyodo–Kato cohomology, *J. Algebra* **249** (2002), 247–265.
- MILNE, J. S., Duality in the flat cohomology of a surface, *Ann. Sci. Éc. Norm. Super. (4)* **9** (1976), 171–201.
- MILNE, J. S., Values of zeta functions of varieties over finite fields, *Amer. J. Math.* **108** (1986), 297–360; Addendum, *Amer. J. Math.*, to appear.
- MILNE, J. S. AND RAMACHANDRAN, N., The de Rham–Witt and \mathbb{Z}_p -cohomologies of an algebraic variety, *Adv. Math.* **198** (2005), 36–42.
- MILNE, J. S. AND RAMACHANDRAN, N., (2013), Motivic complexes and special values of zeta functions. [arXiv:1311.3166](https://arxiv.org/abs/1311.3166).
- NAKKAJIMA, Y., p -adic weight spectral sequences of log varieties, *J. Math. Sci. Univ. Tokyo* **12** (2005), 513–661.
- NAKKAJIMA, Y., Weight filtration and slope filtration on the rigid cohomology of a variety in characteristic $p > 0$, *Mém. Soc. Math. Fr. (N.S.)* (2012), 130–131:vi+250.
- NAKKAJIMA, Y. AND SHIHO, A., Weight filtrations on log crystalline cohomologies of families of open smooth varieties, *Lecture Notes in Mathematics, Volume 1959* (Springer, Berlin, 2008).
- OLSSON, M. C., Crystalline cohomology of algebraic stacks and Hyodo–Kato cohomology, *Astérisque* **316** (2007), 412.
- PETREQUIN, D., Classes de Chern et classes de cycles en cohomologie rigide, *Bull. Soc. Math. France* **131** (2003), 59–121.
- RAPOPORT, M. AND ZINK, T., Über die lokale Zetafunktion von Shimuravarietäten Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik, *Invent. Math.* **68** (1982), 21–101.
- SUH, J., Symmetry and parity in Frobenius action on cohomology, *Compos. Math.* **148** (2012), 295–303.
- SUN, S., Decomposition theorem for perverse sheaves on Artin stacks over finite fields, *Duke Math. J.* **161** (2012), 2297–2310.
- TATE, J. T., On the conjectures of Birch and Swinnerton-Dyer and a geometric analog, *Séminaire Bourbaki* **1965/66** (1966), Exposé 306.
- TSUZUKI, N., Cohomological descent of rigid cohomology for proper coverings, *Invent. Math.* **151** (2003), 101–133.
- VOEVODSKY, V., SUSLIN, A. AND FRIEDLANDER, E. M., Cycles, transfers, and motivic homology theories, *Ann. Math. Stud.*, Volume 143 (Princeton University Press, Princeton, NJ, 2000).