Ergod. Th. & Dynam. Sys., (2022), **42**, 737–776
doi:10.1017/etds.2021.36

Flexibility of Lyapunov exponents with respect to two classes of measures on the torus

ALENA ERCHENKO

Mathematics Department, Stony Brook University,
Simons Center for Geometry and Physics,
Stony Brook, NY, USA
(e-mail: alena.erchenko@stonybrook.edu)

(*Received* 25 *September* 2019 *and accepted in revised form* 12 *August* 2020)

Dedicated to Anatole Katok

Abstract. We consider a smooth area-preserving Anosov diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to an Anosov automorphism L of $\mathbb{T}^2$. It is known that the positive Lyapunov exponent of f with respect to the normalized Lebesgue measure is less than or equal to the topological entropy of L, which, in addition, is less than or equal to the Lyapunov exponent of f with respect to the probability measure of maximal entropy. Moreover, the equalities only occur simultaneously. We show that these are the only restrictions on these two dynamical invariants.

Key words: flexibility, Lyapunov exponents, Anosov diffeomorphism, Markov partition, slow-down deformation
2020 Mathematics Subject Classification: 37D20, 37A35 (Primary); 37C05, 37A05 (Secondary)

Contents

1 Introduction 738
1.1 Anosov volume-preserving diffeomorphisms on tori 738
1.2 Formulation of the result 740
1.3 Outline of the proof 742
1.4 Further questions for Anosov diffeomorphisms of $\mathbb{T}^2$ 743
2 Construction I 744
2.1 Construction I: Anosov twist diffeomorphisms 745
2.2 Estimation of λ_{abs} in Construction I 749
2.3 Estimation of λ_{mme} in Construction I 752
3 Construction II 762

3.1 Construction II: slow-down deformation near a fixed point 763
3.2 Estimation of $\lambda_{\rm abs}$ in Construction II 766
3.3 Estimation of $\lambda_{\rm mme}$ in Construction II 767
Acknowledgements 775
References 775

1. *Introduction*

The aim of the flexibility program is to study natural classes of smooth dynamical systems and to find *constructive tools* to freely manipulate dynamical data inside a fixed class. The result described in this paper is another example demonstrating the flexibility principle in dynamical systems.

1.1. *Anosov volume-preserving diffeomorphisms on tori.*

Consider an n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, $n \geq 2$. Anosov volume-preserving C^∞ (smooth) diffeomorphisms represent a natural class for studying flexibility questions. For any f in this class the tangent bundle of $\mathbb{T}^n$ splits as a direct sum $T\mathbb{T}^n = E^u \oplus E^s$ of two Df-invariant subbundles E^u (unstable) and E^s (stable) such that E^u is uniformly expanded by Df and E^s is uniformly contracted by Df. Moreover, by [**KH95**, Theorem 18.6.1], f is homotopic and topologically conjugate to an Anosov automorphism L given by a hyperbolic matrix in $SL(n, \mathbb{Z})$, that is, a matrix in $SL(n, \mathbb{Z})$ with no eigenvalues on the unit circle.

The *Lyapunov exponent* of f at $x \in \mathbb{T}^n$, $\mathbf{v} \in T_x\mathbb{T}^n \setminus \{\mathbf{0}\}$ is given by

$$\lambda(f, x, \mathbf{v}) = \limsup_{n\to\infty} \frac{\log \|Df_x^n \mathbf{v}\|}{n}. \tag{1.1}$$

Since f is volume-preserving, f preserves the normalized Lebesgue measure Leb which is ergodic [**AS67**]. Moreover, there exists a unique measure of maximal entropy ν_f for f which is ergodic [**Bow71**]. Applying the Oseledets multiplicative ergodic theorem, we obtain that for any f-invariant ergodic Borel probability measure μ the limit

$$\lim_{n\to\infty} \frac{\log \|D_x f^n \mathbf{v}\|}{n}$$

exists for μ-almost every $x \in \mathbb{T}^n$ and all non-zero $\mathbf{v} \in T_x\mathbb{T}^n$. In particular, we obtain a collection of n numbers $\lambda_{1,\mu}(f) \geq \cdots \geq \lambda_{n,\mu}(f)$ as possible limits that are called *Lyapunov exponents of f with respect to μ*. Since f is Anosov, all these numbers are non-zero. We denote the number of positive elements in this list by $u(f)$ and call it the *unstable index of f*. Also, for any f-invariant ergodic Borel probability measure μ,

$$\sum_{i=1}^{u(f)} \lambda_{i,\mu}(f) = \int_{\mathbb{T}^n} \log \|Df|_{E^u}\| \, d\mu. \tag{1.2}$$

The Lyapunov exponents with respect to Leb and ν_f are natural dynamical data to look at in the context of the flexibility program.

We denote by $h_{\rm top}(f)$ and $h_{\rm Leb}(f)$ the topological entropy and the metric entropy with respect to Leb of f, respectively. The following relations for the entropies and Lyapunov exponents considered are known in this setting.

- For an Anosov automorphism $L\colon \mathbb{T}^n \to \mathbb{T}^n$, we have $h_{\text{top}}(L) = h_{\text{Leb}}(L) = \sum_{i=1}^{u(L)} \lambda_{i,\text{Leb}}(L)$.
- Since h_{top} is an invariant of topological conjugacy [**KH95**, Corollary 3.1.4], we obtain that if f is homotopic to an Anosov automorphism L, then $h_{\text{top}}(f) = h_{\text{top}}(L)$.
- Since f is volume-preserving, we have $\sum_{i=1}^{n} \lambda_{i,\text{Leb}}(f) = \sum_{i=1}^{n} \lambda_{i,\nu_f}(f) = 0$.
- Variational principle for entropies [**KH95**, Theorem 4.5.3]: $h_{\text{Leb}}(f) \le h_{\text{top}}(f)$.
- Ruelle's inequality [**Rue78a**]: $h_{\text{top}}(f) \le \sum_{i=1}^{u(f)} \lambda_{i,\nu_f}(f)$.
- Pesin's entropy formula [**BP07**, Theorem 10.4.1]: $h_{\text{Leb}}(f) = \sum_{i=1}^{u(f)} \lambda_{i,\text{Leb}}(f)$.

Thus, some representative questions (ordered by increasing number of requirements) concerning flexibility of Lyapunov exponents for Anosov volume-preserving diffeomorphisms on $\mathbb{T}^n$ are as follows.

Question 1. (Conjecture 1.4 in [**BKRH19**]: weak flexibility for one measure) Given any list of non-zero numbers $\xi_1 \ge \cdots \ge \xi_n$ such that $\sum_{i=1}^{n} \xi_i = 0$, does there exist a smooth volume-preserving Anosov diffeomorphism f of $\mathbb{T}^n$ such that

$$(\lambda_{1,\text{Leb}}(f), \ldots, \lambda_{n,\text{Leb}}(f)) = (\xi_1, \ldots, \xi_n)?$$

Question 2. (Problem 1.3 in [**BKRH19**]: strong flexibility for one measure) Let $L\colon \mathbb{T}^n \to \mathbb{T}^n$, $n \ge 2$, be a volume-preserving Anosov automorphism with the unstable index u. Given any list of numbers

$$\xi_1 \ge \cdots \ge \xi_u > 0 > \xi_{u+1} \ge \cdots \xi_n$$

such that

$$\sum_{i=1}^{n} \xi_i = 0 \quad \text{and} \quad \sum_{i=1}^{u} \xi_i \le h_{\text{top}}(L),$$

does there exist a smooth volume-preserving Anosov diffeomorphism $f\colon \mathbb{T}^n \to \mathbb{T}^n$ homotopic to L such that

$$(\lambda_{1,\text{Leb}}(f), \ldots, \lambda_{n,\text{Leb}}(f)) = (\xi_1, \ldots, \xi_n)?$$

Question 3. (Weak flexibility for two measures) Let $n \in \mathbb{N} \setminus \{1\}$ and $u \in \mathbb{N} \cap [1, n)$. Given any two lists of numbers

$$\xi_1 \ge \cdots \ge \xi_u > 0 > \xi_{u+1} \ge \cdots \xi_n \quad \text{and} \quad \eta_1 \ge \cdots \ge \eta_u > 0 > \eta_{u+1} \ge \cdots \ge \eta_n$$

such that

$$\sum_{i=1}^{n} \xi_i = 0, \quad \sum_{i=1}^{n} \eta_i = 0, \quad \text{and} \quad \sum_{i=1}^{u} \xi_i \le \sum_{i=1}^{u} \eta_i,$$

does there exist a smooth volume-preserving Anosov diffeomorphism $f\colon \mathbb{T}^n \to \mathbb{T}^n$ such that

$$(\lambda_{1,\text{Leb}}(f), \ldots, \lambda_{n,\text{Leb}}(f)) = (\xi_1, \ldots, \xi_n) \quad \text{and}$$
$$(\lambda_{1,\nu_f}(f), \ldots, \lambda_{n,\nu_f}(f)) = (\eta_1, \ldots, \eta_n)?$$

Question 4. (Strong flexibility for two measures) Let $L\colon \mathbb{T}^n \to \mathbb{T}^n$, $n \geq 2$, be a volume-preserving Anosov automorphism with the unstable index u. Given any two lists of numbers

$$\xi_1 \geq \cdots \geq \xi_u > 0 > \xi_{u+1} \geq \cdots \xi_n \quad \text{and} \quad \eta_1 \geq \cdots \geq \eta_u > 0 > \eta_{u+1} \geq \cdots \geq \eta_n$$

such that

$$\sum_{i=1}^{n} \xi_i = 0, \quad \sum_{i=1}^{n} \eta_i = 0, \quad \text{and} \quad \sum_{i=1}^{u} \xi_i < h_{\text{top}}(L) < \sum_{i=1}^{u} \eta_i,$$

does there exist a smooth volume-preserving Anosov diffeomorphism $f\colon \mathbb{T}^n \to \mathbb{T}^n$ homotopic to L such that

$$(\lambda_{1,\text{Leb}}(f), \ldots, \lambda_{n,\text{Leb}}(f)) = (\xi_1, \ldots, \xi_n) \quad \text{and}$$
$$(\lambda_{1,\nu_f}(f), \ldots, \lambda_{n,\nu_f}(f)) = (\eta_1, \ldots, \eta_n)?$$

Corollary 1.6 in [**BKRH19**] gives a positive answer to Question 1 when formulated with strict inequalities among the given numbers. For $n = 2$, it is folklore among specialists that the positive answer to Question 2 was already known by A. Katok using a fairly straightforward global twist construction. The positive answer for $n = 2$ and partial answer for $n > 2$ follow from [**BKRH19**, Theorem 1.5]. Moreover, [**BKRH19**, Theorem 1.7] provides the full solution of Question 2 for $\mathbb{T}^3$ with additional restrictions and the requirement of simple dominated splitting.

In this paper we study Question 4 and provide a positive answer for $n = 2$ (see Theorem A). This result can be considered as the two-dimensional version of [**Erc19**]. All in all, this work differs from [**BKRH19**] by considering flexibility for a pair of exponents instead of a single exponent and by using a more explicit construction. The main difficulty here lies in controlling the measure of maximal entropy. Essentially, the only way to estimate the measures of sets with respect to the measure of maximal entropy is to use Markov partitions which are difficult to understand explicitly for a general Anosov diffeomorphism.

1.2. *Formulation of the result.* Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. Let f be a smooth area-preserving Anosov diffeomorphism on $\mathbb{T}^2$ homotopic to an Anosov automorphism L. We denote by $\lambda_{\text{abs}}(f)$ and $\lambda_{\text{mme}}(f)$ the positive Lyapunov exponents of f with respect to Leb and to the measure of maximal entropy ν_f, respectively.

Thus, summarizing §1.1 with $n = 2$, we have two possibilities: either

$$0 < \lambda_{\text{abs}}(f) < h_{\text{top}}(L) < \lambda_{\text{mme}}(f)$$

or

$$\lambda_{\text{abs}}(f) = \lambda_{\text{mme}}(f) = h_{\text{top}}(L).$$

Question 4 asks if the above relations are the only relations between λ_{abs} and λ_{mme}. Our main theorem shows that this is indeed the case.

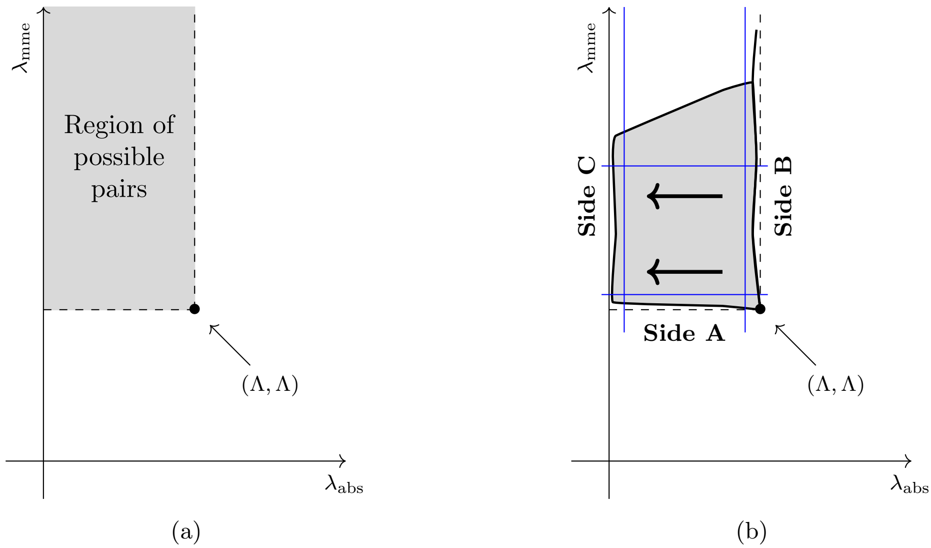

FIGURE 1. (a) Possible values of $(\lambda_{\text{abs}}, \lambda_{\text{mme}})$, (b) Constructions from the proof.

THEOREM A. *Suppose L_A is an Anosov automorphism on $\mathbb{T}^2$ which is induced by a matrix $A \in SL(2, \mathbb{Z})$ with $|\text{trace}(A)| > 2$. Let $\Lambda = h_{\text{top}}(L_A)$. For any $\Lambda_{\text{abs}}, \Lambda_{\text{mme}} \in \mathbb{R}$ such that $0 < \Lambda_{\text{abs}} < \Lambda < \Lambda_{\text{mme}}$, there exists a smooth area-preserving Anosov diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to L_A such that $\lambda_{\text{abs}}(f) = \Lambda_{\text{abs}}$ and $\lambda_{\text{mme}}(f) = \Lambda_{\text{mme}}$.*

The shaded area plus its lower right corner in Figure 1 shows the set of all possible values for pairs $(\lambda_{\text{abs}}, \lambda_{\text{mme}})$ in the setting of Theorem A.

Using that there is no retraction of a square onto its boundary and continuity of the Lyapunov exponents in the constructed family (see Remark 1.1), Theorem A can be reduced to the following result.

THEOREM B. *Suppose L_A is an Anosov automorphism on $\mathbb{T}^2$ which is induced by a matrix $A \in SL(2, \mathbb{Z})$ with $|\text{trace}(A)| > 2$. Let $\Lambda = h_{\text{top}}(L_A)$. For any positive numbers γ, S and T such that $\Lambda < S < T$, there exists a smooth family $\{f_{s,t}\}$, where $(s, t) \in [0, 1] \times [0, 1]$, of area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ homotopic to L_A such that the following hold:*

(1) $\Lambda - \gamma < \lambda_{\text{abs}}(f_{s,0}) \leq \Lambda$ *for all* $s \in [0, 1]$;
(2) $\lambda_{\text{abs}}(f_{s,1}) < \gamma$ *for all* $s \in [0, 1]$;
(3) $\lambda_{\text{mme}}(f_{0,t}) < S$ *for all* $t \in [0, 1]$;
(4) $\lambda_{\text{mme}}(f_{1,t}) > T$ *for all* $t \in [0, 1]$.

Remark 1.1. In a smooth family of Anosov area-preserving diffeomorphisms on $\mathbb{T}^2$, the Lyapunov exponents with respect to Leb and the measure of maximal entropy vary continuously. For the Lebesgue measure, this follows immediately from (1.2) and the fact that the unstable distribution E^u varies continuously. In addition, the measure of maximal entropy depends continuously on the dynamics in the weak* topology. To see this, we can use the fact [**Mos69**, Theorem 1] that for a smooth family of Anosov area-preserving diffeomorphisms, the topological conjugacy to L_A is continuous in the parameters of the family. Moreover, the measure of maximal entropy is mapped to the measure of maximal entropy by the conjugacy. For the families we consider, the continuity of the measure of maximal entropy can alternatively be seen directly using the constructed Markov partition (see §§2.3 and 3.3).

1.3. *Outline of the proof.* To prove Theorem B, we construct (large) smooth (area-preserving) homotopic deformations of Anosov automorphisms, that is, deformations preserving the homotopy class, and estimate λ_{abs} and λ_{mme} of the resulting Anosov diffeomorphisms. We refer to Figure 1(b) in what follows.

- Without loss of generality, we assume that $\mathrm{trace}(A) > 2$. It is enough to prove Theorem B in that case because of the following argument. Assume that $B \in SL(2, \mathbb{Z})$ with $\mathrm{trace}(B) < -2$. Then $-B \in SL(2, \mathbb{Z})$, $\mathrm{trace}(-B) > 2$, and $h_{\mathrm{top}}(L_B) = h_{\mathrm{top}}(L_{-B}) = \Lambda$. Assume that $\Lambda_{\mathrm{abs}}, \Lambda_{\mathrm{mme}} \in \mathbb{R}$ such that $0 < \Lambda_{\mathrm{abs}} < \Lambda < \Lambda_{\mathrm{mme}}$ and there exists an area-preserving Anosov diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ homotopic to L_{-B} such that $\lambda_{\mathrm{abs}}(f) = \Lambda_{\mathrm{abs}}$ and $\lambda_{\mathrm{mme}}(f) = \Lambda_{\mathrm{mme}}$. Let $-I = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Then $L_{-I} \circ f : \mathbb{T}^2 \to \mathbb{T}^2$ is an area-preserving Anosov diffeomorphism homotopic to L_B such that $\lambda_{\mathrm{abs}}(L_{-I} \circ f) = \Lambda_{\mathrm{abs}}$ and $\lambda_{\mathrm{mme}}(L_{-I} \circ f) = \Lambda_{\mathrm{mme}}$.
- In §2 we describe a homotopic deformation that produces for any given positive number H and any small positive number γ a curve in the set of possible values of $(\lambda_{\mathrm{abs}}, \lambda_{\mathrm{mme}})$ γ-close to side B with one endpoint at (Λ, Λ) and the other endpoint corresponding to a smooth area-preserving Anosov diffeomorphism with λ_{mme} larger than H. The resulting diffeomorphisms have a large twist on a thin strip. In §2.2, we find a lower bound on λ_{abs}. In §2.3 we provide a lower bound on λ_{mme} by controlling the measure of maximal entropy of the constructed diffeomorphisms through Markov partitions. The following theorem summarizes the result in §2.

THEOREM C. (Theorem 2.1) *Suppose L_A is an Anosov automorphism on $\mathbb{T}^2$ which is induced by a matrix $A \in SL(2, \mathbb{Z})$ with* $\mathrm{trace}(A) > 2$. *Let $\Lambda = h_{\mathrm{top}}(L_A)$ and $H > 0$. For any positive number γ, there exists a smooth family $\{g_s\}_{s\in[0,1]}$ of area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ homotopic to L_A such that $g_0 = L_A$ and the following assertions hold:*

(A) *$g_s = L_A$ in a neighborhood of $(0, 0)$ for all $s \in [0, 1]$;*
(B) *$\Lambda - \gamma < \lambda_{\mathrm{abs}}(g_s) \le \Lambda$ for all $s \in [0, 1]$;*
(C) *$\lambda_{\mathrm{mme}}(g_1) > H$.*

- In §3, starting from the Anosov diffeomorphisms from the first construction with λ_{mme} in some range of values, we modify them in a smooth way by a slow-down deformation near a fixed point to get Anosov diffeomorphisms realizing a curve in the set of possible pairs $(\lambda_{\mathrm{abs}}, \lambda_{\mathrm{mme}})$ arbitrarily close to side C (see Lemma 3.5). In this construction, we are able to keep the lower boundary of the realized pairs of exponents arbitrarily close to side A (see Lemma 3.7) and the upper boundary above a line $\lambda_{\mathrm{mme}} = T$ (see Lemma 3.10), where T depends on the initial range of values of λ_{mme} for the diffeomorphisms coming from the first construction. As a result, we produce a two-parametric family of Anosov diffeomorphisms coming from homotopic deformations covering any given rectangle within the semi-infinite strip that is the set of possible values of $(\lambda_{\mathrm{abs}}, \lambda_{\mathrm{mme}})$. The following theorem summarizes the result in §3.

THEOREM D. (Theorem 3.1) *Suppose L_A and Λ are as in Theorem C. For any H such that $\Lambda < H$ and positive number γ, let $\{g_s\}_{s\in[0,1]}$ be a smooth family of area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ homotopic to L_A from Theorem C applied for γ and H with lower bound on $\lambda_{\mathrm{mme}}(g_1)$ coming from Lemma 2.7 being larger than H. Then there exists a constant $\tilde{C}$ such that for any $\sigma > 0$, $S > \Lambda$ there exists a smooth family $\{f_{s,t}\}_{(s,t)\in[0,1]\times[0,1]}$ of Anosov diffeomorphisms on $\mathbb{T}^2$ homotopic to L_A such that:*

(A) *$f_{s,0} = g_s$ for all $s \in [0, 1]$;*
(B) *$f_{s,t}$ preserves a probability measure $\mu_{s,t}$ which is absolutely continuous with respect to the Lebesgue measure;*
(C) *$\lambda_{\mathrm{abs}}(f_{s,1}) < \gamma$ for all $s \in [0, 1]$;*
(D) *$\lambda_{\mathrm{mme}}(f_{0,t}) < S$ for all $t \in [0, 1]$;*
(E) *$\lambda_{\mathrm{mme}}(f_{1,t}) \geq H + \tilde{C}\sigma$.*

- Let $\{f_{s,t}\}_{(s,t)\in[0,1]\times[0,1]}$ be the family of Anosov diffeomorphisms in Theorem D. By the Dacorogna–Moser theorem [**HJJ17**, Theorem, Appendix A], there exists a C^∞ family $\{\Psi_{s,t}\}$ of C^∞ diffeomorphisms of $\mathbb{T}^2$ satisfying $\Psi^*_{s,t}\mu_{s,t} = \mathrm{Leb}$. Let $\tilde{f}_{s,t} = \Psi_{s,t} f_{s,t} \Psi^{-1}_{s,t}$. Then $\{\tilde{f}_{s,t}\}$ is a C^∞ family of Anosov diffeomorphisms on $\mathbb{T}^2$ that preserve Leb. Since the conjugacy is smooth, we obtain $\lambda_{\mathrm{abs}}(\tilde{f}_{s,t}) = \lambda_{\mathrm{abs}}(f_{s,t})$ and $\lambda_{\mathrm{mme}}(\tilde{f}_{s,t}) = \lambda_{\mathrm{mme}}(f_{s,t})$. Therefore, we obtain Theorem B if we apply Theorem D for $H = 2T$ and sufficiently small σ.

1.4. *Further questions for Anosov diffeomorphisms of* $\mathbb{T}^2$. Interestingly, Theorem A can be reformulated as a statement on flexibility for the pressure function among smooth area-preserving Anosov diffeomorphisms homotopic to a fixed Anosov automorphism as follows.

Let $\phi_t^f(x) = -t \log |Df|_{E_u(x)}|$ for any $x \in \mathbb{T}^2$. This is called the *geometric potential*. The *pressure function* for the potential ϕ_t^f is defined by

$$P(\phi_t^f) = \sup_\mu \bigg(h_\mu(f) + \int_{\mathbb{T}^2} \phi_t^f \, d\mu\bigg),$$

where the supremum is taken over all f-invariant probability measures on $\mathbb{T}^2$ and $h_\mu(f)$ denotes the measure-theoretical entropy of f with respect to μ. It is known that $P(\phi_0^f) = h_{\mathrm{top}}(f)$ and $P(\phi_1^f) = 0$. Also, $P(\phi_t^f)$ is a convex real analytic function of t (see, for example, [**Rue78b**, §§0.2 and 4.6] and [**BG14**]). Since $\int_{\mathbb{T}^2} \phi_t^f \, d\mu$ becomes a dominated term as t tends to $\pm\infty$, we have that $P(\phi_t^f)$ has asymptotes as $t \to \pm\infty$. Moreover, $(d/dt)P(\phi_t^f)|_{t=0} = -\lambda_{\mathrm{mme}}(f)$ and $(d/dt)P(\phi_t^f)|_{t=1} = -\lambda_{\mathrm{abs}}(f)$. Thus, Theorem A shows that we can vary the derivatives of the pressure function at $t = 0$ and $t = 1$. As a result, a more general flexibility question can be formulated in the setting of Theorem A.

Question 5. Let L be an Anosov automorphism on $\mathbb{T}^2$. Given a strictly convex real analytic function $F: \mathbb{R} \to \mathbb{R}$ such that $F(0) = h_{\text{top}}(L)$, $F(1) = 0$, $(dF/dt)|_{t=0} < -h_{\text{top}}(L)$, $(dF/dt)|_{t=1} \in (-h_{\text{top}}(L), 0)$, and $F(t)$ has asymptotes as $t \to \pm\infty$. Does there exist a smooth area-preserving Anosov diffeomorphism f homotopic to L such that $P(\phi_t^f) = F(t)$?

The answer to the above question will require different techniques than presented in this paper. If the answer is negative then it would be interesting to determine which extra conditions on the function must be satisfied. For example, do the higher derivatives of the pressure function [**KS01**] provide any additional restrictions? Is there a finite list of conditions that must be added in order to obtain flexibility?

We can also consider a rigidity problem connected to the pressure function. Let f be a smooth area-preserving Anosov diffeomorphism homotopic to an Anosov automorphism L. By work of de la Llave [**dlL87**] and of Marco and Moriyón [**MM87**], we have that if $\lambda_{\text{abs}}(f) = h_{\text{top}}(L)$, then f and L are C^∞ conjugate. By the properties of the pressure functions discussed above, we have that if $P(\phi_t^f) = P(\phi_t^L)$ for all $t \in \mathbb{R}$, then f and L are C^∞ conjugate. A natural question is whether L can be replaced by any smooth area-preserving Anosov diffeomorphism.

Question 6. Let f and g be smooth area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ that are homotopic. Assume $P(\phi_t^f) = P(\phi_t^g)$ for all $t \in \mathbb{R}$. Does this imply that f and g are C^∞ conjugate?

2. *Construction I*

In this section we prove Theorem 2.1 using several lemmas. We begin by showing how to deduce the theorem from these lemmas before stating and proving the lemmas themselves.

THEOREM 2.1. *Suppose L_A is an Anosov automorphism on $\mathbb{T}^2$ which is induced by a matrix $A \in SL(2, \mathbb{Z})$ with* $\text{trace}(A) > 2$. *Let $\Lambda = h_{\text{top}}(L_A)$ and $H > 0$. For any positive number γ, there exists a smooth family $\{g_s\}_{s\in[0,1]}$ of area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ homotopic to L_A such that $g_0 = L_A$ and the following assertions hold:*

(A) $g_s = L_A$ *in a neighborhood of* $(0, 0)$ *for all* $s \in [0, 1]$*;*
(B) $\Lambda - \gamma < \lambda_{\text{abs}}(g_s) \le \Lambda$ *for all* $s \in [0, 1]$*;*
(C) $\lambda_{\text{mme}}(g_1) > H$.

Proof. Fix $m \in (0, 1)$. Let $l \in (0, l_\gamma)$ where l_γ comes from Lemma 2.6 applied for $\varepsilon = \gamma$. Let $\beta = \beta_0$ from Lemma 2.7. Choose $\tilde{\delta}$ small enough such that

$$Q \log \mu^+((\beta_0 l + \tilde{\delta}(1 - \beta_0))/\tilde{\delta}) + (1 - Q) \log C > H,$$

where $Q, C, \mu^+(\cdot)$ are as in Lemma 2.7. Note that this is possible because $\mu^+(t) \to \infty$ as $t \to \infty$. Also, let $w \in (0, w_0(\tilde{\delta}))$. Then the family of maps $F_{l,\delta}^w$ defined in (2.4) where δ varies in $[\tilde{\delta}, l]$ is the desired family of maps. □

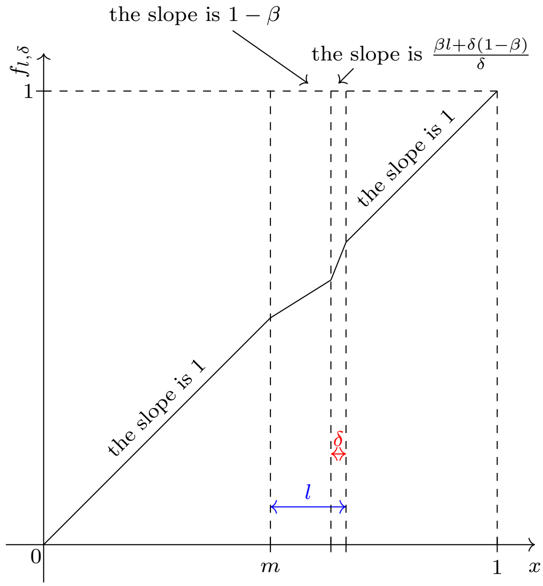

FIGURE 2. The graph of $f_{l,\delta}$.

2.1. *Construction I: Anosov twist diffeomorphisms.* Here we give an explicit formula for a family of smooth area-preserving twist diffeomorphisms (2.4) that are Anosov for an appropriate choice of parameters (see Lemma 2.4). A smooth subfamily of these diffeomorphisms is used to prove Theorem 2.1.

Suppose L_A is an Anosov automorphism on $\mathbb{T}^2$ which is induced by a matrix $A \in SL(2,\mathbb{Z})$ such that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and $a+d>2$.

Fix $m \in (0,1)$, let $l \in (0, 1-m)$ and $\delta \in (0,l)$. Choose $\beta \in (0,1)$ such that $a+d-|b|\beta > 2$. Notice that such β exists because $a+d>2$.

Define a function $f_{l,\delta}\colon [0,1] \to [0,1]$ (see Figure 2) in the following way:

$$f_{l,\delta}(x) = \begin{cases} (1-\beta)x+\beta m & \text{if } m < x \le m+l-\delta, \\ \dfrac{\beta l+\delta(1-\beta)}{\delta}x - \dfrac{\beta(m+l)(l-\delta)}{\delta} & \text{if } m+l-\delta < x \le m+l, \\ x & \text{otherwise.} \end{cases} \tag{2.1}$$

The following lemma allows us to obtain a smooth function that coincides with the given continuous piecewise linear function outside small neighborhoods of points of non-smoothness.

LEMMA 2.2. *Let $f_{l,\delta}(x)$ be as in (2.1) and $w \in (0, \delta/4)$. Denote by $\hat{f}_{l,\delta}(x)$ the continuous map on $\mathbb{R}$ that is a lift of $f_{l,\delta}(x)$ such that $\hat{f}_{l,\delta}(0) = f_{l,\delta}(0)$. Let $\theta_w(x)$ be a smooth positive even function on $\mathbb{R}$ such that $\int_{\mathbb{R}} \theta_w(y)dy = 1$ and $\theta_w(x) = 0$ if $x \notin (-w,w)$,* *where $w > 0$ is sufficiently small. Define*

$$\hat{f}^w_{l,\delta}(x) = \int_{\mathbb{R}} \hat{f}_{l,\delta}(x-y)\theta_w(y)dy \text{ for any } x \in \mathbb{R} \quad \text{and} \quad f^w_{l,\delta}(x) = \hat{f}^w_{l,\delta}(x) \pmod 1. \tag{2.2}$$

Then we have that $f^w_{l,\delta}$ is a smooth function on $\mathbb{R}/\mathbb{Z}$. Moreover, $f^w_{l,\delta}(x) = f_{l,\delta}(x)$ outside of w-neighborhoods of the points $x = m,\ m+l-\delta,\ m+l$.

In particular,

$$\begin{cases} 1-\beta \leq D_x f^w_{l,\delta} \leq 1 & \text{if } m-w < x \leq m+w, \\ D_x f^w_{l,\delta} = 1-\beta & \text{if } m+w < x \leq m+l-\delta-w, \\ 1-\beta \leq D_x f^w_{l,\delta} \leq \dfrac{\beta l + \delta(1-\beta)}{\delta} & \text{if } m+l-\delta-w < x \leq m+l-\delta+w, \\ D_x f^w_{l,\delta} = \dfrac{\beta l + \delta(1-\beta)}{\delta} & \text{if } m+l-\delta+w < x \leq m+l-w, \\ 1 \leq D_x f^w_{l,\delta} \leq \dfrac{\beta l + \delta(1-\beta)}{\delta} & \text{if } m+l-w < x \leq m+l+w, \\ D_x f^w_{l,\delta} = 1 & \text{otherwise.} \end{cases} \tag{2.3}$$

Proof. See the proof of [Erc19, Lemma 3.1]. □

We define $f^0_{l,\delta} = f_{l,\delta}$. For any sufficiently small $w \geq 0$, we consider a family of maps $F^w_{l,\delta}\colon \mathbb{T}^2 \to \mathbb{T}^2$, where

$$F^w_{l,\delta}(x,y) = \begin{pmatrix} (a-|b|)x + by + |b| f^w_{l,\delta}(x) \\ (c-\operatorname{sgn}(b)d)x + dy + \operatorname{sgn}(b)d f^w_{l,\delta}(x) \end{pmatrix} \mod 1. \tag{2.4}$$

In particular, we have $F^w_{l,\delta} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on $\{(x,y) \in \mathbb{T}^2 | x \in [0, m-w] \cup [m+l+w, 1)\}$.

By the choice of β, Lemma 2.2, and Lemma 2.4 below, we have that for all $l \in (0, 1-m)$, $\delta \in (0, l)$, and sufficiently small $w > 0$ the map $F^w_{l,\delta}$ is an area-preserving Anosov diffeomorphism homotopic to L_A.

We will need the following general facts.

Proposition 2.3. Let

$$A(t) = \begin{pmatrix} a + |b|(t-1) & b \\ c + \operatorname{sgn}(b)d(t-1) & d \end{pmatrix} \tag{2.5}$$

where $a, b, c, d, t \in \mathbb{R}$, $ad - bc = 1$, and $a + d + (t-1)|b| > 2$. Then $\det(A) = 1$ and the eigenvalues of $A(t)$ are

$$\mu^{\pm}(t) = \tfrac{1}{2}\left(a + d + |b|(t-1) \pm \sqrt{(a+d+|b|(t-1))^2 - 4}\right) \tag{2.6}$$

with corresponding eigenvectors

$$\mathbf{e}^{\pm}(t) = \begin{pmatrix} 2b \\ \phi^{\pm}(a+d+|b|(t-1)) \end{pmatrix}, \tag{2.7}$$

where $\phi^{\pm}(u) = 2d - u \pm \sqrt{u^2-4}$. In particular, $\mu^+(t) > 1$ and $0 < \mu^-(t) < 1$.

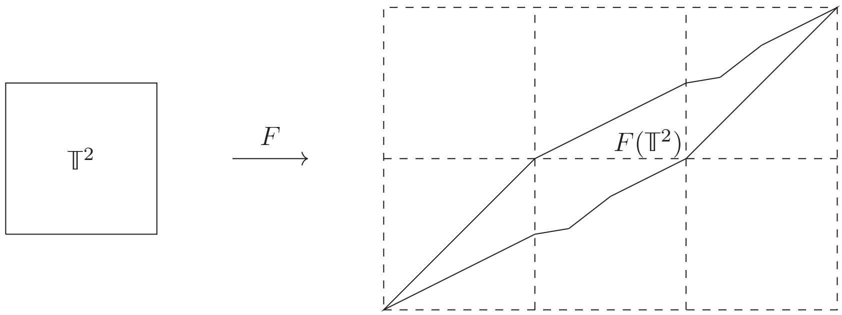

FIGURE 3. The image of $\mathbb{T}^2$ under F obtained from $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $f = f^w_{1/4,1/16}$ with parameters $w > 0$ and $m = \frac{1}{2}$.

Proof. Follows from straightforward computations. $\square$

LEMMA 2.4. *Suppose L_A is an Anosov area-preserving automorphism on $\mathbb{T}^2$ which is induced by a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ with* trace$(A) > 2$. *Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a smooth diffeomorphism such that $f(0) = 0$, $0 < \alpha_1 \le D_x f$ for all $x \in \mathbb{R}/\mathbb{Z}$, and $a + d + (\alpha_1 - 1)|b| > 2$. Define a map $F : \mathbb{T}^2 \to \mathbb{T}^2$ in the following way:*

$$F(x, y) = \begin{pmatrix} (a - |b|)x + by + |b| f(x) \\ (c - \mathrm{sgn}(b)d)x + dy + \mathrm{sgn}(b)df(x) \end{pmatrix} \mod 1.$$

Then F is a smooth area-preserving Anosov diffeomorphism on $\mathbb{T}^2$.

Moreover, let $\tilde{\alpha} > 0$ such that $\tilde{\alpha} < \alpha_1$ and $a + d + (\tilde{\alpha} - 1)|b| > 2$. Define the following vectors:

$$\mathbf{v}^+_{\min}(t) = \begin{pmatrix} 2b \\ \phi^+(a + d + |b|t) \end{pmatrix}, \quad \mathbf{v}^+_{\max} = \begin{pmatrix} b \\ d \end{pmatrix},$$
$$\mathbf{v}^-_{\min}(t) = \begin{pmatrix} 2b \\ \phi^-(a + d + |b|t) \end{pmatrix}, \quad \text{and} \quad \mathbf{v}^-_{\max} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \tag{2.8}$$

where $\phi^\pm$ as in Proposition 2.3 and $t > (2 - (a + d))/|b|$. Let $\mathcal{C}^\pm$ be the union of the positive cone spanned by $\mathbf{v}^\pm_{\min}(\tilde{\alpha} - 1)$ and $\mathbf{v}^\pm_{\max}$ and its symmetric complement with vertex at $(0, 0)$ in $\mathbb{R}^2$, respectively. Then the cones $\mathcal{C}^\pm$ in $T_{(x,y)}\mathbb{T}^2$ for all $(x, y) \in \mathbb{T}^2$ define a system of invariant cones for F. Also, there exist $\mu > 1 > \nu > 0$ that depend only on the entries of A and $\tilde{\alpha}$ such that for all $\mathbf{v} \in \mathcal{C}^+$ we have $\|D_{(x,y)}F\mathbf{v}\| \ge \mu\|\mathbf{v}\|$ and for all $\mathbf{v} \in \mathcal{C}^-$ we have $\|D_{(x,y)}F^{-1}\mathbf{v}\| \ge \nu^{-1}\|\mathbf{v}\|$ for any $(x, y) \in \mathbb{T}^2$.

Remark 2.5. We apply Lemma 2.4 for the family of functions $f^w_{l,\delta}$ (see (2.2)) which satisfy $f^w_{l,\delta}(0) = 0$ and $D_x f^w_{l,\delta} \ge 1 - \beta$ for all $x \in \mathbb{R}/\mathbb{Z}$ (see (2.3)). Recall that we only work with $\beta \in (0, 1)$ such that $a + d - |b|\beta > 2$.

Proof. Notice that F is a smooth map from $\mathbb{T}^2$ to $\mathbb{T}^2$ with Jacobian equal to 1, so it is an area-preserving diffeomorphism.

We show that F is Anosov using invariant cones [**KH95**, Corollary 6.4.8].

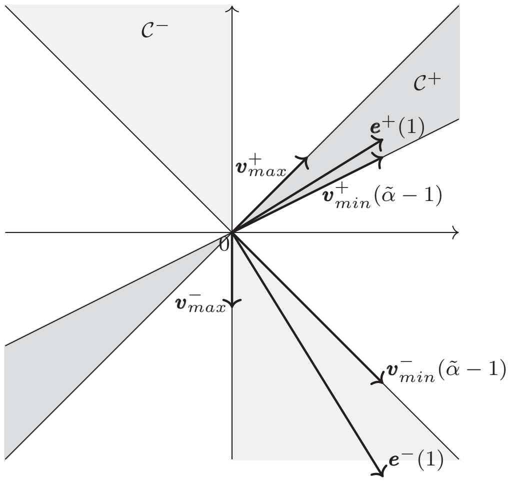

FIGURE 4. The cones $\mathcal{C}^\pm$ for F obtained from $A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ with $\tilde\alpha = 0.5$. Here $\mathbf{e}^\pm(1)$ are eigenvectors of A.

Define $A(t) = \left(\begin{smallmatrix} a+|b|(t-1) & b \\ c+\mathrm{sgn}(b)d(t-1) & d \end{smallmatrix}\right)$. In particular, we have $D_{(x,y)}F = A(D_x f)$ for any $(x, y) \in \mathbb{T}^2$.

We point out some properties of $\phi^\pm$. First, we have that $\phi^+(u)$ is monotonically increasing and $\phi^-(u)$ is monotonically decreasing for $u > 2$. In particular, $2(d-1) \le \phi^+(u) < 2d$ for $u > 2$. Also, both functions are smooth for $u > 2$. Moreover, $\lim_{u\to+\infty} \phi^+(u) = 2d$ and $\lim_{u\to+\infty} \phi^-(u) = -\infty$. Therefore, we obtain that $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$. Moreover, for any $t \ge \alpha_1 > \tilde\alpha$, we have

$$A(t)\mathcal{C}^+ \subset \mathrm{Int}\,(\mathcal{C}^+) \quad \text{and} \quad A(t)^{-1}\mathcal{C}^- \subset \mathrm{Int}\,(\mathcal{C}^-), \tag{2.9}$$

where Int stands for the interior of the set. See Figure 4 for an example of $\mathcal{C}^\pm$.

We consider the inner product of $\mathbf{v}^+_{\min}(\tilde\alpha-1)$ and $\mathbf{v}^+_{\max}$:

$$\langle \mathbf{v}^+_{\min}(\tilde\alpha-1), \mathbf{v}^+_{\max}\rangle = 2b^2 + d\phi^+(a+d+|b|(\tilde\alpha-1)) \ge \begin{cases} 2b^2+2d(d-1) & \text{if } d>0, \\ 2b^2+2d^2 & \text{if } d \le 0. \end{cases} \tag{2.10}$$

By (2.10), using the fact that $d \in \mathbb{Z}$, we have that $\langle \mathbf{v}^+_{\min}(\tilde\alpha-1), \mathbf{v}^+_{\max}\rangle > 0$.

Recall that $\mathbf{v}^+_{\min}(\tilde\alpha-1)$ is an eigenvector of $A(\tilde\alpha)$ with an eigenvalue $\mu^+(\tilde\alpha) > 1$. Therefore,

$$\begin{aligned} \|A(t)\mathbf{v}^+_{\min}(\tilde\alpha-1)\|^2 &= \left\| \left[A(\tilde\alpha) + \begin{pmatrix} |b|(t-\tilde\alpha) & 0 \\ \mathrm{sgn}(b)d(t-\tilde\alpha) & 0 \end{pmatrix} \right] \mathbf{v}^+_{\min}(\tilde\alpha-1) \right\|^2 \\ &= \left\| A(\tilde\alpha)\mathbf{v}^+_{\min}(\tilde\alpha-1) + 2(t-\tilde\alpha)b \begin{pmatrix} |b| \\ \mathrm{sgn}(b)d \end{pmatrix} \right\|^2 \\ &= \|\mu^+(\tilde\alpha)\mathbf{v}^+_{\min}(\tilde\alpha-1) + 2|b|(t-\tilde\alpha)\mathbf{v}^+_{\max}\|^2 \\ &= (\mu^+(\tilde\alpha)\|\mathbf{v}^+_{\min}(\tilde\alpha-1)\|)^2 + 4\mu^+(\tilde\alpha)(t-\tilde\alpha)|b|\langle \mathbf{v}^+_{\min}(\tilde\alpha-1), \mathbf{v}^+_{\max}\rangle \\ &\quad + (2(t-\tilde\alpha)|b|\|\mathbf{v}^+_{\max}\|)^2. \end{aligned}$$

Therefore, since $\mu^+(\tilde{\alpha}) > 1$ and $\langle \mathbf{v}^+_{\min}(\tilde{\alpha}-1), \mathbf{v}^+_{\max}\rangle > 0$, for any $t > \tilde{\alpha}$ we have $\|A(t)\mathbf{v}^+_{\min}(\tilde{\alpha}-1)\| \geq \mu^+(\tilde{\alpha})\|\mathbf{v}^+_{\min}(\tilde{\alpha}-1)\|$.

Moreover, using that $ad - bc = 1$, $d \in \mathbb{Z}$, and $a + |b|(t-1) + d > 2$ for any $t > \tilde{\alpha}$, we have

$$\begin{aligned}
\|A(t)\mathbf{v}^+_{\max}\|^2 &= \left\| \begin{pmatrix} a+|b|(t-1) & b \\ c + \operatorname{sgn}(b)d(t-1) & d \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} b(a+|b|(t-1)+d) \\ bc + |b|d(t-1) + d^2 \end{pmatrix} \right\|^2 \\
&= \left\| \begin{pmatrix} b(a+|b|(t-1)+d) \\ -1 + d(a+|b|(t-1)+d) \end{pmatrix} \right\|^2 \\
&= b^2(a+d+|b|(t-1))^2 + (d(a+d+|b|(t-1)) - 1)^2 \geq 4b^2 + (2d-1)^2 \\
&= \left(1 + \frac{3b^2 + 3d^2 - 4d + 1}{b^2 + d^2}\right)\|\mathbf{v}^+_{\max}\|^2.
\end{aligned}$$

In particular, $\|A(t)\mathbf{v}^+_{\max}\| \geq \mu^+_{\max}\|\mathbf{v}^+_{\max}\|$, where $\mu^+_{\max} = 1 + (3b^2 + 3d^2 - 4d + 1)/(b^2 + d^2) > 1$ because $3d^2 - 4d + 1 \geq 0$ for $d \in \mathbb{Z}$ and $b \neq 0$ as $a, d \in \mathbb{Z}$, $ad - bc = 1$, and $\operatorname{trace}(A) > 2$.

Let $\mu = \min\{\mu^+(\tilde{\alpha}), \mu^+_{\max}\} > 1$. By the properties of ϕ^+ and the fact that $\operatorname{trace}(A(t)) > 2$ for $t > \tilde{\alpha}$, we have that the expanding eigenvectors of $A(t)$ for any $t > \tilde{\alpha}$ belong to $\mathcal{C}^+$ (see (2.7)). Moreover, for any $t > \tilde{\alpha}$, if the set of all vectors in $\mathbb{R}^2$ that expand at least μ times by $A(t)$ is non-empty, then it is a cone containing an expanding eigenvector of $A(t)$. As shown above, $\mathbf{v}^+_{\min}(\tilde{\alpha}-1)$ and $\mathbf{v}^+_{\max}$ expand at least μ times by $A(t)$ for any $t > \tilde{\alpha}$. Since $\mathcal{C}^+$ is the cone spanned by $\mathbf{v}^+_{\min}(\tilde{\alpha}-1)$ and $\mathbf{v}^+_{\max}$, we have $\|A(t)\mathbf{v}\| \geq \mu\|\mathbf{v}\|$ for all $\mathbf{v} \in \mathcal{C}^+$ and for all $t > \tilde{\alpha}$.

Similarly, it can be shown that there exists $\nu \in (0, 1)$ such that for all $\mathbf{v} \in \mathcal{C}^-$, we have $\|A(t)^{-1}\mathbf{v}\| \geq \nu^{-1}\|\mathbf{v}\|$ for all $t > \tilde{\alpha}$.

Since $D_{(x,y)}F = A(D_x f)$ for any $(x, y) \in \mathbb{T}^2$ and $D_x f \geq \alpha_1 > \tilde{\alpha}$, by the criterion for a map to be Anosov using invariant cones, we obtain that F is Anosov. □

2.2. *Estimation of λ_{abs} in Construction I.* The goal of this section is to prove the following lemma.

LEMMA 2.6. *Consider the smooth area-preserving Anosov diffeomorphisms $F^w_{l,\delta} : \mathbb{T}^2 \to \mathbb{T}^2$ defined in (2.4) (see also Lemma 2.2) using $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2, \mathbb{Z})$ with $a + d > 2$. Then for any $\varepsilon > 0$ there exists $l_\varepsilon = l_\varepsilon(A, \varepsilon) > 0$ such that for any $0 < l < l_\varepsilon$, any $0 < \beta < (a + d - 2)/|b|$, any $\delta \in (0, l)$, and any $w \in (0, \delta/4)$, we have $\lambda_{\text{abs}}(F^w_{l,\delta}) > \Lambda - \varepsilon$, where $\Lambda = h_{\text{top}}(L_A) = \log(\mu^+(1))$ (see (2.6)).*

Proof. By Lemma 2.4, we have the following. For each point $p \in \mathbb{T}^2$, let $\mathcal{C}^\pm_p$ be the union of the positive cone in the tangent space at p spanned by $\mathbf{v}^\pm_{\min}(-\beta)$ and $\mathbf{v}^\pm_{\max}$ (see (2.8)) and its symmetric complement. Then $\mathcal{C}^+_p \cap \mathcal{C}^-_p = \emptyset$,

$$D_p(F^w_{l,\delta})\mathcal{C}^+_p \subset \mathcal{C}^+_{F^w_{l,\delta}(p)}, \quad \text{and} \quad D_p(F^w_{l,\delta})^{-1}\mathcal{C}^-_p \subset \mathcal{C}^-_{(F^w_{l,\delta})^{-1}(p)}. \tag{2.11}$$

Moreover, there exists $\mu > 1$ that depends only on A and β such that

$$\|D_p(F^w_{l,\delta})\mathbf{v}\| \geq \mu\|\mathbf{v}\| \quad \text{for any } \mathbf{v} \in \mathcal{C}^+_p. \tag{2.12}$$

Let $\mathbf{v}^u = \mathbf{e}^+(1)$ and $\mathbf{v}^s = \mathbf{e}^-(1)$ (see (2.7)). Then

$$\begin{aligned}
\mathbf{v}^+_{\max} &= c^u_{\max}\mathbf{v}^u + c^s_{\max}\mathbf{v}^s,\\
&\text{where } \begin{pmatrix} c^u_{\max} \\ c^s_{\max}\end{pmatrix} = \frac{1}{4\sqrt{(a+d)^2-4}}\begin{pmatrix} (a+d)+\sqrt{(a+d)^2-4} \\ -(a+d)+\sqrt{(a+d)^2-4}\end{pmatrix},\\
\mathbf{v}^+_{\min}(-\beta) &= c^u_{\min}\mathbf{v}^u + c^s_{\min}\mathbf{v}^s,\\
&\text{where } \begin{pmatrix} c^u_{\min} \\ c^s_{\min}\end{pmatrix} = \frac{1}{2\sqrt{(a+d)^2-4}}\begin{pmatrix} \phi^+(a+d-|b|\beta)-\phi^-(a+d) \\ \phi^+(a+d)-\phi^+(a+d-|b|\beta)\end{pmatrix},
\end{aligned} \tag{2.13}$$

where ϕ^+, ϕ^- are as in Proposition 2.3.

Moreover, any $\mathbf{v} \in \mathcal{C}^+_p$ can be written in the form $\mathbf{v} = \alpha_1\mathbf{v}^+_{\max} + \alpha_2\mathbf{v}^+_{\min}(-\beta)$, where $\alpha_1\alpha_2 \geq 0$. In particular, for any $n \in \mathbb{N}$, we have

$$\frac{\|A^n\mathbf{v}\|}{\|\mathbf{v}\|} \geq \min\left\{\frac{\|A^n\mathbf{v}^+_{\max}\|}{\|\mathbf{v}^+_{\max}\|}, \frac{\|A^n\mathbf{v}^+_{\min}(-\beta)\|}{\|\mathbf{v}^+_{\min}(-\beta)\|}\right\}$$

if $\mathbf{v} \neq \mathbf{0}$. Thus, for any $\mathbf{v} \in \mathcal{C}^+_p$,

$$\|A^n\mathbf{v}\| \geq e^{\Lambda n}\|\mathbf{v}^u\||\sin\angle(\mathbf{v}^u, \mathbf{v}^s)|\min\left\{\frac{c^u_{\max}}{\|\mathbf{v}^+_{\max}\|}, \frac{c^u_{\min}}{\|\mathbf{v}^+_{\min}(-\beta)\|}\right\}\|\mathbf{v}\|.$$

Notice that $c^u_{\max}/\|\mathbf{v}^+_{\max}\|$ depends only on A. Also, we have

$$\frac{c^u_{\min}}{\|\mathbf{v}^+_{\min}(-\beta)\|} = \|\mathbf{v}^u\|^{-1}\left(1 + 2\frac{c^s_{\min}}{c^u_{\min}}\cdot\frac{\|\mathbf{v}^s\|}{\|\mathbf{v}^u\|}\cos\angle(\mathbf{v}^u, \mathbf{v}^s) + \left(\frac{c^s_{\min}}{c^u_{\min}}\cdot\frac{\|\mathbf{v}^s\|}{\|\mathbf{v}^u\|}\right)^2\right)^{-\frac{1}{2}},$$

where

$$\frac{c^s_{\min}}{c^u_{\min}} = \frac{2\sqrt{(a+d)^2-4}}{\phi^+(a+d-|b|\beta)-\phi^-(a+d)} - 1.$$

Since $\phi^+(u)$ is monotonically increasing and $2(d-1) \leq \phi^+(u) < 2d$ for $u > 2$, we have that

$$\frac{2\sqrt{(a+d)^2-4}}{(a+d)+\sqrt{(a+d)^2-4}} - 1 < \frac{c^s_{\min}}{c^u_{\min}} \leq \frac{2\sqrt{(a+d)^2-4}}{(a+d)-2+\sqrt{(a+d)^2-4}} - 1.$$

As a result, there exists a constant $C > 0$ that depends only on A such that

$$\|A^n\mathbf{v}\| \geq e^{\Lambda n}C\|\mathbf{v}\|. \tag{2.14}$$

By the Oseledets multiplicative ergodic theorem, we obtain that

$$\lambda_{\mathrm{abs}}(F^w_{l,\delta}) = \lim_{n\to\infty}\frac{\log\|D_p(F^w_{l,\delta})^n\mathbf{v}\|}{n} \tag{2.15}$$

for almost every $p \in \mathbb{T}^2$ with respect to the Lebesgue measure and $\mathbf{v} \in \mathcal{C}^+_p$.

Let $\mathcal{S} = \{(x, y) \in \mathbb{T}^2 | m - w \le x \le m + l + w\}$. We have $\mathrm{Leb}(\mathcal{S}) = l + 2w$, where Leb is the normalized Lebesgue measure. Moreover, we recall that $F^w_{l,\delta} = L_A$ on $\mathbb{T}^2 \setminus \mathcal{S}$, in particular, $D_x F^w_{l,\delta} = A$ for all $x \in \mathbb{T}^2 \setminus \mathcal{S}$. Consider $p \in \mathbb{T}^2$ and a natural number n. We write

$$n = \sum_{j=1}^{s} n_j,$$

where $n_1 \in \{0\} \cup \mathbb{N}$ and the numbers $n_j \in \mathbb{N}$ for $j = 1, \ldots, s$ are chosen in the following way.

(1) The number n_1 is the first moment when $(F^w_{l,\delta})^{n_1}(p) \in \mathcal{S}$.

(2) The number n_2 is such that the number $n_1 + n_2$ is the first moment when $(F^w_{l,\delta})^{n_1+n_2}(p) \in \mathbb{T}^2 \setminus \mathcal{S}$.

(3) The rest of the numbers are defined in the same way. For any $k \in \mathbb{N}$, the number n_{2k+1} is such that the number $\sum_{j=1}^{2k+1} n_j$ is the first moment when $(F^w_{l,\delta})^{\sum_{j=1}^{2k+1} n_j}(p) \in \mathcal{S}$, and the number n_{2k+2} is such that the number $\sum_{j=1}^{2k+2} n_j$ is the first moment when $(F^w_{l,\delta})^{\sum_{j=1}^{2k+2} n_j}(p) \in \mathbb{T}^2 \setminus \mathcal{S}$.

Let $\mathbf{v} \in \mathcal{C}^+_p$ and $\|\mathbf{v}\| = 1$. Then we have

$$\log \|D_p (F^w_{l,\delta})^n \mathbf{v}\| = \sum_{j=1}^{s} \log \|D_{(F^w_{l,\delta})^{n_1+n_2+\cdots+n_{j-1}}(p)} (F^w_{l,\delta}) l^{n_j} \mathbf{v}_j\|,$$

where

$$\mathbf{v}_1 = \mathbf{v},\ \ \mathbf{v}_2 = \frac{D_p (F^w_{l,\delta})^{n_1} \mathbf{v}_1}{\|D_p (F^w_{l,\delta})^{n_1} \mathbf{v}_1\|}, \quad \text{and} \quad \mathbf{v}_j = \frac{D_{(F^w_{l,\delta})^{n_1+n_2+\cdots+n_{j-2}}(p)} (F^w_{l,\delta})^{n_{j-1}} \mathbf{v}_{j-1}}{\|D_{(F^w_{l,\delta})^{n_1+n_2+\cdots+n_{j-2}}(p)} (F^w_{l,\delta})^{n_{j-1}} \mathbf{v}_{j-1}\|}, \quad \text{for } j = 3, \ldots, s.$$

In particular, $\|\mathbf{v}_j\| = 1$ for $j = 1, \ldots, s$.

Using (2.12) and (2.14), we obtain for $k \in \mathbb{N}$,

$$\|D_{(F^w_{l,\delta})^{n_1+n_2+\cdots+n_{j-1}}(p)} (F^w_{l,\delta})^{n_j} \mathbf{v}_j\| \ge \begin{cases} e^{\Lambda n_j} C & \text{if } j = 2k - 1, \\ \mu^{n_j} & \text{if } j = 2k. \end{cases}$$

As a result,

$$\log \|D_p (F^w_{l,\delta})^n \mathbf{v}\| \ge \left[\frac{s}{2}\right] \log C + \Lambda \sum_{k=1}^{[s/2]} n_{2k-1} + (\log \mu) \sum_{k=1}^{[s/2]} n_{2k}.$$

Since $F^w_{l,\delta}$ is a smooth Anosov diffeomorphism, by Birkhoff's ergodic theorem we obtain that

$$\frac{1}{n} \sum_{k=1}^{[s/2]} n_{2k-1} \to (1 - l - 2w) \quad \text{and} \quad \frac{1}{n} \sum_{k=1}^{[s/2]} n_{2k} \to (l + 2w) \quad \text{as } n \to \infty.$$

Moreover, each visit to $\mathcal{S}$ is at least one iterate, so $\limsup_{n\to\infty}([s/2]/n) \le (l + 2w)$.

Therefore, since $\mu > 1$,

$$\lambda_{\mathrm{abs}}(F^w_{l,\delta}) \ge (l + 2w) \min\{\log(C\mu), \log \mu\} + \Lambda(1 - l - 2w)$$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

$$\geq (l+2w)\min\{\log(C),0\} + \Lambda(1-l-2w)$$
$$\to \Lambda \quad \text{as } l\to 0 \text{ since } 0<w<\frac{l}{4}. \qquad \square$$

2.3. *Estimation of* λ_{mme} *in Construction I.* The results of this section can be summarized in the following lemma.

LEMMA 2.7. *Suppose L_A is an Anosov area-preserving automorphism on $\mathbb{T}^2$ which is induced by a matrix $A = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ with* $\operatorname{trace}(A) > 2$. *Fix $m\in(0,1)$ and let $l \in (0, 1-m)$ (see the setting of §2.1). There exists $\beta_0 \in (0,(a+d-2)/|b|)$ such that there exist C and $\delta_0 \in (0,l)$ such that there exists $Q>0$ with the following property. Let $\delta\in(0,\delta_0)$. Then there exists $w_0 = w_0(\delta)$ such that for any $w\in(0,w_0]$ a smooth area-preserving Anosov diffeomorphism $F^w_{l,\delta}\colon \mathbb{T}^2\to\mathbb{T}^2$ defined in (2.4) with the parameter $\beta=\beta_0$ has the following properties:*

(1) *$F^w_{l,\delta} = A((\beta_0 l + \delta(1-\beta_0))/\delta)$ in $S_2^{w_0}(\delta)$ where A is defined in (2.5) and $S_2^{w_0}(\delta)$ is defined in (2.24);*
(2) *$\nu_{F^w_{l,\delta}}(S_2^{w_0}(\delta)) \geq Q$ where $\nu_{F^w_{l,\delta}}$ is the measure of maximal entropy of $F^w_{l,\delta}$;*
(3) *$\lambda_{\mathrm{mme}}(F^w_{l,\delta}) \geq Q\log\mu^+((\beta_0 l+\delta(1-\beta_0))/\delta) + (1-Q)\log C$ where $\mu^+(\cdot)$ is defined in (2.6).*

The key ingredient to estimate λ_{mme} is the construction of a Markov partition, which allows us to represent dynamical systems by symbolic systems (see [**KH95**, §18.7] for more details). We use the Adler–Weiss construction of a Markov partition [**AW67**] to construct a Markov partition of $F^w_{l,\delta}$.

Let $p\in\mathbb{T}^2$. Then

$$W^s_w(p) = \{z\in\mathbb{T}^2 \mid \lim_{n\to\infty} \operatorname{dist}((F^w_{l,\delta})^n(z),(F^w_{l,\delta})^n(p)) = 0\}$$

and

$$W^u_w(p) = \{z\in\mathbb{T}^2 \mid \lim_{n\to\infty} \operatorname{dist}((F^w_{l,\delta})^{-n}(z),(F^w_{l,\delta})^{-n}(p)) = 0\}$$

are the stable and unstable manifolds of $F^w_{l,\delta}$ at p, respectively. Moreover, for any $\varepsilon>0$ let $W^i_w(p,\varepsilon)$ be the ε-neighborhood of p in $W^i_w(p)$, where $i=u,s$. We denote $F_{l,\delta} = F^0_{l,\delta}$, and $W^i(p) = W^i_0(p)$ for $i=u,s$.

Let $\mathbf{v}^u = \mathbf{e}^+(1)$ and $\mathbf{v}^s = \mathbf{e}^-(1)$ (see (2.7)). Since $F^w_{l,\delta} = L_A$ in a neighborhood of $(0,0)$ by the construction of $F^w_{l,\delta}$, there exists $\kappa>0$ such that for any sufficiently small $w\geq 0$, we have

$$W^i_w((0,0),\kappa) = \left\{(x,y) \mid \operatorname{dist}((x,y),(0,0)) \leq \kappa, \left\langle \begin{pmatrix} -y \\ x\end{pmatrix}, \mathbf{v}^i\right\rangle = 0\right\} \quad \text{for } i=u,s.$$

Moreover, since $(0,0)$ is a fixed point for $F^w_{l,\delta}$, we have

$$W^s_w((0,0)) = \bigcup_{n\in\mathbb{N}} (F^w_{l,\delta})^{-n}(W^s_w((0,0),\kappa)) \quad \text{and}$$
$$W^u_w((0,0)) = \bigcup_{n\in\mathbb{N}} (F^w_{l,\delta})^{n}(W^u_w((0,0),\kappa)). \tag{2.16}$$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

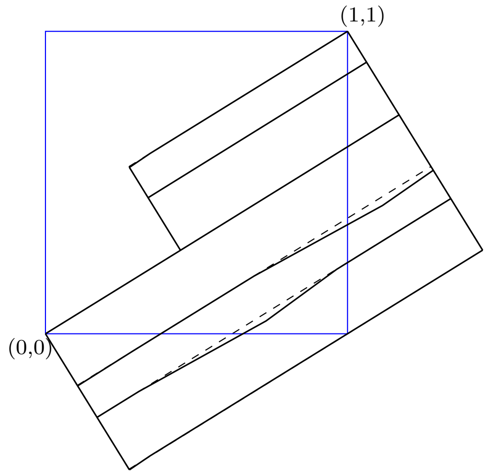

FIGURE 5. The image of the partition $\mathcal{R}$ (in black, solid) for $F_{l,\delta}$ which is a perturbation of L_A where $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Some boundaries are compared to the partition of L_A (in black, dashed). The square (in blue) is $\mathbb{T}^2$.

2.3.1. *Markov partition for $F_{l,\delta}$.* Let us draw segments of $W^u((0,0))$ and $W^s((0,0))$ until they cross sufficiently many times and separate $\mathbb{T}^2$ into two disjoint (curvilinear) rectangles R_1, R_2. Define $\mathcal{R}$ to be the partition into rectangles determined by $R_i \cap F_{l,\delta}(R_j)$, where $i, j = 1, 2$. See Figure 5 for an example. For $n \in \mathbb{N}$ let $\mathcal{R}^n$ be the partition into rectangles generated by $(F_{l,\delta})^i(R') \cap (F_{l,\delta})^j(R'')$, where $R', R'' \in \mathcal{R}$ and $i, j = -n, -n+1, \ldots, n-1, n$. Let $\mathcal{R}^0 = \mathcal{R}$. Note that the $\mathcal{R}^n$ depend on A, β, m, l, and δ even though we do not emphasize this in the notation. By the construction, for each $R \in \mathcal{R}^n$, we have that two opposite boundaries of R are contained in $W^u((0,0))$ and the other two are contained in $W^s((0,0))$.

We have the following property for the partition constructed.

LEMMA 2.8. *There exists $\beta_0 \in (0, (a+d-2)/|b|)$ and there exists $\delta_0(\beta_0) \in (0, l)$ such that there exists $n_0 \in \mathbb{N}$ with the following property. Let $\delta \in (0, \delta_0)$ and $\mathcal{R}$ be the Markov partition for $F_{l,\delta}$ (described above) with $\beta = \beta_0$. Then there exists $R \in \mathcal{R}^{n_0}$ such that $R \subset \{(x, y) \in \mathbb{T}^2 | x \in (m+l-\delta, m+l)\}$.*

Lemma 2.8 will follow from the lemmas below. First, we introduce some notation and definitions.

Denote

$$\begin{aligned} S_1(\delta) &= \{(x, y) \in \mathbb{T}^2 | x \in [m, m+l-\delta]\}, \\ S_2(\delta) &= \{(x, y) \in \mathbb{T}^2 | x \in [m+l-\delta, m+l]\}, \\ S_3 &= \mathbb{T}^2 \setminus \{(x, y) \in \mathbb{T}^2 | x \in (m, m+l)\}. \end{aligned} \tag{2.17}$$

Definition 2.9. Let $n \in \mathbb{N}$ and $\mathcal{R}^n$ be as described above. Let $R \in \mathcal{R}^n$. The *s-size of* R, denoted by $d_s(R)$, is the distance in the $\mathbf{v}^s$ direction between the two opposite boundaries (or their extensions) of R that are contained in $W^u((0,0))$. The *s-size of* $\mathcal{R}^n$ is $\tilde{d}_s(\mathcal{R}^n) = \max_{R \in \mathcal{R}^n} d_s(R)$.

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

Definition 2.10. Let $n \in \mathbb{N}$ and $\mathcal{R}^n$ be as described above. Let $R \in \mathcal{R}^n$. The *u-size of R*, denoted by $d_u(R)$, is the distance in the $\mathbf{v}^u$ direction between the two opposite boundaries (or their extensions) of R that are contained in $W^s((0, 0))$. The *u-size of* $\mathcal{R}^n$ is $\tilde{d}_u(\mathcal{R}^n) = \max_{R \in \mathcal{R}^n} d_u(R)$.

LEMMA 2.11. *Consider the setting above. Then there exists* $\beta_s \in (0, (a + d - 2)/|b|)$ *such that for any* $\beta \in (0, \beta_s)$ *there exists* $\delta_s = \delta_s(\beta) \in (0, l)$ *such that there exists a constant* $\nu_s \in (0, 1)$ *with the following properties. Let* $\delta \in (0, \delta_s)$ *and* $\mathcal{R}$ *be the partition for* $F_{l,\delta}$. *Then for any* $n \in \mathbb{N}$, *and any* $R \in \mathcal{R}^n$ *we have*

$$d_s(F_{l,\delta}(R)) < \nu_s d_s(\mathcal{R}^n).$$

Proof. Let $\delta \in (0, l)$ and $\beta \in (0, (a + d - 2)/|b|)$. Consider the partition $\mathcal{R}$ for $F_{l,\delta}$. Let $R \in \mathcal{R}^n$ and $d_s(R) = r$. Define $p_1, p_2 \in W^u((0, 0))$ as the points on the opposite boundaries of $F_{l,\delta}(R)$ such that the segment $[p_1, p_2]$ has direction $\mathbf{v}^s$. Partition $[p_1, p_2]$ into the minimal number of segments such that each segment is fully contained in one of the regions $F_{l,\delta}(S_1)$, $F_{l,\delta}(S_2)$, and $F_{l,\delta}(S_3)$.

Recall that $F_{l,\delta}$ is linear in S_1, S_2, and S_3, so it takes a piece of a line to a piece of a line.

First, we find the directions of lines in S_1 and S_2 that are taken to lines with the direction $\mathbf{v}^s$. Let $A(t)$ be as in Proposition 2.3. By the definition of $F_{l,\delta}$, we have $DF_{l,\delta} = A(1 - \beta)$ on S_1 and $DF_{l,\delta} = A((\beta l + \delta(1 - \beta))/\delta)$ on S_2.

It is easy to see that $A(t)\mathbf{u}(t) = \mathbf{v}^s$ if and only if

$$\mathbf{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = (A(t))^{-1}\mathbf{v}^s = e^{\Lambda}\mathbf{v}^s + \operatorname{sgn}(b)(t - 1) \begin{pmatrix} 0 \\ -dv_1^s + bv_2^s \end{pmatrix},$$

where $\mathbf{v}^s = \begin{pmatrix} v_1^s \\ v_2^s \end{pmatrix}$ and $\Lambda = h_{\text{top}}(L_A)$.

As a result, for any $\varepsilon_1 > 0$ there exists $\beta_1 = \beta_1(\varepsilon_1, A) \in (0, (a + d - 2)/|b|)$ such that for any $\beta \in (0, \beta_1(\varepsilon_1))$ we have $\|\mathbf{v}^s\|/(\|\mathbf{u}(1 - \beta)\|) < e^{-\Lambda} + \varepsilon_1$. Moreover, for any $\varepsilon_2 > 0$ there exists $\delta_1 = \delta_1(\varepsilon_2, A, l, \beta) \in (0, l)$ such that for any $\delta \in (0, \delta_1(\varepsilon_2))$ we have $\|\mathbf{v}^s\|/(\|\mathbf{u}((\beta l + \delta(1 - \beta))/\delta)\|) < \varepsilon_2$.

Also,

$$\begin{aligned} \frac{u_2(t)}{u_1(t)} &= \frac{v_2^s}{v_1^s} + \operatorname{sgn}(b)(t - 1)e^{-\Lambda}\bigg(- d + b\frac{v_2^s}{v_1^s}\bigg) \\ &= \frac{v_2^s}{v_1^s} - \operatorname{sgn}(b)(t - 1)\frac{1}{4}(d + a - \sqrt{(a + d)^2 - 4})(d + a + \sqrt{(a + d)^2 - 4}) \\ &= \frac{v_2^s}{v_1^s} - \operatorname{sgn}(b)(t - 1). \end{aligned} \tag{2.18}$$

According to the definition of $F_{l,\delta}$, the construction of $\mathcal{R}^n$, and (2.11), $W^u((0, 0))$ is a piecewise linear curve with linear pieces having directions in the cone spanned by $\mathbf{v}^+_{\min}(-\beta)$ and $\mathbf{v}^+_{\max}$.

Denote by $L(I)$ the length of $I \subset \mathbb{R}^2$ in the standard Euclidean metric. We estimate $L([p_1, p_2])$. In the following discussion the phrase 'a side has direction $\mathbf{v}$' means 'a side is contained in a line with direction $\mathbf{v}$'.

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

Case 1. Assume $[p_1, p_2] \subset F_{l,\delta}(S_3)$. Then $(F_{l,\delta})^{-1}([p_1, p_2])$ has direction $\mathbf{v}^s$ and $F_{l,\delta} = L_A$ on S_3, so $L([p_1, p_2]) \leq e^{-\Lambda} d_s(\mathcal{R}^n)$.

Case 2. Assume $[p_1, p_2] \subset F_{l,\delta}(S_1)$. Let $\Delta D_1D_2D_3$ be a triangle such that D_1D_2 has direction $\mathbf{u}(1-\beta)$, D_1D_3 has direction $\mathbf{v}^s$, $L(D_1D_3) = d_s(\mathcal{R}^n)$, and D_2D_3 has direction $\mathbf{v}^+_{\min}(-\beta)$. Then, for sufficiently small β which depends only on A,

$$L(D_1D_2) = d_s(\mathcal{R}^n)\bigg(1 + \frac{2|b|\beta}{\sqrt{(a+d)^2-4} + \sqrt{(a+d-|b|\beta)^2-4} - |b|\beta}\bigg) \cdot \sqrt{\frac{1+((v^s_2/v^s_1)+\operatorname{sgn}(b)\beta)^2}{1+(v^s_2/v^s_1)^2}}.$$

Let $\Delta T_1T_2T_3$ be a triangle such that T_1T_2 has direction $\mathbf{u}(1-\beta)$, T_1T_3 has direction $\mathbf{v}^s$, $L(T_1T_3) = d_s(\mathcal{R}^n)$, and T_2T_3 has direction $\mathbf{v}^+_{\max}$. Then, for sufficiently small β which depends only on A,

$$L(T_1T_2) = d_s(\mathcal{R}^n)\bigg(1 + \frac{2|b|\beta}{(a+d) + \sqrt{(a+d)^2-4} - 2|b|\beta}\bigg)\sqrt{\frac{1+((v^s_2/v^s_1)+\operatorname{sgn}(b)\beta)^2}{1+(v^s_2/v^s_1)^2}}.$$

Using (2.18), we obtain that $L((F_{l,\delta})^{-1}([p_1, p_2])) \leq \max\{L(D_1D_2), L(T_1T_2)\}$ and

$$L([p_1, p_2]) \leq \max\{L(D_1D_2), L(T_1T_2)\}\frac{\|\mathbf{v}^s\|}{\|\mathbf{u}(1-\beta)\|},$$

where $(\max\{L(D_1D_2), L(T_1T_2)\})/d_s(\mathcal{R}^n) \to 1$ as $\beta \to 0$. Therefore, there exists $\beta_2 = \beta_2(A) \in (0, (a+d-2)/|b|)$ such that there exists $\nu_2 = \nu_2(\beta_2) \in (0, 1)$ such that for any $\beta \in (0, \beta)$ we have $L([p_1, p_2]) \leq \nu_2 d_s(\mathcal{R}^n)$.

Case 3. Assume $[p_1, p_2] \subset F_{l,\delta}(S_2)$. Let $\Delta D_1D_2D_3$ be a triangle such that the following hold: D_1D_2 has direction $\mathbf{u}((\beta l + \delta(1-\beta))/\delta)$, D_1D_3 has direction $\mathbf{v}^s$, $L(D_1D_3) = d_s(\mathcal{R}^n)$, and D_2D_3 has direction $\mathbf{v}^+_{\min}(-\beta)$. Then

$$L(D_1D_2) = d_s(\mathcal{R}^n)\bigg|1 - \frac{(\operatorname{sgn}(b)\beta(l-\delta))/\delta}{(\sqrt{(a+d)^2-4} + \sqrt{(a+d-|b|\beta)^2-4} + |b|\beta)/2b + (\operatorname{sgn}(b)\beta(l-\delta))/\delta}\bigg| \cdot \sqrt{\frac{1+((v^s_2/v^s_1)-\operatorname{sgn}(b)\beta(l-\delta)/\delta)^2}{1+(v^s_2/v^s_1)^2}}.$$

Let $\Delta T_1T_2T_3$ be a triangle such that T_1T_2 has direction $\mathbf{u}((\beta l + \delta(1-\beta))/\delta)$, T_1T_3 has direction $\mathbf{v}^s$, $L(D_1D_3) = d_s(\mathcal{R}^n)$, and D_2D_3 has direction $\mathbf{v}^+_{\max}$. Then

$$L(T_1T_2) = d_s(\mathcal{R}^n)\bigg|1 - \frac{(\operatorname{sgn}(b)\beta(l-\delta))/\delta}{(d/b) - (v^s_2/v^s_1) + (\operatorname{sgn}(b)\beta(l-\delta))/\delta}\bigg| \cdot \sqrt{\frac{1+((v^s_2/v^s_1)-(\operatorname{sgn}(b)\beta(l-\delta))/\delta)^2}{1+(v^s_2/v^s_1)^2}}.$$

Using (2.18), we obtain that $L((F_{l,\delta})^{-1}([p_1, p_2])) \leq \max\{L(D_1D_2), L(T_1T_2)\}$. In particular, for any fixed β, there exists $\delta_3 = \delta_3(\beta, A, l) \in (0, l)$ such that for any $\delta \in (0, \delta_3)$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

we have

$$L([p_1,p_2]) = L((F_{l,\delta})^{-1}([p_1,p_2]))\frac{\|\mathbf{v}^s\|}{\|\mathbf{u}((\beta l+\delta(1-\beta))/\delta)\|} \le e^{-\Lambda}d_s(\mathcal{R}^n).$$

Case 4. Assume $[p_1,p_2]\subset F_{l,\delta}(S_1)\cup F_{l,\delta}(S_3)$ and $[p_1,p_2]\cap \operatorname{Int}(F_{l,\delta}(S_i))\neq\emptyset$ for $i=1,3$. Combining cases 1 and 2, we obtain that there exist $\beta_2\in(0,(a+d-2)/|b|)$ and $\nu_2=\nu_2(\beta_2)\in(0,1)$ such that for any $\beta\in(0,\beta_2)$ we have $L([p_1,p_2])\le \max\{e^{-\Lambda},\nu_2\}d_s(\mathcal{R}^n)$.

Case 5. Assume $[p_1,p_2]\subset F_{l,\delta}(S_2)\cup F_{l,\delta}(S_3)$ and $[p_1,p_2]\cap \operatorname{Int}(F_{l,\delta}(S_i))\neq\emptyset$ for $i=2,3$. Combining Cases 1 and 3, we obtain that for any $\beta\in(0,\beta_2)$ (where β_2 defined in Case 4) we have that there exists $\delta_3=\delta_3(\beta,A,l)\in(0,l)$ such that for any $\delta\in(0,\delta_3)$ we have $L([p_1,p_2])\le e^{-\Lambda}d_s(\mathcal{R}^n)$.

Case 6. Assume $[p_1,p_2]\subset F_{l,\delta}(S_1)\cup F_{l,\delta}(S_2)$ and $[p_1,p_2]\cap \operatorname{Int}(F_{l,\delta}(S_i))\neq\emptyset$ for $i=1,2$. There is a piece of a line with direction $\mathbf{v}^s$ that intersects two opposite boundaries of a rectangle in $\mathcal{R}^n$ that are contained in $W^u((0,0))$ and passes through a point of $(F_{l,\delta})^{-1}([p_1,p_2])$ that belongs to the line $x=m+l-\delta$ (on $\mathbb{T}^2$). Then, using cases 2 and 3, we obtain that $L([p_1,p_2])\le\max\{\nu_2,e^{-\Lambda}\}d_s(\mathcal{R}^n)$.

Case 7. Assume $[p_1,p_2]\subset F_{l,\delta}(S_1)\cup F_{l,\delta}(S_2)\cup F_{l,\delta}(S_3)$ and $[p_1,p_2]\cap\operatorname{Int}(S_i)\neq\emptyset$ for $i=1,2,3$. Let $q_j=(F_{l,\delta})^{-1}(p_j)$ where $j=1,2$

Subcase 7.1. Assume $q_j\in S_3$ where $j=1,2$. Recall that since $F_{l,\delta}=L_A$ on S_3 and $F_{l,\delta}$ is a piecewise linear map, if two segments $\gamma_1,\gamma_2\subset F_{l,\delta}(S_3)\cap[p_1,p_2]$, then the segments $(F_{l,\delta})^{-1}(\gamma_1)$ and $(F_{l,\delta})^{-1}(\gamma_2)$ are subsets of a line with direction $\mathbf{v}^s$ connecting the opposite boundaries of a rectangle in $\mathcal{R}^n$ that are contained in $W^u((0,0))$.

Let Q_i be the maximal segment in $[q_1,q_2]$ such that $Q_i\subset S_i$, where $i=1,2$. We denote $r_1=L(Q_1)$ and $r_2=L(Q_2)$. Let Q_i' be the maximal segment in $(F_{l,\delta})^{-1}([p_1,p_2])$ such that $Q_i'\subset S_i$, where $i=1,2$. We denote $r_1'=L(Q_1')$ and $r_2'=L(Q_2')$. By the definition of $F_{l,\delta}$ and (2.18), we have

$$r_1=(l-\delta)\sqrt{1+\left(\frac{v_2^s}{v_1^s}\right)^2},\qquad r_2=\delta\sqrt{1+\left(\frac{v_2^s}{v_1^s}\right)^2},$$

$$r_1'=(l-\delta)\sqrt{1+\left(\frac{v_2^s}{v_1^s}+\operatorname{sgn}(b)\beta\right)^2}=r_1\frac{\sqrt{1+((v_2^s/v_1^s)+\operatorname{sgn}(b)\beta)^2}}{\sqrt{1+(v_2^s/v_1^s)^2}},$$

$$r_2'=\delta\sqrt{1+\left(\frac{v_2^s}{v_1^s}-C_\delta\right)^2}=r_2\frac{\sqrt{1+((v_2^s/v_1^s)-C_\delta)^2}}{\sqrt{1+(v_2^s/v_1^s)^2}},$$

where $C_\delta=\operatorname{sgn}(b)\beta(l-\delta)/\delta$. Then

$$L(F_{l,\delta}(Q_1'))=r_1'\frac{\|\mathbf{v}^s\|}{\|\mathbf{u}(1-\beta)\|}=r_1e^{-\Lambda}\sqrt{\frac{1+((v_2^s/v_1^s)+\operatorname{sgn}(b)\beta)^2}{1+((v_2^s/v_1^s)-e^{\Lambda}\beta\operatorname{sgn}(b)(-d+b(v_2^s/v_1^s)))^2}} \tag{2.19}$$

and

$$L(F_{l,\delta}(Q_2')) = r_2' \frac{\|\mathbf{v}^s\|}{\|\mathbf{u}(\beta\ell/\delta + (1-\beta))\|}$$
$$= r_2 e^{-\Lambda} \sqrt{\frac{1 + ((v_2^s/v_1^s) - C_\delta)^2}{1 + ((v_2^s/v_1^s) + C_\delta e^{-\Lambda}(-d + bv_2^s/v_1^s))^2}}. \quad (2.20)$$

Choose $K > 0$ such that $e^{-\Lambda}(1+K) < 1$ and

$$\bigg(-d + b\frac{v_2^s}{v_1^s}\bigg)^{-1} + K = \frac{2}{(a + d + \sqrt{(a+d)^2 - 4})} + K < 1$$

which is possible since $\Lambda > 0$ and $a + d > 2$. Let $\nu_7 = \max\{e^{-\Lambda}(1+K), 2/(a+d+\sqrt{(a+d)^2-4}) + K\}$. Then there exists $\beta_7 = \beta_7(A, K) \in (0, (a+d-2)/|b|)$ such that for any $\beta \in (0, \beta_7)$ there exists $\delta_7 = \delta_7(\beta, K, A, l) \in (0, l)$ such that for any $\delta \in (0, \delta_7)$ we have $L(F_{l,\delta}(Q_i')) \le r_i \nu_7$ for $i = 1, 2$ because we have (2.19) and (2.20). Combining this with case 1 and the fact that $L([q_1, q_2]) \le d_s(\mathcal{R}^n)$, we obtain that for the specified choice of β_7 and δ_7, we have that $L([p_1, p_2]) \le \nu_7 d_s(\mathcal{R}^n)$ for any $\delta \in (0, \delta_7)$.

Subcases 7.2–7.3. Let Q_i' be the maximal piece of $(F_{l,\delta})^{-1}([p_1, p_2])$ in S_i for $i = 1, 2, 3$. Notice that Q_3' has direction $\mathbf{v}^s$. Let Q be a piece of a line with direction $\mathbf{v}^s$ connecting the opposite boundaries of a rectangle in $\mathcal{R}^n$ that are contained in $W^u((0,0))$ which passes through a point of $(F_{l,\delta})^{-1}([p_1, p_2])$ that belongs to the line $x = m + l - \delta$.

Subcase 7.2. Assume $q_1 \in S_3$ and $q_2 \in S_2$. Let Q_1 be the maximal piece of Q in $\{(x, y) \in \mathbb{T}^2 | 0 \le x \le m + l - \delta\}$ and $Q_2 = Q \setminus Q_1$. Let $D_1D_2D_3D_4(\mathbf{z})$ be a trapezoid such that D_1D_2 and D_3D_4 have direction $\mathbf{v}^s$, $L(D_1D_2) = L(Q_1)$, D_2D_3 has direction $\mathbf{u}(1-\beta)$,

$$L(D_2D_3) = (l - \delta)\sqrt{1 + (v_2^s/v_1^s + \operatorname{sgn}(b)\beta)^2},$$

and $D_1D_4(\mathbf{z})$ has direction $\mathbf{z} = \binom{z_1}{z_2}$. In our case, we consider $\mathbf{z} = \mathbf{v}^+_{\max}, \mathbf{v}^+_{\min}(-\beta)$. Moreover, to have a trapezoid for our choice of $\mathbf{z}$, we should have

$$\frac{L(Q_1)}{\sqrt{1 + (v_2^s/v_1^s)^2}} \ge (l - \delta)\left(1 - \frac{\operatorname{sgn}(b)\beta}{z_2/z_1 - v_2^s/v_1^s}\right).$$

Then we obtain

$$L(D_2D_3) + L(D_3D_4) = L(Q_1) + (l-\delta)\Bigg(\sqrt{1 + \bigg(\frac{v_2^s}{v_1^s} + \operatorname{sgn}(b)\beta\bigg)^2}$$
$$- \bigg(1 - \frac{\operatorname{sgn}(b)\beta}{(z_2/z_1) - (v_2^s/v_1^s)}\bigg)\sqrt{1 + \bigg(\frac{v_2^s}{v_1^s}\bigg)^2}\Bigg).$$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

In particular, for sufficiently small $\beta > 0$ (depending on A), we have

$$L(D_2D_3) + L(D_3D_4) \le L(Q_1)\max\left\{1, \max_{\mathbf{z}\in\{\mathbf{v}^+_{\max},\mathbf{v}^+_{\min}(-\beta)\}}\left\{\frac{\sqrt{1+((v^s_2/v^s_1)+\operatorname{sgn}(b)\beta)^2}}{(1-\operatorname{sgn}(b)\beta/(z_2/z_1-v^s_2/v^s_1))\sqrt{1+(v^s_2/v^s_1)^2}}\right\}\right\}. \tag{2.21}$$

Moreover,

$$\begin{aligned} L(F_{l,\delta}(Q'_3\cup Q'_1)) &= e^{-\Lambda}L(Q'_3) + \frac{\|\mathbf{v}^s\|}{\|\mathbf{u}(1-\beta)\|}L(Q'_1) \\ &\le \max\left\{e^{-\Lambda}, \frac{\|\mathbf{v}^s\|}{\|\mathbf{u}(1-\beta)\|}\right\}L(Q'_3\cup Q'_1) \\ &\le \max\left\{e^{-\Lambda}, \frac{\|\mathbf{v}^s\|}{\|\mathbf{u}(1-\beta)\|}\right\}(L(D_2D_3)+L(D_3D_4)). \end{aligned} \tag{2.22}$$

Let $\nu'_7 \in (e^{-\Lambda}, 1)$. Combining (2.21) and (2.22), we obtain that there exists $\beta'_7 = \beta'_7(A, \nu'_7)$ such that for any $\beta \in (0, \beta'_7)$, $L(F_{l,\delta}(Q'_3 \cup Q'_1)) \le \nu'_7 L(Q_1)$.

Also, for any $\beta \in (0, \beta'_7)$ there exists $\delta_3 = \delta_3(\beta, A, l) \in (0, l)$ such that for any $\delta \in (0, \delta_3)$ we have $L(F_{l,\delta}(Q'_2)) \le e^{-\Lambda}L(Q_2)$ (see case 3). Then for the above choice of parameters we have

$$L([p_1, p_2]) \le \nu'_7 L(Q_1) + e^{-\Lambda}L(Q_2) \le \nu'_7 L(Q_1 \cup Q_2) \le \nu'_7 d_s(\mathcal{R}^n).$$

Subcase 7.3. Assume $q_1 \in S_1$ and $q_2 \in S_3$. Let Q_2 be the maximal piece of Q in $\{(x, y) \in \mathbb{T}^2 | m + l - \delta \le x \le 1\}$ and $Q_1 = Q \setminus Q_2$. Let $D_1D_2D_3D_4(\mathbf{z})$ be a trapezoid such that D_1D_2 and D_3D_4 have directions $\mathbf{v}^s$, $L(D_1D_2) = L(Q_2)$, D_2D_3 has direction $\mathbf{u}((\beta l + \delta(1-\beta))/\delta)$ and $L(D_2D_3) = \delta\sqrt{1 + ((v^s_2/v^s_1) - (\operatorname{sgn}(b)\beta(l-\delta))/\delta)^2}$, and $D_1D_4(\mathbf{z})$ has direction $\mathbf{z} = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$. In our case, we consider $\mathbf{z} = \mathbf{v}^+_{\max}, \mathbf{v}^+_{\min}(-\beta)$. Moreover, to have a trapezoid for our choice of $\mathbf{z}$, we should have $L(Q_2)/\sqrt{1 + (v^s_2/v^s_1)^2} \ge (\operatorname{sgn}(b)\beta(l-\delta))/((z_2/z_1) - (v^s_2/v^s_1)) + \delta$. Moreover, we have

$$\operatorname{sgn}(b)\left(\frac{2d}{2b} - \frac{v^s_2}{v^s_1}\right) = \frac{(a+d)+\sqrt{(a+d)^2-4}}{2|b|} > 0$$

and

$$\begin{aligned} &\operatorname{sgn}(b)\left(\frac{2d-(a+d-|b|\beta)+\sqrt{(a+d-|b|\beta)^2-4}}{2b} - \frac{v^s_2}{v^s_1}\right) \\ &\quad = \frac{|b|\beta+\sqrt{(a+d-|b|\beta)^2-4}+\sqrt{(a+d)^2-4}}{2|b|} > 0. \end{aligned}$$

Thus, we obtain

$$L(D_3D_4) = L(Q_2) - \delta\sqrt{1+\left(\frac{v^s_2}{v^s_1}\right)^2}\left(1+\frac{\operatorname{sgn}(b)\beta(l-\delta)}{\delta(z_2/z_1-v^s_2/v^s_1)}\right) \le L(Q_2).$$

Then for the choice of parameters as in case 7.1 we have

$$L(F_{l,\delta}(Q_2')) + L(F_{l,\delta}(Q_3')) \le \frac{\|\mathbf{v}^s\|}{\|\mathbf{u}((\beta l + \delta(1-\beta))/\delta)\|} L(Q_2') + e^{-\Lambda} L(Q_2) \le \nu_7 L(Q_2). \tag{2.23}$$

Let $\nu_7'' = \max\{\nu_7, \nu_2\}$ and $\beta_7'' = \min\{\beta_2, \beta_7\}$. Then for any $\beta \in (0, \beta_7'')$ there exists $\delta_7 = \delta_7(\beta, K, l, A) \in (0, l)$ such that for any $\delta \in (0, \delta_7)$ we have $L([p_1, p_2]) \le \nu_7'' d_s(\mathcal{R}^n)$.

Combining cases 1–7, we obtain the statement of the lemma. □

Similarly to Lemma 2.11, we can obtain the following lemma.

LEMMA 2.12. *Consider the setting above. Then there exists $\beta_u \in (0, (a+d-2)/|b|)$ such that for any $\beta \in (0, \beta_u)$ there exist $\delta_u = \delta_u(\beta) \in (0, l)$ and $\nu_u = \nu_u(\delta_u) \in (0, 1)$ with the following properties. Let $\delta \in (0, \delta_u)$ and $\mathcal{R}$ be the partition for $F_{l,\delta}$. Then for any $n \in \mathbb{N}$ and any $R \in \mathcal{R}^n$ we have*

$$d_u((F_{l,\delta})^{-1}(R)) < \nu_u d_u(\mathcal{R}^n).$$

Proof of Lemma 2.8. First, let $\beta_0 = \frac{1}{2}\min\{\beta_s, \beta_u\}$, where β_s and β_u are as in Lemmas 2.11 and 2.12. Then $\delta' = \frac{1}{2}\min\{\delta_s(\beta_0), \delta_u(\beta_0)\}$.

Let $\delta \in (0, \delta')$. Denote $\tilde{A} = A((\beta_0 l + \delta(1-\beta_0))/\delta)$. Then for any vector $\mathbf{v} = \alpha_1 \mathbf{v}^-_{\min}(-\beta_0) + \alpha_2 \mathbf{v}^-_{\max}$, where $\alpha_1\alpha_2 \ge 0$ and $\alpha_1^2 + \alpha_2^2 \ne 0$, we have

$$\begin{aligned}
\hat{\mathbf{v}} = \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix} &= \tilde{A}^{-1}\mathbf{v} \\
&= \begin{pmatrix} d & -b \\ -c - \operatorname{sgn}(b) d \frac{\beta_0(l-\delta)}{\delta} & a + |b|\frac{\beta_0(l-\delta)}{\delta} \end{pmatrix} \left(\alpha_1 \begin{pmatrix} 2b \\ \phi^-(a+d-|b|\beta_0) \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right) \\
&= \frac{\beta_0(l-\delta)}{\delta} |b| \big(\alpha_1 \big(2d - \phi^-(a+d-|b|\beta_0)\big) + \alpha_2\big) \begin{pmatrix} 0 \\ -1 \end{pmatrix} \\
&\quad + A^{-1} \begin{pmatrix} 2b\alpha_1 \\ \alpha_1\phi^-(a+d-|b|\beta_0) - \alpha_2 \end{pmatrix}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\min\left\{0, \frac{2}{b(2d - \phi^-(a+d-|b|\beta_0))}\right\} &\le \frac{\hat{v}_2}{\hat{v}_1} + \frac{\operatorname{sgn}(b)\beta_0(l-\delta)}{\delta} + \frac{a}{b} \\
&\le \max\left\{0, \frac{2}{b(2d - \phi^-(a+d-|b|\beta_0))}\right\}.
\end{aligned}$$

Therefore, there exists $\delta_0 \in (0, \delta')$ such that the following assertion holds. Let $\delta \in (0, \delta_0)$ and I be a segment of $W^s((0,0))$ contained in $S_2(\delta)$ with endpoints $(m+l-\delta, y_1)$ and $(m+l, y_2)$ on its boundary. Then $|y_2 - y_1| \ge \frac{1}{2}\beta_0 l$.

Let $\mathcal{R}$ be the Markov partition for $F_{l,\delta}$. Denote by $\partial_j(\mathcal{R})$ the boundary of $\mathcal{R}$ that is contained in $W^j((0,0))$, where $j = s, u$. By Lemmas 2.4 and 2.12, there exists $n_1 = n_1(\delta_0) \in \mathbb{N}$ such that there are two distinct segments $I, J \subset \partial_s(\mathcal{R}_1^n) \cap S_2(\delta)$ with endpoints $(m+l-\delta, y_1^I)$, $(m+l, y_2^I)$ and $(m+l-\delta, y_1^J)$, $(m+l, y_2^J)$, respectively, with the property that $\min\{|y_1^I - y_2^J|, |y_1^J - y_2^I|\} \ge \frac{1}{4}\beta_0 l$. Then, using Lemma 2.11, there

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

exists also $n_2 = n_2(\delta_0) \in \mathbb{N}$ such that there are two distinct segments $W, V \subset \partial_u(\mathcal{R}_2^n) \cap S_2(\delta)$ with endpoints $(m + l - \delta, y_1^W)$, $(m + l, y_2^W)$ and $(m + l - \delta, y_1^V)$, $(m + l, y_2^V)$, respectively, with the property that $W \cap I \neq \emptyset$, $W \cap J \neq \emptyset$, $V \cap I \neq \emptyset$, $V \cap J \neq \emptyset$, and the intersection does not contain the endpoints of I, J, W, V. Thus for $n_0 = \max\{n_1, n_2\}$ there exists $R \in \mathcal{R}^{n_0}$ such that $R \subset \{(x, y) \in \mathbb{T}^2 | x \in (m + l - \delta, m + l)\}$. □

2.3.2. *Markov partition for $F^w_{l,\delta}$ when $w > 0$.* We construct the Markov partition for $F^w_{l,\delta}$ in the same way as for $F_{l,\delta}$. We quickly recall it while introducing some notation. Let us draw segments of $W^u_w((0, 0))$ and $W^s_w((0, 0))$ until they cross sufficiently many times and separate $\mathbb{T}^2$ into two disjoint (curvilinear) rectangles R_1, R_2. Define $\mathcal{R}_w$ to be the partition into rectangles determined by $R_i \cap F^w_{l,\delta}(R_j)$, where $i, j = 1, 2$. For $n \in \mathbb{N}$ let $\mathcal{R}^n_w$ be the partition into rectangles generated by $(F^w_{l,\delta})^i(R) \cap (F^w_{l,\delta})^j(T)$, where $S, T \in \mathcal{R}_w$ and $i, j = -n, -n + 1, \ldots, n - 1, n$. Let $\mathcal{R}^0 = \mathcal{R}$.

LEMMA 2.13. *Let β_0, δ_0, and n_0 be as in Lemma 2.8. Let*

$$S^w_2(\delta) = \{(x, y) \in \mathbb{T}^2 | x \in (m + l - \delta + w, m + l - w)\}. \tag{2.24}$$

Then for any $\delta \in (0, \delta_0)$ there exists $w_0 = w_0(\delta) \in (0, \delta/4)$ such that there exists $R \in \mathcal{R}^{n_0}_{w_0}$ such that $R \subset S^{w_0}_2(\delta)$, where $\mathcal{R}_{w_0}$ is the Markov partition described above for $F^{w_0}_{l,\delta}$ with $\beta = \beta_0$. In particular, there exists $Q = Q(\beta_0, \delta_0) > 0$ such that if $\nu^{w_0}_{l,\delta}$ is the measure of maximal entropy for $F^{w_0}_{l,\delta}$, then $\nu^{w_0}_{l,\delta}(S^{w_0}_2(\delta)) \geq Q$.

Proof. Recall that in a neighborhood of $(0, 0)$ we have $F_{l,\delta} = F^w_{l,\delta} = L_A$ for $w \in (0, \delta/4)$. In particular, the corresponding stable and unstable manifolds coincide in a neighborhood of $(0, 0)$. Let $\mathcal{R}$ be the constructed Markov partition for $F_{l,\delta}$. From (2.16), there exists $N = N(n_0, \kappa) \in \mathbb{N}$ such that for any point $p \in \partial_j(\mathcal{R}^{n_0})$ for $j = s, u$ there exist $q \in W^j_w((0, 0), \kappa) = W^j_0((0, 0), \kappa)$ and $n \in \mathbb{Z}$ such that $|n| \leq N$ and $p = (F_{l,\delta})^n(q)$. Moreover, $(F^w_{l,\delta})^n(q) \in W^j_w((0, 0))$. By Lemma 2.2, we obtain that $\mathrm{dist}(p, (F^w_{l,\delta})^n(q))$ can be made arbitrarily small by choosing a sufficiently small w, and this choice can be made in a uniform way on compact sets for $|n| \leq N$. Therefore, using Lemma 2.8, we obtain that there exists a sufficiently small $w_0 > 0$ such that there exists $R \in \mathcal{R}^{n_0}_{w_0}$ with $R \subset S^{w_0}_2(\delta)$.

The statement about the measure of maximal entropy follows from the fact that $F^{w_0}_{l,\delta}$ is topologically semiconjugate to the topological Markov chain defined using the Markov partition constructed. Moreover, the semiconjugacy is one-to-one on all periodic points except for the fixed points. In particular, the measure of maximal entropy for $F^{w_0}_{l,\delta}$ is defined by the measure of maximal entropy for the topological Markov shift (Parry measure; see, for example, [**KH95**, Proposition 4.4.2]) using the topological semiconjugacy. □

2.3.3. *Lower bound on $\lambda_{\mathrm{mme}}(F^w_{l,\delta})$.* We now work towards obtaining a lower bound on $\lambda_{\mathrm{mme}}(F^w_{l,\delta})$.

We are in the setting of Lemma 2.13. We will use the same notation as in Proposition 2.3 and Lemma 2.4.

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

Let $\mathbf{v}^u_\delta = \mathbf{e}^+((\beta_0 l + \delta(1-\beta_0))/\delta)$ and $\mathbf{v}^s_\delta = \mathbf{e}^-((\beta_0 l + \delta(1-\beta_0))/\delta)/\|\mathbf{e}^-((\beta_0 l + \delta(1-\beta_0))/\delta)\|$. Denote

$$\hat{\delta} = a + d + |b|\bigg(\frac{\beta_0 l + \delta(1-\beta_0)}{\delta} - 1\bigg) \quad \text{and} \quad \hat{\beta}_0 = a + d - |b|\beta_0.$$

Then

$$\mathbf{v}^+_{\max} = c^{u,\delta}_{\max}\mathbf{v}^u_\delta + c^{s,\delta}_{\max}\mathbf{v}^s_\delta$$

where

$$\begin{pmatrix} c^{u,\delta}_{\max} \\ c^{s,\delta}_{\max} \end{pmatrix} = \frac{1}{4\sqrt{\hat{\delta}^2-4}} \begin{pmatrix} \hat{\delta} + \sqrt{\hat{\delta}^2-4} \\ \big(\sqrt{\hat{\delta}^2-4} - \hat{\delta}\big)\left\|\mathbf{e}^-\left(\frac{\beta_0 l+\delta(1-\beta_0)}{\delta}\right)\right\| \end{pmatrix} \to \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \quad \text{as } \hat{\delta} \to \infty,$$

and

$$\mathbf{v}^+_{\min}(-\beta_0) = c^{u,\delta}_{\min}\mathbf{v}^u_\delta + c^{s,\delta}_{\min}\mathbf{v}^s_\delta$$

where

$$\begin{pmatrix} c^{u,\delta}_{\min} \\ c^{s,\delta}_{\min} \end{pmatrix} = \frac{1}{2\sqrt{\hat{\delta}^2-4}} \begin{pmatrix} \hat{\delta} - \hat{\beta}_0 + \sqrt{\hat{\delta}^2-4} + \sqrt{\hat{\beta}_0^2-4} \\ \Big(\hat{\beta}_0 - \hat{\delta} + \sqrt{\hat{\delta}^2-4} - \sqrt{\hat{\beta}_0^2-4}\Big)\left\|\mathbf{e}^-\left(\frac{\beta_0 l+\delta(1-\beta_0)}{\delta}\right)\right\| \end{pmatrix}$$
$$\to \begin{pmatrix} 1 \\ \hat{\beta}_0 - \sqrt{\hat{\beta}_0^2-4} \end{pmatrix} \quad \text{as } \hat{\delta} \to \infty.$$

As a result, if δ is sufficiently small, then for any $\mathbf{v} = \alpha_1\mathbf{v}^+_{\max} + \alpha_2\mathbf{v}^+_{\min}(-\beta_0)$, where $\alpha_1\alpha_2 \geq 0$ and $\alpha_1^2 + \alpha_2^2 \neq 0$, we have for any $n \in \mathbb{N}$,

$$\left\|\bigg(A\bigg(\frac{\beta_0 l + \delta(1-\beta_0)}{\delta}\bigg)\bigg)^n \mathbf{v}\right\| \geq \bigg(\mu^+\bigg(\frac{\beta_0 l + \delta(1-\beta_0)}{\delta}\bigg)\bigg)^n C\|\mathbf{v}\|, \tag{2.25}$$

where $C = |b| \min\{1/(2\|\mathbf{v}^+_{\max}\|), 1/\|\mathbf{v}^+_{\min}(-\beta_0)\|\} \in (0, 1)$.

Let $S^{w_0}_2(\delta)$ be as in Lemma 2.13. Then $\nu^{w_0}_{l,\delta}(S^{w_0}_2(\delta)) \geq Q$. We obtain the lower bound on $\lambda_{\mathrm{mme}}(F^w_{l,\delta})$ in a similar way as in §2.2 by replacing $\mathcal{S}$ by $\mathbb{T}^2 \setminus S^{w_0}_2(\delta)$ and Leb by $\nu^{w_0}_{l,\delta}$ and using Birkhoff's ergodic theorem for $\nu^{w_0}_{l,\delta}$. More precisely, consider $p \in \mathbb{T}^2$ and a natural number n. We write

$$n = \sum_{j=1}^{s} n_j,$$

where $n_1 \in \{0\} \cup \mathbb{N}$ and the numbers $n_j \in \mathbb{N}$ for $j = 1, \ldots, s$ are chosen in the following way.

(1) The number n_1 is the first moment when $(F^w_{l,\delta})^{n_1}(p) \in \mathbb{T}^2 \setminus S^{w_0}_2(\delta)$.
(2) The number n_2 is such that the number $n_1 + n_2$ is the first moment when $(F^w_{l,\delta})^{n_1+n_2}(p) \in S^{w_0}_2(\delta)$.
(3) The rest of the numbers are defined in the same way. For any $k \in \mathbb{N}$, the number n_{2k+1} is such that the number $\sum_{j=1}^{2k+1} n_j$ is the first moment when $(F^w_{l,\delta})^{\sum_{j=1}^{2k+1} n_j}(p) \in \mathbb{T}^2 \setminus S^{w_0}_2(\delta)$, and the number n_{2k+2} is such that the number $\sum_{j=1}^{2k+2} n_j$ is the first moment when $(F^w_{l,\delta})^{\sum_{j=1}^{2k+2} n_j}(p) \in S^{w_0}_2(\delta)$.

Let $\mathbf{v} \in \mathcal{C}_p^+$ and $\|\mathbf{v}\| = 1$. Then we have

$$\log \|D_p(F^w_{l,\delta})^n \mathbf{v}\| = \sum_{j=1}^{s} \log \|D_{(F^w_{l,\delta})^{n_1+n_2+\cdots+n_{j-1}}(p)}(F^w_{l,\delta})^{n_j}\mathbf{v}_j\|,$$

where $\mathbf{v}_1 = \mathbf{v}$, $\mathbf{v}_2 = D_p(F^w_{l,\delta})^{n_1}\mathbf{v}_1/\|D_p(F^w_{l,\delta})^{n_1}\mathbf{v}_1\|$, and

$$\mathbf{v}_j = \frac{D_{(F^w_{l,\delta})^{n_1+n_2+\cdots+n_{j-2}}(p)}(F^w_{l,\delta})^{n_{j-1}}\mathbf{v}_{j-1}}{\|D_{(F^w_{l,\delta})^{n_1+n_2+\cdots+n_{j-2}}(p)}(F^w_{l,\delta})^{n_{j-1}}\mathbf{v}_{j-1}\|} \quad \text{for } j = 3, \ldots, s.$$

In particular, $\|\mathbf{v}_j\| = 1$ for $j = 1, \ldots, s$.

Recall that $DF^w_{l,\delta} = A((\beta_0 l + \delta(1-\beta_0))/\delta)$ on $S_2^{w_0}(\delta)$. Thus, using (2.12) and (2.25), we obtain for $k \in \mathbb{N}$,

$$\|D_{(F^w_{l,\delta})^{n_1+n_2+\cdots+n_{j-1}}(p)}(F^w_{l,\delta})^{n_j}\mathbf{v}_j\| \geq \begin{cases} \left(\mu^+\left(\frac{\beta_0 l+\delta(1-\beta_0)}{\delta}\right)\right)^n C & \text{if } j = 2k-1, \\ \mu^{n_j} & \text{if } j = 2k. \end{cases}$$

As a result, using the fact that $\mu > 1$, we have

$$\begin{aligned} \log \|D_p(F^w_{l,\delta})^n \mathbf{v}\| &\geq \left[\frac{s}{2}\right] \log C + \mu^+\left(\frac{\beta_0 l + \delta(1-\beta_0)}{\delta}\right) \sum_{k=1}^{[s/2]} n_{2k-1} + (\log \mu) \sum_{k=1}^{[s/2]} n_{2k} \\ &\geq \left[\frac{s}{2}\right] \log C + \mu^+\left(\frac{\beta_0 l + \delta(1-\beta_0)}{\delta}\right) \sum_{k=1}^{[s/2]} n_{2k-1}. \end{aligned}$$

Since $F^w_{l,\delta}$ is a smooth Anosov diffeomorphism, by Birkhoff's ergodic theorem we obtain that

$$\frac{1}{n}\sum_{k=1}^{[s/2]} n_{2k-1} \to \nu^{w_0}_{l,\delta}(S_2^{w_0}(\delta)) \quad \text{and} \quad \frac{1}{n}\sum_{k=1}^{[s/2]} n_{2k} \to (1 - \nu^{w_0}_{l,\delta}(S_2^{w_0}(\delta))) \quad \text{as } n \to \infty.$$

Moreover, each visit to $\mathbb{T}^2 \setminus S_2^{w_0}(\delta)$ is at least one iterate, so $\limsup_{n\to\infty}([s/2]/n) \leq (1 - \nu^{w_0}_{l,\delta}(S_2^{w_0}(\delta))) \leq 1 - Q$.

Therefore, we obtain

$$\lambda_{\text{mme}}(F^{w_0}_{l,\delta}) \geq Q \log \mu^+\left(\frac{\beta_0 l + \delta(1-\beta_0)}{\delta}\right) + (1-Q)\log C \to +\infty \quad \text{as } \delta \to 0, \tag{2.26}$$

as $C \in (0, 1)$ and $Q = Q(\beta_0, \delta_0) \in (0, 1)$ are independent of δ while w_0 depends on δ.

3. *Construction II*

In this section we show how to decrease the Lyapunov exponent with respect to the Lebesgue probability measure while controlling the Lyapunov exponent with respect to the measure of maximal entropy starting from the Anosov diffeomorphisms in Theorem 2.1. In this section we use the construction described in [**HJJ17**] while providing estimates of the Lyapunov exponents. As before, we first state the main theorem of the section and give its proof before stating and proving the necessary lemmas.

THEOREM 3.1. *Suppose L_A and Λ as in Theorem C. For any H such that $\Lambda < H$ and positive number γ, let $\{g_s\}_{s\in[0,1]}$ be a smooth family of area-preserving Anosov diffeomorphisms on $\mathbb{T}^2$ homotopic to L_A from Theorem C applied for γ and H with lower bound on $\lambda_{\mathrm{mme}}(g_1)$ coming from Lemma 2.7 being larger than H. Then there exists a constant $\tilde{C}$ such that for any $\sigma > 0$, $S > \Lambda$ there exists a smooth family $\{f_{s,t}\}_{(s,t)\in[0,1]\times[0,1]}$ of Anosov diffeomorphisms on $\mathbb{T}^2$ homotopic to L_A such that:*

(A) $f_{s,0} = g_s$ *for all* $s \in [0, 1]$;

(B) $f_{s,t}$ *preserves a probability measure* $\mu_{s,t}$ *which is absolutely continuous with respect to the Lebesgue measure;*

(C) $\lambda_{\mathrm{abs}}(f_{s,1}) < \gamma$ *for all* $s \in [0, 1]$;

(D) $\lambda_{\mathrm{mme}}(f_{0,t}) < S$ *for all* $t \in [0, 1]$;

(E) $\lambda_{\mathrm{mme}}(f_{1,t}) \geq H + \tilde{C}\sigma$.

Proof. We define $f_{s,0} = g_s$ for all $s \in [0, 1]$ so we automatically have (A) in the theorem. Moreover, by Theorem C, we have that $\lambda_{\mathrm{abs}}(f_{s,0}) > \Lambda - \gamma$ and $\lambda_{\mathrm{mme}}(f_{1,0}) > H$ due to the special form of the lower bound on $\lambda_{\mathrm{mme}}(f_{1,0})$ (see the proof of Theorem 2.1). Let $\tilde{C} = \log(K_1 K_2^{-1}) + \log(C)$ be as in Lemma 3.10. Choose r_0 small enough such that Lemmas 3.7 and 3.10 hold for desired estimates. Apply Construction II to the family $\{f_{s,0}\}$ with the chosen r_0 and η changing up to $\eta_1/2$ where η_1 comes from Lemma 3.5. Thus, we have (B) in the theorem. Furthermore, the choice of η_1 guarantees that we have (C) in the theorem. Also, by the choice of r_0 satisfying Lemma 3.7 with appropriate choice of χ, we have (D) in the theorem. Finally, from the form of the lower bound 2 in Lemma 3.10, we obtain (E) in the theorem. □

3.1. *Construction II: slow-down deformation near a fixed point.* We will now describe the slow-down deformation near a fixed point that we will use. The construction comes from [**HJJ17**] but was originally introduced by A. Katok to give an example of a Bernoulli area-preserving smooth diffeomorphism (on the boundary of the set of Anosov diffeomorphism) with non-zero Lyapunov exponents on any surface (see [**Kat79**]). For more explanation, see Remark 3.2.

Recall that the family of diffeomorphisms, $\{f_{s,0}\}_{s\in[0,1]}$, built by Construction I has the properties that each $f_{s,0}$ has $(0, 0)$ as a fixed point and is equal to L_A on $\mathbb{T}^2 \setminus \{(x, y) \in \mathbb{T}^2 | m - w_0 < x < m + l + w_0\}$ where $m \in (0, 1)$, $l \in (0, 1 - m)$, and $w_0 > 0$ (very small). We choose a coordinate system centered at $(0, 0)$ with the basis consisting of eigenvectors $\mathbf{v}^u = \mathbf{e}^+(1)$ and $\mathbf{v}^s = \mathbf{e}^-(1)$ of A (see (2.7)). In this coordinate system, $A = \begin{pmatrix} e^{\Lambda} & 0 \\ 0 & e^{-\Lambda} \end{pmatrix}$.

Let $D_r = \{(s_1, s_2) | s_1^2 + s_2^2 \leq r^2\}$ be a disk of radius r centered at $(0, 0)$. Choose $0 < r_0 < 1$ and set $r_1 = 2r_0\Lambda$. Then we have

$$D_{r_0} \subset \mathrm{Int}\, A(D_{r_1}) \cap \mathrm{Int}\, A^{-1}(D_{r_1}).$$

The linear map $x \mapsto Ax$ is the time-one map of the local flow generated by the following system of differential equations in D_{r_1}:

$$\frac{ds_1}{dt} = s_1\Lambda, \qquad \frac{ds_2}{dt} = -s_2\Lambda. \tag{3.1}$$

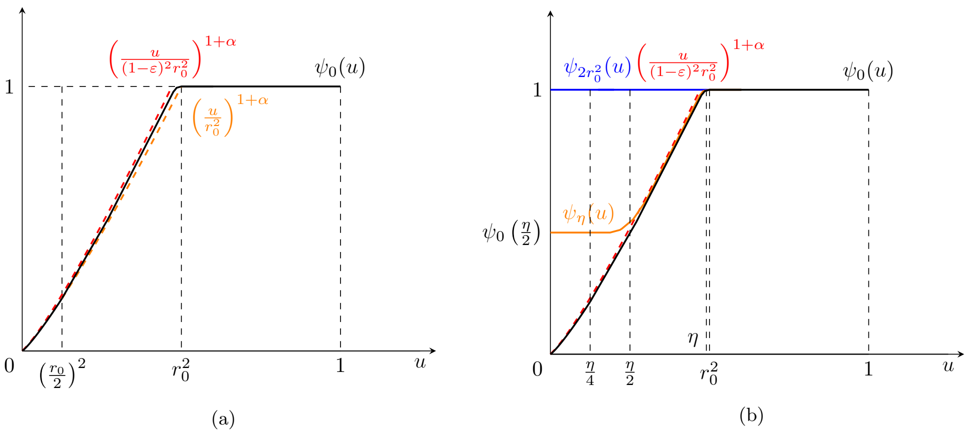

FIGURE 6. (a) An example of $\psi_0(u)$, (b) An example of $\psi_n(u)$.

Let $\alpha \in (0, \frac{1}{3})$ and $\varepsilon \in (0, 1)$ such that $(1+\alpha)/(1-\varepsilon)^{2(1+\alpha)} < \frac{4}{3}$. Choose a C^∞ function $\psi_0 : [0, 1] \to [0, 1]$ satisfying:

(1) $\psi_0(u) = 1$ for $u \geq r_0^2$;

(2) $\psi_0(u) = (u/r_0^2)^{1+\alpha}$ for $0 \leq u \leq (r_0/2)^2$;

(3) $\psi_0(u) > 0$ and $\psi_0'(u) \geq 0$ for $0 < u < r_0^2$;

(4) $\psi_0(u) \geq (u/r_0^2)^{1+\alpha}$ and

$$\psi_0'(u) \leq \frac{1+\alpha}{(1-\varepsilon)^{2(1+\alpha)} r_0^2} \left(\frac{u}{r_0^2}\right)^\alpha$$

for $u \in [(r_0/2)^2, r_0^2]$;

(5) in particular, for the α and ε considered, we have

$$\psi_0'(u) \leq \frac{4}{3r_0^2} \quad \text{when } u \in [(r_0/2)^2, r_0^2].$$

Notice that by (2) the derivative of ψ_0 at $(r_0/2)^2$ is $(1+\alpha)/(2^{2\alpha} r_0^2)$. See Figure 6(a) for an example of $\psi_0(u)$.

Define a one-parameter family of C^∞ functions

$$\psi_\eta : [0, 1] \to [0, 1] \quad \text{where } 0 \leq \eta \leq 2r_0^2,$$

such that:

(1) $\psi_\eta(u) > 0$ and $\psi_\eta'(u) \geq 0$ for $0 < u < r_0^2$;

(2) $\psi_\eta(u) = 1$ for $u \geq r_0^2$;

(3) $\psi_\eta(u) = \psi_0(\eta/2)$ for $0 \leq u \leq \eta/4$;

(4) $\psi_\eta(u) = \psi_0(u)$ for $u \geq \eta$;

(5) if $\eta_1 \leq \eta_2$, then $\psi_{\eta_1}(u) \leq \psi_{\eta_2}(u)$ for every $u \in [0, 1]$;

(6)

$$\psi_\eta'(u) \leq \frac{1+\alpha}{(1-\varepsilon)^{2(1+\alpha)} r_0^2} \left(\frac{u}{r_0^2}\right)^\alpha \leq \frac{4}{3r_0^2} \quad \text{for } u \in (0, 1);$$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

(7) $\psi_\eta(u) \to \psi_0(u)$ as $\eta \to 0$ pointwise on $[0, 1]$;
(8) the map $(\eta, u) \mapsto \psi_\eta(u)$ is C^∞ smooth.

Notice that $\psi_{2r_0^2}(u) \equiv 1$ for $u \in [0, 1]$. See Figure 6(b) for an example of $\psi_\eta(u)$. Also, we have

$$\int_0^1 \frac{1}{\psi_0(u)}\, du \text{ diverges} \quad \text{and} \quad \int_0^1 \frac{1}{\psi_\eta(u)}\, du < \infty \quad \text{for } \eta > 0. \tag{3.2}$$

Remark 3.2. Note that in [**Kat79**] the function ψ_0 (in the notation above) is such that $\int_0^1 (1/\psi_0(u))\, du$ converges. In comparison to [**HJJ17**, **Kat79**], we consider a more explicit choice of ψ_0 which allows for the estimation of Lyapunov exponents. Furthermore, the maps that we work with are Anosov as we are not considering the map coming from ψ_0 itself. This is analogous to [**Kat79**, Corollary 4.2].

Consider the following slow-down deformation of the flow described by the system (3.1) in D_{r_0}:

$$\frac{ds_1}{dt} = s_1 \psi_\eta(s_1^2 + s_2^2)\Lambda, \quad \frac{ds_2}{dt} = -s_2 \psi_\eta(s_1^2 + s_2^2)\Lambda. \tag{3.3}$$

Let g_η be the time-one map of this flow. Observe that g_η is defined and of class C^∞ in D_{r_1}, and it coincides with L_A in a neighborhood of ∂D_{r_1} by the choice of ψ_η, r_0, and r_1. As a result, for sufficiently small r_0 we obtain a C^∞ diffeomorphism

$$G_{s,\eta}(x) = \begin{cases} f_{s,0}(x) & \text{if } x \in \mathbb{T}^2 \setminus D_{r_1}, \\ g_\eta(x) & \text{if } x \in D_{r_1}. \end{cases} \tag{3.4}$$

Using (3.2) for $\eta > 0$, we can define a positive C^∞ function

$$\kappa_\eta(s_1, s_2) := \begin{cases} \left(\psi_\eta(s_1^2 + s_2^2)\right)^{-1} & \text{if } (s_1, s_2) \in D_{r_0}, \\ 1 & \text{otherwise,} \end{cases}$$

and its average

$$K_\eta := \int_{\mathbb{T}^2} \kappa_\eta \, d\text{Leb}.$$

For $\eta > 0$ and $s \in [0, 1]$, the diffeomorphism $G_{s,\eta}$ preserves the probability measure $d\mu_\eta = K_\eta^{-1}\kappa_\eta \, d\text{Leb}$, that is, μ_η is absolutely continuous with respect to the Lebesgue measure. See the paragraph containing equation (3.2) in [**HJJ17**] for the idea of the proof.

Let $p \in \mathbb{T}^2$. Consider the cones

$$\begin{aligned} \mathcal{K}^+(p) &= \{\xi_1 \mathbf{v}^u + \xi_2 \mathbf{v}^s \mid \xi_1, \xi_2 \in \mathbb{R}, |\xi_2| \le |\xi_1|\}, \\ \mathcal{K}^-(p) &= \{\xi_1 \mathbf{v}^u + \xi_2 \mathbf{v}^s \mid \xi_1, \xi_2 \in \mathbb{R}, |\xi_1| \le |\xi_2|\} \end{aligned} \tag{3.5}$$

in $T_p\mathbb{T}^2$.

LEMMA 3.3. (Cf. [**Kat79**, Proposition 4.1]) *For every $s \in [0, 1]$, $\eta > 0$, and $p \in \mathbb{T}^2$ we have*

$$DG_{s,\eta}\mathcal{K}^+(p) \subsetneq \mathcal{K}^+(G_{s,\eta}(p)) \quad \textit{and} \quad DG_{s,\eta}^{-1}\mathcal{K}^-(p) \subsetneq \mathcal{K}^-(G_{s,\eta}^{-1}(p)).$$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

Moreover, $E^+_{s,\eta}(p) = \bigcap_{j=0}^{\infty} DG^j \mathcal{K}^+(G^{-j}(p))$ *and* $E^-_{s,\eta}(p) = \bigcap_{j=0}^{\infty} DG^{-j} \mathcal{K}^-(G^j(p))$ *are one-dimensional subspaces of* $T_p\mathbb{T}^2$.

Proof. The cases for $\mathcal{K}^+(p)$ and $\mathcal{K}^-(p)$ are similar. Thus, we restrict ourselves to the inclusion for $\mathcal{K}^+(p)$.

Notice that the vector $\binom{0}{1}$ in the standard Euclidean coordinates is equal to $(v^u - v^s)/(2\sqrt{(a+d)^2-4})$. As a result, using (2.13), we obtain that $\mathcal{C}_p^+ \subset \mathcal{K}^+(p)$ and $\mathcal{C}_p^- \subset \mathcal{K}^-(p)$ (see Lemma 2.4 for notation). By the constructions of $f_{s,0}$ and $G_{s,\eta}$ and Proposition 2.3, the desired inclusion holds outside of the disk D_{r_1}.

The system of variational equations corresponding to the system (3.3) implies that the following equation holds for the tangent ζ_η:

$$\frac{d\zeta_\eta}{dt} = -2\Lambda((\psi_\eta(s_1^2+s_2^2) + (s_1^2+s_2^2)\psi'_\eta(s_1^2+s_2^2))\zeta_\eta + s_1 s_2 \psi'_\eta(s_1^2+s_2^2)(\zeta_\eta^2+1)). \tag{3.6}$$

Since $\psi_\eta(s_1^2+s_2^2) > 0$ in D_{r_1} for $\eta > 0$, substituting $\zeta_\eta = 1$ and $\zeta_\eta = -1$ in (3.6) gives, respectively,

$$\frac{d\zeta_\eta}{dt} = -2\Lambda(\psi_\eta(s_1^2+s_2^2) + (s_1+s_2)^2\psi'_\eta(s_1^2+s_2^2)) < 0,$$
$$\frac{d\zeta_\eta}{dt} = 2\Lambda(\psi_\eta(s_1^2+s_2^2) + (s_1-s_2)^2\psi'_\eta(s_1^2+s_2^2)) > 0.$$

Thus, the result about the desired inclusion follows.

The statement that $E^+_{s,\eta}(p)$ and $E^-_{s,\eta}(p)$ are one-dimensional subspaces follows from the argument in [**Kat79**, Proposition 4.1]. □

Lemma 3.3 and the fact that $G_{s,\eta}$ preserves a smooth positive measure imply the following corollary.

COROLLARY 3.4. *$G_{s,\eta}$ is a C^∞ Anosov diffeomorphism on $\mathbb{T}^2$ for any $\eta > 0$.*

3.2. *Estimation of λ_{abs} in Construction II.*

We use the notation introduced in §3.1. The estimation of $\lambda_{\mathrm{abs}}(G_{s,\eta})$ follows from the estimation of metric entropy in [**HJJ17**, §3] which we provide here for completeness. From the following lemma, we can guarantee (2) in Theorem B.

LEMMA 3.5. *For any $\gamma > 0$ there exists η_1 such that for any $0 < \eta < \eta_1$ and $s \in [0, 1]$ we have $\lambda_{\mathrm{abs}}(G_{s,\eta}) = \lambda_{\mu_\eta}(G_{s,\eta}) < \gamma$.*

Proof. Let U be any fixed neighborhood of $(0, 0)$ and $U \subset D_{r_1}$. By (3.2) and properties (5) and (7) of the family of positive functions $\{\psi_\eta\}_{\eta\in[0,r_0^2]}$ defined above, applying the monotone convergence theorem, we have

$$\lim_{\eta\to 0}\int_{\mathbb{T}^2} \kappa_\eta \, d\mathrm{Leb} = \int_{\mathbb{T}^2} \lim_{\eta\to 0} \kappa_\eta \, d\mathrm{Leb} = \int_{\mathbb{T}^2} \kappa_0 \, d\mathrm{Leb} = \infty$$

and

$$\lim_{\eta\to 0}\int_{\mathbb{T}^2\setminus U} \kappa_\eta \, d\mathrm{Leb} = \int_{\mathbb{T}^2\setminus U} \lim_{\eta\to 0} \kappa_\eta \, d\mathrm{Leb} = \int_{\mathbb{T}^2\setminus U} \kappa_0 \, d\mathrm{Leb} < \infty.$$

Therefore, we have the following equalities:

$$\lim_{\eta\to 0}\mu_\eta(\mathbb{T}^2\setminus U)=\lim_{\eta\to 0}\int_{\mathbb{T}^2\setminus U}K_\eta^{-1}\kappa_\eta\,d\mathrm{Leb}=\lim_{\eta\to 0}\frac{\int_{\mathbb{T}^2\setminus U}\kappa_\eta\,d\mathrm{Leb}}{\int_{\mathbb{T}^2}\kappa_\eta\,d\mathrm{Leb}}=0 \tag{3.7}$$

and

$$\lim_{\eta\to 0}\mu_\eta(U)=1.$$

By the ergodicity of μ_η for $G_{s,\eta}$ for $\eta>0$, (1.2) implies

$$\lambda_{\mu_\eta}(G_{s,\eta})=\int_{\mathbb{T}^2}\log|DG_{s,\eta}|_{E_x^u(G_{s,\eta})}|\,d\mu_\eta,$$

where $E_x^u(G_{s,\eta})$ is the unstable subspace at x with respect to $G_{s,\eta}$.

By property (2) of ψ_0 and the property (8) of ψ_η, we obtain that for any $\gamma>0$ there exist $\rho>0$ and $\eta_0>0$ such that $DG_{s,\eta}$ is close to the identity map and $\log|DG_{s,\eta}|_{E_x^u(G_{s,\eta})}|<\gamma/2$ in D_ρ for $s\in[0,1]$ and $0<\eta<\eta_0$. Therefore, for any $s\in[0,1]$,

$$\int_{D_\rho}\log|DG_{s,\eta}|_{E_x^u(G_{s,\eta})}|\,d\mu_\eta<\frac{\gamma}{2}. \tag{3.8}$$

Moreover, by property (8), we have that $\log|DG_{s,\eta}|_{E_x^u(G_{s,\eta})}|$ is uniformly bounded. Therefore, there exists $\eta_1<\eta_0$ such that for any $0<\eta<\eta_1$ we have

$$\int_{\mathbb{T}^2\setminus D_\rho}\log|DG_{s,\eta}|_{E_x^u(G_{s,\eta})}|\,d\mu_\eta<\frac{\gamma}{2}. \tag{3.9}$$

By (3.8) and (3.9), we obtain the lemma. □

3.3. *Estimation of* λ_{mme} *in Construction II.* We use the notation introduced in §3.1.

3.3.1. *Upper bound on* λ_{mme}. Here, we prove Lemma 3.7 which provides an upper bound on λ_{mme} for a family of maps in Construction II.

The main ingredient to obtain an upper bound on λ_{mme} is to estimate the consecutive time each trajectory spends in the annulus $D_{2\Lambda r_0}\setminus D_{r_0/(2\Lambda)}$ independent of r_0 and η.

Recall that $\Lambda=h_{\mathrm{top}}(L_A)$ and α is a constant that appears in the second condition on ψ_0 (see §3.1).

LEMMA 3.6. (Cf. [**PSZ19**, Lemma 5.6]) *There exists $T_0>0$ depending only on Λ and α such that for any solution $s_\eta(t)=(s_1(t),s_2(t))_\eta$ of (3.3) with $s_\eta(0)\in D_{r_1}$, we have*

$$\max\{t|s_\eta(t)\in D_{r_1}\setminus D_{r_0/(2\Lambda)}\}<T_0$$

for any $\eta\in[0,2r_0^2]$, where $r_1=2\Lambda r_0$.

Proof. We omit the dependence of s_1 and s_2 on η in the notation below. Let $u=s_1^2+s_2^2$.

Assume $s_1^2\le s_2^2$. Then, by (3.3), we have

$$\frac{du}{dt}=2\Lambda\psi_\eta(u)(s_1^2-s_2^2)=-2\Lambda\psi_\eta(u)(u^2-4s_1^2s_2^2)^{\frac{1}{2}}. \tag{3.10}$$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

For $s_2^2 \le s_1^2$, by (3.3), we have

$$\frac{du}{dt} = 2\Lambda\psi_\eta(u)(s_1^2 - s_2^2) = 2\Lambda\psi_\eta(u)(u^2 - 4s_1^2s_2^2)^{\frac{1}{2}}. \tag{3.11}$$

Recall that by (3.3) we have $s_1(t)s_2(t) = s_1(0)s_2(0)$ for any t.

If $4s_1^2(0)s_2^2(0) \le r_0^4/(32\Lambda^4)$, then, under the assumptions $s_1^2 \le s_2^2$ and $r_0^2/(2\Lambda)^2 < u \le r_1^2$, we have

$$\frac{du}{dt} \le -2\Lambda\psi_\eta(u)\bigg(\bigg(\frac{r_0}{2\Lambda}\bigg)^4 - \frac{r_0^4}{32\Lambda^4}\bigg)^{\frac{1}{2}} = -\Lambda^{-1}\psi_\eta(u)\frac{r_0^2}{2\sqrt{2}} \le -\Lambda^{-1}\psi_0\bigg(\frac{r_0^2}{(2\Lambda)^2}\bigg)\frac{r_0^2}{2\sqrt{2}}$$
$$\le -2^{-4-2\alpha}\Lambda^{-3-2\alpha}r_0^2.$$

Similarly, under the assumption $s_2^2 \le s_1^2$ and $r_0^2/(2\Lambda)^2 < u \le r_1^2$, we have

$$\frac{du}{dt} \ge 2\Lambda\psi_\eta(u)\bigg(\bigg(\frac{r_0}{2\Lambda}\bigg)^4 - \frac{r_0^4}{32\Lambda^4}\bigg)^{\frac{1}{2}} \ge 2^{-4-2\alpha}\Lambda^{-3-2\alpha}r_0^2.$$

Therefore, under the assumptions $s_1^2 \le s_2^2$, starting from $u(0) = r_1^2 = 4\Lambda^2r_0^2$, it takes at most $t = 2^{2+2\alpha}\Lambda^{1+2\alpha}(16\Lambda^4 - 1)$ time to reach $u = r_0^2/(2\Lambda)^2$, unless the assumption $s_1^2 \le s_2^2$ is violated. If the assumption $s_1^2 \le s_2^2$ is violated, then, by symmetry of (3.10) and (3.11), the orbit will leave D_{r_1} in at most $2 \cdot 2^{2+2\alpha}\Lambda^{1+2\alpha}(16\Lambda^4 - 1)$ time. A similar argument works if we start from $u(0) = r_0^2/(2\Lambda)^2$ under the assumption $s_2^2 \le s_1^2$.

Assume that $4s_1^2(0)s_2^2(0) > r_0^4/(32\Lambda^4)$. If the trajectory is in D_{r_1}, then $r_1^2 \ge s_1^2(t) + s_2^2(t) \ge s_2(t)^2$, in particular, $r_0^4/(32\Lambda^4) < 4s_1^2(t)s_2^2(t) \le 4s_1^2(t)r_1^2$, and therefore, $s_1^2(t) > r_0^2/(512\Lambda^6)$. Using (3.3), we obtain

$$\frac{d}{dt}(s_1^2) = 2s_1\frac{d}{dt}s_1 = 2s_1^2\psi_\eta(u)\Lambda > \frac{r_0^2}{\Lambda^{7+2\alpha}}2^{-10-2\alpha}.$$

Therefore, $s_1^2(t)$ will increase to r_1^2 and the orbit will leave D_{r_1} in at most $2^{12+2\alpha}\Lambda^{9+2\alpha}$ time.

Finally, $T_0 = \max\{2^{3+2\alpha}\Lambda^{1+2\alpha}(16\Lambda^4 - 1), 2^{12+2\alpha}\Lambda^{9+2\alpha}\}$. □

The following lemma could be of independent interest as we obtain an upper bound on the forward Lyapunov exponents for $G_{0,\eta}$. We recall that $G_{0,\eta}$ depends on the size of the ball where the slow-down deformation is done, that is, it depends on r_0. Moreover, $G_{0,2r_0^2} = L_A$.

LEMMA 3.7. *For any $\chi > 0$ there exists $r_\chi \in (0, 1)$ such that for any $r_0 \in (0, r_\chi)$, $x \in \mathbb{T}^2$ and $\mathbf{v} \in T_x\mathbb{T}^2$ with $\|v\| \ne 0$,*

$$\lambda(G_{0,\eta}, x, \mathbf{v}) < \Lambda + \chi$$

for all $\eta \in (0, 2r_0^2]$, where $\lambda(G_{0,\eta}, x, \mathbf{v}) = \lim\sup_{n\to\infty}(1/n)\log\|D_xG_{0,\eta}^n\mathbf{v}\|$ is the forward Lyapunov exponent of $(x, \mathbf{v})$ with respect to $G_{0,\eta}$. In particular,

$$\lambda_{\mathrm{mme}}(G_{0,\eta}) < \Lambda + \chi \quad \text{for all } \eta \in (0, 2r_0^2].$$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

Proof. Consider $x \in \mathbb{T}^2$ and $\mathbf{v} \in T_x\mathbb{T}^2$ with $\|v\| \neq 0$. Let n be a natural number. We write

$$n = \sum_{j=1}^{s} n_j,$$

where the numbers $n_j \in \{0\} \cup \mathbb{N}$ are chosen in the following way.

(1) The number n_1 is the first moment when $G_{0,\eta}^{n_1}(x) \in D_{r_1} \setminus D_{r_0/(2\Lambda)}$.

(2) The number n_2 is such that the number $n_1 + n_2$ is the first moment when $G_{0,\eta}^{n_1+n_2}(x) \in D_{r_0/(2\Lambda)}$.

(3) The number n_3 is such that the number $n_1 + n_2 + n_3$ is the first moment when $G_{0,\eta}^{n_1+n_2+n_3}(x) \in D_{r_1} \setminus D_{r_0/(2\Lambda)}$;

(4) The number n_4 is such that the number $n_1 + n_2 + n_3 + n_4$ is the first moment when $G_{0,\eta}^{n_1+n_2+n_3+n_4}(x) \notin D_{r_1}$.

(5) The rest of the numbers are defined following the same pattern.

If $x \in \mathbb{T}^2$ is such that the $G_{0,\eta}$-orbit of x does not enter into D_{r_1}, then $\lambda(G_{0,\eta}, x, \mathbf{v}) \leq \Lambda$ because $G_{0,\eta} = L_A$ outside of D_{r_1}.

Assume the $G_{0,\eta}$-orbit of x enters into D_{r_1}. By the definition of $\lambda(G_{0,\eta}, x, \mathbf{v})$, in that case it is enough to consider the case when $G_{0,\eta}^{-1}(x) \in D_{r_1}$ but $x \notin D_{r_1}$.

We have

$$\log \|D_x G_{0,\eta}^n \mathbf{v}\| = \log \|\mathbf{v}\| + \sum_{j=1}^{s} \log \|D_{G_{0,\eta}^{n_1+n_2+\cdots+n_{j-1}}(x)} G_{0,\eta}^{n_j} \mathbf{v}_j\|, \tag{3.12}$$

where $\mathbf{v}_1 = \mathbf{v}/\|\mathbf{v}\|$, $\mathbf{v}_2 = D_x G_{0,\eta}^{n_1}\mathbf{v}_1/\|D_x G_{0,\eta}^{n_1}\mathbf{v}_1\|$, and

$$\mathbf{v}_j = \frac{D_{G_{0,\eta}^{n_1+n_2+\cdots+n_{j-2}}(x)} G_{0,\eta}^{n_{j-1}} \mathbf{v}_{j-1}}{\|D_{G_{0,\eta}^{n_1+n_2+\cdots+n_{j-2}}(x)} G_{0,\eta}^{n_{j-1}} \mathbf{v}_{j-1}\|} \quad \text{for } j = 3, \ldots, s.$$

In particular, $\|\mathbf{v}_j\| = 1$ for $j = 1, \ldots, s$.

Recall that $G_{0,\eta}$ coincides with L_A in $\mathbb{T}^2 \setminus D_{r_1}$ (see (3.4)). Thus, for any $N \in \mathbb{N}$, there exists a positive number $\theta = \theta(N, \Lambda)$ such that if $r_1 < \theta$, that is, $r_0 < \theta/(2\Lambda)$, then for any y such that $G_{0,\eta}^{-1}(y) \in D_{r_1}$ but $y \notin D_{r_1}$ we have $G_{0,\eta}^{n}(y) \notin D_{r_1}$ for any $\eta \in (0, 2r_0^2]$ and $0 \leq n \leq N$. Therefore, if r_0 is sufficiently small, then $n_1 \geq N$.

The coefficient matrix of the variational equation (3.3) is

$$C_\eta(s_1(t), s_2(t)) = \Lambda \begin{pmatrix} \psi_\eta(s_1^2(t) + s_2^2(t)) + 2s_1^2(t)\psi_\eta'(s_1^2(t) + s_2^2(t)) & 2s_1(t)s_2(t)\psi_\eta'(s_1^2(t) + s_2^2(t)) \\ -2s_1(t)s_2(t)\psi_\eta'(s_1^2(t) + s_2^2(t)) & -\psi_\eta(s_1^2(t) + s_2^2(t)) \\ & -2s_2^2(t)\psi_\eta'(s_1^2(t) + s_2^2(t)) \end{pmatrix}. \tag{3.13}$$

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

Let $s_\eta(t) = (s_1(t), s_2(t))_\eta$ be the solution of (3.3) with initial condition $s_\eta(0) = x$. Denote by $A_\eta(t)$ a 2×2 matrix solving the variational equation

$$\frac{dA_\eta(t)}{dt} = C_\eta(s_\eta(t))A_\eta(t) \tag{3.14}$$

with initial condition $A_\eta(0) = \mathrm{Id}$. Then $A_\eta(1) = D_x G_{0,\eta}$.

Moreover, by (3.14) and the Cauchy–Schwarz inequality, we have for any vector $\mathbf{v}$,

$$\frac{d\|A_\eta(t)\mathbf{v}\|}{dt} \leq \left\| \frac{d[A_\eta(t)\mathbf{v}]}{dt} \right\| = \|C_\eta(s_\eta(t))A_\eta(t)\mathbf{v}\| \leq \|C_\eta(s_\eta(t))\|_{\mathrm{op}} \|A_\eta(t)\mathbf{v}\|, \tag{3.15}$$

where $\|\cdot\|_{\mathrm{op}}$ is the operator norm.

By (3.15), (3.13), the definition of ψ_η, and property (5) of ψ_0, we have that there exists a positive constant M independent of r_0 and η such that $\|A_\eta(t)\mathbf{v}\| \leq e^{Mt}\|\mathbf{v}\|$ for any t and any vector $\mathbf{v}$. In particular, $\|D_x G_{0,\eta}\|_{\mathrm{op}} \leq e^M$ if $x \in D_{r_1} \setminus D_{r_0/(2\Lambda)}$.

Consider $x \in D_{r_0/(2\Lambda)}$. Note that, by (3.3), we have that for any $t \in [0, 1]$ and $\eta \in (0, 2r_0^2]$ the image of x under the time-t map of the flow (3.3) is in $D_{r_0/2}$. In particular, $G_{0,\eta}(D_{r_0/(2\Lambda)}) \subseteq D_{r_0/2}$ for any $\eta \in (0, 2r_0^2]$.

Recall that if $u \in [0, (r_0/2)^2]$, then $\psi_0(u) = (u/r_0^2)^{1+\alpha}$ and $\psi_0'(u) = (u/r_0^2)^\alpha(1+\alpha)/r_0^2$. Therefore, by the choice of $\psi_\eta(u)$ for $u \in [\eta^2/r_0^2, \eta]$, we can guarantee that $0 \leq \psi_\eta(u) \leq 2^{-2-2\alpha}$ and $0 \leq \psi_\eta'(u) \leq 2^{-2\alpha}(1+\alpha)/r_0^2$ for $u \in [0, (r_0/2)^2]$ and $\eta \in (0, 2r_0^2]$. Then $\|C_\eta(s_\eta(t))\|_{\mathrm{op}} \leq \Lambda$ if $s_\eta(t) \in D_{r_0/2}$ because $2^{-2-2\alpha}(3+2\alpha) \in (0, \frac{3}{4}]$ and $2^{-1-2\alpha}(1+\alpha) \in (0, \frac{1}{2}]$ for $\alpha > 0$. Therefore, if $x \in D_{r_0/(2\Lambda)}$, then $\|D_x G_{0,\eta}\|_{\mathrm{op}} \leq e^\Lambda$. Thus, using (3.12) and Lemma 3.6, we obtain that for any $\chi > 0$ there exists sufficiently small r_χ such that for any $r_0 \in (0, r_\chi)$, $\eta \in (0, 2r_0^2]$, $x \in \mathbb{T}^2$, and $\mathbf{v} \in T_x\mathbb{T}^2$ with $\|\mathbf{v}\| \neq 0$,

$$\lambda(G_{0,\eta}, x, \mathbf{v}) \leq \Lambda + \frac{2T_0 M}{N} \leq \Lambda + \chi. \qquad \square$$

3.3.2. *Lower bound on λ_{mme}.* Our next and final goal is to prove Lemma 3.10 which gives a lower bound on λ_{mme} for the maps in Construction II.

LEMMA 3.8. *For any $\alpha \in (0, \frac{1}{3})$ and $\varepsilon \in (0, 1)$ such that $(1+\alpha)/(1-\varepsilon)^{2(1+\alpha)} < \frac{4}{3}$ there exists $\rho = \rho(\alpha, \varepsilon) \in (0, 1)$ such that for every $s \in [0, 1]$, $\eta > 0$, and $p \in D_{r_1}$ we have*

$$DG_{s,\eta}\mathcal{K}_\rho^+(p) \subset \mathcal{K}_\rho^+(G_{s,\eta}(p)),$$

where $\mathcal{K}_\rho^+(p)$ is the cone of size ρ in $T_p\mathbb{T}^2$, that is,

$$\mathcal{K}_\rho^+(p) = \{\xi_1\mathbf{v}^u + \xi_2\mathbf{v}^s \mid \xi_1, \xi_2 \in \mathbb{R}, |\xi_2| \leq \rho|\xi_1|\}. \tag{3.16}$$

Moreover, ρ can be expressed as

$$\rho(\alpha, \varepsilon) = \frac{-((1-\varepsilon)^{2(1+\alpha)} + 1 + \alpha) + \sqrt{((1-\varepsilon)^{2(1+\alpha)} + 1 + \alpha)^2 - (1+\alpha)^2}}{(1+\alpha)}. \tag{3.17}$$

Proof. As in Lemma 3.3, using the system of variational equations corresponding to the system (3.3), we look at the following equation for the tangent ζ_η:

$$\frac{d\zeta_\eta}{dt} = -2\Lambda((\psi_\eta(s_1^2+s_2^2) + (s_1^2+s_2^2)\psi_\eta'(s_1^2+s_2^2))\zeta_\eta + s_1s_2\psi_\eta'(s_1^2+s_2^2)(\zeta_\eta^2+1)). \tag{3.18}$$

First, observe that if $(s_1, s_2) \in (D_{r_1} \setminus D_{r_0})$ then ψ_η is constant, in particular, ζ_η is decreasing when $\zeta_\eta > 0$ and increasing when $\zeta_\eta < 0$. Also, if $s_1s_2 = 0$, then we have the same conclusion about ζ_η. Thus, we can assume in the consideration of the further cases that $s_1s_2 \neq 0$. Due to symmetry, it is enough to analyze the case $s_1, s_2 > 0$.

Let $s_1, s_2 > 0$. Then ζ_η is decreasing when $\zeta_\eta > 0$, so we consider the case $\zeta_\eta < 0$. Moreover, let $k = s_1s_2/(s_1^2+s_2^2)$. Notice that $k \in (0, \frac{1}{2}]$.

Assume $(s_1, s_2) \in D_{r_0}$. By properties (5) and (6) of ψ_η, we have

$$\psi_\eta(s_1^2+s_2^2) \geq \left(\frac{s_1^2+s_2^2}{r_0^2}\right)^{1+\alpha} \quad \text{and} \quad \psi_\eta'(s_1^2+s_2^2) \leq \frac{1+\alpha}{(1-\varepsilon)^{2(1+\alpha)}r_0^2}\left(\frac{s_1^2+s_2^2}{r_0^2}\right)^{\alpha}.$$

Thus, plugging this expression into (3.18), we obtain

$$\frac{d\zeta_\eta}{dt} \geq -2\Lambda\psi_\eta'(s_1^2+s_2^2)(s_1^2+s_2^2)\left(\left(\frac{(1-\varepsilon)^{2(1+\alpha)}}{1+\alpha}+1\right)\zeta_\eta + k(\zeta_\eta^2+1)\right).$$

It is easy to see that $(d\zeta_\eta/dt) \geq 0$ if $\zeta_\eta \in [\zeta^-(k), \zeta^+(k)]$, where

$$\zeta^\pm(k) = \frac{-((1-\varepsilon)^{2(1+\alpha)}+1+\alpha) \pm \sqrt{((1-\varepsilon)^{2(1+\alpha)}+1+\alpha)^2 - 4k^2(1+\alpha)^2}}{2k(1+\alpha)}.$$

Also, $\zeta^+(k) \geq \zeta^+(\frac{1}{2})$ and $\zeta^-(k) \leq \zeta^-(\frac{1}{2})$ for $k \in (0, \frac{1}{2}]$. Thus, ζ_η is non-decreasing for $\zeta_\eta \in [\zeta^-(\frac{1}{2}), \zeta^+(\frac{1}{2})]$.

As a result, using that ζ_η is smooth and the above analysis, we obtain that $\rho(\alpha, \varepsilon) = \zeta^+(\frac{1}{2})$ gives the desired cone. □

Let $p \in \mathbb{T}^2$ and $\mathbf{v} \in T_p\mathbb{T}^2$. Then $\mathbf{v} = \xi_1\mathbf{v}^u + \xi_2\mathbf{v}^s$. Denote by $\|\cdot\|_{u,s}$ the norm in $\mathbb{R}^2$ such that $\|\mathbf{v}\|_{u,s}^2 = \xi_1^2 + \xi_2^2$.

LEMMA 3.9. *Assume we are in the setting of Lemma 3.8. For any $p \in D_{r_1}$, and any $\mathbf{v} \in \mathcal{K}^+_{\rho(\alpha,\varepsilon)}(p)$ (see (3.16)), we have*

$$\|DG_{s,\eta}\mathbf{v}\|_{u,s} \geq \|\mathbf{v}\|_{u,s}.$$

In particular, for any $p \in D_{r_1}$, and any $\mathbf{v} \in \mathcal{K}^+_{\rho(\alpha,\varepsilon)}(p)$,

$$\|DG_{s,\eta}\mathbf{v}\| \geq K_1K_2^{-1}\|\mathbf{v}\|,$$

where K_1, K_2 are constants coming from the equivalence of norms $\|\cdot\|$ and $\|\cdot\|_{u,s}$ in $\mathbb{R}^2$, that is, $0 < K_1 \leq 1 \leq K_2$, $K_1\|\mathbf{u}\|_{u,s} \leq \|\mathbf{u}\| \leq K_2\|\mathbf{u}\|_{u,s}$ for any $\mathbf{u} \in \mathbb{R}^2$.

Moreover, let $\mathcal{C}_p^+$ be the union of the positive cone in the tangent space at $p \in \mathbb{T}^2$ spanned by $\mathbf{v}^+_{\min}(-\beta)$ and $\mathbf{v}^+_{\max}$ (see (2.8)) and its symmetric complement. There exists $N \in \mathbb{N}$ such that for each $p \in \mathbb{T}^2$ we have

$$(DL_A)^N(\mathcal{C}_p^+) \subset \mathcal{K}^+_{\rho(\alpha,\varepsilon)}(L_A^N(p)) \quad \text{and} \quad (DL_A)^N(\mathcal{K}^+_{\rho(\alpha,\varepsilon)}(p)) \subset \mathcal{C}^+_{L_A^N(p)}.$$

Proof. Let $p \in D_{r_1}$ and $\mathbf{v}(0) \in \mathcal{K}^+_\rho(p)$. Moreover, $\mathbf{v}(t)$ is the evolution of $\mathbf{v}(0)$ along the flow. In particular, $\mathbf{v}(t) = \xi_1(t)\mathbf{v}^u + \xi_2(t)\mathbf{v}^s$ where $|\xi_2(t)| \le \rho|\xi_1(t)|$ for any $t > 0$.

By properties of ψ_η, we have

$$\frac{\psi'_\eta(s_1^2 + s_2^2)}{\psi_\eta(s_1^2 + s_2^2)} \le \frac{1+\alpha}{(1-\varepsilon)^{2(1+\alpha)}} \cdot \frac{1}{s_1^2 + s_2^2}.$$

Using (3.3), for any $\alpha \in (0, \frac{1}{3})$ and $\varepsilon \in (0, 1)$ such that $(1+\alpha)/(1-\varepsilon)^{2(1+\alpha)} < \frac{4}{3}$ we have

$$\begin{aligned}
&\frac{d}{dt}(\xi_1^2 + \xi_2^2)\\
&\quad = 2\Lambda((\psi_\eta(s_1^2+s_2^2) + 2s_1^2\psi'_\eta(s_1^2+s_2^2))\xi_1^2 - (\psi_\eta(s_1^2+s_2^2) + 2s_2^2\psi'_\eta(s_1^2+s_2^2))\xi_2^2)\\
&\quad = 2\Lambda(\psi_\eta(s_1^2+s_2^2) + 2s_1^2\psi'_\eta(s_1^2+s_2^2))\bigg(\xi_1^2 - \frac{\psi_\eta(s_1^2+s_2^2) + 2s_2^2\psi'_\eta(s_1^2+s_2^2)}{\psi_\eta(s_1^2+s_2^2) + 2s_1^2\psi'_\eta(s_1^2+s_2^2)}\xi_2^2\bigg)\\
&\quad \ge 2\Lambda(\psi_\eta(s_1^2+s_2^2) + 2s_1^2\psi'_\eta(s_1^2+s_2^2))\xi_2^2\bigg(\rho^{-2}(\alpha,\varepsilon) - \bigg(1 + 2\frac{1+\alpha}{(1-\varepsilon)^{2(1+\alpha)}}\bigg)\bigg) \ge 0.
\end{aligned}$$

Let $z \in (0, 1)$. Then for any $n \in \mathbb{N}$ we have $(DL_A)^n(\mathcal{K}^+_z(p)) = \mathcal{K}^+_{e^{-2n\Lambda}z}(L^n_A(p))$. Thus, we obtain the statement about the cone inclusions. □

Recall that $\bar\beta$, $\bar\delta$, and $\bar w$ are the values of the parameters in the construction of $f_{1,0}$ (see (2.4)). Let D_r be a disk of radius r centered at $(0, 0)$ and

$$\bar{\mathcal{S}} = \{(x, y) \in \mathbb{T}^2 | x \in (m + l - \bar\delta + \bar w, m + l - \bar w)\},$$

that is, the region of the perturbation described in Construction I where $DG_{1,\eta} = A((\bar\beta + \bar\delta(1 - \bar\delta))/\bar\delta)$ (see Proposition 2.3). Denote by $\nu_{G_{1,\eta}}$ the measure of maximal entropy for $G_{1,\eta}$.

LEMMA 3.10. *Let $\alpha \in (0, \frac{1}{3})$ and $\varepsilon \in (0, 1)$ such that $(1+\alpha)/(1-\varepsilon)^{2(1+\alpha)} < \frac{4}{3}$. Let $\bar Q = \nu_{f_{1,0}}(\bar{\mathcal{S}})$. Then for any $\sigma > 0$ there exists r_σ such that for all sufficiently small $r_0 \in (0, r_\sigma)$ and for all $\eta \in (0, 2r_0^2]$ the following assertions hold for $G_{1,\eta}$ obtained in Construction II with the parameters α, ε, r_0, and η:*

(1) $\nu_{G_{1,\eta}}(D_{r_\sigma}) \le \sigma$;

(2)

$$\begin{aligned}
\lambda_{\mathrm{mme}}(G_{1,\eta}) \ge{}& (\log(K_1K_2^{-1}) + \log(C))\sigma + \log(C)(1 - \bar Q)\\
&+ \log\mu^+((\bar\beta + \bar\delta(1-\bar\delta))/\bar\delta)(\bar Q - \sigma),
\end{aligned}$$

where C is a constant that depends only on the matrix A and $\bar\beta$ (see (2.25)), and K_1, K_2 are constants in Lemma 3.9.

Proof. Recall that in Construction II, for any sufficiently small $r_0 \in (0, 1)$ we have that $f_{1,0} = G_{1,2r_0^2}$ and for any $\eta \in (0, 2r_0^2]$, $G_{1,\eta}(x) = f_{1,0}(x)$ if $x \in \mathbb{T}^2 \setminus D_{r_1}$, where

https://doi.org/10.1017/etds.2021.36 Published online by Cambridge University Press

$r_1 = 2r_0\Lambda$. Also, there exists $\bar{r} > 0$ such that for any $r \in (0, \bar{r})$, $f_{1,0}(\bar{\mathcal{S}}) \cap D_r = \emptyset$ and $f_{1,0}(D_r) \cap \bar{\mathcal{S}} = \emptyset$.

Consider a periodic point q of $f_{1,0}$ other than $(0, 0)$. Build a Markov partition $\mathcal{MP}$ for $f_{1,0}$ using the point q and its stable and unstable manifolds, $W^s(q)$ and $W^u(q)$, respectively (Adler–Weiss construction). Since $(0, 0)$ is a fixed point for $f_{1,0}$, it follows that $(0, 0) \notin W^s(q) \cap W^u(q)$. In particular, there is a refinement $\overline{\mathcal{MP}}$ of $\mathcal{MP}$ such that:

- if $\mathcal{P}\bar{\mathcal{S}} = \{R \in \overline{\mathcal{MP}} | R \subset \bar{\mathcal{S}}\}$, then $\nu_{f_{1,0}}(\bigcup_{R\in\mathcal{P}\bar{\mathcal{S}}} R) \geq \bar{Q} - \sigma$;
- there exists $R_\sigma \in \overline{\mathcal{MP}}$ such that $(0, 0)$ is in the interior of R_σ and $\nu_{f_{1,0}}(R_\sigma) < \sigma$.

Choose $r_\sigma < \bar{r}$ such that $D_{r_\sigma} \subset R_\sigma$.

Let $\rho(\alpha, \varepsilon)$ be as in Lemma 3.8. Let $\mathcal{C}^+_p$ be the union of the positive cone in the tangent space at $p \in \mathbb{T}^2$ spanned by $\mathbf{v}^+_{\min}(-\bar{\beta})$ and $\mathbf{v}^+_{\max}$ (see (2.8)) and its symmetric complement, where $\bar{\beta}$ is the value of β in the construction of $f_{1,0}$. By Lemma 3.9, we can choose $N \in \mathbb{N}$ such that for each $p \in \mathbb{T}^2$ we have

$$(DL_A)^N(\mathcal{C}^+_p) \subset \mathcal{K}^+_{\rho(\alpha,\varepsilon)}(L^N_A(p)) \quad \text{and} \quad (DL_A)^N(\mathcal{K}^+_{\rho(\alpha,\varepsilon)}(p)) \subset \mathcal{C}^+_{L^N_A(p)}.$$

Let $r_0 > 0$ such that $D_{r_1} \subset D_{r_\sigma}$. Recall that $r_1 = 2r_0\Lambda$. Since $f_{1,0} = L_A$ in a neighborhood of $(0, 0)$, we can choose sufficiently small r_0 such that the following two facts hold.

- Let $p \in \mathbb{T}^2$ and $k, n \in \{0\} \cup \mathbb{N}$ be such that $f^k_{1,0}(p) \notin D_{r_\sigma}$, $f^{k+j}_{1,0}(p) \in D_{r_\sigma} \setminus D_{r_1}$ for $j = 1, 2, \ldots, n$, and $f^{k+n+1}_{1,0}(p) \in D_{r_1}$. Then $n \geq N$.
- Let $p \in \mathbb{T}^2$ and $k, n \in \{0\} \cup \mathbb{N}$ be such that $f^k_{1,0}(p) \in D_{r_1}$, $f^{k+j}_{1,0}(p) \in D_{r_\sigma} \setminus D_{r_1}$ for $j = 1, 2, \ldots, n$, and $f^{k+n+1}_{1,0}(p) \notin D_{r_\sigma}$. Then $n \geq N$.

Let $G_{1,\eta}$ be an Anosov diffeomorphism obtained from $f_{1,0}$ using Construction II with the parameter r_0 as above. Then for any $\eta \in (0, 2r_0^2]$ we have the following assertions.

- $\nu_{G_{1,\eta}}(D_{r_\sigma}) < \sigma$.
- $\nu_{G_{1,\eta}}(\bar{\mathcal{S}}) \geq \bar{Q} - \sigma$.
- Let $p \in \mathbb{T}^2$ and $k, n \in \{0\} \cup \mathbb{N}$ be such that $G^k_{1,\eta}(p) \notin D_{r_\sigma}$, $G^{k+j}_{1,\eta}(p) \in D_{r_\sigma} \setminus D_{r_1}$ for $j = 1, 2, \ldots, n$, and $G^{k+n+1}_{1,\eta}(p) \in D_{r_1}$. Then $n \geq N$.
- Let $p \in \mathbb{T}^2$ and $k, n \in \{0\} \cup \mathbb{N}$ be such that $G^k_{1,\eta}(p) \in D_{r_1}$, $G^{k+j}_{1,\eta}(p) \in D_{r_\sigma} \setminus D_{r_1}$ for $j = 1, 2, \ldots, n$, and $G^{k+n+1}_{1,\eta}(p) \notin D_{r_\sigma}$. Then $n \geq N$.

Consider $p \in \mathbb{T}^2 \setminus D_{r_\sigma}$ and a natural number n. We write $n = \sum_{j=1}^s n_j$, where the numbers $n_j \in \{0\} \cup \mathbb{N}$ are chosen in the following way (see Figure 7 for an example).

(1) The number n_1 is the first moment when $(G_{1,\eta})^{n_1}(p) \in \bar{\mathcal{S}} \cup D_{r_\sigma}$.

(2) The number n_2 is such that the number $n_1 + n_2$ is the first moment when $(G_{1,\eta})^{n_1+n_2}(p) \in \mathbb{T}^2 \setminus (\bar{\mathcal{S}} \cup D_{r_\sigma})$.

(3) The rest of the numbers are defined following this pattern as done in the proof of Lemma 3.7.

Notice that by the choice of r_σ, for any $j = 1, 3, 5, \ldots$ we have that either

$$(G_{1,\eta})^{n_1+n_2+\cdots+n_j+k}(p) \in \bar{\mathcal{S}} \quad \text{for all } k \in \mathbb{Z} \cap [0, n_{j+1})$$

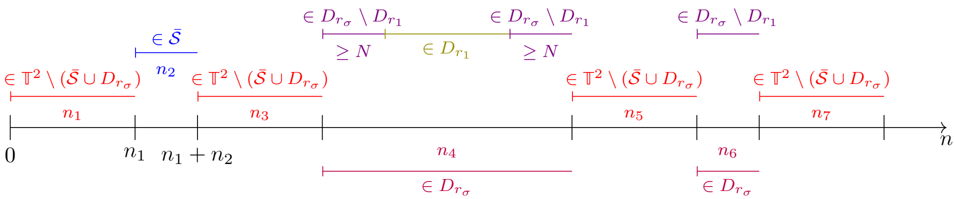

FIGURE 7. Partition of the orbit of p under $G_{1,\eta}$.

or

$$(G_{1,\eta})^{n_1+n_2+\cdots+n_j+k}(p) \in D_{r_\sigma} \quad \text{for all } k \in \mathbb{Z} \cap [0, n_{j+1}).$$

Let $\mathbf{v} \in \mathcal{C}_p^+$ and $\|\mathbf{v}\| = 1$. Then we have

$$\log \|D_p(G_{1,\eta})^n \mathbf{v}\| = \sum_{j=1}^{s} \log \|D_{(G_{1,\eta})^{n_1+n_2+\cdots+n_{j-1}}(p)}(G_{1,\eta})^{n_j}\mathbf{v}_j\|,$$

where $\mathbf{v}_1 = \mathbf{v}$, $\mathbf{v}_2 = D_p(G_{l,\eta})^{n_1}\mathbf{v}_1/\|D_p(G_{1,\eta})^{n_1}\mathbf{v}_1\|$, and

$$\mathbf{v}_j = \frac{D_{(G_{1,\eta})^{n_1+n_2+\cdots+n_{j-2}}(p)}(G_{1,\eta})^{n_{j-1}}\mathbf{v}_{j-1}}{\|D_{(G_{1,\eta})^{n_1+n_2+\cdots+n_{j-2}}(p)}(G_{1,\eta})^{n_{j-1}}\mathbf{v}_{j-1}\|} \quad \text{for } j = 3, \ldots, s.$$

In particular, $\|\mathbf{v}_j\| = 1$ for $j = 1, \ldots, s$.

Using (2.12), (2.25), and Lemma 3.9, we obtain for $k \in \mathbb{N}$ that

$$\|D_{(G_{1,\eta})^{n_1+n_2+\cdots+n_{j-1}}(p)}(G_{1,\eta})^{n_j}\mathbf{v}_j\| \geq \begin{cases} K_1K_2^{-1} & \text{if } j = 2k,\ (G_{1,\eta})^{n_1+n_2+\cdots+n_{j-1}}(p) \in D_{r_\sigma}, \\ \left(\mu^+\left(\frac{\bar\beta+\bar\delta(1-\bar\delta)}{\bar\delta}\right)\right)^{n_j} C & \text{if } j = 2k,\ (G_{1,\eta})^{n_1+n_2+\cdots+n_{j-1}}(p) \in \bar{\mathcal{S}}, \\ \mu^{n_j} & \text{if } j = 2k-1, \end{cases}$$

where C is a constant that depends only on A and $\bar\beta$ (see (2.25)).

As a result,

$$\begin{aligned} \log \|D_p(G_{1,\eta})^n \mathbf{v}\| \geq{}& \log(K_1K_2^{-1}) \sum_{k=1}^{[s/2]} \mathbb{1}_{D_{r_\sigma}}((G_{1,\eta})^{n_1+n_2+\cdots+n_{2k-1}}(p)) \\ &+ \log(C) \sum_{k=1}^{[s/2]} \mathbb{1}_{\bar{\mathcal{S}}}((G_{1,\eta})^{n_1+n_2+\cdots+n_{2k-1}}(p)) \\ &+ \log \mu^+\left(\frac{\bar\beta+\bar\delta(1-\bar\delta)}{\bar\delta}\right) \sum_{k=1}^{[s/2]} n_{2k}\mathbb{1}_{\bar{\mathcal{S}}}((G_{1,\eta})^{n_1+n_2+\cdots+n_{2k-1}}(p)) \\ &+ (\log \mu) \sum_{k=1}^{[s/2]} n_{2k-1}, \end{aligned}$$

where $\mathbb{1}_{D_{r_\sigma}}$ and $\mathbb{1}_{\bar{\mathcal{S}}}$ are the characteristic functions of the corresponding sets.

Since $G_{1,\eta}$ is a smooth Anosov diffeomorphism, using Birkhoff's ergodic theorem, we obtain, as $n \to \infty$,

$$\frac{1}{n}\sum_{k=1}^{[s/2]} n_{2k}\mathbb{1}_{\bar{\mathcal{S}}}((G_{1,\eta})^{n_1+n_2+\cdots+n_{2k-1}}(p)) \to \nu_{G_{1,\eta}}(\bar{\mathcal{S}})$$

and

$$\frac{1}{n}\sum_{k=1}^{[s/2]} n_{2k-1} \to \nu_{G_{1,\eta}}(\mathbb{T}^2 \setminus (D_{r_\sigma} \cup \bar{\mathcal{S}})).$$

Moreover, since $n_{2k} \geq 1$ for $k = 1, 2, \ldots, [s/2]$, we have

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{[s/2]} \mathbb{1}_{D_{r_\sigma}}((G_{1,\eta})^{n_1+n_2+\cdots+n_{2k-1}}(p)) \\
&\quad \leq \lim_{n\to\infty}\sum_{k=1}^{[s/2]} n_{2k}\mathbb{1}_{D_{r_\sigma}}((G_{1,\eta})^{n_1+n_2+\cdots+n_{2k-1}}(p)) = \nu_{G_{1,\eta}}(D_{r_\sigma}).
\end{aligned}$$

Also, notice that each visit to $\mathbb{T}^2 \setminus (D_{r_\sigma} \cup \bar{\mathcal{S}})$ is at least one iterate, so

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{[s/2]} \mathbb{1}_{\bar{\mathcal{S}}}((G_{1,\eta})^{n_1+n_2+\cdots+n_{2k-1}}(p)) &\leq \lim_{n\to\infty}\frac{1}{n}[s/2] \leq \nu_{G_{1,\eta}}(\mathbb{T}^2 \setminus (D_{r_\sigma} \cup \bar{\mathcal{S}})) \\
&= 1 - \nu_{G_{1,\eta}}(\bar{\mathcal{S}}) - \nu_{G_{1,\eta}}(D_{r_\sigma}).
\end{aligned}$$

Thus, since $K_1K_2^{-1}, C \in (0, 1)$,

$$\begin{aligned}
\lambda_{\text{mme}}(G_{1,\eta}) &\geq \log(K_1K_2^{-1})\nu_{G_{1,\eta}}(D_{r_\sigma}) + \log(C)(1 - \nu_{G_{1,\eta}}(\bar{\mathcal{S}}) - \nu_{G_{1,\eta}}(D_{r_\sigma})) \\
&\quad + \log \mu^+\left(\frac{\bar{\beta} + \bar{\delta}(1-\bar{\delta})}{\bar{\delta}}\right)\nu_{G_{1,\eta}(\bar{\mathcal{S}})} + (\log \mu)\nu_{G_{1,\eta}}(\mathbb{T}^2 \setminus (D_{r_\sigma} \cup \bar{\mathcal{S}})) \\
&\geq \log(K_1K_2^{-1})\sigma + \log(C)(1 - \bar{Q} + \sigma) + (\bar{Q} - \sigma)\log \mu^+\left(\frac{\bar{\beta} + \bar{\delta}(1-\bar{\delta})}{\bar{\delta}}\right). \quad \square
\end{aligned}$$

Acknowledgements. The author would like to thank Andrey Gogolev for helpful discussions and Vaughn Climenhaga for pointing out some references. Question 6 was originally posed by Federico Rodriguez Hertz. The author is also grateful to Anatole Katok for suggesting the problem. Much of this work was completed while the author was at The Ohio State University. She is also grateful for NSF grant 1642548, which supported travel to present the result in this paper. Additionally, the author was partially supported by NSF grant 1547145. The author would like to thank the anonymous referee for thoroughly reading the initial version of the paper and for suggestions to improve the presentation.

REFERENCES

[AS67] D. Anosov and Y. Sinai. Certain smooth ergodic systems. *Russian Math. Surveys* **22** (1967), 103–167.

[AW67] R. Adler and B. Weiss. Entropy, a complete metric invariant for automorphisms of the torus. *Proc. Natl. Acad. Sci. USA* **57** (1967), 1573–1576.

[BG14] K. Burns and K. Gelfert. Lyapunov spectrum for geodesic flows of rank 1 surfaces. *Discrete Contin. Dyn. Syst.* **34**(5) (2014), 1841–1872.

[BKRH19] J. Bochi, A. Katok and F. R. Hertz. Flexibility of Lyapunov exponents. *Ergod. Th. & Dynam. Sys.* this special issue.

[Bow71] R. Bowen. Periodic points and measures for Axiom A diffeomorphisms. *Trans. Amer. Math. Soc.* **154** (1971), 377–397.

[BP07] L. Barreira and Y. Pesin. *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents (Encyclopedia of Mathematics and its Applications, 115)*. Cambridge University Press, Cambridge, 2007.

[dlL87] R. de la Llave. Invariant for smooth conjugacy of hyperbolic dynamical systems II. *Comm. Math. Phys.* **109** (1987), 369–378.

[Erc19] A. Erchenko. Flexibility of Lyapunov exponents for expanding circle maps. *Discrete Contin. Dyn. Syst.* **39**(5) (2019), 2325–2342.

[HJJ17] H. Hu, M. Jiang and Y. Jiang. Infimum of the metric entropy of volume preserving Anosov systems. *Discrete Contin. Dyn. Syst.* **37**(9) (2017), 4767–4783.

[Kat79] A. Katok. Bernoulli diffeomorphisms on surfaces. *Ann. of Math. (2)* **110**(3) (1979), 529–547.

[KH95] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, 54)*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and L. Mendoza.

[KS01] M. Kotani and T. Sunada. The pressure and higher correlations for an Anosov diffeomorphism. *Ergod. Th. & Dynam. Sys.* **21** (2001), 807–821.

[MM87] J. M. Marco and R. Moriyón. Invariants for smooth conjugacy of hyperbolic dynamical systems, III. *Comm. Math. Phys.* **112**(2) (1987), 317–333.

[Mos69] J. Moser. On a theorem of Anosov. *J. Differential Equations* **5** (1969), 411–440.

[PSZ19] Y. Pesin, S. Senti and K. Zhang. Thermodynamics of the Katok map. *Ergod. Th. & Dynam. Sys.* **39** (2019), 764–794.

[Rue78a] D. Ruelle. An inequality for the entropy of differentiable maps. *Bull. Braz. Math. Soc.* **9** (1978), 83–87.

[Rue78b] D. Ruelle. *Thermodynamic Formalism: The Mathematical Structure of Equilibrium Statistical Mechanics (Cambridge Mathematical Library)*, 2nd edn. Cambridge University Press, Cambridge, 2004.