



Assouad Spectrum Thresholds for Some Random Constructions

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Abstract. The Assouad dimension of a metric space determines its extremal scaling properties. The derived notion of the Assouad spectrum fixes relative scales by a scaling function to obtain interpolation behaviour between the quasi-Assouad and the box-counting dimensions. While the quasi-Assouad and Assouad dimensions often coincide, they generally differ in random constructions. In this paper we consider a generalised Assouad spectrum that interpolates between the quasi-Assouad and the Assouad dimension. For common models of random fractal sets, we obtain a dichotomy of its behaviour by finding a threshold function where the quasi-Assouad behaviour transitions to the Assouad dimension. This threshold can be considered a phase transition, and we compute the threshold for the Gromov boundary of Galton–Watson trees and one-variable random self-similar and self-affine constructions. We describe how the stochastically self-similar model can be derived from the Galton–Watson tree result.

1 Introduction

The Assouad dimension is an important notion in embedding theory due to the famous Assouad embedding theorem [As77, As79] and its invariance under bi-Lipschitz maps. The latter implies that a metric space X cannot be embedded by a bi-Lipschitz map into \mathbb{R}^d for any d less than the Assouad dimension of X . The Assouad embedding theorem provides a partial converse. The Assouad dimension is therefore a good indicator of thickness in a metric space and is an upper bound to most notions of dimension in use today [Fr14, Roll]. In particular, it is an upper bound to the Hausdorff, box-counting, and packing dimensions. Heuristically, the Assouad dimension “searches” for the thickest part of a space relative to two scales $0 < r < R$ by finding the minimal exponent s such that the every R -ball can be covered by at most $(R/r)^s$ balls of diameter r .

Over the last few years much progress has been made towards our understanding of this dimension, and it is now a crucial part of fractal geometry; see *e.g.*, [Ch19, Fr14, FMT18, GHM16, KR16, Tr19] and references therein. Several other notions of dimension were derived from its definition, and this family of Assouad-type dimensions has attracted much interest. An important notion is the θ -Assouad spectrum introduced by Fraser and Yu [FY18], which aims to interpolate between the upper box-counting and the Assouad dimension to give fine information on the structure of metric spaces;

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see [Fr19] for a recent survey. It analyses sets by fixing the relation $r = R^{1/\theta}$ in the definition of the Assouad dimension for parameters $\theta \in (0, 1)$.

It turns out that the Assouad spectrum interpolates between the upper box-counting dimension and the quasi-Assouad dimension introduced by Lü and Xi [LX16]. That is, for $\theta \rightarrow 0$, the θ -Assouad spectrum tends to the upper box-counting dimension, whereas for $\theta \rightarrow 1$, it approaches the quasi-Assouad dimension; see [FHH+19]. In fact, the quasi-Assouad dimension could be defined in terms of the Assouad spectrum.

In many cases the quasi-Assouad dimension and Assouad dimension coincide, and the Assouad spectrum gives best relative scaling information. However, in many stochastic settings they differ. This can be explained by the Assouad dimension picking up very extreme behaviour that is almost surely lost over all geometric scales [FMT18].

In their landmark paper [FY18], Fraser and Yu discuss the possibility of extending the definition of the Assouad spectrum to analyse the case when quasi-Assouad and Assouad dimensions differ. These general spectra, which we shall also refer to as *generalised Assouad spectra*, would then shed some light on the behaviour of “in-between” scales. This is done by changing the relation $r = R^{1/\theta}$ to a general dimension function $r = \varphi(R)$. García, Hare, and Mendivil studied this notion of spectrum (including their natural dual, the lower Assouad dimension spectrum) and obtain interpolation results similar to those for the Assouad spectra; see [GHM19]. A common observation is that the intermediate spectrum is constant and equal to either the quasi-Assouad or Assouad dimension around a *threshold function*. That is, there exists a function $\phi(x)$ such that $\dim_A^\varphi F = \dim_{qA} F$ for $\varphi(x) = \omega(\phi(x))$ and $\dim_A^\varphi F = \dim_A F$ for $\varphi(x) = o(\phi(x))$, where we have used the standard little omega- and *o*-notations.¹ The standard examples where quasi-Assouad and Assouad dimensions differ are random constructions, and this threshold can be considered a phase transition in the underlying stochastic process. In this paper we will explore this threshold function for various random models.

García *et al.* [GHM19a] considered the following random construction. Let (l_i) be a non-increasing sequence such that $\sum l_i = 1$. For each i , let U_i be an i.i.d. copy of U , the random variable that is uniformly distributed in $[0, 1]$. Note that, almost surely, $U_i \neq U_j$ for all $i \neq j$. Therefore, almost surely, there is a total ordering of the (U_i) . The complementary set of the random arrangement E is defined as the complement of arranging open intervals of length l_i in the order induced by (U_i) . That is,

$$E = \bigcup_{y \in [0,1]} \left\{ x = \sum_{V_y} l_i : V_y = \{i : U_i < y\} \right\}.$$

Almost surely, this set is uncountable and has a Cantor-like structure. García *et al.* previously determined the (quasi-)Assouad dimensions of deterministic realisations [GHM16], where the order is taken as $U_i < U_{i+1}$ as well as the Cantor arrangement, when U_i is equal to the right-hand end point of the canonical construction intervals of the Cantor middle-third set (ignoring repeats). They confirmed in [GHM19a] that the quasi-Assouad dimension of E is almost surely equal to the quasi-Assouad

¹A function satisfies $f(x) = o(g(x))$ if $f/g \rightarrow 0$ as $x \rightarrow 0$. Similarly, $f(x) = \omega(g(x))$ if $g(x) = o(f(x))$.

dimension of the Cantor arrangement C , whereas the Assouad dimension takes the value 1. They further gave the threshold function, which they computed as $\phi(x) = \log|\log x|/|\log x|$.

Theorem 1.1 (García et al. [GHM19a]) *Let $\phi(x) = \log|\log x|/|\log x|$ and let l_i be a decreasing sequence such that $\sum l_i = 1$. Assume further that there exists $\varepsilon > 0$ such that*

$$\varepsilon < \frac{\sum_{j \geq 2^{n+1}} l_j}{\sum_{j \geq 2^n} l_j} < 1 - \varepsilon$$

for all $n \in \mathbb{N}$. Then, almost surely, $\dim_A^\varphi E = \dim_A^\varphi C$ for $\varphi = \omega(\phi(x))$ and $\dim_A^\varphi E = 1$ for $\varphi = o(\phi(x))$.

In this article, we give elementary proofs of the threshold dimension functions for several canonical random sets. Under separation conditions we obtain the threshold for one-variable random iterated function systems with self-similar maps and self-affine maps of Bedford–McMullen type. We also determine the threshold for the Gromov boundary of Galton–Watson trees. While we do not state it explicitly, using the methods found in [Tr19], our result for Galton–Watson trees directly applies to stochastically self-similar and self-conformal sets as well as fractal percolation sets.

Our proofs rely on the theory of large deviations as well as a dynamical version of the Borel–Cantelli lemmas.

2 Definitions and Results

Let $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$. We say that ϕ is a *dimension function* if $\phi(x)$ and $\phi(x)|\log x|$ are monotone. Let $N_r(X)$ be the minimal number of sets of diameter at most r needed to cover X . The generalised Assouad spectrum (or intermediate Assouad spectrum) with respect to ϕ is given by

$$\dim_A^\phi F = \inf \left\{ s : (\exists C > 0)(\forall 0 < r = R^{1+\phi(R)} < R < 1) \sup_{x \in F} N_r(F \cap B(x, R)) \leq C \left(\frac{R}{r}\right)^s = R^{-\phi(R)s} \right\}.$$

We will also refer to this quantity as the ϕ -Assouad dimension of F . Many other variants of the Assouad dimension can now be obtained by restricting ϕ in some way. The Assouad spectrum \dim_A^θ considered by Fraser and Yu can be obtained by setting $\phi(R) = 1/\theta - 1$. The quasi-Assouad dimension $\dim_{qA} F$ is given by the limit $\dim_{qA} F = \lim_{\theta \rightarrow 1} \dim_A^\theta F$, whereas the Assouad dimension is obtained by letting $\phi(R) = 0$ and allowing $r \leq R$. We note that we define the generalised Assouad spectrum slightly differently than in [FY18]. Instead of requiring $r = \varphi(R)$, we consider the dimension function ϕ setting $r = R^{1+\phi(R)}$. Hence, $\varphi(R)/R = R^{\phi(R)}$. We use this notation as it is slightly more convenient to use.

In fractal geometry, two canonical models are used to obtain random fractal sets: stochastically self-similar sets and one-variable random sets. We first compute the

generalised Assouad spectrum for Galton–Watson processes from which the stochastically self-similar case will follow. We will then move on to one-variable random constructions and analyse random self-similar constructions as well as a randomisation of Bedford–McMullen carpets.

2.1 Galton–Watson Trees and Stochastically Self-similar Sets

Let X be a random variable that takes values in $\{0, 1, \dots\}$ and write $\theta_j = \mathbb{P}\{X = j\}$ for the probability that X takes value j . The Galton–Watson process Z_i is defined inductively by letting $Z_0 = 1$ and $Z_{k+1} = \sum_{j=1}^{Z_k} X_j$, where each summand X_j is an i.i.d. copy of X . The Galton–Watson tree is obtained by considering a tree with single root and determining ancestors for every node with law X , independent of all other nodes. The number of nodes at level k is then given by Z_k and, conditioned on non-extinction, this process generates a (random) infinite tree. We endow the set of infinite descending paths starting at the root with the standard metric $d(x, y) = 2^{-|x \wedge y|}$, where $|x \wedge y|$ is the level of the least common ancestor. This gives rise to the Gromov boundary of the random tree that we refer to as F_τ .

Throughout, we assume that we are in the supercritical case, *i.e.*,

$$m = \mathbb{E}(X) = \sum_{k=1}^N \theta_k k > 1.$$

We will also assume that the Galton–Watson process has bounded offspring distribution, meaning that there exists N such that $\theta_k = 0$ for all $k > N$.

The normalised Galton–Watson process is defined by $W_k = Z_k/m^k$. It is a standard application of the martingale convergence theorem to show that $W_k \rightarrow W$ almost surely. Conditioned on non-extinction, we additionally have $W \in (0, \infty)$ almost surely. We refer the reader to [Li00, LPP95] for some other fundamental dimension theoretic results of Galton–Watson processes.

It was established in [FMT18] that the Gromov boundary, using the standard metric, has Assouad dimension $\log N/\log 2$ almost surely. In [Tr19], the quasi-Assouad dimension was computed as

$$\dim_{\text{qA}} F_\tau = \frac{\log \mathbb{E}(X)}{\log 2} = \dim_{\text{B}} F_\tau = \dim_{\text{H}} F_\tau$$

almost surely. This means, in particular, that the θ -Assouad spectrum is constant and equal to the Hausdorff dimension.² This is in fact the typical behaviour for all dimension functions $\phi(x) \leq C(\log|\log x|/|\log x|)$, where $C > 0$ is some constant. Conversely, for $\phi(x) = \omega(\log|\log x|/|\log x|)$ we recover the Assouad dimension; cf. Theorem 1.1.

Theorem 2.1 *Let F_τ be the Gromov boundary of a supercritical Galton–Watson tree with bounded offspring distribution. Let*

$$\phi(x) > \frac{\log|\log x|}{|\log x|} (\varepsilon \log m)^{-1}$$

²In all models considered in this paper (except the self-affine construction), the Hausdorff and box-counting dimensions coincide almost surely, and all instances of \dim_{B} can be replaced by \dim_{H} .

be a dimension function. Then

$$\dim_A^\phi F_\tau \leq (1 + \varepsilon) \dim_B F_\tau$$

almost surely.

Note that we trivially have $\dim_A^\phi F \geq \dim_B F$, and so we have the following corollary.

Corollary 2.2 *Let $\phi(x) = \omega(\log|\log x|/|\log x|)$ be a dimension function. Then $\dim_A^\phi F_\tau = \dim_B F_\tau = \dim_H F_\tau$ almost surely.*

For the reverse direction, we obtain the following bound.

Theorem 2.3 *Let F_τ be the Gromov boundary of a supercritical Galton–Watson tree with bounded offspring distribution. Let N be the maximal integer s.t. $\theta_N > 0$. Further, let*

$$(2.1) \quad \phi(x) < \frac{\log|\log x|}{|\log x|} \frac{\log(N/m)}{\varepsilon \log m \log N}$$

be a dimension function. Then,

$$\dim_A^\phi F_\tau \geq \min\{(1 + \varepsilon) \dim_B F_\tau, \dim_A F_\tau\}$$

almost surely.

Assume that $m < N$ and so $\dim_B F_\tau < \dim_A F_\tau$ almost surely. Under the hypothesis of Theorem 2.3, for ε satisfying $(1 + \varepsilon) = \dim_A F_\tau / \dim_B F_\tau = \log N / m$, we obtain $\dim_A^\phi F_\tau \geq \dim_A F_\tau$ almost surely. It follows that for

$$\phi(x) < \frac{\log|\log x|}{\log(1/x)} \frac{\log(N/m)}{(N/m) \log m \log N} = \frac{m}{N} \left(\frac{1}{\log m} - \frac{1}{\log N} \right) \frac{\log|\log x|}{|\log x|},$$

we get $\dim_A^\phi F_\tau = \dim_A F_\tau$.

Corollary 2.4 *There exists $C > 0$ such that, almost surely, $\dim_A^\phi F_\tau = \dim_A F_\tau$ for all dimension functions $\phi(x) \leq C \log|\log x|/|\log x|$.*

We will prove both theorems in Section 3. These two bounds are not optimal in the sense that there is a slight gap between the upper and the lower bound. This gap, after rearranging, is of order $1 - \log m / \log N = 1 - \dim_B F_\tau / \dim_A F_\tau$. Let ϕ_ε be equal to the right-hand side of (2.1). We can combine Theorems 2.1 and 2.3 to give

$$(1 + \varepsilon) \dim_B F_\tau \leq \dim_A^{\phi_\varepsilon} F_\tau \leq (1 + \varepsilon) \frac{\dim_A F_\tau}{\dim_A F_\tau - \dim_B F_\tau} \dim_B F_\tau$$

for an appropriate range of $\varepsilon > 0$.

2.1.1 Stochastically Self-similar Sets

Theorems 2.1 and 2.3 can also be applied in the setting of stochastically self-similar sets that were first studied by Falconer [Fal86] and Graf [Gr87]. Since we do not exclude the case where there is no descendant, the analysis also applies to fractal percolation in the sense of Falconer and Jin [FJ15]. In the case of Mandelbrot percolation, where a d -dimensional cube is split into n^d equal subcubes of sidelength $1/n$ and is kept with probability $p > 0$, the number of subcubes is a Galton–Watson process, and the surviving subcubes at level k can be modelled by a Galton–Watson tree. Since subcubes at the same level have the same diameters, the limit set is almost surely bi-Lipschitz to the Gromov boundary of an appropriately set up Galton–Watson tree with the small caveat that the graph metric needs to be changed from $d(x, y) = 2^{-|x \wedge y|}$ to $d'(x, y) = n^{-|x \wedge y|}$. This change, however, just affects the results above by a constant and not their asymptotic behaviour. For non-homogeneous self-similar sets, where the size may vary at a given generation, one needs to set up a Galton–Watson tree that models the set. This is described in full detail in [Tr19], and we omit its derivation here. Using those methods, Theorems 2.1 and 2.3 become the following corollary.

Corollary 2.5 *Let F_τ be a stochastically self-similar set arising from finitely many self-similar IFS that satisfy the uniform open set condition. Then if $\phi(x) = \omega(\log|\log x|/|\log x|)$, we obtain $\dim_\Lambda^\phi F_\tau = \dim_B F_\tau$ almost surely. Conversely, if $\phi(x) = o(\log|\log x|/|\log x|)$, then $\dim_\Lambda^\phi F_\tau = \dim_A F_\tau$ almost surely.*

2.2 One-variable Random Sets

A different popular model for random fractal sets is the one-variable model. It is sometimes also referred to as a homogeneously random construction. We will avoid the latter term to avoid ambiguity with homogeneous iterated function systems. Let $\Lambda \subset \mathbb{R}^n$ be a compact set. With each $\lambda \in \Lambda$, we associate an iterated function system $\mathbb{I}_\lambda = \{f_1^\lambda, \dots, f_{N_\lambda}^\lambda\}$, where each f_i^λ is a strictly contracting diffeomorphism on some non-empty open set V . Throughout this section we make the standing assumption that $\sup_\lambda N_\lambda < \infty$, that $0 < \inf_{\lambda, i, x} |(f_i^\lambda)'(x)| \leq \sup_{\lambda, i, x} |(f_i^\lambda)'(x)| < 1$, and that there exists a non-empty compact set $\Delta \subset V$ such that $f_i^\lambda(\Delta) \subseteq \Delta$ for all $\lambda \in \Lambda$ and $1 \leq i \leq N_\lambda$. With each $\omega \in \Omega = \Lambda^{\mathbb{N}}$, we associate the set F_ω given by

$$F_\omega = \bigcap_{k=1}^\infty \bigcup_{\substack{1 \leq i_j \leq N_{\lambda_j} \\ 1 \leq j \leq k}} f_{i_k}^{\lambda_k} \circ \dots \circ f_{i_1}^{\lambda_1}(\Delta).$$

Let μ be a Borel probability measure supported on Λ and let $\mathbb{P} = \mu^{(\mathbb{N})}$ be the product measure on $\Omega = \Lambda^{\mathbb{N}}$. We write

$$\mathbb{E}(X(\omega)) = \int_\Omega X(\omega) d\mathbb{P}(\omega) \quad \text{and} \quad \mathbb{E}^g(X(\omega)) = \exp \int_\Omega \log X(\omega) d\mathbb{P}(\omega).$$

The one-variable random attractor F_ω is then obtained by choosing $\omega \in \Omega$ according to the law \mathbb{P} .

2.2.1 One-variable Random Self-similar Sets

To make useful dimension estimates, we have to restrict the class of functions. The simplest model is that of self-similar sets, where we restrict f_i^λ to similarities. That is, $|f_i^\lambda(x) - f_i^\lambda(y)| = c_i^\lambda|x - y|$ for all $x, y \in V$ and some $c_i^\lambda > 0$. It is well known that for self-similar maps and our standing assumptions, the Hausdorff and box-counting dimensions are bounded above by the unique s satisfying

$$(2.2) \quad \mathbb{E}^g \left(\sum_{1 \leq i \leq N_{\omega_1}} (c_i^{\omega_1})^s \right) = 1.$$

If one further assumes that there exists a non-empty open set U such that the union $\bigcup_{i=1}^{N_\lambda} f_i^\lambda(U)$ is disjoint for all λ and $f_i^\lambda(U) \subseteq U$, we say that the uniform open set condition holds. Under this assumption the unique s in (2.2) coincides with the Hausdorff, box-counting, and quasi-Assouad dimension of F_ω for \mathbb{P} -almost all ω ; see e.g., [Tr17] and references therein. Since we refer to the sum above quite frequently, we write $\mathfrak{S}_\lambda^t = \sum (c_i^\lambda)^t$. To avoid the trivial case when the Assouad dimension coincides with the Hausdorff dimension, and there is nothing to prove as the generalised Assouad dimension coincides with this common value, we make the assumption that the system is not almost deterministic. That is, $\mathbb{P}(\mathfrak{S}_{\omega_1}^s = 1) \neq 1$, where s is the almost sure Hausdorff dimension. In particular, this implies that the Assouad dimension is strictly larger than the Hausdorff (and upper box-counting) dimension. In fact, the Assouad dimension of F_ω is, almost surely, given by

$$\dim_A F_\omega = \sup\{s : \mu(\{\lambda \in \Lambda : \mathfrak{S}_\lambda^s \geq 1\}) > 0\};$$

see [FMT18, Tr17]. To not obscure the result with needless technicality, we only analyse the case when the iterated function systems are homogeneous, i.e., $c_i^\lambda = c_j^\lambda$ for all λ . We write $c(\lambda)$ for the common value; then $\mathfrak{S}_\lambda^s = N_\lambda c(\lambda)^s$.

Theorem 2.6 *Let F_ω be a one-variable random self-similar set generated by homogeneous iterated functions systems satisfying the uniform open set condition. Then the following dichotomy holds: Let $\phi(x)$ be a dimension function such that*

$$\sum_{k=1}^\infty e^{-\phi(e^{-k})k} < \infty.$$

Then $\dim_A^\phi F_\omega = \dim_B F_\omega$ for dimension functions $\phi(x) = \omega(\phi(x))$, almost surely. Conversely, let $\phi(x)$ be a dimension function such that

$$\sum_{k=1}^\infty e^{-\phi(e^{-k})k} = \infty.$$

Then $\dim_A^\phi F_\omega = \dim_A F_\omega$ for dimension functions $\phi(x) = o(\phi(x))$, almost surely. Additionally, assume there exists $\Lambda' \subset \Lambda$ such that $\mathfrak{S}_\lambda^{s_A} = 1$ for all $\lambda \in \Lambda'$, where s_A is the almost sure Assouad dimension of F_ω , and $\mu(\Lambda') > 0$. Then the above result holds for all $\phi(x) \leq C(\phi(x))$, where $C > 0$ is some constant.

We prove this in Section 4. The methods we have used rely on the individual iterated function systems being homogeneous, but we need not have made this argument.

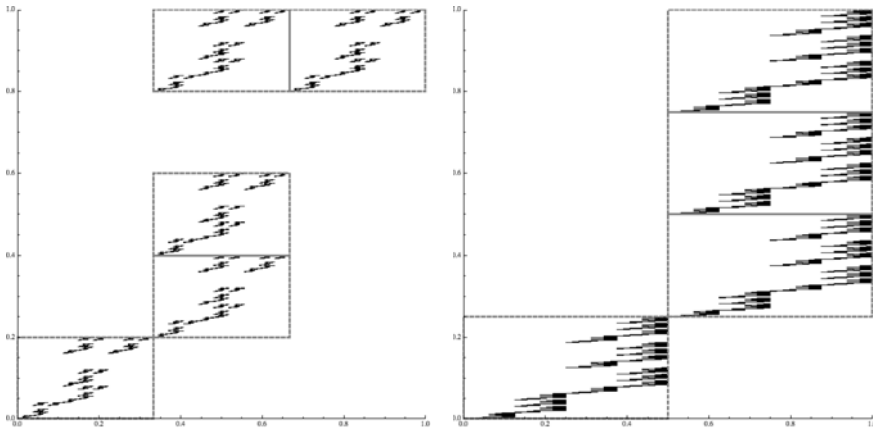


Figure 1: Two Bedford–McMullen carpets with their generating affine maps. The left example F_1 has $m_1 = 3$, $n_1 = 5$, $C_1 = 3$, $B_1 = 3$, and $N_1 = 5$. The right example F_2 has $m_2 = 2$, $n_2 = 4$, $C_2 = 3$, $B_2 = 2$, and $N_2 = 4$.

One can construct random graph-directed attractors that approximate the random set to arbitrary precision; see [Tr17, §2]. A similar argument to that of Theorem 2.6 should give an analogous result. Since this is somewhat more technical, we leave this case open.

2.2.2 One-variable Random Bedford–McMullen Carpets

Bedford–McMullen carpets were first studied in [Be84, Mc84] and are simple self-affine iterated function systems that are often the easiest to give as counterexamples to the self-similar theory. They consist of non-overlapping images of the unit square with fixed horizontal and vertical contraction of $1/m$ and $1/n$, respectively, that align in an $m \times n$ grid of the unit square. See Figure 1 for two examples.

We randomise the construction in the same one-variable random fashion by choosing different sub rectangles at every step from the finite collection of possible arrangements. We denote these in the same way and write $2 \leq m_\lambda < n_\lambda \leq N < \infty$ for the subdivisions. Then, $f_i^\lambda(x)$ are of the form

$$f_i^\lambda(x) = \begin{pmatrix} 1/m_\lambda & 0 \\ 0 & 1/n_\lambda \end{pmatrix} x + \begin{pmatrix} a_i^\lambda \\ b_i^\lambda \end{pmatrix},$$

where $a_i^\lambda \in \{0, \dots, m_\lambda - 1\}$ and $b_i^\lambda \in \{0, \dots, n_\lambda - 1\}$. The boundedness of n_λ implies that there are only finitely many IFSSs with finitely many maps. Hence, Λ is finite and μ is finitely supported. We write $p_\lambda = \mu(\lambda)$.

We rely heavily on results in [FT18], where the θ -Assouad spectrum of these attractors are found. In fact, this part can be considered an extension of [FT18], in the sense that the previous work gave a complete characterisation of the spectrum between the upper box-counting and the quasi-Assouad dimension, whereas we extend this to the Assouad dimension. Let C_λ be the maximal number of maps that align

in a column and let B_λ be the maximal number of non-empty columns. Further, let N_λ be the number of maps in the IFS indexed by λ ; see also Figure 1 for an example. Write $\bar{n} = \mathbb{E}^g(n_\lambda)$ and $\bar{m} = \mathbb{E}^g(m_\lambda)$. Similarly, let \bar{B} , \bar{C} , and \bar{N} denote their respective geometric means. The almost sure Assouad spectrum was found in [FT18] to be

$$\dim_A^\theta F_\omega = \begin{cases} \frac{1}{1-\theta} \left(\frac{\log(\bar{B}\bar{C}^\theta\bar{N}^{-\theta})}{\log \bar{m}} + \frac{\log(\bar{N}\bar{B}^{-1}\bar{C}^{-\theta})}{\sum_\lambda p_\lambda \log n_\lambda} \right), & 0 < \theta \leq \frac{\log \bar{m}}{\log \bar{n}}, \\ \frac{\log \bar{B}}{\log \bar{m}} + \frac{\log \bar{C}}{\log \bar{n}}, & \frac{\log \bar{m}}{\log \bar{n}} < \theta < 1. \end{cases}$$

From this we can further deduce that, almost surely,

$$\dim_B F_\omega = \frac{\log \bar{B}}{\log \bar{m}} + \frac{\log \bar{N} \bar{B}^{-1}}{\log \bar{n}} \quad \text{and} \quad \dim_{qA} F_\omega = \frac{\log \bar{B}}{\log \bar{m}} + \frac{\log \bar{C}}{\log \bar{n}}.$$

However, the almost sure Assouad dimension is generally distinct from the quasi-Assouad dimension and given by

$$\dim_A F_\omega = \max_{\lambda \in \Lambda} \frac{\log B_\lambda}{\log m_\lambda} + \max_{\lambda \in \Lambda} \frac{\log C_\lambda}{\log n_\lambda};$$

see [FMT18]. Note, in particular, that the dimension does not depend on the exact form of μ , provided it is supported on Λ . Our main result in this section is bridging this gap with a similar dichotomy as for self-similar sets.

Theorem 2.7 *Let F_ω be a one-variable random Bedford-McMullen carpet. Then the following dichotomy holds. Let $\phi(x) \leq \log \bar{n} / \log \bar{m} - 1$ be a dimension function such that*

$$\sum_{k=1}^\infty e^{-\phi(e^{-k})k} < \infty.$$

Then $\dim_A^\phi F_\omega = \dim_{qA} F_\omega$ for dimension functions $\varphi(x) = \omega(\phi(x))$, almost surely.

Conversely, let $\phi(x)$ be a dimension function such that

$$\sum_{k=1}^\infty e^{-\phi(e^{-k})k} = \infty.$$

There exists $C > 0$ such that $\dim_A^\phi F_\omega = \dim_A F_\omega$ for all dimension functions $\varphi(x) \leq C\phi(x)$, almost surely.

Remark 2.8 The dichotomies, or phase transitions, observed in Theorems 2.1, 2.3, 2.6, and 2.7 can be seen as a form of random mass transference principle, as described in [AT19, §5]. In the theorems described there, no assumptions are being made on the overlaps, and it would be interesting to know if any separation condition assumptions are needed in our results at all.

3 Proofs for Galton–Watson Trees

In this section we prove Theorems 2.1 and 2.3. We rely on the following lemma.

Lemma 3.1 *Let Z_k be a Galton–Watson process with finitely supported offspring distribution. Then there exists $\tau > 0$ and $K > 0$ such that for all $\varepsilon > 0$,*

$$\mathbb{P}(Z_k \geq m^{(1+\varepsilon)k}) \leq K \exp(-\tau m^{\varepsilon k})$$

for all $k \in \mathbb{N}$.

The lemma follows easily from a standard Chernoff bound combined with a simplified probability generating function due to the finitely supported offspring distribution. For a more general result with unbounded support, see Athreya [At94, Theorem 4] and [Tr19, §3].

Proof of Lemma 3.1 Note that

$$\begin{aligned} \mathbb{P}(Z_k \geq m^{(1+\varepsilon)k}) &= \mathbb{P}(W_k \geq m^{\varepsilon k}) = \mathbb{P}(\exp(\tau W_k) \geq \exp(\tau m^{\varepsilon k})) \\ &\leq \exp(-\tau m^{\varepsilon k}) \mathbb{E}(\exp(\tau W_k)) \end{aligned}$$

by Markov’s inequality.

It remains to bound $\sup_k \mathbb{E}(\exp(\tau W_k))$ for some $\tau > 0$. Let $F_X(s) = \mathbb{E}(s^X)$ be the probability generating function of the random variable X . Then

$$\begin{aligned} F_{Z_k}(s) &= \mathbb{E}(s^{\sum_{i=1}^{Z_{k-1}} X}) = \mathbb{E}(\mathbb{E}(s^{\sum_{i=1}^{Z_{k-1}} X} \mid Z_{k-1} = z)) \\ &= \mathbb{E}(\mathbb{E}(s^X)^{Z_{k-1}}) = F_{Z_{k-1}} \circ F_X(s), \end{aligned}$$

and so

$$F_{Z_k}(s) = \underbrace{F_X \circ \dots \circ F_X}_{k \text{ times}}(s) =: F_X^{(k)}(s).$$

We also obtain $F_{W_k}(s) = F_X^{(k)}(s^{1/m^k})$, and bounding $\sup_k \mathbb{E}(\exp(\tau W_k))$ is equivalent to finding $s > 1$ such that $\sup_k F_{W_k}(s) < \infty$. Since X is finitely supported, $F_X(s) = \sum_{i=0}^N \theta_i s^i$ is a polynomial of degree N . It is easy to check that $F_X(1) = 1$ and $F'_X(1) = \sum_{i=1}^N \theta_i i = \mathbb{E}(X) = m$. Therefore, $F_X(s)$ is a C^2 diffeomorphism with derivative bounded away from 1 and ∞ in $B(1, \delta_1)$ for some $\delta_1 > 0$. Its inverse $F_X^{(-1)}(s)$, therefore, is a strict contraction with derivative $1/m$ at $s = 1$ with fixed point $F_X^{(-1)}(1) = 1$.

Let $G(s) = s^{1/m}$. Note that $G'(1) = 1/m$ and $G(1) = 1$. Further, there exists $\delta_2 > 0$ such that $G(s)$ is a C^2 diffeomorphism with derivative bounded away from 0 and 1. Note that $F_{W_k}(s) = F_X^{(k)} \circ G^{(k)}(s)$, and, using the chain rule and smoothness, it is easy to check that

$$\begin{aligned} \frac{d}{ds} F_X^{(-k)}(s) &\leq C_1 \frac{d}{ds} F_X^{(-k)}(t) \quad (\forall s, t \in B(1, \delta_1), \quad \forall k \in \mathbb{N}), \\ \frac{d}{ds} G^{(k)}(s) &\leq C_2 \frac{d}{ds} G^{(k)}(t) \quad (\forall s, t \in B(1, \delta_2), \quad \forall k \in \mathbb{N}), \end{aligned}$$

and

$$\frac{d}{ds} F_X^{(-k)}(1) = \frac{d}{ds} G^{(k)}(1) = m^{-k}.$$

Observe that $B(1, \delta_1 / (C_1 m^k)) \subset F_X^{(-k)}(B(1, \delta_1))$, and set $0 < \varepsilon \leq \min\{\delta_2, \delta_1 / (C_1 C_2)\}$. It follows that $\text{diam}(G^{(k)}(B(x, \varepsilon))) \leq 2C_2 \varepsilon / m^k \leq \delta_1 / (C_1 m^k)$, and so

$$G^{(k)}(B(1, \varepsilon)) \subseteq B(1, \delta_1 / (C_1 m^k)) \subseteq F_X^{(-k)}(B(1, \delta_1)).$$

Therefore, $F_{W_k}(1 + \varepsilon) = F_X^{(k)} \circ G^{(k)}(1 + \varepsilon) \leq 1 + \delta_1$. Letting $K = 1 + \delta_1$, our claim holds. ■

Note that a ball of size r in the metric on F_τ with centre $x \in F_\tau$ is simply the unique subtree containing x that starts at level k satisfying $2^{-k} \leq r < 2^{-(k-1)}$. Hence, for two scales $r < R$ the quantity $N_r(B(x, R))$ is equal to the number of nodes at level l satisfying $2^{-l} \leq r < 2^{-(l-1)}$ that share a common ancestor with x at level k satisfying $2^{-k} \leq R < 2^{-(k-1)}$. Using independence, this is an independent copy of Z_{l-k+1} and so $N_r(B(x, R)) \sim Z_{\log_2(1/r) - \log_2(1/R)}$.

Proof of Theorem 2.1 Fix $\varepsilon > 0$. By Lemma 3.1, we have

$$\mathbb{P}\{Z_k \geq m^{(1+\varepsilon)k} \text{ for some } k \geq l\} \leq K \sum_{j=l}^\infty \exp(-\tau m^{\varepsilon k}) \lesssim \exp(-\tau m^{\varepsilon l}).$$

For large enough k , there exists A (depending on the realisation) such that $Z_k \leq Am^k$, almost surely. Thus, the probability P_k that there is a node at generation k that exceeds the average from generation $(1 + \phi(e^{-k}))k$ onwards satisfies

$$\begin{aligned} P_k &\lesssim m^k \exp(-\tau m^{\varepsilon \phi(e^{-k})k}) = \exp(k \log m - \tau m^{\varepsilon \phi(e^{-k})k}) \\ &\leq \exp(k \log m - \tau m^{(1+\delta) \log k / \log m}) \end{aligned}$$

for some $\delta > 0$ by assumption on ϕ . Finally, as $k \log m - \tau k^{1+\delta} < -k^{1+\delta/2}$ for large enough k , we have

$$\sum_{k=1}^\infty P_k \lesssim \sum_{k=1}^\infty \exp(k \log m - \tau k^{1+\delta}) \lesssim \sum_{k=1}^\infty \exp(-\tau k^{1+\delta/2}) < \infty.$$

An application of the Borel–Cantelli lemma shows that, almost surely, there exists a level k_0 from which no node at level $k \geq k_0$ in the Galton–Watson tree will have more than $m^{(1+\varepsilon)l}$ many descendants for $l \geq \phi(e^{-k})k$. Geometrically, this means that for all $r \sim 2^{-l} < R^{(1+\phi(e^{-k}))k} \sim 2^{-(1+\phi(e^{-k}))k}$ small enough, we obtain

$$N_r(B(x, R)) \sim Z_{l-k} \lesssim m^{(1+\varepsilon)(\log_2(1/r) - \log_2(1/R))} = \left(\frac{R}{r}\right)^{(1+\varepsilon) \log m / \log 2},$$

which gives the required result. ■

Proof of Theorem 2.3 Let $N = \max\{i : \theta_i > 0\}$ and assume that $N > m$, as otherwise there is nothing to prove. There exists $q > 0$ and $k_0 \geq 1$ such that $\mathbb{P}(Z_k > m^k) \geq q$ for all $k \geq k_0$ by the martingale convergence theorem. Write $p = \theta_N$. Fix k such that $k - l_k > k_0$, where $l_k = \varepsilon k \log m / \log(N/m)$. Consider the probability that the maximal branching is chosen in the first l_k levels after the root. Then, at level l_k , there are N^{l_k} descendants. This occurs with probability $p p^N p^{N^2} \dots p^{N^{l_k-1}}$. Each descendant

has m^{k-l_k} descendants at level k with probability at least q and therefore the probability that $Z_k \geq m^{(1+\varepsilon)k}$ is bounded below by

$$(3.1) \quad p \cdots p^{N^{l_{k-1}}} q^{N^{l_k}} \geq \rho^{N^{l_{k+1}}} = \exp(N^{\varepsilon k} \log m / \log(N/m)^{+1} \cdot \log \rho)$$

for some $\rho > 0$. Let (k_i) be a sequence such that $(1 + \phi(e^{-k_i}))k_i < k_{i+1}$. Then

$$(3.2) \quad \tilde{P}_i = \mathbb{P} \left(\exists \text{ node at level } k_i \text{ s.t. } Z_{n_i} \geq m^{(1+\varepsilon)n_i} \right. \\ \left. \text{for } n_i = \phi(e^{-k_i})k_i \geq (1 - (1 - \rho^{N^{l_{k_i}}})^{Am^k}) \right)$$

by independence and the fact that there are at least Am^k nodes at level k . Note that by combining (2.1) with (3.1), one obtains

$$\rho^{N^{l_{k+1}}} \geq \exp(Nk^{1-\delta} \log \rho)$$

for some $\delta > 0$. Since further $Am^k = A \exp(k \log m)$, we obtain

$$\exp \log(1 - \rho^{N^{l_{k_i}}})^{Am^k} \leq \exp(-Am^k \rho^{N^{l_{k_i}}}) \\ \leq \exp(-A \exp(k \log m - Nk^{1-\delta} \log(1/\rho))) \rightarrow 0.$$

Therefore, for large enough k , the quantity in (3.2) is bounded below by $1/2$ and

$$\sum_i \tilde{P}_i \gtrsim \sum_i 1/2 = \infty.$$

The disjointness, combined with the Borel–Cantelli lemma, therefore posits the existence of infinitely many i for which such a maximal chain exists. The dimension result directly follows by taking $R_i = 2^{-k_i}$ and $r_i = 2^{-n_i}$ to give a sequence of $N_{r_i}(B(x_i, R_i))$ such that

$$N_{r_i}(B(x_i, R_i)) \gtrsim m^{(1+\varepsilon)(n_i-k_i)} = \left(\frac{R}{r}\right)^{(1+\varepsilon) \log m / \log 2}. \quad \blacksquare$$

4 One-variable Proofs

4.1 Cramér’s Theorem for i.i.d. Variables

Cramér’s theorem is a fundamental result in large deviations concerning the error of sums of i.i.d. random variables. Given a sequence of i.i.d. random variables $(X_i)_i$, we write $S_n = \sum_{i=1}^n X_i$. The rate function of this process is defined by the Legendre transform of the moment generating function of the random variable. That is, the moment generating function is $M(\theta) = \mathbb{E}(\exp(\theta X_1))$, and its Legendre transform is $I(x) = \sup_{\theta \in \mathbb{R}} \theta x - \log M(\theta)$.

Theorem 4.1 *Let $(X_i)_i$ be a sequence of centred i.i.d. random variables with common finite moment generating function $M(\theta) = \mathbb{E}(\exp(\theta X_1))$. Then if $M(\theta)$ is well-defined for all θ , the following hold:*

- (i) for any closed set $F \subseteq \mathbb{R}$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(S_n \in F) \leq - \inf_{x \in F} I(x);$$

(ii) for any open set $U \subseteq \mathbb{R}$,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \mathbb{P}(S_n \in U) \geq - \inf_{x \in U} I(x).$$

Letting $F = [a, \infty)$ and $U = (a, \infty)$ for $0 < a < \text{ess sup } X_1$, we have $I(a) > 0$, and for all $\delta > 0$, there exists $N_\delta \in \mathbb{N}$ such that

$$\begin{aligned} - \inf_{x \in U} I(x) - \delta &\leq \frac{1}{n} \log \mathbb{P}\{S_n \in U\} \leq \frac{1}{n} \mathbb{P}\{S_n \in F\} \leq - \inf I(x) + \delta, \\ e^{-(I(a)+\delta)n} &\leq \mathbb{P}\left\{\sum_{i=1}^n X_i \geq an\right\} \leq e^{-(I(a)-\delta)n} \end{aligned}$$

for all $n \geq N_\delta$, since I is non-decreasing. Note that this holds for any n large enough, and so in particular, even if n depends on the stochastic process.

4.2 A Dynamical Borel–Cantelli Lemma

To establish the strong dichotomy of the almost sure existence of extreme events, we will need a theorem slightly stronger than the second Borel–Cantelli lemma.

Let E_n be a sequence of events such that $\sum \mathbb{P}(E_n) = \infty$. If those events were independent, the second Borel–Cantelli lemma would assert that almost every $\omega \in \Omega$ is contained in $\omega \in E_n$ for infinitely many n , *i.e.*,

$$\mathbb{P}\left(\bigcap_{K=1}^{\infty} \bigcup_{k=K}^{\infty} E_k\right) = 1.$$

Since we will be dealing with events that are not independent, we will use a stronger version. Define the correlation by

$$\mathcal{R}_{n,m} = \mathbb{P}(E_n \cap E_m) - \mathbb{P}(E_n)\mathbb{P}(E_m).$$

The following theorem can be derived from the work of Sprindžuk [Sp79]; see also [CK01, Theorem 1.4].

Theorem 4.2 *Let E_n be a sequence of events such that $\sum \mathbb{P}(E_n) = \infty$. Assume there exists $C > 0$ such that*

$$\sum_{N \leq n, m \leq M} \mathcal{R}_{n,m} \leq C \sum_{i=N}^M \mathbb{P}(E_i)$$

for all $1 \leq N < M < \infty$. Then, for \mathbb{P} -almost every ω , $\omega \in E_n$ for infinitely many n .

Proof This is a direct application of [Sp79, §7, Lemma 10] with $f_k(\omega) = \chi_{E_k}(\omega)$, $f_k = \varphi_k = \mathbb{P}(E_k)$, and the conclusion that $\sum \chi_{E_k}(\omega)$ diverges. ■

4.3 Proof of Theorem 2.6: Self-similar Sets

Let $X_i = \log \mathfrak{S}_{\omega_i}^s = \log N_{\omega_i} c(\omega_i)^s$. Note that $\mathbb{E}(X_i) = 0$ and $\text{ess sup}|X_\lambda| < \infty$. Hence, the moment generating function of X_i is well defined for all θ , and we can apply

Cramér’s theorem. Let $r < R^{1+\varphi(R)} < R < R_0$, and set $k(R)$ and $k(r)$ such that

$$\prod_{i=1}^{k(R)} c(\omega_i) \sim R \quad \text{and} \quad \prod_{i=1}^{k(r)} c(\omega_i) \sim r.$$

Since $\phi(e^{-x})x$ is non-increasing we can, without loss of generality, take R_0 (depending on the realisation) small enough such that Cramér’s theorem holds for $k(r) - k(R)$. Thus, for all $\varepsilon > 0$,

$$\mathbb{P} \left\{ \sum_{i=k(R)}^{k(r)} X_i \geq \varepsilon(k(r) - k(R)) \right\} \leq e^{-(I(\varepsilon) - \delta)(k(r) - k(R))}.$$

Therefore, the probability $P(R)$ that there exists r satisfying $r < R^{1+\varphi(R)} < R < R_0$ for a given R is bounded by

$$\begin{aligned} P(R) &\leq \sum_{l=k(R^{1+\varphi(R)})}^{\infty} \mathbb{P} \left\{ \sum_{i=k(R)}^l X_i \geq \varepsilon(l - k(R)) \right\} \\ &\leq \sum_{l=k(R^{1+\varphi(R)})}^{\infty} e^{-\tau(l - k(R))} \lesssim e^{-\tau(k(R^{1+\varphi(R)}) - k(R))}, \end{aligned}$$

where we have written $\tau = I(\varepsilon) - \delta$ to ease notation. Without loss of generality, using Cramér’s theorem we can assume that R_0 is also chosen small enough such that

$$\mathbb{E}^g(c(\lambda))^{(1+\delta)(k(R^{1+\varphi(R)}) - k(R))} \leq \frac{k(R^{1+\varphi(R)})}{\prod_{i=k(R)} c(\omega_i)} \sim \frac{R^{1+\varphi(R)}}{R} = R^{\varphi(R)}.$$

Thus,

$$k(R^{1+\varphi(R)}) - k(R) \geq -\tau' \varphi \left(\prod_{i=1}^{k(R)} c(\omega_i) \right) \cdot \log \left(\prod_{i=1}^{k(R)} c(\omega_i) \right)$$

for some $\tau' > 0$. Now, for any given ω , the number of levels such that $\log(R) = n$ is uniformly bounded. Further, the number of products of $\prod c(\omega_i)$ that are comparable to e^{-n} is uniformly bounded. Therefore, the sum over the probabilities that there exists a ball $B(x, R)$ at level $k(R)$ such that X_i exceeds the mean by more than ε is bounded by

$$\begin{aligned} \sum_{\log R=1}^{\infty} P(R) &\leq C \sum_{n=1}^{\infty} e^{-\tau\tau' \varphi(R) \log 1/R} \\ &\leq C' \sum_{n=1}^{\infty} e^{-\tau\tau' \varphi(e^{-n})n} < C'' \sum_{n=1}^{\infty} e^{-\phi(e^{-n})n} < \infty. \end{aligned}$$

By the Borel–Cantelli lemma, this happens only finitely many times, almost surely. Finally, we can conclude that almost surely for small enough R (depending on the realisation) there are no pairs $r < R^{1+\varphi(R)}$ such that $\sum_{i=k(R)}^{k(r)} X_i > \varepsilon(k(r) - k(R))$. Then

$$\sum_{i=k(R)}^{k(r)} \log N_{\omega_i} c(\omega_i)^s \leq \varepsilon(k(r) - k(R)).$$

Observe that the number of r coverings is comparable to the number of descendants of the $B(x, R)$ cylinder. Therefore,

$$N_r(B(x, R) \cap F_\omega) \sim \prod_{i=k(R)}^{k(r)} N_{\omega_i} \lesssim \prod_{i=k(R)}^{k(r)} \frac{e^\varepsilon}{c(\omega_i)^s} \lesssim \left(\frac{R}{r}\right)^{s+\varepsilon'}$$

for some ε' such that $\varepsilon' \rightarrow 0$ as $\varepsilon \rightarrow 0$. Since ε was arbitrary, we have the desired conclusion for the first part.

We now prove the second half of the theorem. Recall that the almost sure Assouad dimension s_A of F_ω is given by $s_A = \sup_{\lambda \in \text{supp } \mu} \{-\log N_\lambda / \log c(\lambda)\}$. Let $\varepsilon > 0$ and take

$$T_\varepsilon = \left\{ \lambda \in \Lambda : \frac{-\log N_\lambda}{\log c(\lambda)} \geq s_A - \varepsilon \right\} \quad \text{and} \quad p_\varepsilon = \mu(T_\varepsilon).$$

Define $c_{\text{sup}} = \sup_\lambda c(\lambda)$ and $c_{\text{inf}} = \inf_\lambda c(\lambda)$. Let $\psi(n) = \varphi((c_{\text{sup}})^n)\gamma$, where φ is given as $\varphi(R) = o(\phi(R))$ and $\gamma = \log c_{\text{inf}} / \log c_{\text{sup}}$. Recall that ψ is non-increasing and consider the events

$$E_n = \{ \omega \in \Omega : \omega_i \in T_\varepsilon \text{ for } n \leq i < n + \psi(n)n \}.$$

Clearly, $\mathbb{P}(E_n) = p_\varepsilon^{\psi(n)n}$. The event $E_n \cap E_m$ for $n \leq m$ has probability

$$\mathbb{P}(E_n \cap E_m) = p_\varepsilon^{\psi(m)m+m-n}$$

due to the overlap of $[n, \psi(n)n] \cap [m, \psi(m)m]$. The correlation is

$$\mathcal{R}_{n,m} = \mathbb{P}(E_n \cap E_m) - \mathbb{P}(E_n)\mathbb{P}(E_m) = p_\varepsilon^{\psi(m)m+m-n} - p_\varepsilon^{\psi(n)n+\psi(m)m}.$$

Therefore,

$$\begin{aligned} \sum_{N \leq n, m \leq M} \mathcal{R}_{n,m} &\leq 2 \sum_{m=N}^M \sum_{n=N}^m \mathcal{R}_{n,m} \leq 2 \sum_{m=N}^M p_\varepsilon^{\psi(m)m} \sum_{n=N}^m p_\varepsilon^{m-n} \\ &\leq 2 \sum_{m=N}^M p_\varepsilon^{\psi(m)m} \sum_{i=0}^\infty p_\varepsilon^i \leq C_\varepsilon \sum_{m=N}^M p_\varepsilon^{\psi(m)m} \leq C_\varepsilon \sum_{m=N}^M \mathbb{P}(E_m) \end{aligned}$$

for all $1 \leq N < M < \infty$. To use Theorem 4.2, it remains to check divergence the latter sum. As we are in the diverging case, $\varphi(x) \rightarrow 0$ as $x \rightarrow 0$ and $\varphi(x) = o(\phi(x))$ as $x \rightarrow 0$. Then

$$\begin{aligned} \sum_{n=1}^\infty \mathbb{P}(E_m) &= \sum_{n=1}^\infty p_\varepsilon^{\psi(n)n} = \sum_{n=1}^\infty e^{-\varphi((c_{\text{sup}})^n)\gamma n \log(1/p_\varepsilon)} \\ (4.1) \quad &= \sum_{n=1}^\infty \exp\left(-\varphi(e^{-n \log(1/c_{\text{sup}})})n \log(1/c_{\text{sup}})\gamma \frac{\log(1/p_\varepsilon)}{\log(1/c_{\text{sup}})}\right) \end{aligned}$$

$$(4.2) \quad \gtrsim \sum_{n=1}^\infty \exp\left(-\phi(e^{-n \log(1/c_{\text{sup}})})n \log(1/c_{\text{sup}})\right)$$

$$(4.3) \quad \sim \sum_{n=1}^\infty \exp\left(-\phi(e^{-n})n\right) = \infty,$$

where we have used the integration test and the substitution rule to obtain (4.3). We have obtained (4.2) by $\varphi(x) = o(\phi(x))$ to combat the final fraction in (4.1). However, if p_ε is bounded away from 0 as $\varepsilon \rightarrow 0$, we can sharpen this to $\varphi(x) \leq C\phi(x)$ by

taking $\varepsilon = 0$ and using the bound on p_ε . This can happen when there exists $\Lambda' \subset \Lambda$ with $\mu(\Lambda') > 0$ that maximises $-\log N_\lambda / \log c(\lambda)$, i.e., when $\mathfrak{S}_\lambda^{s_A} = 1$ for all $\lambda \in \Lambda'$.

Application of Theorem 4.2 gives us that $\omega \in E_n$ for infinitely many n , almost surely. That is, given a generic $\omega \in \Omega$, there are infinitely many n such that $\omega_k \in T_\varepsilon$ for $n \leq k \leq \psi(n)n$. Therefore, considering the ball $f_{\omega|n}(\Delta) \cap F_\omega$ of diameter $R \sim \prod_{k=1}^n c(\omega_k)$, we can use the fact that the interiors are separated and standard arguments (see e.g., [Tr19, Lemma 3.2]) to claim that this ball must be covered by at least $C \prod_{k=n}^{(1+\psi(n))n} N_{\omega_k}$ many balls of radius $r \sim \prod_{k=1}^{(1+\psi(n))n} c(\omega_k)$. Therefore, there exist x, r, R such that

$$N_r(B(x, R) \cap F_\omega) \gtrsim \prod_{k=n}^{(1+\psi(n))n} N_{\omega_k} \geq \prod_{k=n}^{(1+\psi(n))n} c(\omega_k)^{-(s-\varepsilon)} \sim \left(\frac{R}{r}\right)^{s_A-\varepsilon}.$$

Finally, we check that $r \lesssim R^{1+\varphi(R)}$. This is equivalent to verifying that $r/R \lesssim R^{\varphi(R)}$, which we can readily check by the estimates obtained above:

$$\begin{aligned} \frac{r}{R} &\sim \prod_{k=n}^{(1+\psi(n))n} c(\omega_k) \leq (c_{\text{sup}})^{\psi(n)n} = \exp\left(\varphi((c_{\text{sup}})^n)n\gamma \log(1/c_{\text{sup}})\right) \\ &= (c_{\text{inf}})^{\varphi((c_{\text{sup}})^n)n} \leq R^{\varphi(R)}. \end{aligned}$$

Therefore, $\dim_\Lambda^\varphi F_\omega \geq s_A - \varepsilon$ almost surely. Since $\varepsilon > 0$ was arbitrary (or can in cases be chosen to be 0), we obtain the required result. ■

4.4 Proof of Theorem 2.7 Bedford–McMullen Carpets

We define the random variables $k_1^\omega(R), k_2^\omega(R)$ as the levels when the rectangles in the construction have base length R and height R , respectively, that is,

$$\prod_{i=1}^{k_2^\omega(R)} n_{\omega_i}^{-1} \sim R \quad \text{and} \quad \prod_{i=1}^{k_1^\omega(R)} m_{\omega_i}^{-1} \sim R.$$

It follows from the estimates in [FT18] that

$$(4.4) \quad N_{R^{1+\varphi(R)}}(B(x, R) \cap F_\omega) \sim \prod_{l=k_2^\omega(R)}^{k_2^\omega(R^{1+\varphi(R)})} C_{\omega_l} \prod_{l=k_1^\omega(R)}^{k_1^\omega(R^{1+\varphi(R)})} B_{\omega_l}.$$

Let $X_i = \log C_{\omega_i} - \log \bar{C}$ and $Y_i = \log B_{\omega_i} - \log \bar{B}$, where $\bar{C} = \mathbb{E}^{\mathcal{G}}(C_\lambda)$ and $\bar{B} = \mathbb{E}^{\mathcal{G}}(B_\lambda)$. As in the self-similar case, we have $E(X_i) = E(Y_i) = 0$, and due to the finiteness of Λ , the moment generating function exists for all θ . Hence we can apply Cramér’s theorem. Let $r < R^{1+\varphi(R)} < R < R_0$, where R_0 is chosen small enough such that Cramér’s theorem holds for $k_i^\omega(R) - k_i^\omega(r)$, ($i = 1, 2$). Thus, for all $\varepsilon > 0$,

$$\begin{aligned} &\mathbb{P}\left\{\sum_{i=k_2^\omega(R)}^{k_2^\omega(r)} X_i + \sum_{i=k_1^\omega(R)}^{k_1^\omega(r)} Y_i \geq \varepsilon(k_2^\omega(r) - k_2^\omega(R) + k_1^\omega(r) - k_1^\omega(R))\right\} \leq \\ &\mathbb{P}\left\{\sum_{i=k_2^\omega(R)}^{k_2^\omega(r)} X_i \geq \varepsilon(k_2^\omega(r) - k_2^\omega(R))\right\} + \mathbb{P}\left\{\sum_{i=k_1^\omega(R)}^{k_1^\omega(r)} Y_i \geq \varepsilon(k_1^\omega(r) - k_1^\omega(R))\right\}. \end{aligned}$$

Applying Cramér’s theorem to both probabilities and analogous to the self-similar case, we obtain

$$\sum_{i=k_2^\omega(R)}^{k_2^\omega(r)} X_i + \sum_{i=k_1^\omega(R)}^{k_1^\omega(r)} Y_i < \varepsilon \left(k_2^\omega(r) - k_2^\omega(R) + k_1^\omega(r) - k_1^\omega(R) \right).$$

almost surely, for all $r < R^{1+\varphi(R)}$ with R small enough (and depending on the realisation). Thus, following the same argument as in [FT18], and using (4.4), gives

$$\begin{aligned} N_{R^{1+\varphi(R)}}(B(x, R) \cap F_\omega) &\lesssim \overline{C}^{(1+\varepsilon')(k_2^\omega(R^{1+\varphi(R)})-k_2^\omega(R))} \overline{B}^{(1+\varepsilon')(k_1^\omega(R^{1+\varphi(R)})-k_1^\omega(R))} \\ &\lesssim \left(\frac{R}{r}\right)^{(1+\varepsilon'')s_q}, \end{aligned}$$

where $s_q = \text{ess dim}_{qA} F_\omega$ and $\varepsilon'' \rightarrow 0$ as $\varepsilon \rightarrow 0$. As $\varepsilon > 0$ was arbitrary, we conclude that $\dim_A^\varphi F_\omega = \dim_{qA} F_\omega$ almost surely. This finishes the first part.

For the second part, recall that the almost sure Assouad dimension s_A of F_ω is given by

$$s_A = \max_{\lambda \in \Lambda} \frac{\log B_\lambda}{\log m_\lambda} + \max_{\lambda \in \Lambda} \frac{\log C_\lambda}{\log n_\lambda}.$$

Without loss of generality, assume the first summand is maximised by $1 \in \Lambda$, whereas the second is maximised by $2 \in \Lambda$ (where we can identify $1 \sim 2$ if necessary). Define $\psi(j) = \varphi(n_{\min}^j) \log n_2 / \log n_{\max}$, where $\varphi(x) \leq \gamma \phi(x)$ with

$$\gamma = \frac{\log n_{\min} \log n_{\max}}{(\log(1/p_2) + \kappa \log(1/p_1)) \log n_2} \quad \text{and} \quad \kappa = \log n_2 / \log m_1.$$

Consider the events

$$E_l = \left\{ \omega \in \Omega : \omega_i = 2 \text{ for } l \leq i < \psi(l)l \text{ and } \omega_i = 1 \text{ for } l' \leq i < l' + \kappa\psi(l)l \right. \\ \left. \text{where } l' = k_1^\omega(R) \text{ and } R \text{ is the least value satisfying } k_2^\omega(R) = l \right\}.$$

Almost surely, $k_1^\omega(R) - k_2^\omega(R) \gg \psi(l)l$ for large enough l . Therefore E_l consists of two fixed strings of letters 2 and 1 of lengths $\psi(l)l$ and $\kappa\psi(l)l$, respectively, that do not overlap. This further gives $\mathbb{P}(E_l) \sim p_2^{\psi(l)l} p_1^{\kappa\psi(l)l}$. Considering an $\omega \in E_l \cap E_j$ for $j \leq l$ there are only two cases that appear (with large probability):

- $[j, j + \psi(j)j]$ intersects $[l, l + \psi(l)l]$ and $[j', j' + \kappa\psi(j)j]$ intersects $[l', l' + \kappa\psi(l)l]$;
- $[j', j' + \psi(j)j]$ intersects $[l, l + \psi(l)l]$.

In the first case, the probability is given by

$$(4.5) \quad \mathbb{P}(E_j \cap E_l) \sim p_2^{\psi(l)l+j} p_1^{\kappa\psi(l)l+l'-j} \leq p_2^{\psi(l)l+l-j} p_1^{\kappa\psi(l)l+\tau(l-j)} \sim \mathbb{P}(E_l) p_2^{l-j} p_1^{\tau(l-j)}.$$

The estimate that $l - j \lesssim l' - j'$ arises from the observation that the R_l associated with l is related to R_j by $R_l/R_j \leq n_{\min}^{-(l-j)}$ and $R_l/R_j \geq m_{\max}^{-(l'-j')}$. This gives $\tau \geq \log n_{\min} / \log m_{\max}$. Note further that the last term in (4.5) implies uniform summability over $1 \leq j \leq l$.

The second case can only occur if the maximal letters are identical, that is, $1 \equiv 2$ and $p_1 = p_2$. This gives

$$\mathbb{P}(E_j \cap E_l) \sim p_2^{\kappa\psi(l)l + \psi(l)l - j' + \psi(j)j} \sim \mathbb{P}(E_l)p_2^{l - j' + \psi(j)j},$$

which is also uniformly summable over $1 \leq j \leq l$.

We can conclude that for any $1 \leq J \leq l$,

$$\sum_{j=J}^l \mathcal{R}_{j,l} \leq \sum_{j=J}^l \mathbb{P}(E_j \cap E_l) \leq \sum_{j=1}^l \mathbb{P}(E_j \cap E_l) \lesssim \mathbb{P}(E_l),$$

and so for $1 \leq J \leq L$,

$$\sum_{J \leq j, l \leq L} \mathcal{R}_{j,l} \leq 2 \sum_{l=J}^L \sum_{j=J}^l \mathbb{P}(E_j \cap E_l) \lesssim \sum_{l=J}^L \mathbb{P}(E_l).$$

To use Theorem 4.2, we need to show that $\sum \mathbb{P}(E_j) = \infty$. This is similar to proving divergence in Theorem 2.6:

$$\begin{aligned} \sum_{j=1}^{\infty} \mathbb{P}(E_j) &\sim \sum_{j=1}^{\infty} p_2^{\psi(l)l} p_1^{\kappa\psi(l)l} = \sum_{j=1}^{\infty} \exp(-(\log(1/p_2) + \kappa \log(1/p_1))\psi(l)l) \\ &= \sum_{j=1}^{\infty} \exp(-\phi((n_{\min})^{-j})j \log n_{\min}) \\ &\sim \sum_{j=1}^{\infty} \exp(-\phi(e^{-j})j) = \infty. \end{aligned}$$

Hence the conditions of Theorem 4.2 are satisfied and E_l happens infinitely often.

As $\omega \in E_j$ for infinitely many j , almost surely, we have $k_2^\omega(R) = j$ and $k_1^\omega(R) = j' > (1 + \psi(j))j$ for arbitrarily large j . Then, by the definitions of k_2 and ψ , we have $R^{1+\varphi(R)} \geq R n_2^{-\psi(j)j}$ and $R^{\varphi(R)} \geq n_2^{-\psi(j)j}$. Similarly, $R^{\varphi(R)} \geq m_1^{-\kappa\psi(j)j} = n_2^{-\psi(j)j}$, and thus by the estimate (4.4),

$$\begin{aligned} N_{R^{1+\varphi(R)}}(B(x, R) \cap F_\omega) &\sim \prod_{l=k_2^\omega(R)}^{k_2^\omega(R^{1+\varphi(R)})} C_{\omega_l} \prod_{l=k_1^\omega(R)}^{k_1^\omega(R^{1+\varphi(R)})} B_{\omega_l} \\ &= C_2^{k_2^\omega(R^{1+\varphi(R)}) - k_2^\omega(R)} B_1^{k_1^\omega(R^{1+\varphi(R)}) - k_1^\omega(R)} \\ &= R^{-\varphi(R)(\log C_2 / \log n_2 + \log B_1 / \log m_1)} \end{aligned}$$

for infinitely many pairs (x_i, R_i) almost surely. Therefore, the almost sure generalised Assouad spectrum with respect to φ is equal to the almost sure Assouad dimension. ■

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